On a distributed control law with an application to the control of unsteady flow around a cylinder

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Contents

Abstract

A new class of feedback laws for distributed control of dynamical systems is presented. The approach is based on instantaneous control, i.e. control at every time step to the underlying dynamical system. The closed-loop controllers obtained in this way can be proved to be stable in the distributed control case for dynamical systems in finite dimensions, provided that the parameters of the controller are suitably adjusted [4]. As an application a distributed controller for the unsteady flow around a cylinder is numerically investigated. As the numerical results show the approach presented is capable of stabilizing systems with very large dimensions.

1 Introduction

In this work we consider the problem of steering a dynamical system to, or into a small tube around, a given state. This problem may be formulated mathematically as a constraint minimization problem for a cost function of tracking type, where the subsidiary conditions are given by the dynamical system that describes the state. The construction of optimal closed-loop controllers is in the general non-linear case closely connected to the solution of the Hamilton-Jacobi-Bellmann equations. Even in the case of the linear-quadratic regulator problem with quadratic costs and linear constraints the closed-loop controller is characterized by means of a Riccati matrix differential equation whose numerical solution for large scale problems necessitates an enormous amount of computational work.

In this note we present an suboptimal control approach that leads to practically implementable and reliable control laws. It is based on an instantaneous control approach to linear and non-linear dynamical systems. For this purpose the dynamical system is discretized in time. Then, at every time instance a stationary control problem related to the one given in advance is solved either exactly, or approximately using gradient-type methods. It is shown that each of these methods may be interpreted as a special stable time discretization of a class of closed-loop control laws, provided the parameters involved in the approach are adjusted in a proper way. This is of particular importance for the construction of practically relevant controllers for fluid flows, as is pointed out in [3].

The paper is organized as follows. In Section 2 we briefly describe the model problem, which is chosen to be the linear-quadratic regulator problem in its simplest form and state some well known results concerning solutions and numerical solution techniques. In Section 3 we introduce the concept of instantaneous stabilization. This approach turns out to yield practically realizable feedback laws for which we provide some theoretical results in the finite dimensional case. In Section 4 we discuss the approach at hand of the well known inverted pendulum equations. It turns out that the qualitative properties of the instantaneous controller are comparable to to that of the optimal stationary Riccati controller. We close with Section 5, where we present the numerical results of instantaneous stabilization applied to the control of unsteady flow around a cylinder.

2 The model problem

In this section we line out the derivation of control laws at hand of the linear-quadratic regulator problem (LQR problem) in its simplest form. To make the ideas of the approach as transparent as possible we do not concern about the questions of controllability and observability. These topics in connection with the approach presented here are discussed in [11]. Thus, the model problem will be formulated as follows: Given the desired state vector $\bar{x} \in \mathbb{R}^n$ and an initial state $x_0 \in \mathbb{R}^n$, find a control vector $u(t) \in \mathbb{R}^n$ such that the objective function

$$J(u) = \frac{1}{2} \int_0^T \gamma |u(t)|^2 + |x(t) - \bar{x}|^2 \, dt \tag{1}$$

is minimal compared to the value of J at all input vectors $v(t) \in \mathbb{R}^n$ satisfying

$$\begin{aligned} \dot{x}(t) + Ax(t) &= b(t) + v(t), \\ x(0) &= x_0 \end{aligned} \tag{2}$$

Here, $\gamma > 0$ weights the costs for the control input, the matrix $A \in \mathbb{R}^{n \times n}$ is assumed to be regular and the inhomogeneity is denoted by b. It is well known that this problem admits a unique solution

$$u^* = \frac{1}{\gamma} \mu, \tag{3}$$

where the adjoint state $\mu \in \mathbb{R}^n$ is connected to the state x by the linear forward-backward in time Hamilton equations

$$\begin{aligned} \dot{x} + Ax &= b + \tfrac{1}{\gamma}\mu, \\ -\dot{\mu} + A^*\mu &= -(x - \bar{x}), \\ x(0) &= x_0, \\ \mu(T) &= 0. \end{aligned} \tag{4}$$

Thus, the control u^* in (3) for $\bar{x}$ independent of time may be computed with the help of the solution of the two-point boundary value problem

$$\begin{aligned} \ddot{\mu} - (A^* - A)\,\dot{\mu} - \left(AA^* + \tfrac{1}{\gamma}A\right)\mu &= b - A\bar{x} \\ A^*\mu(0) - \dot{\mu}(0) &= \bar{x} - x_0 \\ \mu(T) &= 0. \end{aligned} \tag{5}$$

Choosing for the control the Ansatz

$$u(t) = \frac{1}{\gamma}\{-P(t)(x - \bar{x}) + p(t)\} \tag{6}$$

leads to the time-variant feedback matrix P and a time-dependent vector field p satisfying the Riccati matrix differential equations

$$\begin{aligned} \dot{P} - A^*P - PA + I - \tfrac{1}{\gamma}P^2 &= 0 \\ \dot{p} - \tfrac{1}{\gamma}Pp - A^*p &= -PA\bar{x} \\ P(T) &= 0 \\ p(T) &= 0. \end{aligned} \tag{7}$$

In practical applications the use of an numerical approximation of the optimal control law (6) amounts to the storage of an $n \times n$-matrix and an n-vector at every time instance. Especially for large time intervals $T \gg 0$ or/and large system dimension the application of this law is not workable, not to mention the enormous amount of computational work necessary to provide the numerical solution of (7). Concerning storage the problems arising for an forward-backward iterative solution approach to Eq.(4) (compare [5]) or for a finite-difference method applied to (5) are similar.

For infinite time horizon $T = \infty$ the optimal control law is also given by (6), where now the feedback matrix P and the vector p are the suitable time-invariant stationary solution of Eqs. 7, i.e. P and p satisfy the quadratic system

$$\begin{aligned} A^*P - PA + I - \tfrac{1}{\gamma}P^2 &= 0 \\ -\tfrac{1}{\gamma}Pp - A^*p &= -PA\bar{x}. \end{aligned} \tag{8}$$

Investigations on algorithms for the numerical solution of this problem are discussed, for example, by Rauter and Sachs in [6]. However, for large system dimensions as they arise from the discretization of parabolic equations, say the effort to solve Eqs. 8 numerically is enormous. It is one goal of this paper to present an approach which provides numerically computable control laws even for systems of very large dimension.

In the next section we introduce an approach to the design of time invariant control laws that is completely different to the method sketched above as it relies on consistent time integration of the underlying dynamical system.

3 Instantaneous control strategy

As pointed out in the previous section the construction and numerical implementation of optimal control laws suffer from practical implementability, at least when the problems under consideration have large system dimension. In order to overcome these difficulties, we now suggest a different way of constructing closed-loop controllers for dynamical systems.

The approach that we take is based on a time discretization of the state equation given in (2) with $v = 0$. For this purpose let $0 = t_0 < t_1 < \ldots < t_m = T$ denote a equidistant grid on the time interval $[0, T]$ with step size $h = \frac{T}{m}$. At each discrete time level t_i a stationary control problem is solved approximately for an approximate optimal control u_i^* and this control is used to steer the system from t_i to t_{i+1}, where a new approximate optimal control is determined. It cannot be claimed that the controls obtained in this manner approximate the optimal control for (1),(2) as the discretization parameter tends to zero. However, this procedure will be justified by the effectiveness that it exhibits for numerical examples and the interpretation that it allows as closed-loop control law.

As time discretization method we choose the implicit Euler method and as cost function an instantaneous version of (1) is taken. The optimization problem in every time step then has the form

$$(P_i)\left\{\begin{array}{l} \min J(u^{j+1}, x^{j+1}) = \frac{\gamma}{2}|u^{j+1}|^2 + \frac{1}{2}|x^{j+1} - \bar{x}^j|^2 \\ \text{s.t.} \\ \quad (I + hA)x^{j+1} \quad = \quad x^j + hb^j + u^{j+1}, \end{array}\right. \tag{9}$$

which, due to the quadratic character of the cost function, admits a unique solution (x^{j+1}, μ^{j+1}). This tuple solves the corresponding first order optimality conditions given by

$$\left.\begin{array}{rcl} (I + hA)x^{j+1} & = & x^j + hb^j + u^{j+1} \\ (I + hA^*)\mu^{j+1} & = & -(x^{j+1} - \bar{x}^j) \\ \gamma u^{j+1} - \mu^{j+1} & = & 0, \end{array}\right\} \tag{10}$$

see [4]. Having in mind to treat cost functions involving more general nonlinearities than quadratic ones suitable methods to solve for (x^{j+1}, μ^{j+1}) in (9) are gradient-type methods. The approach presented in the Algorithm following next uses exactly one gradient step at each time level t_i to approximate the solution of problem (9). As is shown later this leads to a control law which is capable of steering the state x of the dynamical system to the desired state $\bar{x}$.

Algorithm 3.1 One-gradient-step stabilization (ogss)

1. Given initial values x^0, set $j = 0$, $t_0 = 0$

2. Given u_0^j, solve

$$\begin{array}{rcl} (I + hA)x & = & x^j + hb^j + u_0^j \\ (I + hA^*)\mu & = & -(x - \bar{x}^j) \end{array}$$

3. Set $\nabla J(u_0^j) = \gamma u_0^j - \mu$
4. Given $\rho > 0$, set $u^{j+1} = u_0^j - \rho \nabla J(u_0^j)$
5. Solve
$$(I + hA)x^{j+1} = x^j + hb^j + u^{j+1}$$
6. Set $t_{j+1} = t_j + h$, $j = j + 1$
7. If $t_j < T$ goto 2.

Here we note, that the gradient of J at u in direction v is given by

$$\nabla J(u)(v) \;=\; (\gamma u - \mu, v), \tag{11}$$

so that the additional computational effort at every time step in comparison with the uncontrolled approach consists in the two solves in step 2. The optimal control problem (4),(2) has thus been replaced by a sequence of stationary problems (P_i) which can be realized on workstations even for systems with large dimensions.

Remark 3.1 One may also use the exact solution of problem (9) to obtain an algorithm similar to Algorithm 3.1, see [4, Algorithm 3.3]. In numerical applications promising results are obtained with a near to optimal gradient descent $\rho* \approx \operatorname{argmin}_{\rho \geq 0}\ J(u_0^j - \rho \nabla J(u_0^j))$ in step 4. of the above algorithm, compare also the numerical results presented in Section 5 and [3].

The major ingredient in the convergence proof for Algorithm 3.1 is its interpretation as conditionally stable semi-implicit discretization scheme of a dynamical system related to the one given in eq. (2). For this purpose we abbreviate

$$B^* := (I + hA)^{-1}, \quad B := (I + hA^*)^{-1},$$

recall $\bar{x} \in \mathbb{R}^n$ to be the desired state, set $f := A\bar{x}$ and require $h \max\{\|A\|, \|A^*\|\} < 1$, so that the matrices B and B^* are well defined.

Theorem 3.2 Algorithm 3.1 is equivalent to the semi-implicit time discretization

$$(I + hA)x^{j+1} = x^j + hb^j - \rho BB^*(x^j - \bar{x}^j) - h\rho BB^*(b^j - f^j), \quad x^0 := x_0, \tag{12}$$

of the dynamical system

$$\dot{x} + Ax = b - \frac{\rho}{h} BB^*(x - \bar{x}) - \rho BB^*(b - f), \quad x(0) = x_0, \tag{13}$$

provided $u_0^j := 0$ for every $j \in \mathbb{N}$.

Proof. $u_0^j = 0$ in step 4. gives $u^j = \rho\mu$. Solving first for x and then for μ in step 2 gives, under consideration of $B\bar{x}^j = BB^*\bar{x}^j + hBB^*f^j$ (since $f^j = A\bar{x}^j$) and step 5, the desired result. □

Remark 3.3 The differential equation in (13) admits a unique smooth solution. This , as well as the result of Theorem 3.2 holds true also in the infinite dimensional case, provided that the operator $A : X \to X$ is linear with real-part of the spectra bounded from below, say, where X denotes a Banach space.

As the next step we prove that the difference scheme (12) is stable for certain configurations of the parameters involved. For this purpose it is sufficient to ensure that the matrix $C_I := (I + hA)^{-1}(I - \rho BB^*)$ has spectral radius lower than or equal to 1. A sufficient condition for this to hold is given in

Theorem 3.4 Let the parameters h and $0 < \rho < 2$ satisfy the relation

$$\frac{1}{h} \ \geq \ \frac{(14\rho + 2) \max \{\|A\|, \|A^*\|\}}{1 - |1 - \rho|}. \tag{14}$$

Then the discretization (12) of (13) is stable.

Proof. A detailed proof of the theorem is given in [4, Theorem 5.1]. For the convenience of the reader we present a sketch. To begin with note, that the matrix C_I may be written as

$$C_I = I \underbrace{-\rho BB^* - hA}_{=:hC_A} + \rho hABB^* + \left(\sum_{k=2}^{\infty} (-h)^k A^k \right) (I - \rho BB^*) \,. \tag{15}$$

Using the Neumann expansions of B and B^* and restricting h to satisfy $2h$ $\max \{\|A\|, \|A^*\|\} < 1$, rough estimation yields the desired condition, see [4]. $\square$

Here and in the following we use the item stability as it is defined in the book [8, Definition 5.1.1].

As an immediate consequence of the proof of this theorem we obtain a condition for exponential stability of the differential equation (13).

Theorem 3.5 Let the assumptions of Theorem 3.4 be satisfied. Furthermore, let condition (14) hold with strict inequality sign. Then, the differential equation (13) is exponentially stable.

Proof. Again, we present a sketch. The matrix C_A is the system matrix of the differential equation (13). Following the lines of the proof of the previous theorem in [4] one concludes that the spectral radius r_σ of the matrix $I + hC_A$ satisfies

$$r_\sigma (I + hC_A) \ \leq \ \|I + hC_A\| < \ 1 \quad \text{(for all } h \text{ satisfying (14))}.$$

This implies $\mathrm{Re}(\lambda(C_A)) < 0$ for all eigenvalues $\lambda(C_A)$ of C_A. $\square$

In view of what follows we denote the set of m-times continuously differentiable functions with bounded derivatives up to order m on the infinite time interval $[0, \infty)$ and values in $\mathbb{R}^n$ by $C_b^m([0, \infty), \mathbb{R}^n)$.

In the next theorem we prove decay of the solution of the differential equation (13) and of the iterates $\{x^j\}_{j \in \mathbb{N}}$ of Algorithm 3.1 into a tube around $\bar{x}$ of size proportional to the parameter h.

Theorem 3.6 Let $\bar{x}$ be independent of time, let the assumptions of Theorem 3.2, 3.4 and 3.5 be satisfied. Furthermore, let the inequality in Eq. 14 be strict and assume that $f - b \in C_b^0([0,\infty), \mathbb{R}^n)$. Then, the iterates $\{x^j\}_{j\in\mathbb{N}}$ and the controls $\{u^j\}_{j\in\mathbb{N}}$ of Algorithm 3.1 satisfy

$$\sup_{j\in\mathbb{N}, j\geq j_0} |x^j - \bar{x}| \;\leq\; C\,h, \qquad \sup_{j\in\mathbb{N}, j\geq j_0} |u^j| \;\leq\; C\,h \tag{16}$$

with some fixed $j_0(h, \rho, x^0)$ and some positive constant C independent of the parameter h.

If, in addition $\rho = 1$, there exists a finite time t_0 such that

$$\sup_{j\in\mathbb{N}, t\geq t_0} |x(t) - \bar{x}| \;\leq\; C\,h. \tag{17}$$

Proof. We start with proving the second claim. To begin with, set $\Phi : x - \bar{x}$ and observe that Φ solves the stable differential equation

$$\dot{\Phi} \;=\; \underbrace{-(A + \frac{\rho}{h}BB^*)}_{=:C_A}\,\Phi \underbrace{-(\rho BB^* - I)}_{=:C_B}(b - f), \tag{18}$$

which admits the unique solution

$$\Phi(t) \;=\; e^{-C_A t}\Phi(0) + \int_0^t e^{-C_A(t-s)}C_B(b - f)(s)\,ds.$$

Using the expansion $BB^* = I - h\,(A + A^*) + O(h^2)$ $(h \to 0)$, the regularity requirements on the difference $b - f$, $\rho = 1$ and the exponential stability of the differential equation (18) one obtains for some $t_0 \geq 0$

$$|\Phi(t)| \;\leq\; Ch \quad \forall t \geq t_0,$$

where the positive constant can be chosen independent of h.

In order to prove the first claim we exploit the stability properties of the semi-implicit Euler discretization of (18) given by

$$(I + hA)\Phi^{j+1} \;=\; \Phi^j - \rho BB^*\Phi^j - h(\rho BB^* - I)(b^j - f^j). \tag{19}$$

Since strict inequality in condition (14) implies $\|C_I\| < 1$ we can conclude

$$|\Phi^j| \;\leq\; \|C_I\|^j\,|x^0 - \bar{x}| + \frac{h}{1 - \|C_I\|}\,\|b - f\|_\infty \;\leq\; C\,h\ (j \geq j_0).$$

This is the desired result for the iterates x^j. For the controls u^j the result immediately follows from the representation

$$u^{j+1} \;=\; -\rho BB^*(x^j - \bar{x}^j) - h\rho BB^*(b^j - f^j),$$

the regularity requirements on the difference $b - f$ and the the estimate for the iterates x_j. $\square$

Remark 3.7 Note, that if b is chosen as to satisfy $b = A\bar{x}$, the choice $\rho = 1$ in Theorem 3.6 is not necessary to obtain the results in (17).

4 An example

As example we investigate the linearization of the pendulum equation

$$\ddot{\varphi} + \sin\varphi \;=\; 0$$

around the unstable stationary solution $\varphi = \pi, \dot{\varphi} = 0$:

$$\dot{x} + \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{=:A} x = \underbrace{\begin{bmatrix} 0 \\ -\pi \end{bmatrix}}_{=:f}.$$

The state $\bar{x}$ to be observed is given by $\bar{x} = [\pi, 0]^t$. Further we set $b = [0, -\pi]$, so that $A\bar{x} = b$ holds true.

We compute the stationary Riccati solution in order to compare it to the results obtained by the control law given by Algorithm 3.1. Using the stationary Riccati solutions P and p from (8) to stabilize the linearized pendulum equations leads to the feedback law

$$\dot{x} \;=\; -\left(A + \tfrac{1}{\gamma}P\right)x + \left(f - \tfrac{1}{\gamma}p\right),$$

where the positive definite matrix P and the vector p are given by

$$P = \begin{bmatrix} \sqrt{\gamma^2+\gamma} & \gamma \\ \gamma & \sqrt{\gamma^2+\gamma} \end{bmatrix}, \quad p = \begin{bmatrix} -\sqrt{\gamma^2+\gamma}\pi \\ -\gamma\pi \end{bmatrix}.$$

Using this, a short calculation gives the optimal state for the infinite horizon problem,

$$x = \begin{bmatrix} \pi \\ 0 \end{bmatrix} + e^{-t\cdot\sqrt{1+\frac{1}{\gamma}}} \begin{bmatrix} a \\ b \end{bmatrix},$$

where a and b depend on the initial state of the differential equation. The control has the form $u = -\frac{1}{\gamma}(Px + p)$ and is given by

$$u = -\tfrac{1}{\gamma} \begin{bmatrix} a\sqrt{\gamma^2+\gamma} + b\gamma \\ a\gamma + b\sqrt{\gamma^2+\gamma} \end{bmatrix} e^{-t\cdot\sqrt{1+\frac{1}{\gamma}}}.$$

Two more lines of computation finally yield that the cost $J = \displaystyle\int_0^\infty \|x - \bar{x}\|^2 + \gamma\|u\|^2\,dt$ is given by

$$J = \sqrt{\gamma^2+\gamma}\left(a^2 + 2ab\sqrt{\tfrac{\gamma}{\gamma+1}} + b^2\right) \;=\; O(\gamma)\ (\gamma \to 0).$$

Now we proceed similar for the instantaneous control law. To begin with, let $\frac{1}{h} > \|A\| = 1$. Then, there holds $B = B^* = 1 - hA + O(h^2)$ $(h \to 0)$ and $BB^* = 1 - 2hA + O(h^2)$ $(h \to 0)$. For the matrix $C_A = -(A + \frac{\rho}{h}BB^*)$ (see (18)) we obtain the approximation

$$\tilde{C}_A \;\doteq\; \begin{bmatrix} -\frac{\rho}{h} & 1-2\rho \\ 1-2\rho & -\frac{\rho}{h} \end{bmatrix},$$

which is of order $O(h)$ $(h \to 0)$. Using this approximations it is sufficient to investigate the feedback control given by

$$\dot{x} = \tilde{C}_A x + \tfrac{\rho}{h}\,(I - 2hA)\,\bar{x} + b - \rho\,(I - 2hA)\,(b - f)$$

since its exact solution

$$x \;=\; ae^{\lambda_1 t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + be^{\lambda_2 t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

deviates from that of the original control law (13) only by a term of size $O(h)$ $(h \to 0)$. The eigenvalues $\lambda_{1/2}$ of the matrix $\tilde{C}_A$ are given by

$$\lambda_{1/2} \;=\; -\frac{\rho}{h} \pm \operatorname{sign}(1 - 2\rho)|1 - 2\rho|,$$

so that negative eigenvalues are guaranteed, provided the parameter h is sufficiently small. Finally, we compute the control and the costs. For the control we obtain

$$u = -\left(ae^{\lambda_1 t}\left(\tfrac{\rho}{h} + 2\rho\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + be^{\lambda_2 t}\left(\tfrac{\rho}{h} + 2\rho\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \longrightarrow 0 \; (t \;\to\; \infty),$$

the value of the costs J are given by

$$J = \frac{a^2}{\lambda_1}\left(1 + \gamma\left(\frac{\rho}{h} + 2\rho\right)^2\right) + \frac{b^2}{\lambda_2}\left(1 + \gamma\left(\frac{\rho}{h} + 2\rho\right)^2\right) \le C\left(\frac{h}{\rho} + \frac{\gamma\rho}{h}\right)$$

with some positive constant C independent of ρ, γ and h. As a result we obtain that both, the optimal Riccati controller and the controller of Algorithm 3.1 have the same qualitative behaviour.

We conclude with presenting the numerical results for the distributed control law when it is applied to the regulation of the flow around a cylinder.

5 Distributed control of flow around a cylinder

Here we present numerical results related to the control of laminar unsteady flow around a cylinder in a two-dimensional spatial domain Ω. The instantaneous stabilization Algorithm 3.1 is applied to reconstruct the Stokes flow in an observation volume Ω_s behind the cylinder. Using this observations a volume force is computed in every time step in order to minimize the difference in the L^2-norm between the flow and the Stokes flow.

5.1 The distributed control problem

We now put this problem into the framework developed in Section 3. For this purpose we consider the control problem

$$
(P) \left\{ \begin{array}{rcll} \multicolumn{4}{l}{\min J(f) = \frac{\gamma}{2} \int\limits_0^T \left[\int_{\Omega_s} |u - \bar{u}|^2 \, d\Omega \ + \ \frac{\gamma}{2} \int_\Omega |f|^2 \, d\Omega \right] dt} \\ \multicolumn{4}{l}{\text{s.t.}} \\ u_t - \frac{1}{\mathrm{Re}} \Delta u + (u \cdot \nabla) u + \nabla p & = & f & \text{in } Q \\ -\operatorname{div} u & = & 0 & \text{in } Q \\ u & = & g & \text{on } \Gamma_d \times (0,T] \\ \frac{1}{\mathrm{Re}} \partial_\eta u & = & p\eta & \text{on } \Gamma_{out} \times (0,T] \\ u(0) & = & u_0, & \end{array} \right. \tag{20}
$$

with the instationary incompressible Navier-Stokes equations as subsidiary conditions. Here, f denotes the control force acting on the whole of the spatial domain Ω, $Q := \Omega \times (0,T)$ denotes the space-time cylinder, g are boundary values at the Dirichlet boundary Γ_d and u_0 denotes the initial state. On the outflow boundary Γ_{out} natural boundary conditions are prescribed.

Next, we formally bring the Navier-Stokes equations into the form (2). To begin with we first introduce their variational formulation based on the Leray projection of the stationary Stokes problem, see [10, Lemma 2.1]. We set $V := \{v \in H^1(\Omega, \mathbb{R}^2);\, v_{|\Gamma_d} = 0, \operatorname{div} v = 0\}$ the set of two-dimensional divergence-free vector fields and denote by $(\cdot,\cdot)$ the scalar product in $L^2(\Omega, \mathbb{R}^2)$ $(L^2(\Omega, \mathbb{R}^{2\times 2}))$. The vector field u is called variational solution of the Navier-Stokes equations iff $u(t) \in g(t) + V$, $u(0) = u_0$ and

$$
(u_t, v) + \frac{1}{\mathrm{Re}} (\nabla u, \nabla v) + ((u\nabla)u, v) \ = \ (f, v) \quad \forall\, v \in V, \text{ a.e. in } (0,T). \tag{21}
$$

Thus, using the Stokes operator S, this variational formulation may be rewritten as Burgers equation in the space V, i.e. find $u(t) \in g(t) + V$ with $u(0) = u_0$ and

$$
u_t + Su + (u\nabla)u \ = \ f. \tag{22}
$$

If finally we set $b = b(u, \nabla u) := -(u\nabla)u$, problem (P) has been rewritten in the form (1), (2).

In this setting Algorithm 3.1 applied to the control of flow around a cylinder is equivalent to the semi-implicit time discretization (12 of the dynamical system

$$
\dot{u} + Su = b - \frac{\rho}{h} BB^*(u - \bar{u}) - \rho BB^*(b - f), \quad u(0) = u_0, \tag{23}
$$

where $f := S\bar{u}$. We note that the operator B is the self-adjoint solution operator of the quasi-Stokes problem in V, i.e.

$$
w \ = \ Bh \quad \Longleftrightarrow \quad (I + hS)w = h \quad \text{in } V.
$$

5.2 Numerical results

We now present numerical results obtained by the application of Algorithm 3.1 in order to reconstruct the Stokes flow in a given observation volume $\Omega_s \subset \Omega$ behind the cylinder, see Fig. 1. As computational domain we chose a configuration that coincides with the domain used for the two-dimensional benchmark computations in [7]. The Reynolds number is given by the product of the diameter d of the cylinder and the bulk velocity U at the inlet, divided by the kinematic viscosity ν of the fluid. The boundary conditions are no-slip at the top and the bottom walls and *do-nothing* at the outflow boundary. At the inlet a parabolic inflow profile is described. For the spatial discretization the Taylor-Hood finite element is used on a grid containing 5374 triangles, 2840 pressure and 11054 velocity nodes with a-priori local refinement in a neighbourhood of the cylinder. For the discretization of the quasi-Stokes problems the Taylor-Hood finite element is used. Numerical solutions of various problems are performed using an extension of the Navier-Stokes solver developed by Bänsch in [1].

For the computations that follow the Reynolds number is chosen as $\mathrm{Re} = 100$. Furthermore, we use the parameter values $\gamma = 1.e-2$ and $h = 0.0075$. The observation volume is taken as $\Omega_s = [4,6] \times [-1,1]$. The time histories of control and of the descent parameters ρ used in Algorithm 3.1 are documented in Tab. 1. In this table ρ^* denotes an approximation of the optimal descent parameter, see Remark 3.1. In Fig. 2 the values of costs (a) and control costs (b) are

Time	$0 \le t \le 1$	$1 \le t \le 5$	$5 \le t \le 10$	$10 \le t \le 20$
ρ	no control	0.1	$\rho \approx \rho^*$	no control
Time	$20 \le t \le 40$	$40 \le t \le 100$	$100 \le t \le 120$	$120 \le t \le 200$
ρ	$\rho \approx \rho^*$	no control	$\rho \approx \rho^*$	no control

Table 1: Time history of control and the gradient descent parameters ρ used in Algorithm 3.1

shown. When control is switched on the costs decrease immediately. If for ρ an approximation of the optimal gradient step-size is used the additional decrease of cost is enormous. As control is relaxed the flow evolves to the uncontrolled state, whose periodic behaviour is very good reflected by the periodic difference to the steady Stokes flow shown in Fig. 1(a). As Fig. 2(b) shows switching on control is accompanied by a large rise of the control costs. Once control is invoked their cost numerically converge to a steady state as is predicted by Theorem 3.6. Note, that in contrast to what is shown in Fig. 2(b), there is no control cost in time intervals without control force applied to the flow. Since the scale of the plot is logarithmic the control cost are set to their last value $\neq 0$.

Fig 1 shows the Stokes flow (a) together with the uncontrolled Navier-Stokes flow (b). The same figure (c) shows the controlled flow together with the control force (d) at $t = 2$. As one can see the recirculation in a neighbourhood of the observation volume Ω_s vanishes completely. This effect is further amplified when control action is kept on.

Remark 5.1 Further numerical investigations on the control of flow around a cylinder can be found in the papers of Choi [2] and Tang & al. [9]. A numerical comparison of the method presented here and the open-loop optimal control approach for backward-facing step flows is presented in [5]. As is shown there, the cost reduction of the method worked out here applied to backward-facing step flows nearly coincides with that obtained by the open-loop optimal control approach, whereas the amount of numerical work for the suboptimal method is substantially less than that for the open-loop optimal approach.

Acknowledgments

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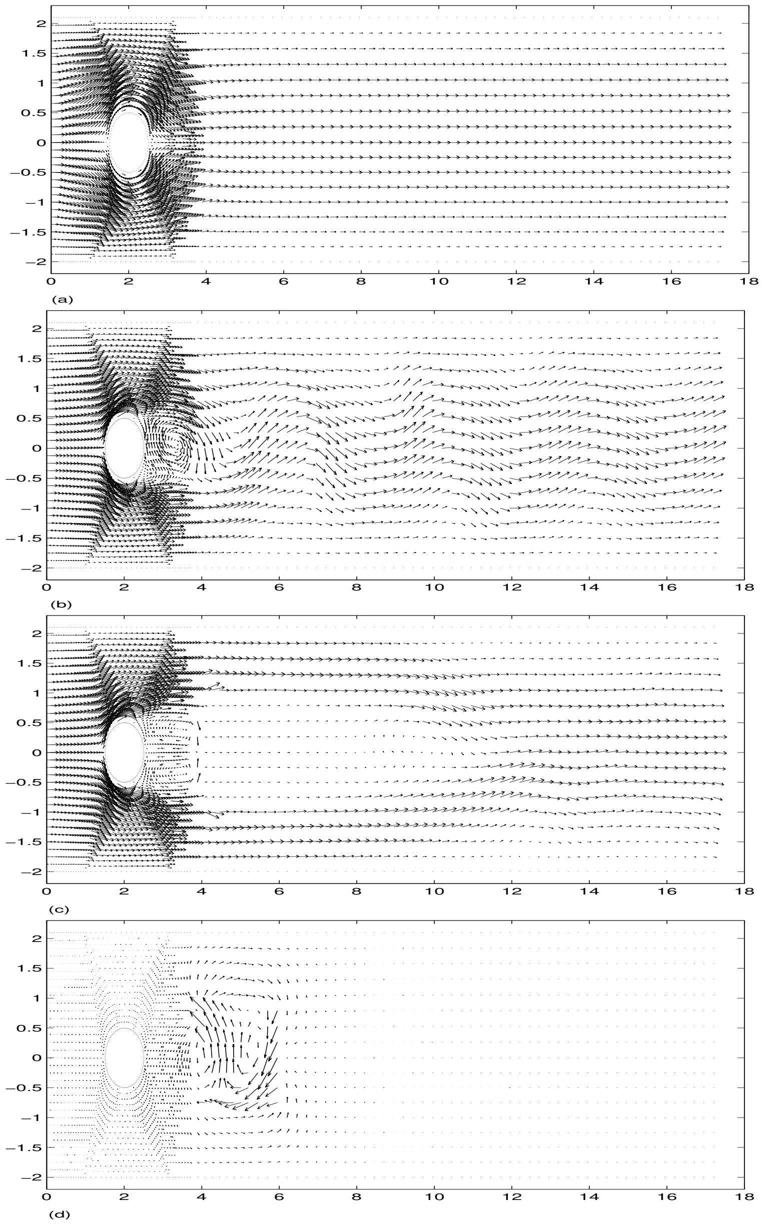

Figure 1: Distributed control at Re=100: (a) Stokes flow;(b) Navier-Stokes flow; (c) Controlled flow; (d) Control force

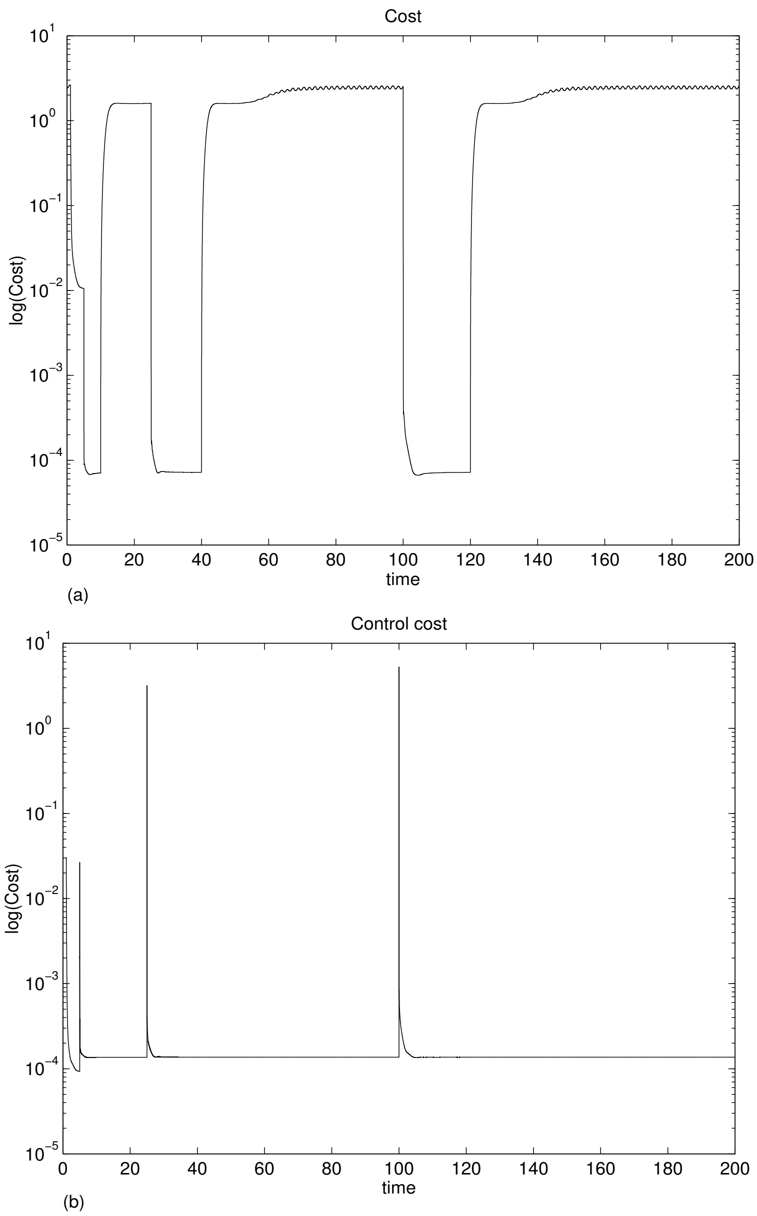

Figure 2: Distributed control at Re=100. Observation in $[4,6] \times [-1,1]$: (a) Cost; (b) Control Cost

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