Suboptimal control strategies for backward facing step flows

M. Hinze
TU Berlin, FB Mathematik
Straße des 17. Juni 136, D-10623 Berlin, Germany
e-mail: hinze@math.tu-berlin.de
K. Kunisch
Universität Graz, Institut für Mathematik
Heinrichstraße 36, A-8010 Graz, Austria
e-mail: karl.kunisch@kfunigraz.ac.at

Keywords: Optimal control, Navier-Stokes equations, backward-facing-step flows

ABSTRACT

In this note suboptimal distributed and boundary control strategies for fluid flows governed by the Navier-Stokes equation (NSE) together with aspects of the numerical and practical implementation of the underlying algorithms are presented. The objective of our numerical investigations is the development of practically relevant feedback control laws for flows over the backward facing step.

For what follows let $\Omega \subset \mathbb{R}^2(\mathbb{R}^3)$ denote the flow region with boundary $\Gamma = \Gamma_c \cup \Gamma_d$, where in the case of boundary control Γ_c denotes the part of the boundary where control is applied and Γ_d is the remaining part of the boundary part where fixed Dirichlet data are prescribed. Furthermore let $\Omega_{obs} \subset \Omega$ denote the observation volume. For the case of boundary control and volume observation the control problem is given by

$$(P) \quad \left\{ \begin{array}{l} \min J(\Phi,u) = J(\Phi) = \frac{l}{2} \int_0^T \int_{\Gamma_c} \Phi^2 dS dt + \int_0^T \int_{\Omega_{obs}} G(u) dx dt \\ \text{s.t.} \\ \begin{array}{rcl} u_t - \frac{1}{Re}\Delta u + (u\nabla)u + \nabla p &=& f \text{ in } \Omega \times (0,T] \\ -div\, u &=& 0 \text{ in } \Omega \times (0,T] \\ u &=& \left\{ \begin{array}{ll} \Phi & \text{on } \Gamma_c \times (0,T] \\ g & \text{on } \Gamma_d \times (0,T] \end{array} \right. \\ u(0) &=& u_0, \end{array} \end{array} \right. \tag{1}$$

where $G(z) = G(z_1, z_2)$ is some sufficiently smooth, scalar valued function, u_0 denotes the initial value of the flow, Φ denotes the control and l is a positive weight. Applying a Lagrangian approach to problem (P) leads to an optimality system involving the NSE, a backwards in time parabolic linear system for the adjoint variables and the optimality condition for Φ (see [1]). To solve (1) numerically for realistic flows necessitates a computing power that is beyond the presently available resources.

Instead of looking for an optimal control for problem (P) and thus for an optimal cost reduction, we apply a time integration scheme to the NSE and solve at each time step an optimization problem with an instantaneous cost function related to the one of problem (1). As we shall demonstrate this allows a significant reduction of the costs with readily available computing power. In order to explain the approach consider the first order semi implicit time integration scheme

$$\begin{array}{rl} u_i - \frac{\delta t}{Re}\Delta u_i + \nabla p_i = \delta t(f_i - (u_{i-1}\nabla)u_{i-1}) + u_{i-1} =: R_i & \text{in } \Omega \\ -div\, u_i = 0 & \text{in } \Omega \\ u_i = \left\{ \begin{array}{ll} \Phi_i & \text{on } \Gamma_c \\ g_i & \text{on } \Gamma_d, \end{array} \right. & \end{array} \tag{2}$$

where $u_0 = u(0)$. The resulting minimization problem at each time instance is given by

$$(P_i) \quad \begin{cases} \min J(\Phi, u_i) = J(\Phi_i) = \frac{l}{2} \int_{\Gamma_c} \Phi_i^2 dS + \int_{\Omega_{obs}} G(u_i) dx \\ \text{s.t. (2).} \end{cases} \tag{3}$$

The Lagrangian corresponding to problem (1) turns out to be (subscripts are dropped)

$$L(u, p, \phi; \xi, \pi, \mu) =$$
$$\frac{l}{2} \int_{\Gamma_c} \Phi^2 \, dS + \int_{\Omega_{obs}} G(u) \, dx + \int_{\Omega} (u - c\Delta u + \delta t \nabla p - R)\xi \, dV - \int_{\Omega} \pi div u \, dV + \int_{\Gamma_c} (u - \Phi)\mu \, dS,$$

where ξ, π, μ denote the adjoint velocity, pressure and boundary control variables, respectively and $c := \frac{\delta t}{Re}$. The optimality system can be deduced from computing formally the derivatives of L w.r.t. u, p and Φ. Having in mind the application of a gradient-type algorithm to problem (3) one requires the derivatives of J at Φ in directions χ. Using the Lagrangian one obtains by a formal calculation

$$\nabla J(\Phi)(\chi) = \nabla_{\Phi} L(\chi) = \int_{\Gamma_c} (l\Phi + c\nabla\xi\eta - \pi\eta)\chi \, dS, \tag{4}$$

where η denotes the outer normal to Ω, ξ is the adjoint velocity and π represents the adjoint pressure. The adjoint variables are determined by the following system of quasi-Stokes equations

$$\begin{aligned} u - c\Delta u + \delta t \, \nabla p &= R \text{ in } \Omega \\ -div \, u &= 0 \text{ in } \Omega \\ u &= \begin{cases} \Phi & \text{on } \Gamma_c \\ g & \text{on } \Gamma_d, \end{cases} \\ \xi - c\Delta\xi + \nabla\pi &= -DG(u)\chi_{\{\Omega_{obs}\}} \text{ in } \Omega \\ -\delta t \, div \, \xi &= 0 \text{ in } \Omega \\ \xi &= 0 \text{ on } \Gamma. \end{aligned} \tag{5}$$

With the gradient of J available the following gradient algorithm can be applied to reduce the value of the cost functional J.

1. Set $k = 0$ and choose Φ^0
2. $\rho^* = \text{argmin}_{\rho \geq 0} H(\rho) := J(\Phi^k - \rho \nabla J(\Phi^k))$
3. $\Phi^{k+1} = \Phi^k - \rho^* \nabla J(\Phi^k)$
4. $k = k + 1$, goto 2.

In this algorithm the expensive part is step 2. Every evaluation $H(\rho)$ involves the solution of system (5) with $\Phi = \Phi^k - \rho \nabla J(\Phi^k)$, so that an efficient inexact line-search should be utilized. At this stage the structure of problem (P) must be exploited. The optimal step-size in the minimum problem of step 2. can again be characterized by a Lagrangian approach. For the problems under consideration a suitable estimate of the step-size is given by

$$\rho^* = -\frac{H'(0)}{H''(0)}, \tag{6}$$

where

$$H'(0) = -\int_{\Gamma_c} |\nabla J(\Phi)|^2 \, dS, \quad H''(0) = \int_{\Gamma_c} (l\nabla J(\Phi) - c\partial_{\eta}\tilde{\xi} + \tilde{\pi}\eta)\nabla J(\Phi) \, dS$$

and $(\tilde{\xi}, \tilde{\pi})$ is the solution of the system of quasi-Stokes equations

$$\begin{aligned}
\tilde{u} - c\Delta\tilde{u} + \delta t\ \nabla\tilde{p} &= 0 \text{ in } \Omega \\
-div\,\tilde{u} &= 0 \text{ in } \Omega \\
\tilde{u} &= \begin{cases} \nabla J(\Phi) & \text{on } \Gamma_c \\ 0 & \text{on } \Gamma_d, \end{cases} \\
\tilde{\xi} - c\Delta\tilde{\xi} + \nabla\tilde{\pi} &= -D^2G(u)\tilde{u}\chi_{\{\Omega_{obs}\}} \text{ in } \Omega \\
-\delta t\, div\,\tilde{\xi} &= 0 \text{ in } \Omega \\
\tilde{\xi} &= 0 \text{ on } \Gamma.
\end{aligned} \tag{7}$$

In the case that the functional J is convex in u and the time-integration scheme (2) is explicit w.r.t. the nonlinear part of the NSE $H''(0)$ is always positive. If in addition J is quadratic in u the value of ρ^* in (6) is exact.

In practical applications only a few steps of the gradient algorithm are applied at each time step. Note that the adjoint equation in (5) together with step 3. of the gradient algorithm provides a feedback-control law based on volume observations of the flow and it is thus not well suited for engineering applications. A general class of feedback-control laws can be derived by applying the approach described above to cost functionals involving boundary control and boundary observation terms only. More precisely let

$$J(\Phi, u) := \frac{l}{2}\int_{\Gamma_c}\Phi^2 dS + \int_{\Gamma_{obs}} F(\nabla u) dS,$$

where Γ_{obs} denotes the observation boundary and $F(z) = F(z_1, \ldots, z_4)$ is some sufficiently smooth, scalar valued function. The optimization problem (3) with this cost functional yields the control law

$$\begin{aligned}
&\Phi^{new} = \Phi - \rho^*\nabla J(\Phi), \text{ with} \\
&\xi - c\Delta\xi + \nabla\pi = 0 \text{ in } \Omega \\
&-\delta t\, div\,\xi = 0 \text{ in } \Omega \\
&\xi = \begin{cases} \frac{1}{c}DF(\nabla u)\eta & \text{on } \Gamma_{obs} \\ 0 & \text{on } \Gamma_d \cup \Gamma_c, \end{cases}
\end{aligned}$$

where u denotes the primal variable in the quasi-Stokes system (5), $\nabla J(\Phi)$ is given by (4) and Γ_d is the portion of the boundary where zero-Dirichlet data are prescribed. The application of this control law to fluid flows neither requires knowledge of the initial condition of the flow nor of the flow pattern in the whole region of fluid flow. It is therefore a good candidate for a feedback control law.

Numerical examples

As numerical examples we present computations of suboptimal distributed and boundary controls for tracking-type functionals. The aim of the feedback control procedure is the reduction of the re-attachment-length of the flow. Here this will be achieved by tracking control to the Stokes-flow in a prescribed observation volume behind the step while the uncontrolled flow is laminar at Reynolds-number 300 (see Fig. 1).

1. Distributed control: In Fig. 3 the results for distributed control are shown. The cost function is given by

$$J(f, u) = \frac{l}{2}\int_{\Omega} f^2\ dx + \frac{1}{2}\int_{\Omega_{obs}} |u - u_s|^2\ dx,$$

i.e. $G(u) = \frac{1}{2}|u - u_s|^2$, where f denotes the volume force in (2) (with $\phi = 0$), u_s denotes the Stokes flow to be attained, Ω_{obs} is the rectangle $(4, 6) \times (-1, 0)$ behind the step and the penalty parameter for the control is $l = 0.01$. The gradient of J at f is given by $\nabla J(f) = l\Phi - \xi$, where ξ is the solution of (5) with $R = R(f)$ and $\Phi = 0$. The uncontrolled flow is the Navier-Stokes

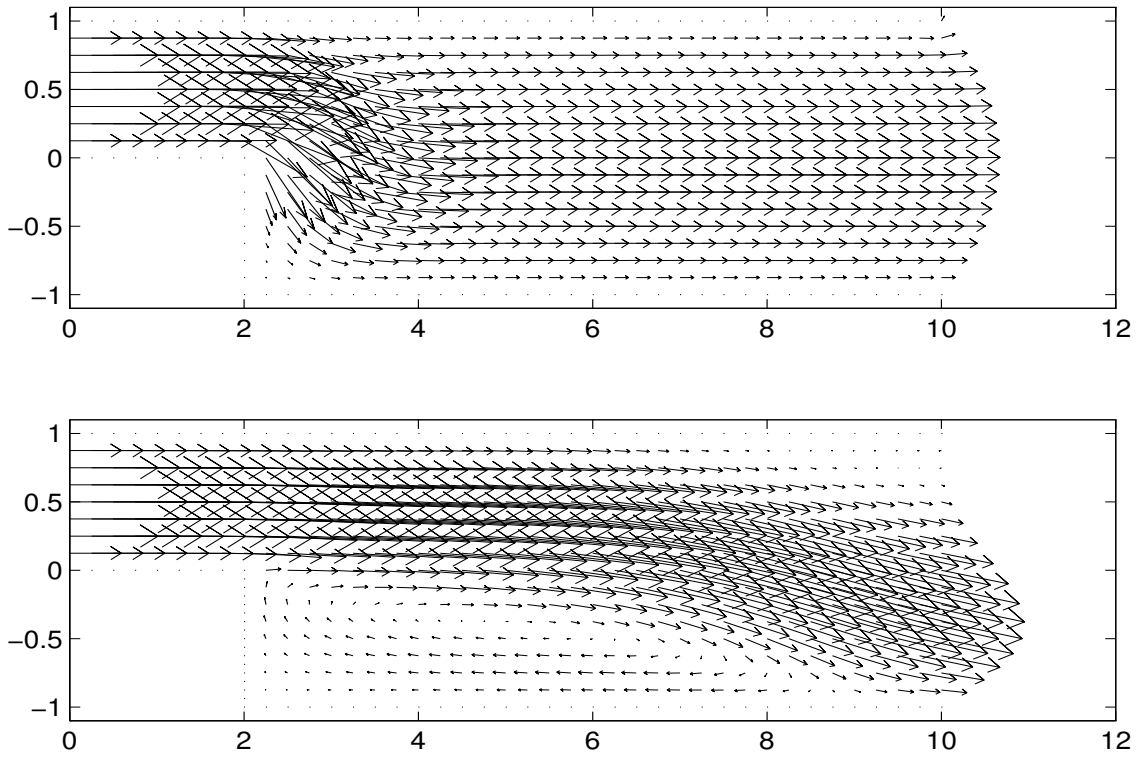

Figure 1: *Stokes flow and Navier-Stokes flow*

flow at Reynolds number 300. Control is invoked at $t = 0.1$ with a fixed step-size $\rho < \rho^*$ up to $t = 20$. At each time instant one step of the gradient algorithm is performed. For $20 \leq t \leq 50$ the optimal step-size ρ^* is applied in the gradient algorithm. This results in further cost reduction. The difference $\|u - u_s\|_0^2$ becomes stationary at an reduction of about 70% w.r.t. the uncontrolled flow. The re-attachment-length reduces as expected from 6.5 to 2. The controlled flow is shown in Fig. 2, top. At $t = 50$ control is switched off. The total costs approach the value of the uncontrolled flow, i.e. the flow evolves into the Navier-Stokes flow at Reynolds number 300.

2. Boundary control: In Fig. 4 the results for boundary control are presented. The cost functional is given by

$$J(\Phi, u) = \frac{l}{2} \int_{\Gamma_c} \Phi^2 \, dS + \frac{1}{2} \int_{\Omega_{obs}} |u - u_s|^2 \, dx,$$

where Φ denotes the boundary control in (2) acting on the upper part of the back wall and again $G(u) = \frac{1}{2}|u - u_s|^2$. In this example the volume force is chosen to be zero. The volume of observation Ω_{obs} as well as the uncontrolled flow are the same as in 1. Control is invoked at $t = 0.1$ with penalty parameter $l = 0.1$ and a fixed step-size $\rho < \rho^*$ up to $t = 20$. For $20 \leq t \leq 100$ the optimal which affects a remarkable reduction of the cost functional (Fig. 4 top) accompanied by increasing control costs (Fig. 4 bottom). The re-attachment-length reduces as for distributed control from 6.5 to 2. The difference $\|u - u_s\|_0^2$ becomes stationary for $t > 40$ (Fig. 4 middle) with an reduction of 40% w.r.t. the uncontrolled flow. The controlled flow is shown in Fig. 2, bottom.

Further literature concerning this class of problems can be found in [2], [3], [4], [5], [6], [7] and [8].

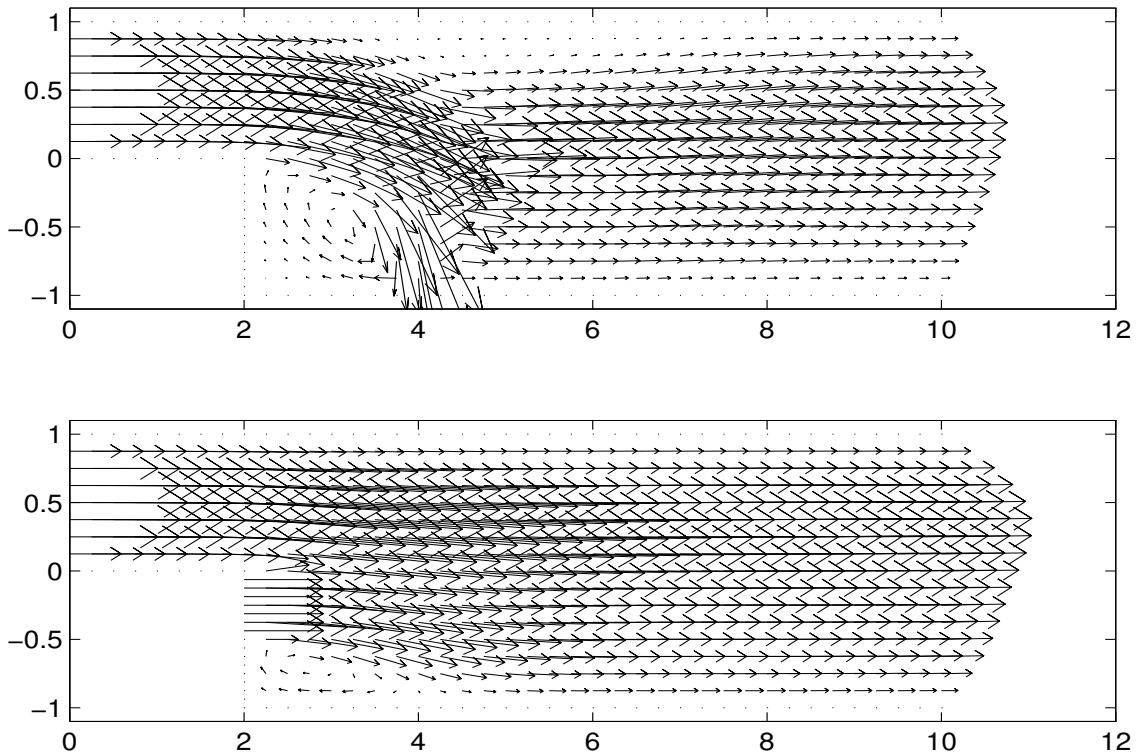

Figure 2: *Simulation results, Top: distributed control, Bottom: boundary control*

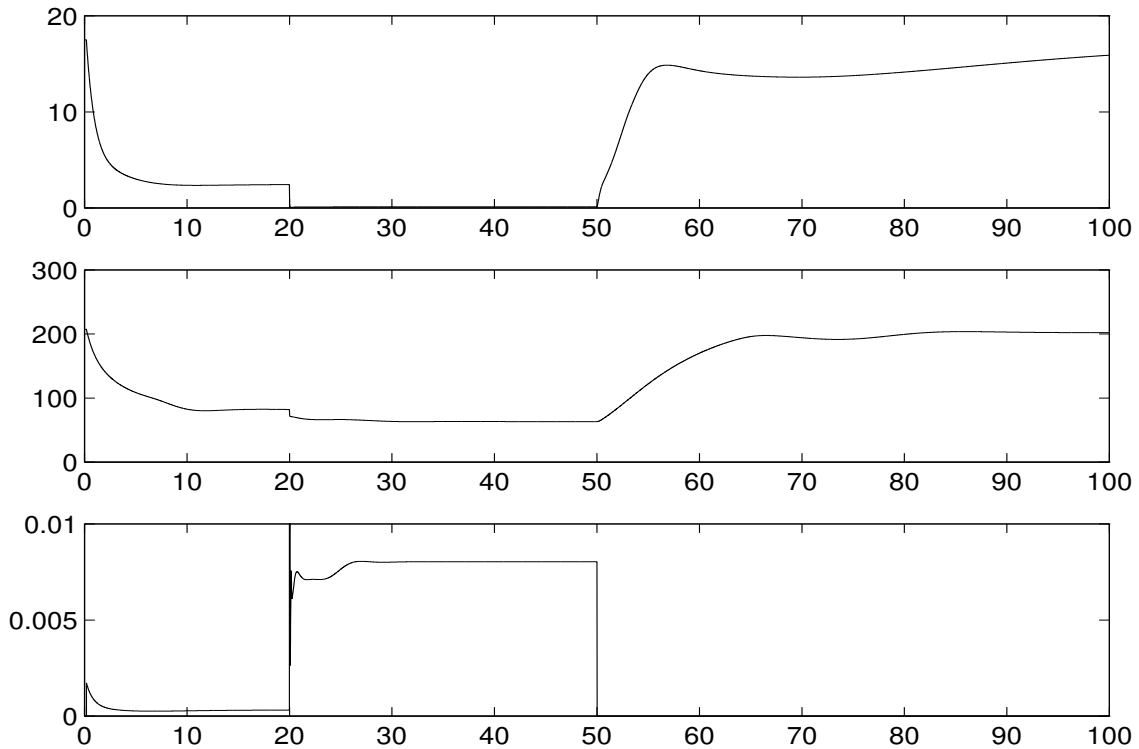

Figure 3: *Distributed control, Top: Total costs, Middle:* $\|u - u_s\|_0^2$*, Bottom: Costs Control*

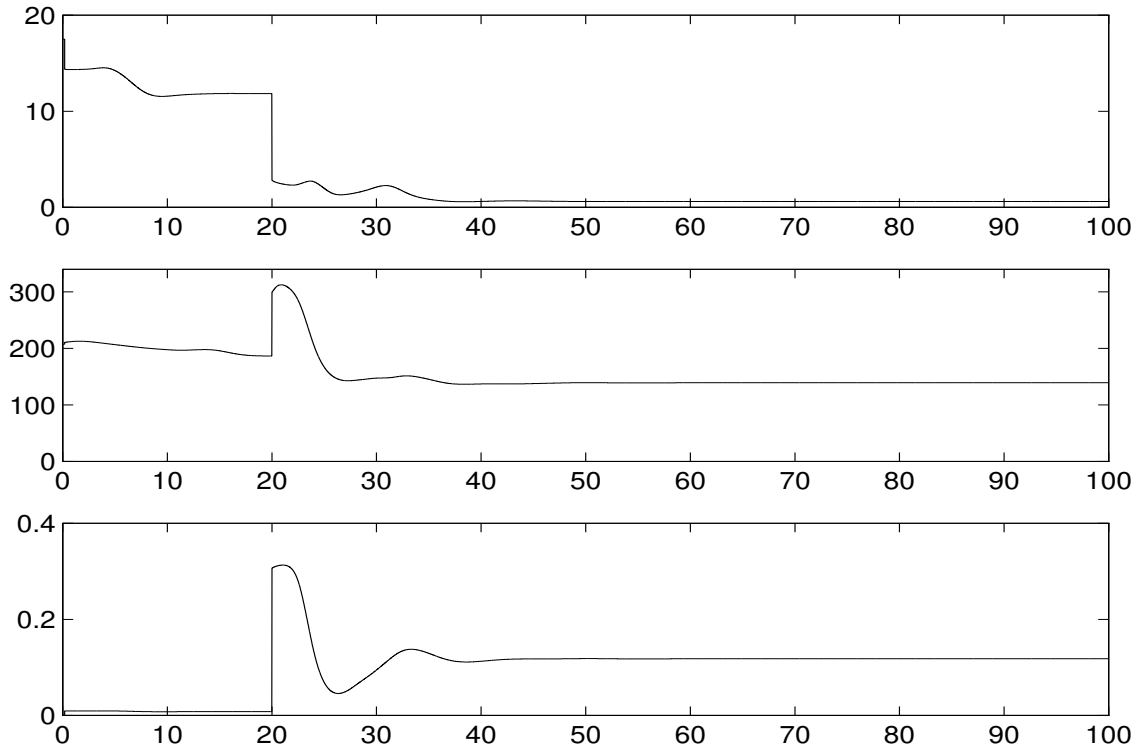

Figure 4: *Boundary control, Top: Total costs, Middle:* $\|u - u_s\|_0^2$*, Bottom: Costs Control*

References

[1] Abergel, F.; Temam,R. *On some Control Problems in Fluid Mechanics*, Theoret. Comput. Fluid Dynamics 1 (303-325), 1990.

[2] Choi, H. *Suboptimal Control of Turbulent Flow Using Control Theory*, Int. Symp. on Math. Modeling of Turbulent Flows, Tokyo, Japan, 1995.

[3] Choi, H.; Hinze, M.; Kunisch, K. *Suboptimal Control of backward-facing-step flows*, Preprint, Technical University Berlin, Germany, in preparation.

[4] Glowinski, R. *Finite Element Methods for the Numerical Simulation of Incompressible Viscous Flow; Introduction to the Control of the Navier-Stokes Equations*, Lectures in Applied Mathematics 28, 1991.

[5] Gunzburger, M.D. *A prehistory of flow control and optimization*, Flow Control, IMA 68, Editor M.D. Gunzburger, Springer 1995

[6] Hou, L.S.; Svobodny, T.P. *Optimization Problems for the Navier-Stokes Equations with Regular Boundary Controls*, J. Math. Anal. Appl. (to appear).

[7] Ito, K.; Kang, S. *A Dissipative Feedback Control Synthesis for Systems Arising in Fluid Dynamics*, Siam J. Control and Optimization 32, No.3 (831-854), 1994.

[8] Temam, R. *Remarks on the control of turbulent flows*, Flow Control, IMA 68, Editor M.D. Gunzburger, Springer 1995.