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Finite element approximation of elliptic control problems with constraints on the gradient

Klaus Deckelnick*, Andreas Günther† & Michael Hinze‡

Abstract: We consider an elliptic optimal control problem with control constraints and pointwise bounds on the gradient of the state. We present a tailored finite element approximation to this optimal control problem, where the cost functional is approximated by a sequence of functionals which are obtained by discretizing the state equation with the help of the lowest order Raviart–Thomas mixed finite element. Pointwise bounds on the gradient variable are enforced in the elements of the triangulation. Controls are not discretized. Error bounds for control and state are obtained in two and three space dimensions. A numerical example confirms our analytical findings.

Mathematics Subject Classification (2000): 49J20, 49K20, 35B37

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1 Introduction

In steel and glass production cooling of melts forms a critical process. In order to accelerate the production process it is highly desirable to speed up the cooling processes while avoiding damage of the products caused by large material stresses. In model based optimization, cooling processes frequently are described by systems of partial differential equations involving the temperature as a system variable, so that large (Von Mises) stresses in the optimization process can be avoided by imposing pointwise bounds on the gradient of the temperature. To solve these kinds of optimization problems numerically it is necessary to use derivative based optimization methods which make use of adjoint variables. This fact then necessitates the development of tailored discrete concepts which take into account the low regularity of adjoint variables and multipliers involved in the optimality conditions of the underlying optimization problem.

In the present work we consider a model problem which involves the optimal control of a linear elliptic pde in the presence of pointwise bounds on the controls and on the gradient of the state. Our aim is to develop and to analyze a finite element concept which is tailored to the numerical treatment of pointwise bounds, and at the same time is able to cope with the low regularity of multipliers. To this purpose we propose an approximation of the state equation using the lowest order Raviart–Thomas mixed finite element, while controls are not

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discretized explicitely, but implicitly through the optimality conditions associated with the discrete approximation to the optimal control problem. Our main result reads

$$\|u - u_h\| + \|y - y_h\| \leq Ch^{\frac{1}{2}} |\log h|^{\frac{1}{2}},$$

and is proved in Theorem 4.1. Here, y, u and y_h, u_h denote the unique solutions of the optimal control problems (2.4) and (3.6), respectively.
Let us briefly comment on related literature. In [3] Casas and Fernandez investigate optimal control of semilinear elliptic pdes with pointwise constraints on the gradient of the state. They provide a complete analysis including results on the structure and on the regularity of multipliers. Numerical analysis for general semilinear elliptic control problems involving finitely many state constraints is provided by Casas and Mateos in [4]. A finite element analysis for elliptic optimal control problems with pointwise bounds on the state is presented by the first and third author in [5], and is extended to the case of general constraints on the control and pointwise constraints on the state in [6]. Meyer in [10] presents a finite element analysis for elliptic optimal control problems in the presence of pointwise bounds on the control and state, where he investigates piecewise constant approximations of the control.
To the best of the authors knowledge this is the first contribution to finite element analysis for elliptic control problems with pointwise bounds on the gradient of the state.

2 Mathematical setting

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with a smooth boundary $\partial\Omega$ and consider the differential operator

$$\mathcal{A}y := -\sum_{i,j=1}^{d} \partial_{x_j}\left(a_{ij} y_{x_i}\right) + a_0 y,$$

where for simplicity the coefficients a_{ij} and a_0 are assumed to be smooth functions on $\bar{\Omega}$. In what follows we assume that $a_{ij} = a_{ji}$, $a_0 \geq 0$ in Ω and that there exists $c_0 > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d \text{ and all } x \in \Omega.$$

From the above assumptions we infer that for a given $u \in L^r(\Omega)$ $(1 < r < \infty)$ the elliptic boundary value problem

$$\begin{aligned} \mathcal{A}y &= u \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

has a unique solution $y \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$. Furthermore,

$$\|y\|_{W^{2,r}} \leq C\|u\|_{L^r}, \tag{2.2}$$

where $\|\cdot\|_{L^r}$ and $\|\cdot\|_{W^{k,r}}$ denote the usual Lebesgue and Sobolev norms. By tracing the dependence on r in the above inequality it is possible to prove that

$$\|y\|_{W^{2,r}} \leq Cr\|u\|_{L^\infty}, \tag{2.3}$$

where C is independent of r.

Next, we formulate the control problem to be considered. Let $\alpha > 0$ and $y_0 \in L^2(\Omega)$ be given and consider

$$\begin{aligned} \min_{u \in K} J(u) = \frac{1}{2}\int_\Omega |y - y_0|^2 + \frac{\alpha}{2}\int_\Omega |u|^2 \\ \text{where } y \text{ solves (2.1) and } \nabla y \in C. \end{aligned} \tag{2.4}$$

Here,

$$K = \{u \in L^2(\Omega) \,|\, a \le u \le b \text{ a.e. in } \Omega\}, \quad C = \{\mathbf{z} \in C^0(\bar{\Omega})^d \,|\, |\mathbf{z}(x)| \le \delta, x \in \bar{\Omega}\},$$

where $a < b$, $\delta > 0$ are constants and $|\cdot|$ denotes the Euclidian norm in $\mathbb{R}^d$. Note that, since $K \subset L^r(\Omega)$ for $r > d$ we have $y \in W^{2,r}(\Omega)$ and hence $\nabla y \in C^0(\bar{\Omega})^d$ by a well–known embedding result.
Finally we suppose that the following Slater condition holds:

$$\exists \hat{u} \in K \quad |\nabla \hat{y}(x)| < \delta,\, x \in \bar{\Omega} \qquad \text{where } \hat{y} \text{ solves (2.1) with } u = \hat{u}. \tag{2.5}$$

Since $\hat{u}$ is feasible for (2.4) we deduce from Theorem 3 in [3], that the above control problem has a unique solution $u \in K$. In order to formulate the optimality conditions we introduce $\mathcal{M}(\bar{\Omega})$ as the space of regular Borel measures, the dual space of $C^0(\bar{\Omega})$. The norm on $\mathcal{M}(\bar{\Omega})$ is given by

$$\|\mu\|_{\mathcal{M}(\bar{\Omega})} = \sup_{f \in C^0(\bar{\Omega}), |f| \le 1} \int_{\bar{\Omega}} f \, d\mu.$$

Theorem 2.1. *An element $u \in K$ is a solution of* (2.4) *if and only if there exist $\boldsymbol{\mu} \in \mathcal{M}(\bar{\Omega})^d$ and $p \in L^t(\Omega)$ $(t < \frac{d}{d-1})$ such that*

$$\int_\Omega p \mathcal{A} z = \int_\Omega (y - y_0) z + \int_{\bar{\Omega}} \nabla z \cdot d\boldsymbol{\mu} \qquad \forall z \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega) \tag{2.6}$$

$$\int_\Omega (p + \alpha u)(\tilde{u} - u) \ge 0 \qquad \forall \tilde{u} \in K \tag{2.7}$$

$$\int_{\bar{\Omega}} (\mathbf{z} - \nabla y) \cdot d\boldsymbol{\mu} \le 0 \qquad \forall \mathbf{z} \in C. \tag{2.8}$$

Here, y is the solution of (2.1) and $\frac{1}{t} + \frac{1}{t'} = 1$.

Remark 2.2. Lemma 1 in [3] shows that the vector valued measure $\boldsymbol{\mu}$ appearing in Theorem 2.1 can be written in the form

$$\boldsymbol{\mu} = \frac{1}{\delta} \nabla y \, \mu,$$

where $\mu \in \mathcal{M}(\bar{\Omega})$ is a nonnegative measure that is concentrated in the set $\{x \in \bar{\Omega} \,|\, |\nabla y(x)| = \delta\}$.

Our aim is to develop and analyze a finite element approximation of problem (2.4). We start by approximating the cost functional J by a sequence of functionals J_h where h is a mesh parameter related to a sequence of triangulations. Since p has very little regularity we propose to use a mixed finite element method based on the Raviart–Thomas element of lowest order. It is a specialty of our approach that it avoids explicit discretization of the controls. This procedure is motivated by the fact that the structure of the discrete analogue to (2.7) already induces a discrete structure on the control through the discretization of the adjoint state p, compare Remark 3.4.

3 Finite element discretization

It is well–known that (2.1) can be written in mixed formulation. To this purpose we introduce $H(\mathrm{div},\Omega) := \{\mathbf{w} \in L^2(\Omega)^d \,|\, \mathrm{div}\mathbf{w} \in L^2(\Omega)\}$ and denote $\mathbf{v} = A\nabla y$, where $A(x) = \big(a_{ij}(x)\big)_{i,j=1}^d$. Then $(y,\mathbf{v})$ satisfies

$$\int_\Omega A^{-1}\mathbf{v}\cdot\mathbf{w} + \int_\Omega y\,\mathrm{div}\mathbf{w} \quad = \quad 0 \qquad \forall \mathbf{w} \in H(\mathrm{div},\Omega) \tag{3.1}$$

$$\int_\Omega z\,\mathrm{div}\mathbf{v} - \int_\Omega a_0 y\, z + \int_\Omega u\, z \quad = \quad 0 \qquad \forall z \in L^2(\Omega). \tag{3.2}$$

In what follows it will be convenient to write $(y,\mathbf{v}) = \mathcal{G}(u)$ for the solution of (3.1), (3.2). Next, let $\mathcal{T}_h$ be a triangulation of Ω with maximum mesh size $h := \max_{T\in\mathcal{T}_h} \mathrm{diam}(T)$. We suppose that $\bar{\Omega}$ is the union of the elements of $\mathcal{T}_h$; boundary elements are allowed to have one curved face. In addition, we assume that the triangulation is quasi-uniform in the sense that there exists a constant $\kappa > 0$ (independent of h) such that each $T \in \mathcal{T}_h$ is contained in a ball of radius $\kappa^{-1}h$ and contains a ball of radius κh. As already mentioned above we use a mixed finite element method based on the lowest order Raviart–Thomas element. Let

$$\mathbf{V_h} := RT_0(\Omega,\mathcal{T}_h) := \{\mathbf{w_h} \in H(\mathrm{div},\Omega) \,|\, \mathbf{w_h}_{|T} \in RT_0(T) \text{ for all } T \in \mathcal{T}_h\},$$

where $RT_0(T) = \{\mathbf{w} : T \to \mathbb{R}^d \,|\, \mathbf{w}(x) = a + \beta x, a \in \mathbb{R}^d, \beta \in \mathbb{R}\}$. Furthermore, let

$$Y_h := \{z_h \in L^2(\Omega) \,|\, z_h \text{ is constant on each } T \in \mathcal{T}_h\}.$$

The variational formulation (3.1), (3.2) gives rise to the following discrete approximation of $\mathcal{G}$. For a given function $u \in L^2(\Omega)$ let $(y_h,\mathbf{v_h}) = \mathcal{G}_h(u) \in Y_h \times \mathbf{V_h}$ be the solution of

$$\int_\Omega A^{-1}\mathbf{v_h}\cdot\mathbf{w_h} + \int_\Omega y_h\,\mathrm{div}\mathbf{w_h} \quad = \quad 0 \qquad \forall \mathbf{w_h} \in \mathbf{V_h} \tag{3.3}$$

$$\int_\Omega z_h\,\mathrm{div}\mathbf{v_h} - \int_\Omega a_0 y_h\, z_h + \int_\Omega u\, z_h \quad = \quad 0 \qquad \forall z_h \in Y_h. \tag{3.4}$$

It is well–known ([2]) that the difference between $(y,\mathbf{v}) = \mathcal{G}(u)$ and $(y_h,\mathbf{v_h}) = \mathcal{G}_h(u)$ can be estimated as follows:

$$\|y - y_h\| + \|\mathbf{v} - \mathbf{v_h}\| \le Ch\big(\|y\|_{H^1} + \|A\nabla y\|_{H^1}\big) \le Ch\|y\|_{H^2} \le Ch\|u\| \tag{3.5}$$

by (2.2). In what follows it will be crucial to control the error between $\mathbf{v}$ and $\mathbf{v_h}$ in $L^\infty(\Omega)$.

Lemma 3.1. *Let $(y,\mathbf{v}) = \mathcal{G}(u)$ and $(y_h,\mathbf{v_h}) = \mathcal{G}_h(u)$. Then*

$$\|y - y_h\|_{L^\infty} + \|\mathbf{v} - \mathbf{v_h}\|_{L^\infty} \le Ch|\log h|\|u\|_{L^\infty}.$$

Proof. see [9], Corollary 5.5, where the result is proved for the model problem $a_{ij} = \delta_{ij}$ and $a_0 = 0$, but it can be extended to the general case using techniques developed in [8]. ∎

Remark 3.2. More recently, localized pointwise error estimates for general second order elliptic equations on smooth domains were proved in [7].

Next define

$$C_h := \{\mathbf{c_h} : \bar{\Omega} \to \mathbb{R}^d \,|\, \mathbf{c_h}_{|T} \text{ is constant and } |\mathbf{c_h}_{|T}| \le \delta,\, T \in \mathcal{T}_h\}.$$

We approximate (2.4) by the following control problem depending on the mesh parameter h:

$$\begin{aligned} &\min_{u\in K} J_h(u) := \frac{1}{2}\int_\Omega |y_h - y_0|^2 + \frac{\alpha}{2}\int_\Omega |u|^2 \\ &\text{subject to } (y_h, \mathbf{v_h}) = \mathcal{G}_h(u) \text{ and } \Big(⨍_T A^{-1}\mathbf{v_h}\Big)_{T\in\mathcal{T}_h} \in C_h. \end{aligned} \tag{3.6}$$

Here, $⨍_T \cdot = \frac{1}{|T|}\int_T \cdot$. We note that the control is not discretized in (3.6). This problem represents a convex infinite–dimensional optimization problem of similar structure as problem (2.4), but with only finitely many constraints on the state.

Lemma 3.3. *There exists $h_0 > 0$ such that problem (3.6) has a unique solution $u_h \in K$ for $0 < h \le h_0$. Furthermore, there are $\boldsymbol{\mu}_T \in \mathbb{R}^d, T \in \mathcal{T}_h$ and $(p_h, \boldsymbol{\chi}_h) \in Y_h \times \mathbf{V_h}$ such that with $(y_h, \mathbf{v_h}) = \mathcal{G}_h(u_h)$ we have*

$$\int_\Omega A^{-1}\boldsymbol{\chi}_h \cdot \mathbf{w_h} + \int_\Omega p_h\, div\mathbf{w_h} + \sum_{T\in\mathcal{T}_h} \boldsymbol{\mu}_T \cdot ⨍_T A^{-1}\mathbf{w_h} = 0 \qquad \forall \mathbf{w_h} \in \mathbf{V_h} \tag{3.7}$$

$$\int_\Omega z_h\, div\boldsymbol{\chi}_h - \int_\Omega a_0 p_h\, z_h + \int_\Omega (y_h - y_0)\, z_h = 0 \qquad \forall z_h \in Y_h. \tag{3.8}$$

$$\int_\Omega (p_h + \alpha u_h)(\tilde{u} - u_h) \ge 0 \qquad \forall \tilde{u} \in K \tag{3.9}$$

$$\sum_{T\in\mathcal{T}_h} \boldsymbol{\mu}_T \cdot \big(\mathbf{c}_{\mathbf{h}|T} - ⨍_T A^{-1}\mathbf{v_h}\big) \le 0 \qquad \forall \mathbf{c_h} \in C_h. \tag{3.10}$$

Proof. We first prove that $\hat{u}$ from (2.5) is feasible for (3.6). Let $(\hat{y}, \hat{\mathbf{v}}) = \mathcal{G}(\hat{u})$ and $(\hat{y}_h, \hat{\mathbf{v}}_\mathbf{h}) = \mathcal{G}_h(\hat{u})$. For $T \in \mathcal{T}_h$ we deduce with the help of Lemma 3.1 and (2.5)

$$\begin{aligned} |⨍_T A^{-1}\hat{\mathbf{v}}_\mathbf{h}| &\le |⨍_T A^{-1}(\hat{\mathbf{v}}_\mathbf{h} - \hat{\mathbf{v}})| + |⨍_T A^{-1}\hat{\mathbf{v}}| \\ &\le C\|\hat{\mathbf{v}} - \hat{\mathbf{v}}_\mathbf{h}\|_{L^\infty} + \max_{x\in\bar\Omega} |\nabla \hat{y}(x)| \\ &\le Ch|\log h| + \max_{x\in\bar\Omega} |\nabla \hat{y}(x)| \le (1-\epsilon)\delta, \end{aligned} \tag{3.11}$$

for some $\epsilon > 0$ and $0 < h \le h_0$, so that $\Big(⨍_T A^{-1}\hat{\mathbf{v}}_\mathbf{h}\Big)_{T\in\mathcal{T}_h} \in C_h$. The result now follows from [3, Theorem 7] with the choices $U = L^2(\Omega)$, $K \subset U$ and $C_h \subset Z := \mathbb{R}^{N_h} \times \mathbb{R}^d$, where N_h is the number of triangles in $\mathcal{T}_h$. ∎

Remark 3.4. We deduce from (3.9) that $u_h = P_K\big(-\frac{p_h}{\alpha}\big)$, where P_K denotes the orthogonal projection in $L^2(\Omega)$ onto K. The structure of K then yields $u_h(x) = P_{[a,b]}\big(-\frac{p_h(x)}{\alpha}\big)$ for $x \in \Omega$, where $P_{[a,b]}$ denotes the pointwise projection onto the interval $[a, b]$. Hence, the discrete solution is also a piecewise constant function.

Similarly to Remark 2.2 we have

Lemma 3.5. *The multiplier $(\boldsymbol{\mu}_T)_{T\in\mathcal{T}_h}$ satisfies*

$$\boldsymbol{\mu}_T = |\boldsymbol{\mu}_T|\frac{1}{\delta}⨍_T A^{-1}\mathbf{v_h}, \quad T \in \mathcal{T}_h.$$

Proof. Fix $T \in \mathcal{T}_h$. The assertion is clear if $\boldsymbol{\mu}_T = 0$. Suppose that $\boldsymbol{\mu}_T \neq 0$ and define $\mathbf{c_h} : \bar{\Omega} \to \mathbb{R}^d$ by

$$\mathbf{c_h}_{|\tilde{\mathbf{T}}} := \begin{cases} \fint_{\tilde{T}} A^{-1}\mathbf{v_h}, & \tilde{T} \neq T, \\ \delta \frac{\boldsymbol{\mu}_T}{|\boldsymbol{\mu}_T|}, & \tilde{T} = T. \end{cases}$$

Clearly, $\mathbf{c_h} \in C_h$ so that (3.10) implies

$$\boldsymbol{\mu}_T \cdot \big(\delta \frac{\boldsymbol{\mu}_T}{|\boldsymbol{\mu}_T|} - \fint_T A^{-1}\mathbf{v_h}\big) \leq 0,$$

and therefore

$$\delta|\boldsymbol{\mu}_T| \leq \boldsymbol{\mu}_T \cdot \fint_T A^{-1}\mathbf{v_h} \leq \delta|\boldsymbol{\mu}_T|.$$

Hence we obtain $\dfrac{\boldsymbol{\mu}_T}{|\boldsymbol{\mu}_T|} = \dfrac{1}{\delta}\fint_T A^{-1}\mathbf{v_h}$ and the lemma is proved. ∎

As a consequence of Lemma 3.5 we immediately infer that

$$|\boldsymbol{\mu}_T| = \boldsymbol{\mu}_T \cdot \frac{1}{\delta}\fint_T A^{-1}\mathbf{v_h}, \qquad T \in \mathcal{T}_h. \tag{3.12}$$

We now use (3.12) in order to derive an important a–priori estimate.

Lemma 3.6. *Let $u_h \in K$ be the optimal solution of (3.6) with corresponding state $(y_h, \mathbf{v_h}) \in Y_h \times \mathbf{V_h}$ and adjoint variables $(p_h, \boldsymbol{\chi}_h) \in Y_h \times \mathbf{V_h}$, $\boldsymbol{\mu}_T, T \in \mathcal{T}_h$. Then*

$$\|y_h\|, \sum_{T \in \mathcal{T}_h} |\boldsymbol{\mu}_T| \leq C \qquad \text{for all } 0 < h \leq h_0.$$

Proof. Combining (3.12) with (3.11) we deduce

$$\boldsymbol{\mu}_T \cdot \fint_T A^{-1}(\mathbf{v_h} - \hat{\mathbf{v}}_\mathbf{h}) \geq \delta|\boldsymbol{\mu}_T| - (1-\epsilon)\delta\,|\boldsymbol{\mu}_T| = \epsilon\delta|\boldsymbol{\mu}_T|.$$

Choosing $\mathbf{w_h} = \mathbf{v_h} - \hat{\mathbf{v}}_\mathbf{h}$ in (3.7) and using the symmetry of A as well as the definition of $\mathcal{G}_h$ we hence obtain

$$\begin{aligned} \epsilon\delta \sum_{T \in \mathcal{T}_h} |\boldsymbol{\mu}_T| &\leq \sum_{T \in \mathcal{T}_h} \boldsymbol{\mu}_T \cdot \fint_T A^{-1}(\mathbf{v_h} - \hat{\mathbf{v}}_\mathbf{h}) \\ &= -\int_\Omega A^{-1}\boldsymbol{\chi}_h \cdot (\mathbf{v_h} - \hat{\mathbf{v}}_\mathbf{h}) - \int_\Omega p_h \operatorname{div}(\mathbf{v_h} - \hat{\mathbf{v}}_\mathbf{h}) \\ &= \int_\Omega (y_h - \hat{y}_h)\mathrm{div}\boldsymbol{\chi}_h - \int_\Omega a_0(y_h - \hat{y}_h)p_h + \int_\Omega (u_h - \hat{u})p_h. \end{aligned}$$

If we use $z_h = y_h - \hat{y}_h$ in (3.8) and $\tilde{u} = \hat{u}$ in (3.9) we finally deduce

$$\begin{aligned} \epsilon\delta \sum_{T \in \mathcal{T}_h} |\boldsymbol{\mu}_T| &\leq -\int_\Omega (y_h - y_0)(y_h - \hat{y}_h) + \alpha \int_\Omega u_h(\hat{u} - u_h) \\ &\leq -\frac{1}{2}\int_\Omega y_h^2 - \frac{\alpha}{2}\int_\Omega u_h^2 + C\int_\Omega \big(y_0^2 + \hat{y}_h^2 + \hat{u}^2\big) \end{aligned}$$

and the result follows. ∎

Remark 3.7. For the measure $\boldsymbol{\mu}_h \in \mathcal{M}(\bar{\Omega})^d$ defined by

$$\int_{\bar{\Omega}} f \cdot d\boldsymbol{\mu}_h := \sum_{T \in \mathcal{T}_h} \boldsymbol{\mu}_T \cdot \fint_T f dx, \qquad f \in C^0(\bar{\Omega})^d,$$

it follows immediately that

$$\|\boldsymbol{\mu}_h\|_{\mathcal{M}(\bar{\Omega})^d} \leq C, \quad 0 < h \leq h_0.$$

4 Error analysis

Theorem 4.1. *Let u and u_h be the solutions of (2.4) and (3.6) with corresponding states y and y_h respectively. Then*

$$\|u-u_h\| + \|y-y_h\| \le Ch^{\frac{1}{2}} |\log h|^{\frac{1}{2}}$$

for all $0 < h \le h_0$.

Proof. Inserting $\tilde{u} = u_h$ into (2.7) and $\tilde{u} = u$ into (3.9) we derive

$$\alpha \int_\Omega |u-u_h|^2 \le \int_\Omega p(u_h - u) + \int_\Omega p_h(u-u_h) \equiv I + II. \tag{4.13}$$

In order to treat the first term we introduce $(y^h, \mathbf{v^h}) = \mathcal{G}(u_h)$ and note that Lemma 3.1 yields

$$\|\mathbf{v^h} - \mathbf{v_h}\|_{L^\infty} \le Ch|\log h| \|u_h\|_{L^\infty} \le Ch|\log h|, \tag{4.14}$$

since $u_h \in K$. Recalling (2.6) we have

$$\begin{aligned} I &= \int_\Omega p\big(\mathcal{A}y^h - \mathcal{A}y\big) \\ &= \int_\Omega (y-y_0)(y^h - y) + \int_{\bar{\Omega}} \big(\nabla y^h - \nabla y\big) \cdot d\boldsymbol{\mu} \\ &= \int_\Omega (y-y_0)(y^h - y) + \int_{\bar{\Omega}} \big(P_\delta(\nabla y^h) - \nabla y\big) \cdot d\boldsymbol{\mu} + \int_{\bar{\Omega}} \big(\nabla y^h - P_\delta(\nabla y^h)\big) \cdot d\boldsymbol{\mu} \end{aligned}$$

where P_δ denotes the orthogonal projection onto $\bar{B}_\delta(0) = \{x \in \mathbb{R}^d \mid |x| \le \delta\}$. Note that

$$|P_\delta(x) - P_\delta(\tilde{x})| \le |x - \tilde{x}| \qquad \forall x, \tilde{x} \in \mathbb{R}^d. \tag{4.15}$$

Since $x \mapsto P_\delta(\nabla y^h(x)) \in C$ we infer from (2.8)

$$I \le \int_\Omega (y-y_0)(y^h - y) + \max_{x \in \bar{\Omega}} |\nabla y^h(x) - P_\delta(\nabla y^h(x))| \|\boldsymbol{\mu}\|_{\mathcal{M}(\bar{\Omega})^d}. \tag{4.16}$$

Let $x \in \bar{\Omega}$, say $x \in T$ for some $T \in \mathcal{T}_h$. Since u_h is feasible for (3.6) we have that $⨍_T A^{-1}\mathbf{v_h} \in \bar{B}_\delta(0)$ so that (4.15) implies

$$\begin{aligned} &\big|\nabla y^h(x) - P_\delta(\nabla y^h(x))\big| \\ &\quad \le \big|\nabla y^h(x) - ⨍_T A^{-1}\mathbf{v_h}\big| + \big|P_\delta(\nabla y^h(x)) - P_\delta\big(⨍_T A^{-1}\mathbf{v_h}\big)\big| \\ &\quad \le 2\big|\nabla y^h(x) - ⨍_T A^{-1}\mathbf{v_h}\big|. \end{aligned} \tag{4.17}$$

Using a well–known interpolation estimate along with (2.3) we obtain

$$\begin{aligned} \big|\nabla y^h(x) - ⨍_T A^{-1}\mathbf{v_h}\big| &\le \big|\nabla y^h(x) - ⨍_T \nabla y^h\big| + \big|⨍_T A^{-1}(\mathbf{v^h} - \mathbf{v_h})\big| \\ &\le Ch^{1-\frac{d}{r}} \|\nabla y^h\|_{W^{1,r}} + C\|\mathbf{v^h} - \mathbf{v_h}\|_{\mathbf{L}^\infty} \\ &\le Crh^{1-\frac{d}{r}} \|u_h\|_{L^\infty} + C\|\mathbf{v^h} - \mathbf{v_h}\|_{\mathbf{L}^\infty} \end{aligned}$$

for $r > d$. Thus, we deduce after choosing $r = |\log h|$ and recalling Lemma 3.1

$$\big|\nabla y^h(x) - \fint_T A^{-1}\mathbf{v_h}\big| \le Ch|\log h|,$$

which combined with (4.16) and (4.17) yields

$$I \le \int_\Omega (y - y_0)(y^h - y) + Ch|\log h|. \tag{4.18}$$

Next, let us introduce $(\tilde{y}_h, \tilde{\mathbf{v}}_\mathbf{h}) := \mathcal{G}_h(u) \in Y_h \times \mathbf{V_h}$. Using (3.4) and (3.7) we infer for the second term

$$\begin{aligned}
II &= -\int_\Omega p_h \mathrm{div}\big(\tilde{\mathbf{v}}_\mathbf{h} - \mathbf{v_h}\big) + \int_\Omega a_0\, p_h\big(\tilde{y}_h - y_h\big) \\
&= \int_\Omega A^{-1}\boldsymbol{\chi}_h \cdot \big(\tilde{\mathbf{v}}_\mathbf{h} - \mathbf{v_h}\big) + \sum_{T\in\mathcal{T}_h} \boldsymbol{\mu}_T \cdot \fint_T A^{-1}\big(\tilde{\mathbf{v}}_\mathbf{h} - \mathbf{v_h}\big) + \int_\Omega a_0\, p_h\big(\tilde{y}_h - y_h\big) \\
&= \int_\Omega A^{-1}\boldsymbol{\chi}_h \cdot \big(\tilde{\mathbf{v}}_\mathbf{h} - \mathbf{v_h}\big) + \int_\Omega a_0\, p_h\big(\tilde{y}_h - y_h\big) \\
&\quad + \sum_{T\in\mathcal{T}_h} \boldsymbol{\mu}_T \cdot \Big(P_\delta\big(\fint_T A^{-1}\tilde{\mathbf{v}}_\mathbf{h}\big) - \fint_T A^{-1}\mathbf{v_h}\Big) \\
&\quad + \sum_{T\in\mathcal{T}_h} \boldsymbol{\mu}_T \cdot \Big(\fint_T A^{-1}\tilde{\mathbf{v}}_\mathbf{h} - P_\delta\big(\fint_T A^{-1}\tilde{\mathbf{v}}_\mathbf{h}\big)\Big).
\end{aligned}$$

Since

$$\Big(P_\delta\big(\fint_T A^{-1}\tilde{\mathbf{v}}_\mathbf{h}\big)\Big)_{T\in\mathcal{T}_h} \in C_h$$

we deduce from (3.10) that

$$\begin{aligned}
II &\le \int_\Omega A^{-1}\boldsymbol{\chi}_h \cdot \big(\tilde{\mathbf{v}}_\mathbf{h} - \mathbf{v_h}\big) + \int_\Omega a_0\, p_h\big(\tilde{y}_h - y_h\big) \\
&\quad + \max_{T\in\mathcal{T}_h} \big|\fint_T A^{-1}\tilde{\mathbf{v}}_\mathbf{h} - P_\delta\big(\fint_T A^{-1}\tilde{\mathbf{v}}_\mathbf{h}\big)\big| \sum_{T\in\mathcal{T}_h} |\boldsymbol{\mu}_T|.
\end{aligned}$$

In order to estimate the last term we note that $\nabla y \in C$ implies that $\big(\fint_T \nabla y\big)_{T\in\mathcal{T}_h} = \big(\fint_T A^{-1}\mathbf{v}\big)_{T\in\mathcal{T}_h} \in C_h$ and hence again by Lemma 3.1

$$\begin{aligned}
\big|\fint_T A^{-1}\tilde{\mathbf{v}}_\mathbf{h} - P_\delta\big(\fint_T A^{-1}\tilde{\mathbf{v}}_\mathbf{h}\big)\big| &\le \big|\fint_T A^{-1}(\tilde{\mathbf{v}}_\mathbf{h} - \mathbf{v})\big| + \big|P_\delta\big(\fint_T A^{-1}(\tilde{\mathbf{v}}_\mathbf{h} - \mathbf{v})\big)\big| \\
&\le C\|\tilde{\mathbf{v}}_\mathbf{h} - \mathbf{v}\|_{L^\infty} \le Ch|\log h|,
\end{aligned}$$

which combined with Lemma 3.6 yields

$$II \le \int_\Omega A^{-1}\boldsymbol{\chi}_h \cdot \big(\tilde{\mathbf{v}}_\mathbf{h} - \mathbf{v_h}\big) + \int_\Omega a_0\, p_h\big(\tilde{y}_h - y_h\big) + Ch|\log h|.$$

The symmetry of A, (3.3) and (3.8) then give

$$\begin{aligned}
II &\le -\int_\Omega \big(\tilde{y}_h - y_h\big)\, \mathrm{div}\boldsymbol{\chi}_h + \int_\Omega a_0\, p_h\big(\tilde{y}_h - y_h\big) + Ch|\log h| \\
&= \int_\Omega (y_h - y_0)\big(\tilde{y}_h - y_h\big) + Ch|\log h|.
\end{aligned} \tag{4.19}$$

Inserting (4.18) and (4.19) into (4.13) we finally obtain

$$\begin{aligned}
\alpha|u-u_h|^2 &\le \int_\Omega (y-y_0)(y^h-y) + \int_\Omega (y_h-y_0)(\tilde{y}_h-y_h) + Ch|\log h| \\
&= -\int_\Omega |y-y_h|^2 + \int_\Omega \big((y_0-y_h)(y-\tilde{y}_h) + (y-y_0)(y^h-y_h)\big) + Ch|\log h| \\
&\le -\int_\Omega |y-y_h|^2 + C\big(\|y-\tilde{y}_h\| + \|y^h-y_h\|\big) + Ch|\log h| \\
&\le -\int_\Omega |y-y_h|^2 + Ch\big(\|u\| + \|u_h\|\big) + Ch|\log h|
\end{aligned}$$

in view of (3.5) and the result follows. ∎

5 Numerical example

We consider (2.4) with the choices $\Omega = B_2(0) \subset \mathbb{R}^2$, $\alpha = 1$,

$$K = \{u \in L^2(\Omega) \,|\, -2 \le u \le 2 \text{ a.e. in } \Omega\}, \quad C = \{\mathbf{z} \in C^0(\bar{\Omega})^2 \,|\, |\mathbf{z}(x)| \le \frac{1}{2}, x \in \bar{\Omega}\}$$

as well as

$$y_0(x) := \begin{cases} \frac{1}{4} + \frac{1}{2}\ln 2 - \frac{1}{4}|x|^2, & 0 \le |x| \le 1, \\ \frac{1}{2}\ln 2 - \frac{1}{2}\ln|x|, & 1 < |x| \le 2. \end{cases}$$

In order to construct a test example we allow an additional right hand side f in the state equation and replace (2.1) by

$$\begin{aligned} -\Delta y &= f + u \quad \text{in } \Omega \\ y &= 0 \qquad\quad \text{on } \partial\Omega, \end{aligned}$$

where

$$f(x) := \begin{cases} 2, & 0 \le |x| \le 1, \\ 0, & 1 < |x| \le 2. \end{cases}$$

The optimization problem then has the unique solution

$$u(x) = \begin{cases} -1 & ,0 \le |x| \le 1 \\ 0 & ,1 < |x| \le 2 \end{cases}$$

with corresponding state $y \equiv y_0$. We note that the bounds on the control are not active, so that we obtain equality in (2.7), i.e. $p = -u$. Furthermore, the measure $\boldsymbol{\mu}$ is given by $\boldsymbol{\mu} = -x\mathcal{L}^1\lfloor\partial B_1(0)$.
For the numerical solution we use the routine fmincon contained in the Matlab optimization toolbox. The state equation was approximated with the help of the Matlab implementation of the lowest order Raviart–Thomas element provided by [1].
For an error functional $E(h)$ we define the experimental order of convergence by

$$\text{EOC} = \frac{\ln E(h_1) - \ln E(h_2)}{\ln h_1 - \ln h_2}.$$

In Table 1 we investigate the error functionals

$$E_u(h) := \|u-u_h\|, \quad E_y(h) := \|y-y_h\|, \text{ and } E_y^P(h) := \|y-y_h^P\|,$$

h	$\|u-u_h\|$	$E_u(h)$	$\|y-y_h\|$	$E_y(h)$	$\|y-y_h^P\|$	$E_y^P(h)$
1.14214	7.29649e-01	-	3.02178e-01	-	1.37428e-001	-
0.60439	3.89627e-01	0.98576	1.53204e-01	1.06726	6.86972e-002	1.08949
0.31017	2.75764e-01	0.51814	7.72993e-02	1.02547	3.29981e-002	1.09918
0.15703	1.96169e-01	0.50034	3.87523e-02	1.01442	1.58055e-002	1.08141

Table 1: Errors and EOCs for the controls, the state and the piecewise linearly post–processed state

h	$\sum_{T\in\mathcal{T}_h}\lvert\boldsymbol{\mu}_T\rvert$
1.14214	5.024
0.60439	5.891
0.31017	6.138
0.15703	6.222

Table 2: Behaviour of the discrete multipliers

where the superscript P is assigned to the piecewise linearly post–processed state associated to u_h. It turns out that the controls show the behaviour predicted by Theorem 4.1, whereas the L^2 Norm of the state seems to converge linearly. The post–processed state shows the same order of convergence, but has a smaller error. In Table 2 we display the values of $\sum_{T\in\mathcal{T}_h}|\boldsymbol{\mu}_T|$, where $(\boldsymbol{\mu}_T)_{T\in\mathcal{T}_h}$ is given by (3.12). These values are expected to converge to $|\boldsymbol{\mu}|(\bar{\Omega}) = 2\pi$ as $h \to 0$.
In Figs. $1-5$ we present the numerical approximations y_h, y_h^P, u_h, v_h and μ_h on a grid containing $m = 1089$ gridpoints. Fig. 5 clearly shows that the support of $\boldsymbol{\mu}$ is concentrated around $|x| = 1$.

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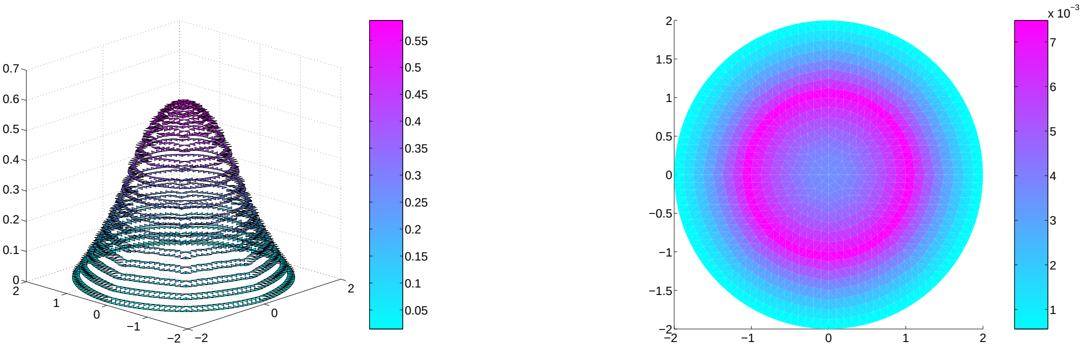

Figure 1: State: Piecewise constant (left), and error (right)

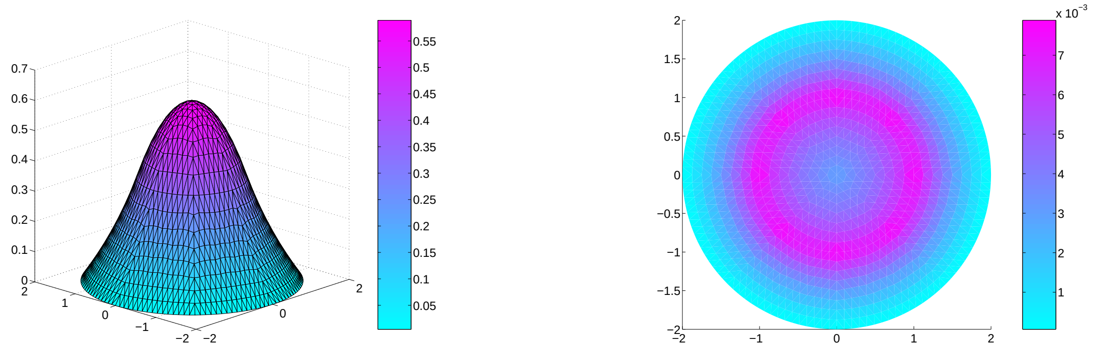

Figure 2: State: Post–processed (left), and error to piecewise constant approximation (right)

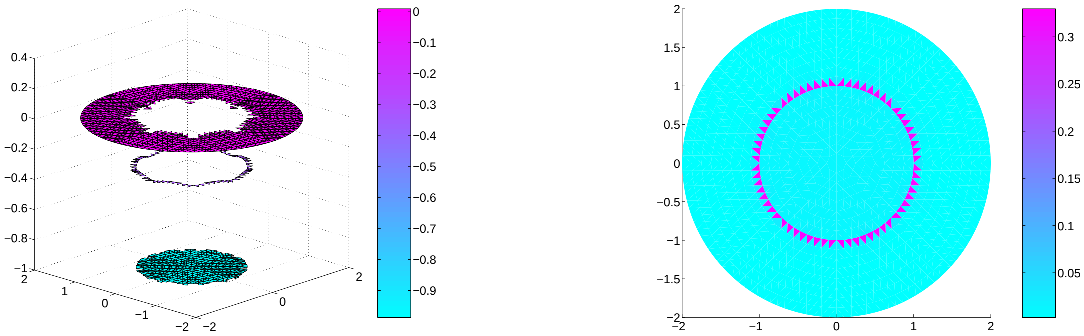

Figure 3: Control: discrete solution (left), error (right)

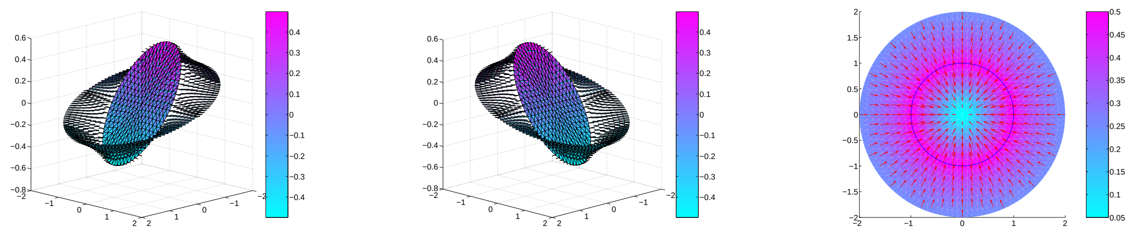

Figure 4: v_h: first component (left), second component (middle) and vector-field (right)

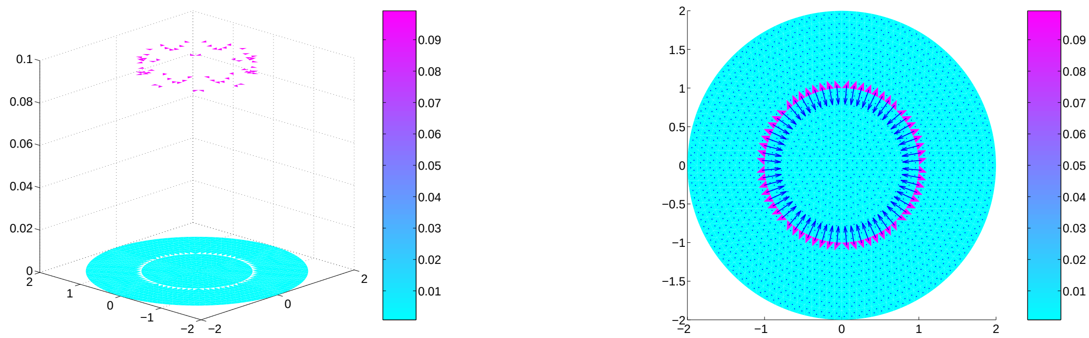

Figure 5: Measure: $|\boldsymbol{\mu}_T|$ (left), $\boldsymbol{\mu}$ (right)

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