Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Basic Concepts of Adaptive Finite Element Methods for Elliptic Boundary Value Problems

Ronald H.W. Hoppe[1,2]

[1] Department of Mathematics, University of Houston
[2] Institute of Mathematics, University of Augsburg

Mathematisches Forschungsinstitut Oberwolfach

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Foundations of AFEM I

For a closed subspace $V \subset H^1(\Omega)$ we assume

$$a(\cdot,\cdot) : V \times V \to \mathbb{R}$$

to be a bounded, V-elliptic bilinear form, i.e.,

$$|a(v,w)| \le C\|v\|_{k,\Omega}\|w\|_{k,\Omega}, \quad v,w \in V, \quad a(v,v) \ge \gamma \|v\|^2_{k,\Omega}, \quad v \in V,$$

for some constants $C > 0$ and $\gamma > 0$. We further assume $\ell \in V^*$ where V^* denotes the algebraic and topological dual of V and consider the variational equation: Find $u \in V$ such that

$$a(u,v) = \ell(v) \quad , \quad v \in V.$$

It is well-known by the Lax-Milgram Lemma that under the above assumptions the variational problem admits a unique solution.

Foundations of AFEM II

Finite element approximations are based on the Ritz-Galerkin approach: Given a finite dimensional subspace $V_h \subset V$ of test/trial functions, find $u_h \in V_h$ such that

$$a(u_h, v_h) \;=\; \ell(v_h), \quad v_h \in V_h.$$

Since $V_h \subset V$, the existence and uniqueness of a discrete solution $u_h \in V_h$ follows readily from the Lax-Milgram Lemma. Moreover, we deduce that the error $e_u := u - u_h$ satisfies the Galerkin orthogonality

$$a(u - u_h, v_h) \;=\; 0, \quad v_h \in V_h,$$

i.e., the approximate solution $u_h \in V_h$ is the projection of the solution $u \in V$ onto V_h with respect to the inner product $a(\cdot,\cdot)$ on V (elliptic projection). Using the Galerkin orthogonality, it is easy to derive the a priori error estimate

$$\|u - u_h\|_{1,\Omega} \;\leq\; M \inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega},$$

where $M := C/\gamma$. This result tells us that the error is of the same order as the best approximation of the solution $u \in V$ by functions from the finite dimensional subspace V_h. It is known as Céa's Lemma.

Foundations of AFEM III

The Ritz-Galerkin method also gives rise to an a posteriori error estimate in terms of the residual $r : V \to \mathbb{R}$

$$r(v) \; := \; \ell(v) \; - \; a(u_h, v), \quad v \in V.$$

In fact, it follows that for any $v \in V$

$$\gamma \|u - u_h\|_{1,\Omega}^2 \le a(u - u_h, u - u_h) = r(u - u_h) \le \|r\|_{-1,\Omega} \; \|u - u_h\|_{1,\Omega},$$

whence

$$\|u - u_h\|_{1,\Omega} \; \le \; \frac{1}{\gamma} \; \sup_{v \in V} \frac{|r(v)|}{\|v\|_{1,\Omega}}.$$

Foundations of AFEM IV

Definition. An error estimator η_h is called reliable, if it provides an upper bound for the error up to data oscillations osc_h^{rel}, i.e., if there exists a constant $C_{rel} > 0$, independent of the mesh size h of the underlying triangulation, such that

$$\|e_u\|_a \leq C_{rel}\, \eta_h + osc_h^{rel} .$$

On the other hand, an estimator η_h is said to be efficient, if up to data oscillations osc_h^{eff} it gives rise to a lower bound for the error, i.e., if there exists a constant $C_{eff} > 0$, independent of the mesh size h of the underlying triangulation, such that

$$\eta_h \leq C_{eff}\, \|e_u\|_a + osc_h^{eff}.$$

Finally, an estimator η_h is called asymptotically exact, if it is both reliable and efficient with $C_{rel} = C_{eff}^{-1}$.

Reliability and Efficiency of Error Estimators II

Remark. The notion 'reliability' is motivated by the use of the error estimator in error control. Given a tolerance tol, an idealized termination criterion would be

$$\|e_u\|_a \ \leq \ tol.$$

Since the error $\|e_u\|_a$ is unknown, we replace it with the upper bound, i.e.,

$$C_{rel}\, \eta_h \ + \ osc_h^{rel} \ \leq \ tol.$$

We note that the termination criterion both requires the knowledge of C_{rel} and the incorporation of the data oscillation term osc_h^{rel}. In the special case $C_{rel} = 1$ and $osc_h^{rel} \equiv 0$, it reduces to

$$\eta_h \ \leq \ tol.$$

An alternative, but less used termination criterion is based on the lower bound, i.e., we require

$$\frac{1}{C_{eff}} \left(\eta_h \ - \ osc_h^{eff} \right) \ \leq \ tol.$$

Typically, this criterion leads to less refinement and thus requires less computational time which motivates to call the estimator efficient.

The Role of the Residual

The error estimate

$$\|u - u_h\|_{1,\Omega} \;\le\; \frac{1}{\gamma} \sup_{v \in V} \frac{|r(v)|}{\|v\|_{1,\Omega}}$$

shows that in order to assess the error $\|e_u\|_a$ we are supposed to evaluate the norm of the residual with respect to the dual space V^*, i.e.,

$$\|r\|_{V^*} \;:=\; \sup_{v \in V \setminus \{0\}} \frac{|r(v)|}{\|v\|_a}.$$

In particular, we have the equality

$$\|r\|_{V^*} \;=\; \|e_u\|_a,$$

whereas for the relative error of $r(v), v \in V$, as an approximation of $\|e_u\|_a$ we obtain

$$\frac{(\|e_u\|_a - r(v))}{\|e_u\|_a} \;=\; \frac{1}{2}\, \|v - \frac{e_u}{\|e_u\|_a}\|_a^2, \quad v \in V \text{ with } \|v\|_a = 1.$$

The goal is to obtain lower and upper bounds for $\|r\|_{V^*}$ at relatively low computational expense.

Model problem: Let Ω be a bounded simply-connected polygonal domain in Euclidean space $\mathbb{R}^2$ with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and consider the elliptic boundary value problem

$$Lu := - \nabla \cdot (a \nabla u) = f \quad \text{in } \Omega ,$$

$$u = 0 \quad \text{on } \Gamma_D , \quad n \cdot a \nabla u = g \quad \text{on } \Gamma_N,$$

where $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$ and $a = (a_{ij})_{i,j=1}^2$ is supposed to be a matrix-valued function with entries $a_{ij} \in L^\infty(\Omega)$, that is symmetric and uniformly positive definite. The vector n denotes the exterior unit normal vector on Γ_N. Setting

$$H^1_{0,\Gamma_D}(\Omega) := \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0 \},$$

the weak formulation is as follows: Find $u \in H^1_{0,\Gamma_D}(\Omega)$ such that

$$a(u,v) = \ell(v) \quad , \quad v \in H^1_{0,\Gamma_D}(\Omega),$$

where

$$a(v,w) := \int_\Omega a \nabla v \cdot \nabla w \, dx, \quad \ell(v) := \int_\Omega f \, v \, dx + \int_{\Gamma_N} g \, v \, d\sigma \quad , \quad v \in H^1_{0,\Gamma_D}(\Omega).$$

FE Approximation: Given a geometrically conforming simplicial triangulation $\mathcal{T}_h$ of Ω, we denote by

$$S_{1,\Gamma_D}(\Omega;\mathcal{T}_h) \ := \ \{ \ v_h \in H^1_{0,\Gamma_D}(\Omega) \ | \ v_h|_T \in P_1(K) \ , \ T \in \mathcal{T}_h \ \}$$

the trial space of continuous, piecewise linear finite elements with respect to $\mathcal{T}_h$. Note that $P_k(T) \ , \ k \geq 0$, denotes the linear space of polynomials of degree $\leq k$ on T. In the sequel we will refer to $\mathcal{N}_h(D)$ and $\mathcal{E}_h(D) \ , \ D \subseteq \bar{\Omega}$ as the sets of vertices and edges of $\mathcal{T}_h$ on D. We further denote by $|T|$ the area, by h_T the diameter of an element $T \in \mathcal{T}_h$, and by $h_E = |E|$ the length of an edge $E \in \mathcal{E}_h(\Omega \cup \Gamma_N)$. We refer to $f_T := |T|^{-1} \int_T f dx$ the integral mean of f with respect to an element $T \in \mathcal{T}_h$ and to $g_E := |E|^{-1} \int_E g ds$ the mean of g with respect to the edge $E \in \mathcal{E}_h(\Gamma_N)$.
The conforming P1 approximation reads as follows: Find $u_h \in S_{1,\Gamma_D}(\Omega;\mathcal{T}_h)$ such that

$$a(u_h, v_h) \ = \ \ell(v_h), \quad v_h \in S_{1,\Gamma_D}(\Omega;\mathcal{T}_h).$$

Representation of the Residual I

The residual r is given by

$$r(v) \ := \int_{\Omega} f\ v\ dx \ + \int_{\Gamma_N} g\ vds \ - \ a(u_h, v) \quad , \quad v \in V.$$

Applying Green's formula elementwise yields

$$a(u_h, v) = \sum_{T \in \mathcal{T}_h} \int_T a\ \nabla u_h \cdot \nabla v\ dx = \sum_{E \in \mathcal{E}_h(\Omega)} \int_E [n \cdot a\ \nabla u_h]\ v\ ds + \sum_{E \in \mathcal{E}_h(\Gamma_N)} \int_E n \cdot a\ \nabla u_h\ v\ ds,$$

where $[n \cdot a\ \nabla u_h]$ denotes the jump of the normal derivative of u_h across $E \in \mathcal{E}_h(\Omega)$ and where we have used that $\Delta u_h \equiv 0$ on $T \in \mathcal{T}_h$, since $u_h|_T \in P_1(T)$. We thus obtain

$$r(v) \ := \sum_{T \in \mathcal{T}_h} r_T(v) \ + \sum_{E \in \mathcal{E}_h(\Omega \cup \Gamma_N)} r_E(v).$$

Representation of the Residual II

Here, the local residuals $r_T(v), T \in \mathcal{T}_h$, are given by

$$r_T(v) \ := \ \int_T (f - Lu_h)v \ dx,$$

whereas for $r_E(v)$ we have

$$r_E(v) \ := \ -\int_E [n \cdot a \ \nabla u_h] v \ ds, \ \ E \in \mathcal{E}_h(\Omega),$$

$$r_E(v) \ := \ \int_E \Big(g - n \cdot a \ \nabla u_h\Big) v \ ds, \ \ E \in \mathcal{E}_h(\Gamma_N).$$

A Posteriori Error Estimator and Data Oscillations

The error estimator η_h consists of element residuals $\eta_T, T \in \mathcal{T}_h$, and edge residuals $\eta_E, E \in \mathcal{E}_H(\Omega \cup \Gamma_N)$, according to

$$\eta_h := \Big(\sum_{T \in \mathcal{T}_h} \eta_T^2 + \sum_{E \in \mathcal{E}_H(\Omega \cup \Gamma_N)} \eta_E^2 \Big)^{1/2},$$

where η_T and η_E are given by

$$\eta_T := h_T \, \|f_T - Lu_h\|_{0,T} \, , \; T \in \mathcal{T}_h,$$

$$\eta_E := \begin{cases} h_E^{1/2} \|[n \cdot a \, \nabla u_h]\|_{0,E} \, , \; E \in \mathcal{E}_h(\Omega), \\ h_E^{1/2} \|g_E - n \cdot a \, \nabla u_h\|_{0,E} \, , \; E \in \mathcal{E}_h(\Gamma_N) \end{cases} .$$

The a posteriori error analysis further invokes the data oscillations

$$\text{osc}_h := \Big(\sum_{T \in \mathcal{T}_h} \text{osc}_T^2(f) + \sum_{E \in \mathcal{E}_h(\Gamma_N)} \text{osc}_E^2(g) \Big)^{1/2},$$

where $\text{osc}_T(f)$ and $\text{osc}_E(g)$ are given by

$$\text{osc}_T(f) := h_T \, \|f - f_T\|_{0,T}, \quad \text{osc}_E(g) := h_E^{1/2} \, \|g - g_E\|_{0,E}.$$

Clément's Quasi-Interpolation Operator I

For $p \in \mathcal{N}_h(\Omega) \cup \mathcal{N}_h(\Gamma_N)$ we denote by φ_p the basis function in $S_{1,\Gamma_D}(\Omega; \mathcal{T}_h)$ with supporting point p, and we refer to D_p as the set

$$D_p \ := \bigcup \{ \ T \in \mathcal{T}_h \ | \ p \in \mathcal{N}_h(T) \ \}.$$

We refer to π_p as the L^2-projection onto $P_1(D_p)$, i.e.,

$$(\pi_p(v), w)_{0,D_p} \ = \ (v, w)_{0,D_p} \quad , \quad w \in P_1(D_p),$$

where $(\cdot,\cdot)_{0,D_p}$ stands for the L^2-inner product on $L^2(D_p) \times L^2(D_p)$. Then, Clément's interpolation operator P_C is defined as follows

$$P_C \ : \ L^2(\Omega) \longrightarrow S_{1,\Gamma_D}(\Omega, \mathcal{T}_h), \quad P_C v := \sum_{p \in \mathcal{N}_h(\Omega) \cup \mathcal{N}_h(\Gamma_N)} \pi_P(v) \ \varphi_P.$$

Clément's Quasi-Interpolation Operator II

Theorem. Let $v \in H^1_{0,\Gamma_D}(\Omega)$. Then, for Clément's interpolation operator there holds

$$\|P_C\, v\|_{0,T} \le C\, \|v\|_{0,D_T^{(1)}}, \quad \|P_C\, v\|_{0,E} \le C\, \|v\|_{0,D_E^{(1)}}, \quad \|\nabla P_C v\|_{0,T} \le C\, \|\nabla v\|_{0,D_T^{(1)}},$$

$$\|v - P_C\, v\|_{0,T} \le C\, h_T\, \|v\|_{1,D_T^{(1)}}, \quad \|v - P_C\, v\|_{0,E} \le C\, h_E^{1/2}\, \|v\|_{1,D_E^{(1)}}.$$

Further, we have

$$\Big(\sum_{T \in \mathcal{T}_h} \|v\|^2_{\mu,D_T^{(1)}} \Big)^{1/2} \le C\, \|v\|_{\mu,\Omega}, \quad 0 \le \mu \le 1,$$

$$\Big(\sum_{E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)} \|v\|^2_{\mu,D_E^{(1)}} \Big)^{1/2} \le C\, \|v\|_{\mu,\Omega}, \quad 0 \le \mu \le 1.$$

where $D_T^{(1)} := \bigcup \{\, T' \in \mathcal{T}_h \mid \mathcal{N}_h(T') \cap \mathcal{N}_h(T) \ne \emptyset \,\}$, $D_E^{(1)} := \bigcup \{\, T' \in \mathcal{T}_h \mid \mathcal{N}_h(E) \cap \mathcal{N}_h(T') \ne \emptyset \,\}$.

Element and Edge Bubble Functions I

The element bubble function ψ_T is defined by means of the barycentric coordinates $\lambda_i^T, 1 \leq i \leq 3$, according to

$$\psi_T \;:=\; 27\,\lambda_1^T\,\lambda_2^T\,\lambda_3^T.$$

Note that $\mathrm{supp}\;\psi_T = T_{int}$, i.e., $\psi_T\,|_{\partial T} = 0\;,\;\;T \in \mathcal{T}_h$. On the other hand, for $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ and $T \in \mathcal{T}_h$ such that $E \subset \partial T$ and $p_i^E \in \mathcal{N}_h(E)\;,\;\;1 \leq i \leq 2$, we introduce the edge-bubble functions ψ_E

$$\psi_E \;:=\; 4\,\lambda_1^T\,\lambda_2^T.$$

Note that $\psi_E\,|_{E'} = 0$ for $E' \in \mathcal{E}_h(T), E' \neq E$.

Element and Edge Bubble Functions II

The bubble functions ψ_T and ψ_E have the following important properties that can be easily verified taking advantage of the affine equivalence of the finite elements:

Lemma. There holds

$$\|p_h\|_{0,T}^2 \le C \int_T p_h^2 \, \psi_T \, dx, \quad p_h \in P_1(T),$$

$$\|p_h\|_{0,E}^2 \le C \int_E p_h^2 \, \psi_E \, d\sigma, \quad p_h \in P_1(E),$$

$$|\, p_h \, \psi_T \,|_{1,T} \le C \, h_T^{-1} \, \|p_h\|_{0,T}, \quad p_h \in P_1(T),$$

$$\|p_h \, \psi_T\|_{0,T} \le C \, \|p_h\|_{0,T}, \quad p_h \in P_1(T),$$

$$\|p_h \, \psi_E\|_{0,E} \le C \, \|p_h\|_{0,E} \quad , \quad p_h \in P_1(E).$$

Element and Edge Bubble Functions III

For functions $p_h \in P_1(E)$, $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ we further need an extension $p_h^E \in L^2(T)$ where $T \in \mathcal{T}_h$ such that $E \subset \partial T$. For this purpose we fix some $E' \subset \partial T$, $E' \neq E$, and for $x \in T$ denote by x_E that point on E such that $(x - x_E) \parallel E'$. For $p_h \in P_1(E)$ we then set

$$p_h^E \; := \; p_h(x_E).$$

Further, for $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ we define $D_E^{(2)}$ as the union of elements $T \in \mathcal{T}_h$ containing E as a common edge

$$D_E^{(2)} \; := \; \bigcup \{ \; T \in \mathcal{T}_h \; | \; E \in \mathcal{E}_h(T) \; \}.$$

Element and Edge Bubble Functions IV

Lemma. There holds

$$| \, p_h^E \, \psi_E \, |_{1,D_E^{(2)}} \leq C \, h_E^{-1/2} \, \|p_h\|_{0,e}, \quad p_h \in P_1(E),$$

$$\|p_h^E \, \psi_E\|_{0,D_E^{(2)}} \leq C \, h_E^{1/2} \, \|p_h\|_{0,E}, \quad p_h \in P_1(E).$$

Further, for all $v \in V$ and $\mu = 0, 1$ there holds

$$\Big(\sum_{E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)} h_E^{1-\mu} \, \|v\|^2_{\mu, D_E^{(2)}} \Big)^{1/2} \leq C \, \Big(\sum_{T \in \mathcal{T}_h} h_T^{1-\mu} \, \|v\|^2_{\mu, T} \Big)^{1/2}.$$

Step MARK of the Adaptive Cycle: Bulk Criterion

Given a universal constant $0 < \Theta < 1$, specify a set $\mathcal{M}_T$ of elements and a set $\mathcal{M}_E$ of edges such that (bulk criterion, Dörfler marking)

$$\Theta \Big(\sum_{T \in \mathcal{T}_H(\Omega)} \eta_T^2 + \sum_{E \in \mathcal{E}_H(\Omega)} \eta_E^2 \Big) \le \sum_{T \in \mathcal{M}_T} \eta_T^2 + \sum_{E \in \mathcal{M}_E} \eta_E^2 \, .$$

Step REFINE of the Adaptive Cycle: Refinement Rules

- Any $T \in \mathcal{M}_T, E \in \mathcal{M}_E$ is refined by bisection.
- Further bisection is used to create a geometrically conforming triangulation $\mathcal{T}_h(\Omega)$.

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Adaptive Finite Element Methods for Unconstrained Optimal Elliptic Control Problems

Ronald H.W. Hoppe[1,2]

[1] Department of Mathematics, University of Houston
[2] Institute of Mathematics, University of Augsburg

Institute for Mathematics and its Applications

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Elliptic Optimal Control Problems: Unconstrained Case

Let Ω be a bounded polygonal domain with boundary $\Gamma = \partial\Omega$. Given a desired state $y^d \in L^2(\Omega)$, $f \in L^2\Omega)$, and $\alpha > 0$, find $(y,u) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\inf_{(y,u)} J(y,u) \;:=\; \frac{1}{2}\int_\Omega |y - y^d|^2 \,dx + \frac{\alpha}{2}\int_\Omega |u|^2 \,dx,$$

$$\begin{aligned} \text{subject to} \quad -\Delta y &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Reduced Optimality Conditions in y and p

Substituting u in the state equation by $p = \alpha u$, we arrive at the following system of two variational equations:

$$a(y,v) - \alpha^{-1}(p,v)_{0,\Omega} = \ell_1(v), \quad v \in V := H_0^1(\Omega) \ ,$$
$$a(p,w) + (y,w)_{0,\Omega} = \ell_2(w), \quad w \in V,$$

where the functionals $\ell_\nu : V \to \mathbb{R}, 1 \le \nu \le 2,$ are given by

$$\ell_1(v) := 0, \ v \in V, \quad \ell_2(w) := (y^d, w)_{0,\Omega}, \ w \in V.$$

The operator-theoretic formulation reads

$$\mathcal{L}(y,p) = (\ell_1, \ell_2)^T,$$

where the operator $\mathcal{L} : V \times V \to V^* \times V^*$ is defined according to

$$(\mathcal{L}(y,p))(v,w) := a(y,v) - \alpha^{-1}(p,v)_{0,\Omega} + a(p,w) + (y,w)_{0,\Omega}.$$

Operator Theoretic Formulation of the Optimality System I

Theorem. The operator $\mathcal{L}$ is a continuous, bijective linear operator. Hence, for any $(\ell_1, \ell_2) \in V^* \times V^*$ the system admits a unique solution $(y, p) \in V \times V$. The solution depends continuously on the data according to

$$\|(y, p)\|_{V \times V} \leq \; C \; \|(\ell_1, \ell_2)\|_{V^* \times V^*}.$$

Proof. The linearity and continuity are straightforward. For the proof of the inf-sup condition, we choose $v = \alpha y - p$ and $w = p + y$. It follows that

$$(\mathcal{L}(y, p))(\alpha y - p, y + p) = \alpha \; a(y, y) + a(p, p) + (y, y)_{0,\Omega} + \alpha^{-1} \; (p, p)_{0,\Omega},$$

which allows to conclude.

Operator Theoretic Formulation of the Optimality System II

Corollary. Let $(y_h, p_h) \in V_h \times V_h$, $V_h \subset V$, be an approximate solution of $(y, p) \in V \times V$. Then, there holds

$$\|(y - y_h, p - p_h)\|_{V \times V} \le C \; \|(\mathcal{R}es_1, \mathcal{R}es_2)\|_{V^* \times V^*},$$

where the residuals $\mathcal{R}es_1 \in V^*$, $\mathcal{R}es_2 \in V^*$ are given by

$$\begin{aligned} \mathcal{R}es_1(v) \; &:= \; \ell_1(v) - a(y_h, v) + \alpha^{-1}(p_h, v)_{0,\Omega}, \;\; v \in V, \\ \mathcal{R}es_2(w) \; &:= \; \ell_2(w) - a(p_h, w) - (y_h, w)_{0,\Omega}, \;\; w \in W. \end{aligned}$$

Proof. The assertion is an immediate consequence of the previous theorem.

Using Galerkin orthogonality and Clément's quasi-interpolation operator P_C, for the first residual Res_1 we find

$$Res_1(v) = \sum_{T\in\mathcal{T}_h(\Omega)} (f, v - P_C v)_{0,T} - \sum_{T\in\mathcal{T}_h(\Omega)} \Big(a(u_h, v - P_C v) + \alpha^{-1}(p_h, v - P_C v)_{0,T}\Big).$$

By an elementwise application of Green's formula and the local approximation properties of P_C it follows that

$$\|Res_1\|_{V^*} \le C\Big(\sum_{T\in\mathcal{T}_h(\Omega)} \eta_{T,1}^2 + \sum_{E\in\mathcal{E}_h(\Omega)} \eta_{E,1}^2\Big)^{1/2},$$

The local residuals are given by

$$\eta_{T,1} := h_T \, \|\Delta y_h + u_h\|_{0,T},$$
$$\eta_{E,1} := h_E^{1/2} \, \|n \cdot [\nabla y_h]\|_{0,E}.$$

Likewise, for the second residual Res_2 we obtain

$$\|\mathrm{Res}_2\|_{V^*} \leq C\Big(\sum_{T\in\mathcal{T}_h(\Omega)} \eta_{T,2}^2 + \sum_{E\in\mathcal{E}_h(\Omega)} \eta_{E,2}^2 \Big)^{1/2},$$

where the local residuals are given by

$$\begin{aligned} \eta_{T,2} &:= h_T \ \|y^d + \Delta p_h - y_h\|_{0,T}, \ T \in \mathcal{T}_h(\Omega), \\ \eta_{E,2} &:= h_E^{1/2} \ \|n \cdot [\nabla p_h]\|_{0,E}, \ E \in \mathcal{E}_h(\Omega). \end{aligned}$$

Reliability of the Residual-Type A Posteriori Error Estimator

Theorem. Let $(y,p) \in V \times V$ and $(y_h,p_h) \in V_h \times V_h$ be the solutions of the continuous and discrete optimality system, respectively. Then, there holds

$$\|(y - y_h, p - p_h)\|_{V \times V} \leq C\eta_h,$$

where the estimator η_h is given by

$$\eta_h := \Big(\sum_{T \in \mathcal{T}_h(\Omega)} (\eta_{T,1}^2 + \eta_{T,2}^2) + \sum_{E \in \mathcal{E}_h(\Omega)} (\eta_{E,1}^2 + \eta_{E,2}^2)\Big)^{1/2}.$$

Efficiency of the Residual-Type A Posteriori Error Estimator I

Lemma. Let $(y,p) \in V \times V$ and $(y_h,p_h) \in V_h \times V_h$ be the solutions of the continuous and discrete optimality system, respectively. Then, there exists a positive constant c depending only on the shape regularity of $\{\mathcal{T}_h(\Omega)\}$ such that for $T \in \mathcal{T}_h(\Omega)$

$$\eta_{T,1}^2 \;\le\; c\;\left(|y - y_h|_{1,T}^2 + h_T^2\;\|u - u_h\|_{0,T}^2\right).$$

Proof. Setting $z_h := u_h|_T \psi_T$ and observing that $\Delta y_h|_T = 0$, Green's formula and the fact that z_h is an admissible test function imply

$$\begin{aligned}
&\eta_{T,1}^2 = h_T^2\;\|u_h\|_{0,T}^2 \le c\;h_T^2\;\;(u_h + \Delta y_h, z_h)_{0,T} = c\;h_T^2\;(-a(y_h,z_h) + (u,z_h)_{0,T}\\
&+(u_h - u, z_h)_{0,T}) = c\;h_T^2\;(a(y - y_h, z_h) + (u_h - u, z_h)_{0,T})\\
&\le\;c(\;h_T^2\;|y - y_h|_{1,T}|z_h|_{1,T} + h_T^2\;\|u - u_h\|_{0,T}\;\|z_h\|_{0,T}).
\end{aligned}$$

Proof cont'd. By the property of the element bubble function

$$| p_h \psi_T |_{1,T} \leq c \, h_T^{-1} \, \|p_h\|_{0,T} \quad , \quad p_h \in P_1(T),$$

and Young's inequality we obtain

$$h_T^2 \, \|u_h\|_{0,T}^2 \leq c(|y - y_h|_{1,T}^2 + h_T^2 \|u - u_h\|_{0,T}^2) + \frac{1}{2} \, h_T^2 \, \|u_h\|_{0,T}^2,$$

which gives the assertion.

Efficiency of the Residual-Type A Posteriori Error Estimator II

Lemma. Let $(y, p) \in V \times V$ and $(y_h, p_h) \in V_h \times V_h$ be the solutions of the continuous and discrete optimality system, respectively. Then, there exists a positive constant c depending only on the shape regularity of $\{\mathcal{T}_h(\Omega)\}$ such that for $T \in \mathcal{T}_h(\Omega)$

$$\eta_{T,2}^2 \;\leq\; c \; (|p - p_h|_{1,T}^2 + h_T^2 \; \|y - y_h\|_{0,T}^2 + osc_T^2),$$

where

$$osc_T := h_T \; \|y^d - y_h^d\|_{0,T}, \quad T \in \mathcal{T}_h(\Omega).$$

Proof. The assertion can be proved along the same lines as in the previous lemma.

Efficiency of the Residual-Type A Posteriori Error Estimator III

Lemma. Let $(y, p) \in V \times V$ and $(y_h, p_h) \in V_h \times V_h$ be the solutions of the continuous and discrete optimality system, respectively. Then, there exists a positive constant c depending only on the shape regularity of $\{\mathcal{T}_h(\Omega)\}$ such that for $E \in \mathcal{E}_h(\Omega)$

$$\eta_{E,1}^2 \;\leq\; c(|y - y_h|_{1,\omega_E}^2 + h_E^2 \;\|u - u_h\|_{0,\omega_E}^2 + \sum_{\nu=1}^{2} \eta_{T_\nu,1}^2) \;.$$

Proof. We set $\zeta_E := (n_E \cdot [\nabla y_h])|_E$ and $z_h := \tilde{\zeta}_E \psi_E$. Then, applying Green's formula and observing that z_h is an admissible test function, we find

$$\eta_{E,1}^2 = h_E \|n_E \cdot [\nabla y_h]\|_{0,E}^2 \leq c \; h_E \; (n_E \cdot [\nabla y_h], \zeta_E \psi_E)_{0,E} = c \; h_E \; \sum_{\nu=1}^{2} (n_{\partial T_\nu} \cdot [\nabla y_h], z_h)_{0,\partial T_\nu}$$

$$= \; c \; h_E \; (a(y_h - y, z_h) + (u - u_h, z_h)_{0,\omega_E} + (f + u_h, z_h)_{0,\omega_E})$$

$$\leq \; c \; h_E^{1/2} \|\nu_E \cdot [\nabla y_h]\|_{0,E} (|y - y_h|_{1,\omega_E} (h_E \; \|u - u_h\|_{0,\omega_E} + (\sum_{\boldsymbol{\nu}=1}^{2} \eta_{T_{\boldsymbol{\nu}},1}^2)^{1/2})),$$

which allows to conclude.

Efficiency of the Residual-Type A Posteriori Error Estimator IV

Lemma. Let $(y,p) \in V \times V$ and $(y_h,p_h) \in V_h \times V_h$ be the solutions of the continuous and discrete optimality system, respectively. Then, there exists a positive constant c depending only on the shape regularity of $\{\mathcal{T}_h(\Omega)\}$ such that for $E \in \mathcal{E}_h(\Omega)$

$$\eta_{E,2}^2 \;\le\; c\Big(|p - p_h|_{1,\omega_E}^2 + h_E^2\,\|y - y_h\|_{0,\omega_E}^2 + \sum_{\nu=1}^{2} \eta_{T_\nu,2}^2\Big)\ .$$

Proof. The proof is similar to the one in the previous lemma.

Efficiency of the Residual-Type A Posteriori Error Estimator V

Theorem. Let $(y,p) \in V \times V$ and $(y_h,p_h) \in V_h \times V_h$ be the solutions of the continuous and discrete optimality system, respectively. Then, there exist positive constants C and c depending only on Ω and the shape regularity of the triangulations such that

$$\|(y-y_h,p-p_h)\|^2_{V\times V} + \|u-u_h\|^2_{0,\Omega} \geq C\ \eta_h^2 - c\ \mathrm{osc}_h^2.$$

where

$$\mathrm{osc}_h^2 := \sum_{T\in\mathcal{T}_h(\Omega)} \mathrm{osc}_T^2.$$

Proof. Combining the results of the previous four lemmas gives the assertion.

Adaptive Finite Element Methods for Control Constrained Optimal Elliptic Control Problems

Ronald H.W. Hoppe[1,2]

[1] Department of Mathematics, University of Houston
[2] Institute of Mathematics, University of Augsburg

Mathematisches Forschungsinstitut Oberwolfach

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Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Model Problem (Distributed Elliptic Control Problem with Control Constraints)

Given a bounded domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\Gamma = \partial\Omega$, functions $y^d, \psi \in L^2(\Omega)$, and $\alpha > 0$, consider the distributed optimal control problem

$$\begin{aligned}
&\text{Minimize} && J(y,u) := \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u\|^2_{0,\Omega}\,,\\
&\text{over} && (y,u) \in H^1_0(\Omega) \times L^2(\Omega)\,,\\
&\text{subject to} && -\Delta\, y = u\,,\\
& && u \in K := \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}\,.
\end{aligned}$$

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Optimality Conditions for the Distributed Control Problem

There exists an adjoint state $p \in H_0^1(\Omega)$ and an adjoint control $\boldsymbol{\lambda} \in L^2(\Omega)$ such that the quadruple $(y, p, u, \boldsymbol{\lambda})$ satisfies

$$
\begin{aligned}
a(y,v) &= (u,v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) \ , \\
a(p,v) &= -\ (y - y^d, v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) \ , \\
p &= \alpha u + \boldsymbol{\lambda}), \\
\boldsymbol{\lambda} &\in \partial I_K(u) \ .
\end{aligned}
$$

In particular, the following complementarity conditions hold true:

$$
\boldsymbol{\lambda} \in L_+^2(\Omega), \quad \psi - u \geq 0, \quad (\boldsymbol{\lambda}, \psi - u)_{0,\Omega} = 0.
$$

Finite Element Approximation of the Distributed Control Problem

Let $\mathcal{T}_H(\Omega)$ be a shape regular, simplicial triangulation of Ω and let

$$V_H := \{ v_H \in C(\Omega) \mid v_H|_T \in P_1(T) \ , \ T \in \mathcal{T}_H(\Omega) \ , \ v_H|_{\partial\Omega} = 0 \}$$

be the FE space of continuous, piecewise linear finite elements.

Consider the following FE Approximation of the distributed control problem

$$\begin{array}{ll} \text{Minimize} & J(y_h, u_h) := \frac{1}{2} \|y_h - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2} \|u_h\|_{0,\Omega}^2 \ , \\ \text{over} & (y_h, u_h) \in V_h \times V_h \ , \\ \text{subject to} & a(y_h, v_h) = (u_h, v_h)_{0,\Omega} \ , \ v_h \in V_h \ , \\ & u_h \in K_h := \{ v_h \in V_h \mid v_h \leq \psi \text{ a.e. in } \Omega \} \ . \end{array}$$

Optimality Conditions for the FE Discretized Control Problem

There exists an adjoint state $p_h \in V_h$ and an adjoint control $\lambda_h \in V_h$ such that the quadraduple $(y_h, p_h, u_h, \lambda_h)$ satisfies

$$
\begin{aligned}
a(y_h, v_h) &= (u_h, v_h)_{0,\Omega} \quad , \quad v_h \in V_h \ , \\
a(p_h, v_h) &= -\ (y_h - y^d, v_h)_{0,\Omega} \quad , \quad v_h \in V_h \ , \\
p_h &= \alpha u_h \ + \ \lambda_h) \ , \\
\lambda_h &\in \partial I_{K_h}(u_h) \ .
\end{aligned}
$$

The following complementarity conditions hold true:

$$
\lambda_h \geq 0, \quad \psi - u_h \geq 0, \quad (\lambda_h, \psi - u_h)_{0,\Omega} = 0.
$$

The A Posteriori Error Estimator

Element and Edge Residuals for the State and the Adjoint State

(i) Element and edge residuals for the state y

$$\eta_y := \Big(\sum_{T \in \mathcal{T}_h(\Omega)} \eta_{y,T}^2 + \sum_{E \in \mathcal{E}_h(\Omega)} \eta_{y,E}^2 \Big)^{1/2}$$

$$\eta_{y,T} := \underbrace{h_T \, \|u_h\|_{0,T}}_{\text{element residuals}} \;,\; T \in \mathcal{T}_h(\Omega) \quad , \quad \eta_{y,E} := \underbrace{h_E^{1/2} \, \|\nu_E \cdot [\nabla y_h]\|_{0,E}}_{\text{edge residuals}} \;,\; E \in \mathcal{E}_h(\Omega)$$

(ii) Element and edge residuals for the adjoint state p

$$\eta_p := \Big(\sum_{T \in \mathcal{T}_h(\Omega)} \eta_{p,T}^2 + \sum_{E \in \mathcal{E}_h(\Omega)} \eta_{p,E}^2 \Big)^{1/2}$$

$$\eta_{p,T}^{(1)} := \underbrace{h_T \, \|y^d - y_h\|_{0,T}}_{\text{element residuals}} \quad , \quad \eta_{p,E} := \underbrace{h_E^{1/2} \, \|\nu_E \cdot [\nabla p_h]\|_{0,E}}_{\text{edge residuals}} \;,\; E \in \mathcal{E}_h(\Omega)$$

Reliability of the Error Estimator

Reliability of the A Posteriori Error Estimator

Theorem Let (y, p, u, λ) be the solution of the distributed control problem and $(y_h, p_h, u_h, \lambda_h)$ be the finite element approximation with respect to the triangulation $\mathcal{T}_h(\Omega)$. Further, let η be the residual type error estimator.

Then, there exists a positive constant C, depending only on α , Ω and on the shape regularity of the triangulation $\mathcal{T}_h(\Omega)$ such that

$$|y - y_h|_{1,\Omega}^2 + |p - p_h|_{1,\Omega}^2 + \|u - u_h\|_{0,\Omega}^2 + \|\lambda - \lambda_h\|_{0,\Omega}^2 \leq C\, \eta^2.$$

Important Tool in the Error Analysis: Intermediate State and Adjoint State

Given a discrete control $u_h \in V_h$, the intermediate state $y(u_h) \in H_0^1(\Omega)$ and the intermediate adjoint state $p(u_h) \in H_0^1(\Omega)$ are the unique solutions of the variational equations

$$\begin{aligned} a(y(u_h),v) &= (u_h,v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) \ , \\ a(p(u_h),v) &= -\,(y(u_h)-y^d,v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) \ . \end{aligned}$$

Lemma. Let $y(u_h)$ and $p(u_h)$ be the intermediate state and adjoint state. Then, we have

$$(p-p(u_h),u-u_h)_{0,\Omega} \ = \ -\ \|y-y(u_h)\|_{0,\Omega}^2 \ \le \ 0 \ .$$

Proof: Obviously, there holds

$$a(y-y(u_h),v_1) \ = \ (u-u_h,v_1)_{0,\Omega} \quad , \quad a(p-p(u_h),v_2) \ = \ (y(u_h)-y,v_2)_{0,\Omega} \quad , \quad v_1,v_2 \in H_0^1(\Omega) \ .$$

The assertion follows readily by choosing $v_1 := p-p(u_h)$ and $v_2 := y-y(u_h)$.

Proof. Since $\mathbf{u} = \alpha^{-1}(\mathbf{p} - \boldsymbol{\lambda}), \ \mathbf{u}_h = \alpha^{-1}(\mathbf{p}_h - \boldsymbol{\lambda}_h)$, we have

$$\alpha \|\mathbf{u} - \mathbf{u}_h\|^2_{0,\Omega} = (\boldsymbol{\lambda}_h - \boldsymbol{\lambda}, \mathbf{u} - \mathbf{u}_h)_{0,\Omega} + (\mathbf{p} - \mathbf{p}_h, \mathbf{u} - \mathbf{u}_h)_{0,\Omega}.$$

Using the complementarity conditions for $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_h$, we find

$$(\boldsymbol{\lambda}_h - \boldsymbol{\lambda}, \mathbf{u} - \mathbf{u}_h)_{0,\Omega} = \underbrace{(\boldsymbol{\lambda}_h, \mathbf{u} - \psi)_{0,\Omega}}_{\leq 0} + \underbrace{(\boldsymbol{\sigma}_h, \psi - \mathbf{u}_H)_{0,\Omega}}_{= 0}$$
$$- \underbrace{(\boldsymbol{\lambda}, \mathbf{u} - \psi)_{0,\Omega}}_{= 0} - \underbrace{(\lambda, \psi - \mathbf{u}_h)_{0,\Omega}}_{\geq 0} \leq 0.$$

Moreover, for the remaining term there holds

$$(\mathbf{p} - \mathbf{p}_h, \mathbf{u} - \mathbf{u}_h)_{0,\Omega} \leq \underbrace{(\mathbf{p} - \mathbf{p}(\mathbf{u}_h), \mathbf{u} - \mathbf{u}_h)_{0,\Omega}}_{\leq 0} + (\mathbf{p}(\mathbf{u}_h) - \mathbf{p}_h, \mathbf{u} - \mathbf{u}_h)_{0,\Omega} \ ,$$

Numerical Example: Distributed Control Problem with Control Constraints

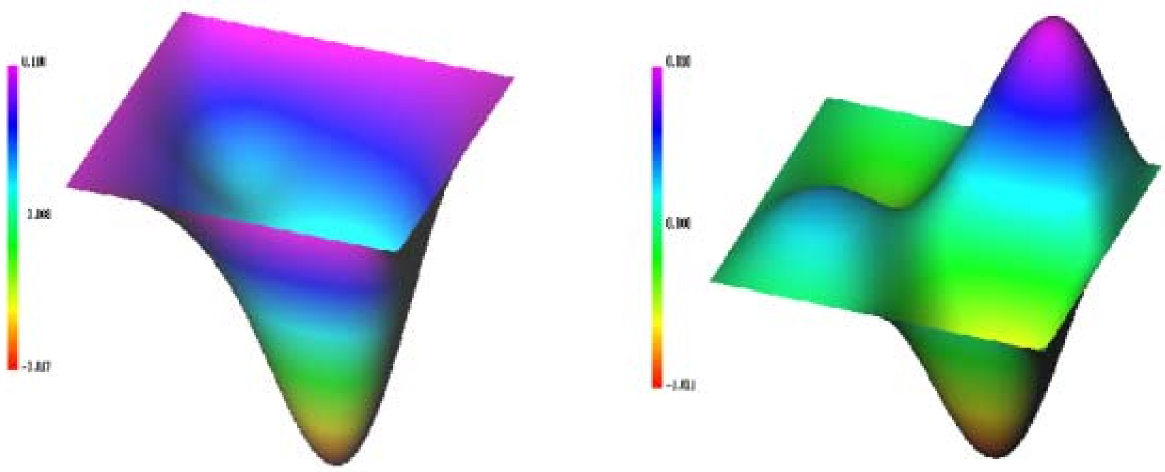
$$\begin{aligned}
&\text{Minimize} && J(y,u) := \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u - u^d\|^2_{0,\Omega}\ , \\
&\text{over} && (y,u) \in H^1_0(\Omega) \times L^2(\Omega)\ , \\
&\text{subject to} && -\Delta\, y = f + u\ , \\
& && u \in K := \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}\ .
\end{aligned}$$

Data:

$$y^d := \sin(2\pi x_1)\,\sin(2\pi x_2)\,\exp(2x_1)/6\ ,$$

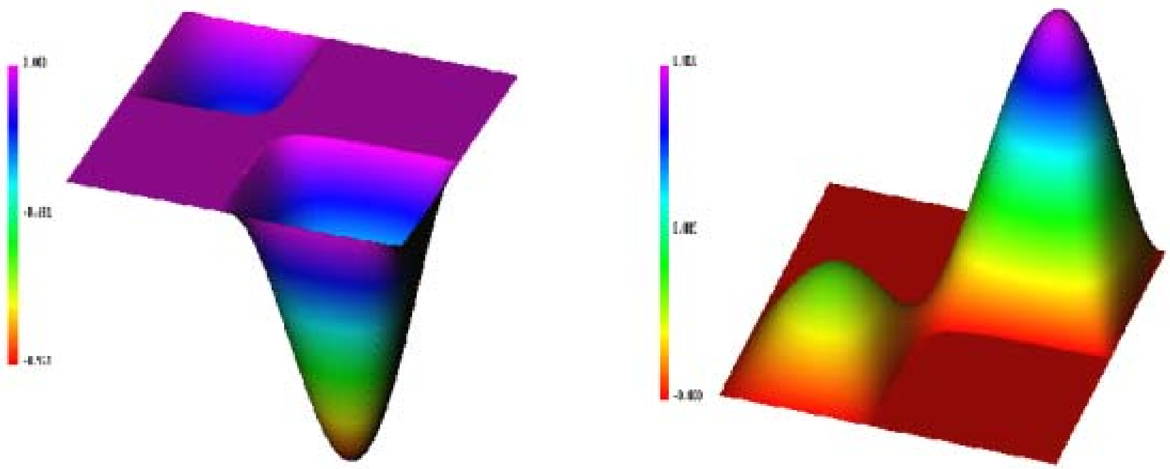
$$\alpha := 0.01\ ,\quad u^d := 0\ ,\quad \psi := 0\ ,\quad f := 0\ .$$

Numerical Results: Distributed Control Problem with Control Constraints I



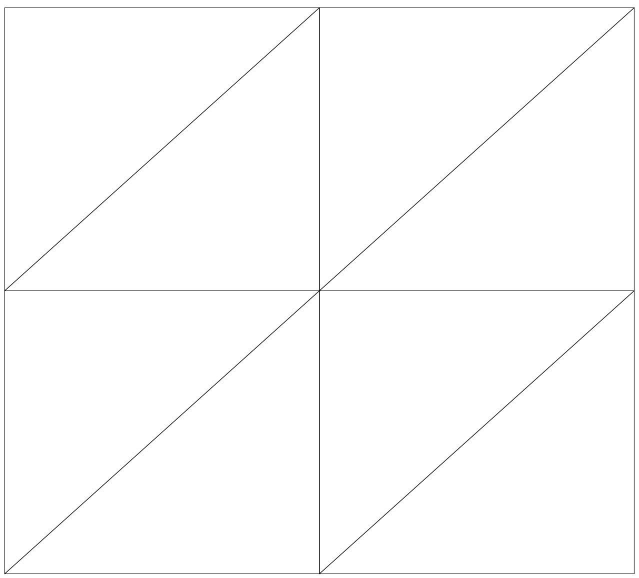
Optimal state (left) and optimal adjoint state (right)

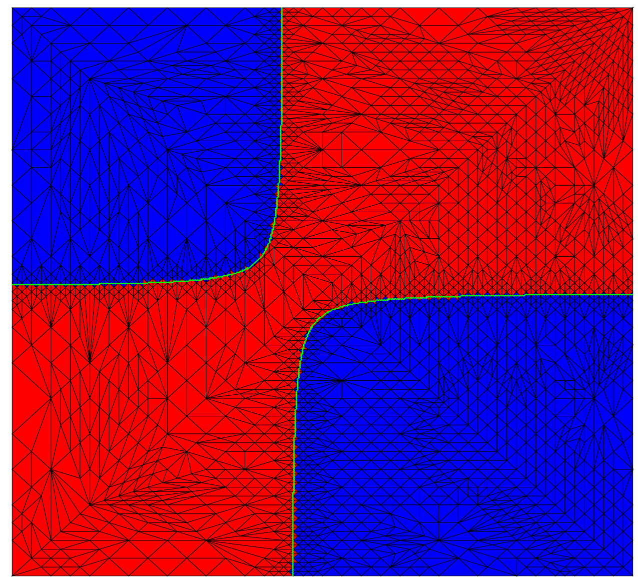
Numerical Results: Distributed Control Problem with Control Constraints I



Optimal control (left) and optimal adjoint control (right)

Numerical Results: Adaptive FEM for a Distributed Control Problem





Initial triangulation and triangulation after 6 refinement steps ($\Theta = 0.6$)

Numerical Results: Distributed Control Problem with Control Constraints I

l	N_{dof}	$\|\|\|z - z_H\|\|\|$	$\|y - y_H\|_1$	$\|p - p_H\|_1$	$\|u - u_H\|_0$	$\|\lambda - \lambda_H\|_0$
0	5	3.24e-01	3.63e-02	3.28e-02	2.52e-01	2.80e-03
1	13	2.27e-01	1.95e-02	1.48e-02	1.91e-01	2.11e-03
2	41	1.24e-01	1.35e-02	1.36e-02	9.59e-02	1.06e-03
3	126	6.19e-02	6.85e-03	7.86e-03	4.68e-02	5.09e-04
4	374	3.57e-02	3.93e-03	4.41e-03	2.65e-02	3.67e-04
5	968	2.50e-02	2.63e-03	2.75e-03	1.88e-02	2.50e-04
6	2553	1.77e-02	1.91e-03	2.32e-03	1.33e-02	1.56e-04
7	5396	1.24e-02	1.30e-03	1.66e-03	9.33e-03	1.16e-04
8	12318	8.60e-03	9.21e-04	1.16e-03	6.45e-03	7.48e-05

Total error, errors in the state, adjoint state, control, adjoint control ($\Theta = 0.7$)

Numerical Results: Distributed Control Problem with Control Constraints I

l	N_{dof}	η_y	η_p	$osc_h(y^d)$
0	5	2.57e-01	4.16e-01	2.83e-01
1	13	1.04e-01	2.04e-01	1.12e-01
2	41	7.95e-02	1.09e-01	2.58e-02
3	126	5.16e-02	6.49e-02	7.12e-03
4	374	3.15e-02	4.10e-02	2.77e-03
5	968	2.13e-02	2.79e-02	1.22e-03
6	2553	1.56e-02	1.92e-02	4.58e-04
7	5396	1.06e-02	1.33e-02	1.87e-04
8	12318	7.56e-03	9.45e-03	8.48e-05

Components of the error estimator and data oscillations ($\Theta = 0.7$)

Numerical Results: Distributed Control Problem with Control Constraints II

$$
\begin{array}{ll}
\text{Minimize} & J(y,u) := \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u - u^d\|^2_{0,\Omega} \\
\text{over} & (y,u) \in H^1_0(\Omega) \times L^2(\Omega) \\
\text{subject to} & -\Delta y = f + u \quad \text{in } \Omega\,, \\
 & u \in K := \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}
\end{array}
$$

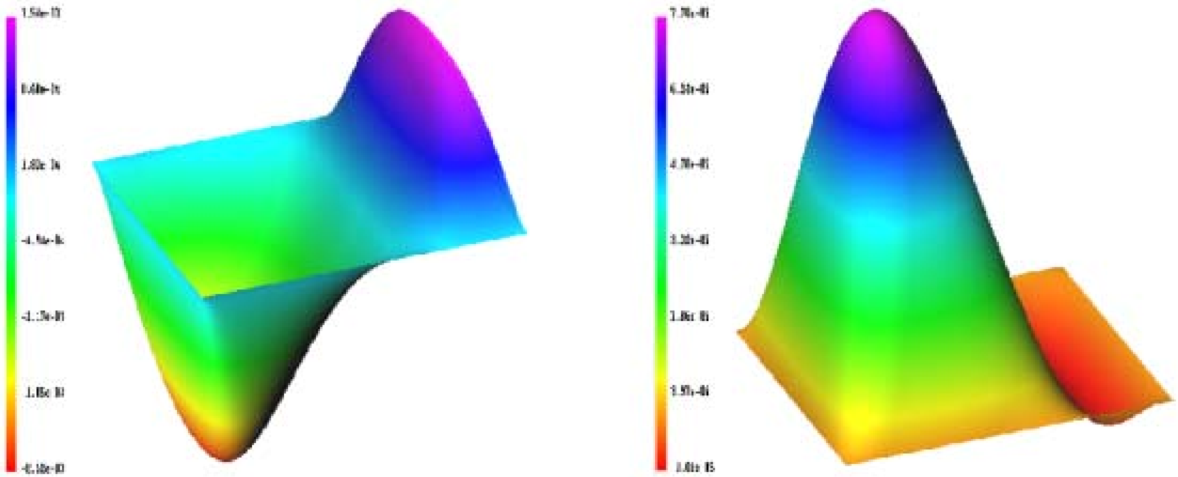
Data: $\Omega := (0,1)^2 \quad , \quad y^d := 0 \quad , \quad u^d := \hat{u} + \alpha^{-1}(\hat{\sigma} + \Delta^{-2}\hat{u})\ ,$

$$
\psi := \begin{cases} (x_1 - 0.5)^8 & , \quad (x_1,x_2) \in \Omega_1, \\ (x_1 - 0.5)^2 & , \quad \text{otherwise} \end{cases} \quad , \quad \alpha := 0.1 \quad , \quad f := 0
$$

$$
\hat{u} := \begin{cases} \psi & , \quad (x_1,x_2) \in \Omega_1 \cup \Omega_2, \\ -1.01\,\psi & , \quad \text{otherwise} \end{cases} \quad , \quad \hat{\sigma} := \begin{cases} 2.25\,(x_1 - 0.75) \cdot 10^{-4} & , \quad (x_1,x_2) \in \Omega_2, \\ 0 & , \quad \text{otherwise} \end{cases} \quad ,
$$

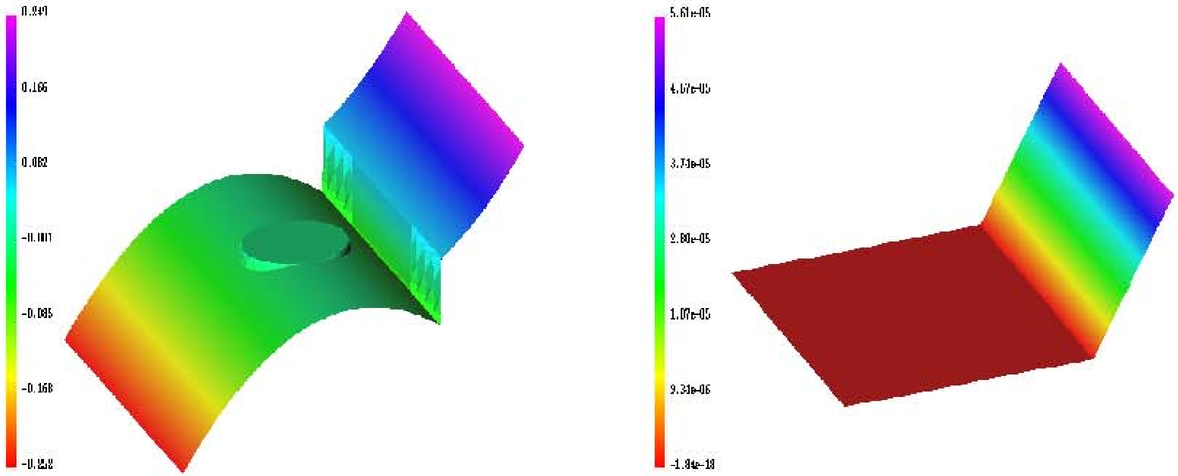
$$
\Omega_1 := \{(x_1,x_2) \in \Omega \mid ((x_1 - 0.5)^2 + (x_2 - 0.5)^2)^{1/2} \le 0.15\} \quad , \quad \Omega_2 := \{(x_1,x_2) \in \Omega \mid x_1 \ge 0.75\}\ .
$$

Numerical Results: Distributed Control Problem with Control Constraints II



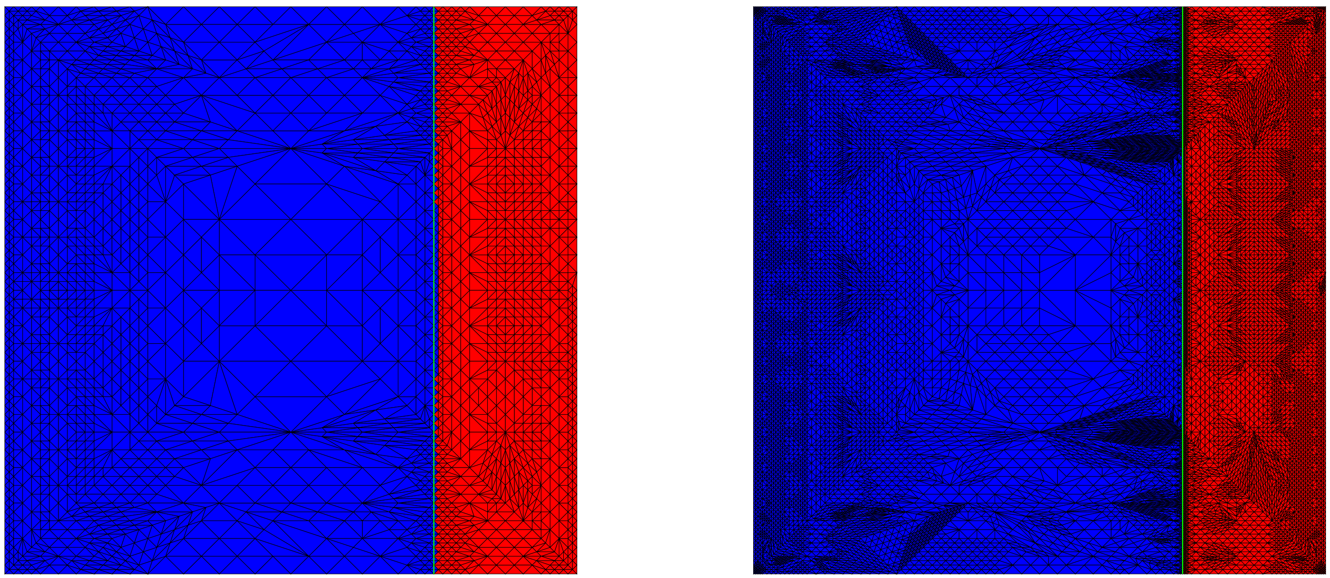
Optimal state (left) and optimal adjoint state (right)

Numerical Results: Distributed Control Problem with Control Constraints II



Optimal control (left) and optimal adjoint control (right)

Numerical Results: Distributed Control Problem with Control Constraints II



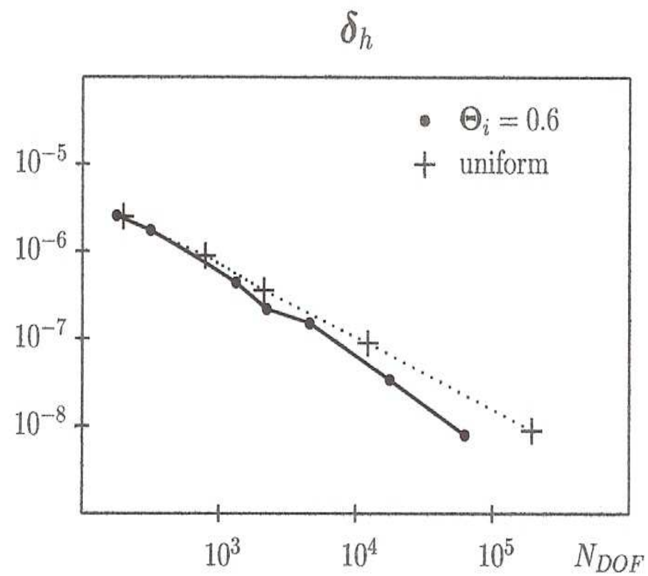
Grid after 6 (left) and 8 (right) refinement steps ($\Theta = 0.6$)

Numerical Results: Distributed Control Problem with Control Constraints II

l	N_{dof}	$\|\|\|z - z_H\|\|\|$	$\|y - y_H\|_1$	$\|p - p_H\|_1$	$\|\|u - u_H\|\|_0$	$\|\|\lambda - \lambda_H\|\|_0$
0	5	8.50e-02	9.31e-03	1.87e-04	7.55e-02	1.31e-05
1	13	5.35e-02	6.87e-03	1.05e-04	4.66e-02	8.86e-06
2	41	3.12e-02	3.84e-03	6.04e-05	2.73e-02	4.62e-06
3	102	2.09e-02	2.39e-03	4.11e-05	1.84e-02	2.28e-06
4	291	1.39e-02	1.58e-03	2.94e-05	1.23e-02	1.38e-06
5	873	9.14e-03	9.71e-04	1.93e-05	8.15e-03	8.35e-07
6	2325	6.08e-03	6.14e-04	1.21e-05	5.46e-03	5.52e-07
7	5813	4.04e-03	3.97e-04	7.56e-06	3.63e-03	3.68e-07
8	14513	2.53e-03	2.60e-04	5.19e-06	2.26e-03	2.32e-07

Total error, errors in the state, adjoint state, control, adjoint control ($\Theta = 0.6$)

Numerical Results: Distributed Control Problem with Control Constraints II



Decrease in the quantity of interest versus total number of DOFs

The Goal Oriented Dual Weighted Approach for State Constrained Elliptic Optimal Control Problems

Ronald H.W. Hoppe[1,2]

[1] Department of Mathematics, University of Houston
[2] Institute of Mathematics, University of Augsburg

Mathematisches Forschungsinstitut Oberwolfach

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Goal-Oriented Dual Weighted Approach

Goal-Oriented Dual Weighted Approach I

The goal oriented dual weighted approach allows to control the error $e_u := u - u_h$ with respect to a rather general error functional or output functional

$$J \ : \ V \subseteq H^1(\Omega) \to \mathbb{R}.$$

The goal oriented dual weighted approach strongly uses the solution $z \in V$ of the associated dual problem

$$a(v, z) \ = \ J(v) \quad , \quad v \in V,$$

and its finite element approximation $z_h \in V_h$, i.e.,

$$a(v_h, z_h) \ = \ J(v_h), \quad v_h \in V_h.$$

Using Galerkin orthogonality, we readily deduce that

$$J(e_u) \ = \ a(e_u, z) \ = \ a(e_u, z - v_h) \ = \ r(z - v_h), \quad v_h \in V_h,$$

where $r(\cdot)$ stands for the residual with respect to the computed finite element approximation u_h.

Goal-Oriented Dual Weighted Approach II

Theorem. Let $u_h \in V_h := S_{1,\Gamma}(\Omega; \mathcal{T}_h(\Omega))$ be the conforming P1 approximation of the solution $u \in H_0^1(\Omega)$ of Poisson's equation with $f \in L^2(\Omega)$ and homogeneous Dirichlet boundary data. Then, the following error representation holds true

$$J(e_u) \;=\; \sum_{T \in \mathcal{T}_h(\Omega)} \Big((r_T, z - v_h)_{0,T} + (r_{\partial T}, z - v_h)_{0,\partial T} \Big), \quad v_h \in V_h,$$

where the element residuals r_T and the edges residuals $r_{\partial T}$ are given by

$$r_T := f, \quad T \in \mathcal{T}_h(\Omega), \quad r_{\partial T}|_E := \begin{cases} \frac{1}{2}\, \nu_E \cdot [\nabla u_h] \;, & E \in \mathcal{E}_h(\partial T \cap \Omega) \\ 0 \;, & E \in \mathcal{E}_h(\partial T \cap \Gamma) \end{cases}$$

Moreover, we have the error estimate

$$|J(e_u)| \;\leq\; \eta_{DW} \;:=\; \sum_{T \in \mathcal{T}_h(\Omega)} \omega_T \; \rho_T,$$

where for $v_h \in V_h$ the element residuals ρ_T and the weights ω_T read

$$\rho_T := \Big(\|r_T\|_{0,T}^2 \;+\; h_T^{-1} \; \|r_{\partial T}\|_{0,\partial T}^2 \Big)^{1/2}, \quad \omega_T := \Big(\|z - v_h\|_{0,T}^2 \;+\; h_T \; \|z - v_h\|_{0,\partial T}^2 \Big)^{1/2}.$$

Goal-Oriented Dual Weighted Approach III

We remark that the previous result is not really a posteriori, since the solution $z \in V$ of the dual solution is not known. Therefore, information about the weights $\omega_T, T \in \mathcal{T}_h(\Omega)$ has to be provided either by means of an a priori analysis or by the numerical solution of the dual problem.

Theorem. Under the assumptions of the previous theorem let the error functional be given by

$$J(v) \; := \; \frac{(\nabla v, \nabla e_u)_{0,\Omega}}{\|\nabla e_u\|_{0,\Omega}}, \quad v \in V.$$

Then, there holds

$$\|\nabla e_u\|_{0,\Omega} \; \leq C \left(\sum_{T \in \mathcal{T}_h(\Omega)} h_T^2 \, \rho_T^2 \right)^{1/2}.$$

Proof. The dual solution $z \in V$ satisfies

$$a(v, z) \; = \; \frac{(\nabla v, \nabla e_u)_{0,\Omega}}{\|\nabla e_u\|_{0,\Omega}}, \quad v \in V,$$

from which we readily deduce the a priori bound

$$\|\nabla z\|_{0,\Omega} \; \leq \; 1.$$

In view of the basic error estimate it follows that

$$J(e_u) = \|\nabla e_u\|_{0,\Omega} \; \leq \Big(\sum_{T \in \mathcal{T}_h(\Omega)} h_T^2 \; \rho_T^2 \Big)^{1/2} \Big(\sum_{T \in \mathcal{T}_h(\Omega)} h_T^{-2} \; \omega_T^2 \Big)^{1/2}.$$

Choosing $v_h = P_C z$, where P_C is Clément's quasi-interpolation operator, we find

$$\inf_{v_h \in V_h} \Big(\sum_{T \in \mathcal{T}_h(\Omega)} (h_T^{-2} \; \|z - v_h\|_{0,T}^2 + h_T^{-1} \; \|z - v_h\|_{0,\partial T}^2 \Big)^{1/2} \; \leq C \; \|\nabla z\|_{0,\Omega}.$$

Using the last inequality in the previous one and observing the error representation gives the assertion.

Goal-Oriented Dual Weighted Approach IV

Theorem. Consider the conforming P1 approximation of Poisson's equation under homogeneous Dirichlet boundary conditions and assume that the solution $u \in V := H_0^1(\Omega)$ is 2-regular. Using the the error functional

$$J(v) := \frac{(v, e_u)_{0,\Omega}}{\|e_u\|_{0,\Omega}}, \quad v \in V,$$

gives rise to the a posteriori error estimate

$$\|e_u\|_{0,\Omega} \le C \left(\sum_{T \in \mathcal{T}_h(\Omega)} h_T^4 \, \rho_T^2 \right)^{1/2}.$$

Goal-Oriented Dual Weighted Approach V

Finally, we apply the goal-oriented dual weighted approach to the pointwise estimation of the error at some point $a \in \Omega$. Given some tolerance $\varepsilon > 0$, we consider the ball

$$K_\varepsilon(a) \; := \; \{x \in \Omega \mid |x - a| < \varepsilon\}$$

around the point a and define the regularized error functional

$$J(v) \; := \; |K_\varepsilon(a)|^{-1} \int\limits_{K_\varepsilon(a)} v \; dx.$$

The dual solution z of $a(v, z) \; = \; J(v)$ behaves like a regularized Green's function

$$z(x) \; \sim \; \log(r(x)) \; , \; r(x) \; := \; \sqrt{|x - a|^2 + \varepsilon^2}.$$

With the residual ρ_T we obtain

$$|(u - u_h)(a)| \; \sim \sum_{T \in \mathcal{T}_h(\Omega)} \frac{h_T^3}{r_T^2} \, \rho_T, \quad r_T := \max_{x \in T} r(x).$$

Department of Mathematics, University of Houston
Institut für Mathematatik, Universität Augsburg

Goal-Oriented Dual Weighted Approach for State Constrained Elliptic Optimal Control Problems

C O N T E N T S

State Constrained Elliptic Control Problems

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Model Problem (Distributed Elliptic Control Problem with State Constraints)

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, and let $A : V \to H^{-1}(\Omega)$, $V := \{v \in H^1(\Omega) \mid v|_{\Gamma_d} = 0\}$, be the linear second order elliptic differential operator $Ay := -\Delta y + cy$, $c \geq 0$, with $c > 0$ or $\mathrm{meas}(\Gamma_D) > 0$. Assume that Ω is such that for each $v \in L^2(\Omega)$ the solution y of $Ay = u$ satisfies $y \in W^{1,r}(\Omega) \cap V$ for some $r > 2$. Moreover, let $u^d, y^d \in L^2(\Omega)$, and $\psi \in W^{1,r}(\Omega)$ such that $\psi|_{\Gamma_D} > 0$ be given functions and let $\alpha > 0$ be a regularization parameter.

Consider the state constrained distributed elliptic control problem

$$\text{Minimize} \quad J(y,u) := \frac{1}{2}\,\|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\,\|u - u^d\|_{0,\Omega}^2 ,$$

$$\text{subject to} \quad Ay = u \text{ in } \Omega \quad , \quad y = 0 \text{ on } \Gamma_D \quad , \quad \nu \cdot \nabla y = 0 \text{ on } \Gamma_N ,$$

$$Iy \in K := \{v \in C(\overline{\Omega}) \mid v(x) \leq \psi(x) \ , \ x \in \overline{\Omega}\} .$$

where I stands for the embedding operator $W^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$.

The Reduced Optimal Control Problem

We introduce the control-to-state map

$$G : L^2(\Omega) \to C(\overline{\Omega}) \quad , \quad y = Gu \ \text{ solves } \ Ay + cy = u \ .$$

We assume that the following Slater condition is satisfied

$$(S) \quad \text{There exists } \ v_0 \in L^2(\Omega) \ \text{ such that } \ Gv_0 \in \text{int}(K) \ .$$

Substituting $y = Gu$ allows to consider the reduced control problem

$$\inf_{u \in U_{ad}} J_{red}(u) \ := \ \frac{1}{2} \, \|Gu - y^d\|^2_{0,\Omega} \ + \ \frac{\alpha}{2} \, \|u - u^d\|^2_{0,\Omega} \ ,$$

$$U_{ad} \ := \ \{v \in L^2(\Omega) \mid (Gv)(x) \leq \psi(x) \ , \ x \in \overline{\Omega}\} \ .$$

Theorem (Existence and uniqueness). The state constrained optimal control problem admits a unique solution $y \in W^{1,r}(\Omega) \cap K$.

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Optimality Conditions for the State Constrained Optimal Control Problem

Theorem. There exists an adjoint state $p \in V^s := \{v \in W^{1,s}(\Omega) \mid v_{\Gamma_D} = 0\}$, where $1/r + 1/s = 1$, and a multiplier $\boldsymbol{\sigma} \in \mathcal{M}_+(\Omega)$ such that

$$\begin{aligned}
(\nabla y, \nabla v)_{0,\Omega} + (cy, v)_{0,\Omega} &= (u, v)_{0,\Omega} \quad , \quad v \in V^s \;, \\
(\nabla p, \nabla w)_{0,\Omega} + (cp, w)_{0,\Omega} &= (y - y^d, w)_{0,\Omega} + \langle \boldsymbol{\sigma}, w \rangle \quad , \quad w \in V^r \;, \\
p + \alpha(u - u^d) &= 0 \;, \\
\langle \boldsymbol{\sigma}, y - \psi \rangle &= 0 \;.
\end{aligned}$$

Proof. The reduced problem can be written in unconstrained form as

$$\inf_{v \in L^2(\Omega)} \widehat{J}(v) \;:=\; J_{red}(v) \;+\; (I_K \circ G)(u)$$

where I_K stands for the indicator function of the constraint set K. The Slater condition and and subdifferential calculus tell us

$$\partial\Big(I_K \circ G\Big)(u) \;=\; G^* \circ \partial I_K(Gu)\ .$$

The optimality condition then reads

$$0 \in \partial\widehat{J}(u) \;=\; J'_{red}(u) \;+\; G^* \circ \partial I_K(Gu)\ .$$

Hence, there exists $\boldsymbol{\sigma} \in \partial I_K(Gu)$ such that

$$\Big(\underbrace{G^*(Gu - y^d + \boldsymbol{\sigma})}_{=:\ p} + \alpha(u - u^d), v\Big)_{0,\Omega} \;=\; 0 \quad , \quad v \in L^2(\Omega)\ .$$

Since $\boldsymbol{\sigma} \in \mathcal{M}(\Omega)$, PDE regularity theory implies $p \in W^{1,s}(\Omega), 1/s + 1/r = 1$.

Finite Element Approximation

Let $\mathcal{T}_\ell(\Omega)$ be a simplicial triangulation of Ω and let

$$V_\ell := \{ v_\ell \in C(\overline{\Omega}) \mid v_\ell|_T \in P_1(T) \, , \, T \in \mathcal{T}_\ell(\Omega) \, , \, v_\ell|_{\Gamma_D} = 0 \}$$

be the FE space of continuous, piecewise linear functions.

Let $u_\ell^d \in V_\ell$ be some approximation of u^d, and let ψ_ℓ be the V_ℓ-interpoland of ψ.

Consider the following FE Approximation of the state constrained control problem

$$\begin{aligned}
&\text{Minimize} && J_\ell(y_\ell, u_\ell) := \frac{1}{2} \|y_\ell - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2} \|u_\ell - u_\ell^d\|_{0,\Omega}^2 \, , \\
&\text{over} && (y_\ell, u_\ell) \in V_\ell \times V_\ell \, , \\
&\text{subject to} && (\nabla y_\ell, \nabla v_\ell)_{0,\Omega} + (c y_\ell, v_\ell)_{0,\Omega} = (u_\ell, v_\ell)_{0,\Omega} \, , \; v_\ell \in V_\ell \, , \\
& && y_\ell \in K_\ell := \{ v_\ell \in V_\ell \mid v_\ell(x) \le \psi_\ell(x) \, , \; x \in \overline{\Omega} \} \, .
\end{aligned}$$

Since the constraints are point constraints associated with the nodal points, the discrete multipliers are chosen from

$$\mathcal{M}_\ell := \{ \mu_\ell \in \mathcal{M}(\Omega) \mid \mu_\ell = \sum_{a \in \mathcal{N}_\ell(\Omega \cup \Gamma_N)} \kappa_a \delta_a \, , \; \kappa_a \in \mathbb{R} \} \, .$$

Representation of the Error in the Quantity of Interest

Primal-Dual Weighted Error Representation I

We set $X := V^r \times L^2(\Omega) \times V^s$ as well as $X_\ell := V_\ell \times V_\ell \times V_\ell$ and introduce the Lagrangians $\mathcal{L} : X \times \mathcal{M}(\Omega) \to \mathbb{R}$ as well as $\mathcal{L}_\ell : X_\ell \times \mathcal{M}_\ell \to \mathbb{R}$ according to

$$\mathcal{L}(x, \boldsymbol{\sigma}) \ := \ J(y, u) \ + \ (\nabla y, \nabla p)_{0,\Omega} \ - \ (u, p)_{0,\Omega} \ + \ \langle \boldsymbol{\sigma}, y - \psi \rangle \ ,$$

$$\mathcal{L}_\ell(x_\ell, \boldsymbol{\sigma}_\ell) \ := \ J_\ell(y_\ell, u_\ell) \ + \ (\nabla y_\ell, \nabla p_\ell)_{0,\Omega} \ - \ (u_\ell, p_\ell)_{0,\Omega} \ + \ \langle \boldsymbol{\sigma}_\ell, y_\ell - \psi_\ell \rangle \ ,$$

where $x := (y, u, p)$ and $x_\ell := (y_\ell, u_\ell, p_\ell)$.

Then, the optimality conditions can be stated as

$$\nabla_x \mathcal{L}(x, \boldsymbol{\sigma})(\varphi) \ = \ 0 \quad , \quad \varphi \in X \ ,$$

$$\nabla_x \mathcal{L}_\ell(x_\ell, \boldsymbol{\sigma}_\ell)(\varphi_\ell) \ = \ 0 \quad , \quad \varphi_\ell \in X_\ell \ .$$

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Primal-Dual Weighted Error Representation II

Theorem. Let $(x, \sigma) \in X$ and $(x_\ell, \sigma_\ell) \in X_\ell$ be the solutions of the continuous and discrete optimality systems, respectively. Then, there holds

$$J(y,u) - J_\ell(y_\ell, u_\ell) \;=\; -\frac{1}{2}\nabla_{xx}\mathcal{L}(x_\ell - x, x_\ell - x) \;+\; \langle \sigma, y_\ell - \psi \rangle \;+\; \mathrm{osc}_\ell^{(1)} \;,$$

where the data oscillations $\mathrm{osc}_\ell^{(1)}$ are given by

$$\mathrm{osc}_\ell^{(1)} \;:=\; \sum_{T \in \mathcal{T}_\ell(\Omega)} \mathrm{osc}_T^{(1)} \;,$$

$$\mathrm{osc}_T^{(1)} \;:=\; (y_\ell - y_\ell^d, y_\ell^d - y^d)_{0,T} \;+\; \frac{1}{2}\,\|y^d - y_\ell^d\|_{0,T}^2 \;+\; \alpha\,(u_\ell - u_\ell^d, u_\ell^d - u^d)_{0,T} \;+\; \frac{\alpha}{2}\,\|u^d - u_\ell^d\|_{0,T}^2 \;.$$

Remark: In the unconstrained case, i.e., $\sigma = \sigma_\ell = 0$, the above result reduces to the error representation in [Becker, Kapp, and Rannacher (2000)].

Interpolation Operators (State Constraints)

We introduce interpolation operators

$$i_\ell^y : V^{\bar{r}} \to V_\ell \ , \ r > \bar{r} > 2 \quad , \quad i_\ell^p : V^{\bar{s}} \to V_\ell \ , \ 0 < \bar{s} < s < 2 \ ,$$

such that for all $y \in V^r$ and $p \in V^s$ there holds

$$\Big(h_T^{r(t-1)} \ \|i_\ell^y y - y\|_{t,r,T}^r \Big)^{1/r} \ \lesssim \ \|y\|_{1,r,D_T} \ , \ 0 \le t \le 1 \ ,$$

$$\Big(h_T^{-r} \ \|i_\ell^y y - y\|_{0,r,T}^r \ + h_T^{-r/2} \ \|i_\ell^y y - y\|_{0,r,\partial T}^r \Big)^{1/r} \ \lesssim \ \|y\|_{1,r,D_T} \ ,$$

$$\Big(h_T^{-s} \ \|i_\ell^p p - p\|_{0,s,T}^s \ + h_T^{-s/2} \ \|i_\ell^p p - p\|_{0,s,\partial T}^s \Big)^{1/s} \ \lesssim \ h_T \ \|p\|_{1,s,D_T} \ ,$$

where $D_T := \{T' \in \mathcal{T}_\ell(\Omega) \mid \mathcal{N}_\ell(T') \cap \mathcal{N}_\ell(T) \neq \emptyset\}$.

Primal-Dual Weighted Error Representation III

Theorem. Under the assumptions of the previous Theorem let $i_\ell^z, z \in \{y,p\}$, be the interpolation operators introduced before. Then, there holds

$$J(y,u) - J_\ell(y_\ell,u_\ell) \;=\; -(r(i_\ell^y y - y) \;+\; r(i_\ell^p p - p)) \;+\; \mu_\ell(x,\sigma) \;+\; osc_\ell^{(1)} \;+\; osc_\ell^{(2)} \;,$$

where $r(i_\ell^y y - y)$ and $r(i_\ell^p p - p)$ stand for the primal-dual weighted residuals

$$r(i_\ell^y y - y) \;:=\; \frac{1}{2}\,\Big((y_\ell - y_\ell^d, i_\ell^y y - y)_{0,\Omega} \;+\; (\nabla(i_\ell^y y - y, \nabla p_\ell)_{0,\Omega} \;+\; \langle \sigma_\ell, i_\ell^y y - y\rangle\Big) \;,$$

$$r(i_\ell^p p - p) \;:=\; \frac{1}{2}\,\Big((\nabla(i_\ell^p p - p, \nabla y_\ell)_{0,\Omega} \;-\; (u_\ell, i_\ell^p p - p)_{0,\Omega}\Big) \;.$$

Moreover, $\mu_\ell(x,\sigma)$ represents the primal-dual mismatch in complementarity

$$\mu_\ell(x,\sigma) \;:=\; \frac{1}{2}\,\Big(\langle \sigma, y_\ell - \psi\rangle \;+\; \langle \sigma_\ell, \psi_\ell - y\rangle\Big) \;,$$

and $osc_\ell^{(2)}$ are further oscillation terms

$$osc_\ell^{(2)} \;:=\; \sum_{T\in\mathcal{T}_\ell(\Omega)} osc_T^{(2)} \quad , \quad osc_T^{(2)} \;:=\; \frac{1}{2}\Big((y^d - y_\ell^d, y_\ell - y)_{0,T} + \alpha\,(u^d - u_\ell^d, u_\ell - u)_{0,T}\Big) \;.$$

Primal-Dual Weighted Residuals

Primal-Dual Weighted Residuals

Theorem. The primal-dual residuals can be estimated according to

$$|r(i_\ell^y y - y)| \le C \sum_{T \in \mathcal{T}_\ell(\Omega)} \left(\omega_T^y \rho_T^y + \omega_T^\sigma \rho_T^\sigma \right) \quad , \quad |r(i_\ell^p p - p)| \le C \sum_{T \in \mathcal{T}_\ell(\Omega)} \omega_T^p \rho_T^p \ .$$

Here, ρ_T^y and ρ_T^p are L^r-norms and L^s-norms of the residuals associated with the state and the adjoint state equation

$$\rho_T^y := \left(\|u_\ell\|_{0,r,T}^r + h_T^{-r/2} \, \|\frac{1}{2}\nu \cdot [\nabla y_\ell]\|_{0,r,\partial T}^r \right)^{1/r} ,$$

$$\rho_T^p := \left(\|y_\ell - y_\ell^d\|_{0,s,T}^s + h_T^{-s/2} \, \|\frac{1}{2}\nu \cdot [\nabla p_\ell]\|_{0,s,\partial T}^s \right)^{1/s} .$$

The corresponding dual weights ω_T^y and ω_T^p are given by

$$\omega_T^y := \left(\|i_\ell^p p - p\|_{0,s,T}^s + h_T^{s/2} \, \|i_\ell^p p - p\|_{0,s,\partial T}^s \right)^{1/s} ,$$

$$\omega_T^p := \left(\|i_\ell^y y - y\|_{0,r,T}^r + h_T^{r/2} \, \|i_\ell^y y - y\|_{0,r,\partial T}^r \right)^{1/r} .$$

The residual ρ_T^σ and its dual weight ω_T^σ are given by

$$\rho_T^\sigma := n_a^{-1} \sum_{a \in \mathcal{N}_\ell(T)} \kappa_a \quad , \quad \omega_T^\sigma := \|i_\ell^y y - y\|_{2/r+\varepsilon,r,T} \ , \ 0 < \varepsilon < (r-2)/r \ .$$

Primal-Dual Mismatch in Complementarity

Primal-Dual Mismatch in Complementarity

The primal-dual mismatch $\mu_\ell(x, \sigma)$ can be made partially a posteriori in the following two particular cases (cf. [Bergounioux/Kunisch (2003)]):

Regular Case

The active set $\mathcal{A}$ is the union of a finite number of mutually disjoint, connected sets $\mathcal{A}_i$, $1 \le i \le m$, with $C^{1,1}$-boundary.

$$p|_{\mathcal{I}} \in H^2(\mathcal{I}) \ , \ p|_{\mathrm{int}(\mathcal{A})} \in H^2(\mathrm{int}(\mathcal{A}))$$

$$-\Delta p = y^d - y \text{ in } \mathcal{I} \ , \ p = -\alpha \Delta \psi \text{ in } \mathcal{A}$$

$$\boldsymbol{\sigma}_{\mathcal{A}} = \begin{cases} 0 & \text{on } \mathcal{I} \\ y^d - \psi - \alpha \Delta^2 \psi & \text{on } \mathcal{A} \end{cases}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathcal{A}} + \boldsymbol{\sigma}_{\mathcal{F}},$$

$$\boldsymbol{\sigma}_{\mathcal{F}} = - \frac{\partial p|_{\mathcal{I}}}{\partial \nu_{\mathcal{I}}} + \alpha \frac{\partial \Delta \psi}{\partial \nu_{\mathcal{A}}}$$

Nonregular Case

The active set $\mathcal{A}$ is a Lipschitzian curve that divides Ω into two connected components Ω_+ and Ω_-.

$$(\nabla p, \nabla w)_{0,\Omega} = (y^d - y, w) - \langle \boldsymbol{\sigma}, w \rangle \ , \ w \in V^r$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathcal{A}} := \boldsymbol{\nu}_{\mathcal{A}} \cdot \nabla p|_{\mathcal{A}_+} - \boldsymbol{\nu}_{\mathcal{A}} \cdot \nabla p|_{\mathcal{A}_-}$$

Primal-Dual Mismatch in Complementarity

The primal-dual mismatch in complementarity has the representations

$$\begin{aligned}
\mu_\ell|_{\mathcal{I}\cap\mathcal{I}_\ell} &= \frac{1}{2}\,(\boldsymbol{\sigma}_{\mathcal{F}}, \mathrm{y}_\ell - \psi)_{0,\mathcal{F}\cap\mathcal{I}_\ell} + \frac{1}{2}\sum_{\mathrm{a}\in\mathcal{N}_\ell(\mathcal{F}_\ell\cap\mathcal{I})} \kappa_{\mathrm{a}}(\mathrm{y}_\ell - \mathrm{y})(\mathrm{a}),\\
\mu_\ell|_{\mathcal{I}\cap\mathcal{A}_\ell} &= \frac{1}{2}\,(\boldsymbol{\sigma}_{\mathcal{F}}, \psi_\ell - \psi)_{0,\mathcal{F}\cap\mathcal{A}_\ell} + \frac{1}{2}\sum_{\mathrm{a}\in\mathcal{N}_\ell(\mathcal{I}\cap\mathcal{A}_\ell)} \kappa_{\mathrm{a}}(\psi_\ell - \mathrm{y})(\mathrm{a}),\\
\mu_\ell|_{\mathcal{A}\cap\mathcal{I}_\ell} &= \frac{1}{2}\,(\boldsymbol{\sigma}_{\mathcal{F}}, \mathrm{y}_\ell - \psi)_{0,\mathcal{F}\cap\mathcal{I}_\ell} + \frac{1}{2}\,(\mathrm{y}^{\mathrm{d}} - \psi - \alpha\Delta^2\psi, \mathrm{y}_\ell - \psi)_{0,\mathcal{A}\cap\mathcal{I}_\ell},\\
\mu_\ell|_{\mathcal{A}\cap\mathcal{A}_\ell} &= \frac{1}{2}\,(\boldsymbol{\sigma}_{\mathcal{F}}, \psi_\ell - \psi)_{0,\mathcal{F}\cap\mathcal{A}_\ell} + \frac{1}{2}(\mathrm{y}^{\mathrm{d}} - \psi - \alpha\Delta^2\psi, \psi_\ell - \psi)_{0,\mathcal{A}\cap\mathcal{A}_\ell}\ .
\end{aligned}$$

Hence, we need appropriate approximations of the continuous coincidence set $\mathcal{A}$, the continuous non-coincidence set $\mathcal{I}$, the continuous free boundary $\mathcal{F}$, and of $\boldsymbol{\sigma}_{\mathcal{F}}$.

Primal-Dual Mismatch in Complementarity (State Constraints)

The coincidence set $\mathcal{A}$ and the non-coincidence set $\mathcal{I}$ will be approximated by

$$\hat{\mathcal{A}}_\ell := \bigcup \{T \in \mathcal{T}_\ell \mid \chi_\ell^{\mathcal{A}}(x) \geq 1 - \kappa h \text{ for all } x \in T\} ,$$
$$\hat{\mathcal{I}}_\ell := \bigcup \{T \in \mathcal{T}_\ell \mid \chi_\ell^{\mathcal{A}}(x) \leq 1 - \kappa h \text{ for some } x \in T\} ,$$

where

$$\chi_\ell^{\mathcal{A}} := I - \frac{\psi - i_\ell^y y_\ell}{\gamma h^r + \psi - i_\ell^y y_\ell} \quad , \quad 0 < \gamma \leq 1 \ , \ r > 0 \ .$$

Note that for $T \subset \mathcal{A}$ we have

$$\|\chi(\mathcal{A}) - \chi_\ell^{\mathcal{A}}\|_{0,T} \leq \min \left(|T|^{1/2}, \gamma^{-1} h^{-r} \|y - i_\ell^y y\|_{0,T} \right) \to 0 \quad \text{for} \quad \|y - i_\ell^y y\|_{0,T} = O(h^q) \ , \ q > r \ .$$

Moreover, $\boldsymbol{\sigma}_{\mathcal{F}}$ will be approximated by

$$\boldsymbol{\sigma}_{\hat{\mathcal{F}}_\ell} := \begin{cases} -\nu_{\hat{\mathcal{I}}_\ell} \cdot \nabla p_\ell|_{\hat{\mathcal{I}}_\ell} + \alpha \ \nu_{\hat{\mathcal{A}}_\ell} \cdot \nabla \Delta \psi & , \quad E \in \partial \mathcal{T}_\ell(\hat{\mathcal{A}}) \cap \partial \mathcal{T}_\ell(\hat{\mathcal{I}}) \\ \nu_{\hat{\mathcal{A}}_\ell} \cdot \nabla p_\ell|_{\hat{\mathcal{A}}_{\ell,+}} - \nu_{\hat{\mathcal{A}}_\ell} \cdot \nabla p_\ell|_{\hat{\mathcal{A}}_{\ell,-}} & , \quad E \in \mathcal{E}_\ell(\hat{\mathcal{A}}) \setminus (\partial \mathcal{T}_\ell(\hat{\mathcal{A}}) \cap \partial \mathcal{T}_\ell(\hat{\mathcal{I}})) \end{cases} .$$

Primal-Dual Mismatch in Complementarity (State Constraints)

The primal-dual mismatch in complementarity can be estimated from above as follows:

$$|\mu_\ell|_{\mathcal{I}\cap\mathcal{I}_\ell}| \le \hat{\mu}_\ell^{(1)} + \hat{\mu}_\ell^{(2)} \ , \ |\mu_\ell|_{\mathcal{I}\cap\mathcal{A}_\ell}| \le \hat{\mu}_\ell^{(1)} + \hat{\mu}_\ell^{(3)} \ , \ |\mu_\ell|_{\mathcal{A}\cap\mathcal{I}_\ell}| \le \hat{\mu}_\ell^{(1)} + \hat{\mu}_\ell^{(4)} \ , \ |\mu_\ell|_{\mathcal{A}\cap\mathcal{A}_\ell}| \le \hat{\mu}_\ell^{(1)} + \hat{\mu}_\ell^{(5)} \ .$$

where

$$\hat{\mu}_\ell^{(1)} := \sum_{E\in\mathcal{E}_\ell(\hat{\mathcal{F}}_\ell)} \hat{\mu}_E^{(1)} \quad , \quad \hat{\mu}_E^{(1)} := \frac{1}{2}\|\sigma_{\hat{\mathcal{F}}_\ell}\|_{0,E}\|y_\ell - \psi\|_{0,E} \ ,$$

$$\hat{\mu}_\ell^{(2)} := \sum_{E\in\mathcal{E}_\ell(\mathcal{F}_\ell\cap\hat{\mathcal{I}}_\ell)} \hat{\mu}_E^{(2)} \quad , \quad \hat{\mu}_E^{(2)} := \frac{1}{2}\sum_{a\in\mathcal{N}_\ell(E)} |(y_\ell - i_\ell^y y_\ell)(a)| \ \kappa_a \ ,$$

$$\hat{\mu}_\ell^{(3)} := \sum_{T\in\mathcal{T}_\ell(\hat{\mathcal{I}}_\ell\cap\mathcal{A}_\ell)} \hat{\mu}_T^{(3)} \quad , \quad \hat{\mu}_T^{(3)} := \frac{1}{2}\sum_{a\in\mathcal{N}_\ell(T)} |y_\ell - i_\ell^y y_\ell)(a)| \ \kappa_a, \ ,$$

$$\hat{\mu}_\ell^{(4)} := \sum_{T\in\mathcal{T}_\ell(\hat{\mathcal{A}}_\ell\cap\mathcal{I}_\ell)} \hat{\mu}_T^{(4)} \quad , \quad \hat{\mu}_T^{(4)} := \frac{1}{2}\|y^d - \psi - \alpha\Delta^2\psi\|_{0,T} \ \|y_\ell - \psi\|_{0,T} \ ,$$

$$\hat{\mu}_\ell^{(5)} := \sum_{T\in\mathcal{T}_\ell(\hat{\mathcal{A}}_\ell\cap\mathcal{A}_\ell)} \hat{\mu}_T^{(5)} \quad , \quad \hat{\mu}_T^{(5)} := \frac{1}{2}\|y^d - \psi - \alpha\Delta^2\psi\|_{0,T} \ \|\psi_\ell - \psi\|_{0,T} \ .$$

Primal-Dual Weighted Data Oscillations

Primal-Dual Weighted Data Oscillations

The data oscillations $\mathrm{osc}_\ell^{(2)}$ as given by

$$\mathrm{osc}_\ell^{(2)} := \sum_{T \in \mathcal{T}_\ell(\Omega)} \mathrm{osc}_T^{(2)} \quad , \quad \mathrm{osc}_T^{(2)} := \frac{1}{2}\Big((y^d - y_\ell^d, y_\ell - y)_{0,T} + \alpha \, (u^d - u_\ell^d, u_\ell - u)_{0,T} \Big) \ ,$$

can be estimated from above according to

$$\mathrm{osc}_\ell^{(2)} \preceq \sum_{T \in \mathcal{T}_\ell(\Omega)} \widehat{\mathrm{osc}}_T^{(2)} \quad , \quad \widehat{\mathrm{osc}}_T^{(2)} := \hat{\omega}_T^p \, \|u^d - u_\ell^d\|_{0,T} \ + \ \hat{\omega}_T^y \, \|y^d - y_\ell^d\|_{0,T} \ + \ \alpha \|u^d - u_\ell^d\|_{0,T}^2 \ ,$$

where the weights $\hat{\omega}_T^p$ and $\hat{\omega}_T^y$ are given by

$$\hat{\omega}_T^p := \|i_\ell^p p_\ell - p_\ell\|_{0,T} \quad , \quad \hat{\omega}_T^y := \|i_\ell^y y_\ell - y_\ell\|_{0,T} \ .$$

State Constraints: Numerical Results

Numerical Results: Distributed Control Problem with State Constraints I

$$\text{Minimize} \quad J(y,u) := \frac{1}{2}\,\|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\,\|u - u^d\|_{0,\Omega}^2 \quad \text{over} \ (y,u) \in H_0^1(\Omega) \times L^2(\Omega)$$

$$\text{subject to} \quad -\Delta y = u \ \text{ in } \Omega \quad , \quad y \in K := \{v \in H_0^1(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}$$

Data: $\Omega := (-2,+2)^2 \quad , \quad y^d(r) := y(r) + \Delta p(r) + \sigma(r) \quad , \quad u^d(r) := u(r) + \alpha^{-1} p(r) \ ,$

$\psi := 0 \quad , \quad \alpha := 0.1 \ ,$

where $y(r), u(r), p(r), \sigma(r)$ is the solution of the problem:

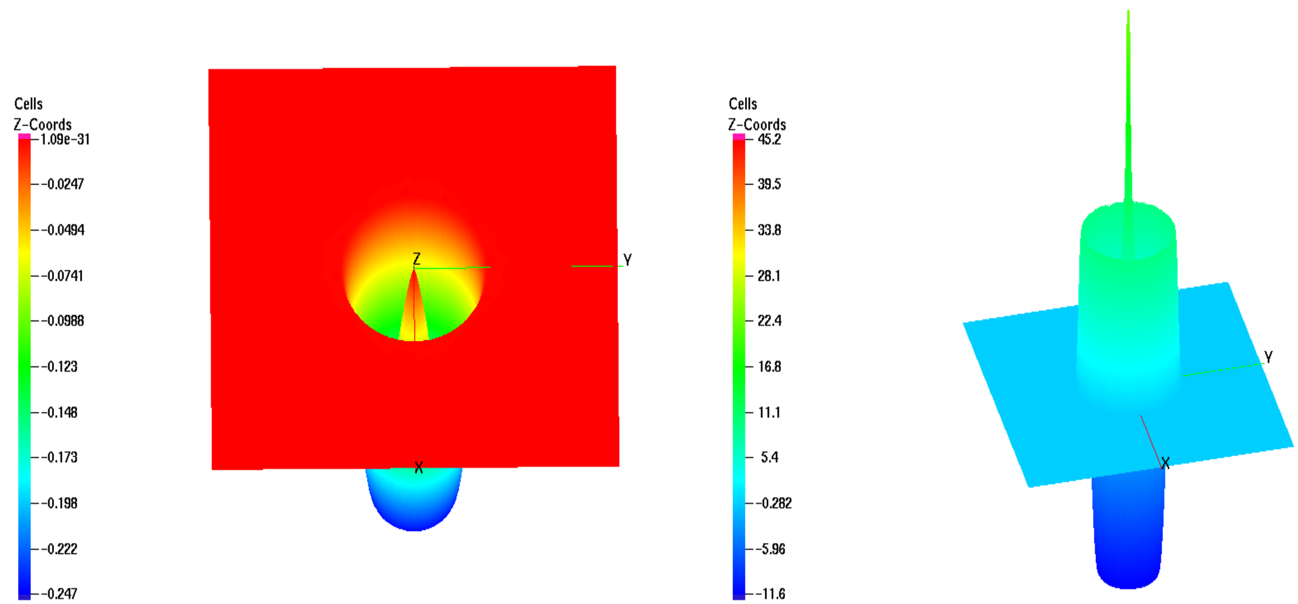
$$y(r) := -r^{4/3} + \gamma_1(r) \ , \ u(r) = -\Delta y(r) \ , \ p(r) = \gamma_2(r) + r^4 - \frac{3}{2}r^3 + \frac{9}{16}r^2 \ , \ \sigma(r) := \begin{cases} 0.0 & , \ r < 0.75 \\ 0.1 & , \ \text{otherwise} \end{cases}$$

,

$$\gamma_1 := \begin{cases} 1 & , \ r < 0.25 \\ -192(r-0.25)^5 + 240(r-0.25)^4 - 80(r-0.25)^3 + 1 & , \ 0.25 < r < 0.75 \\ 0 & , \ \text{otherwise} \end{cases} ,$$

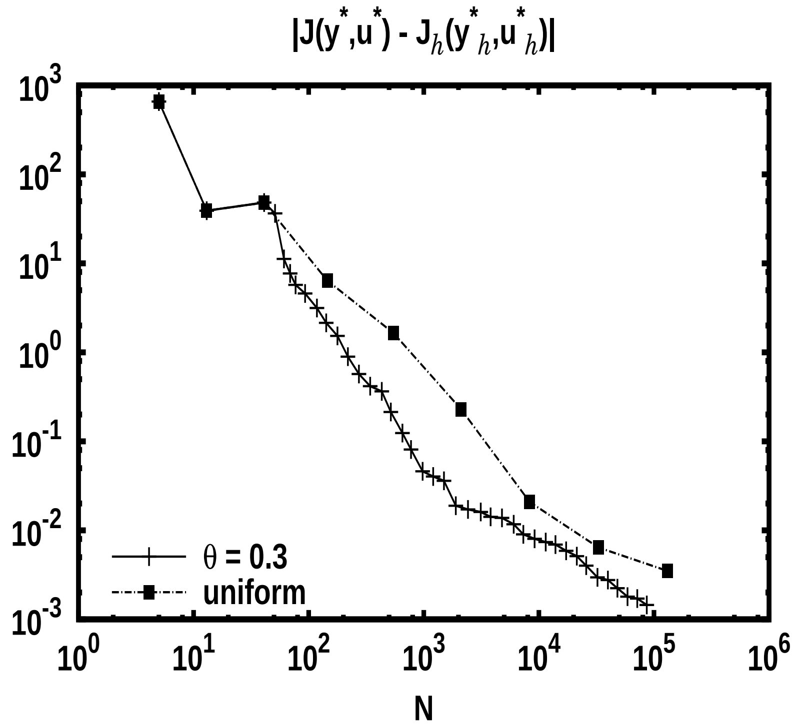
$$\gamma_2 := \begin{cases} 1 & , \ r < 0.75 \\ 0 & , \ \text{otherwise} \end{cases} .$$

Numerical Results: Distributed Control Problem with State Constraints I



Optimal state (left) and optimal control (right)

Numerical Results: Distributed Control Problem with State Constraints I



Decrease in the quantity of interest versus total number of DOFs

Numerical Results: Distributed Control Problem with State Constraints II

$$\text{Minimize} \quad J(y,u) := \frac{1}{2}\,\|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\,\|u - u^d\|_{0,\Omega}^2 \quad \text{over} \ (y,u) \in H^1(\Omega) \times L^2(\Omega)$$

$$\text{subject to} \quad -\Delta y + cy = u \ \text{ in } \Omega \ , \quad y \in K := \{v \in H^1(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}$$

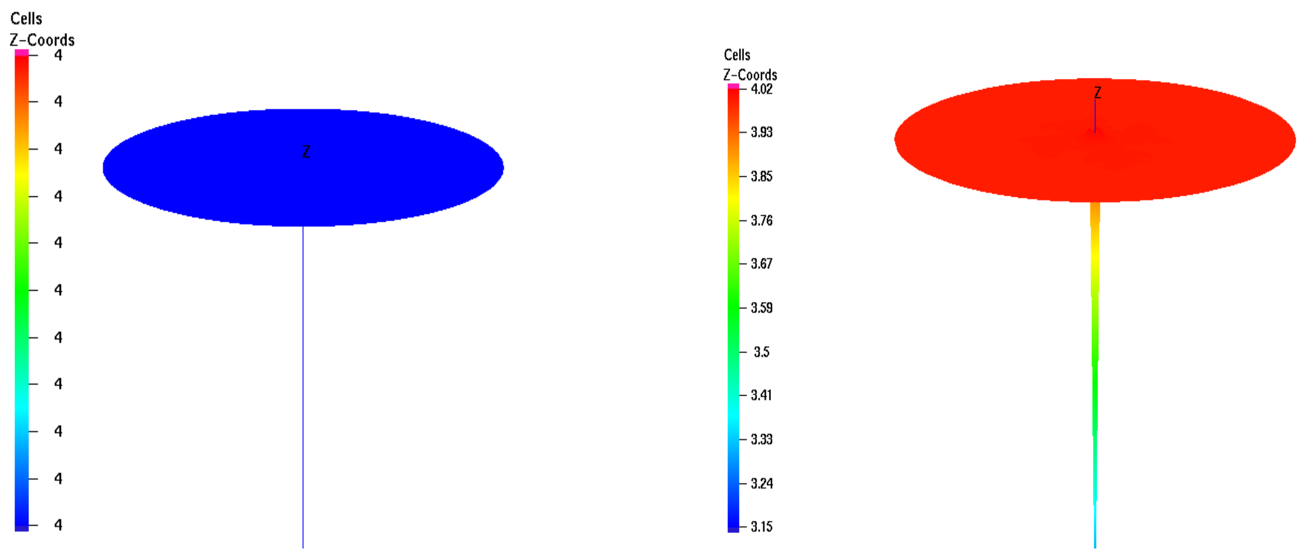
$$\text{Data:} \quad \Omega = B(0,1) := \{x = (x_1,x_2)^T \mid x_1^2 + x_2^2 < 1\} \ , \quad y^d(r) := 4 + \frac{1}{\pi} - \frac{1}{4\pi}r^2 + \frac{1}{2\pi}\ln(r) \ ,$$

$$u^d(r) := 4 + \frac{1}{4\pi}r^2 - \frac{1}{2\pi}\ln(r) \ , \quad \psi := 4 + r \ , \quad \alpha := 1 \ .$$

The solution $y(r), u(r), p(r), \sigma(r)$ of the problem is given by

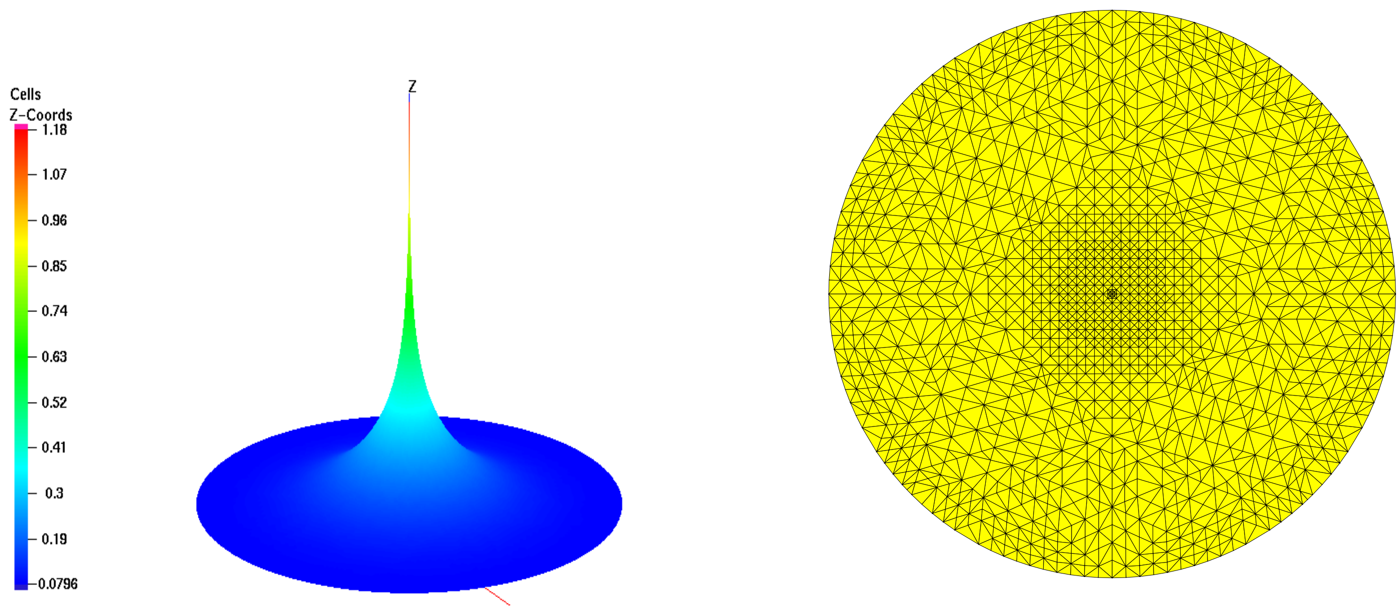
$$y(r) \equiv 4 \ , \quad u(r) \equiv 4 \ , \quad p(r) = \frac{1}{4\pi}r^2 - \frac{1}{2\pi}\ln(r) \ , \quad \sigma(r) = \delta_0 \ .$$

Numerical Results: Distributed Control Problem with State Constraints II



Optimal state (left) and optimal control (right)

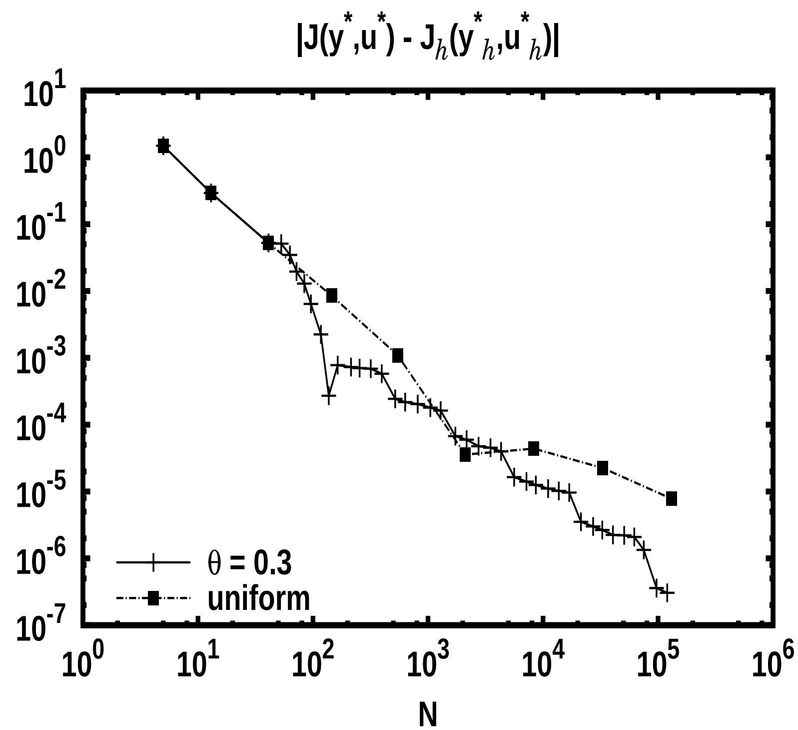
Numerical Results: Distributed Control Problem with State Constraints II



Optimal adjoint state (left) and mesh after 16 adaptive loops (right)

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Numerical Results: Distributed Control Problem with State Constraints II



Decrease in the quantity of interest versus total number of DOFs

Control Constrained Elliptic Control Problems

A Posteriori Error Analysis of AFEM for Optimal Control Problems

(i) Unconstrained problems

R. Becker, H. Kapp, R. Rannacher (2000) R. Becker, R. Rannacher (2001)

(ii) Control constrained problems

W. Liu and N. Yan (2000/01) R. Li, W. Liu, H. Ma, and T. Tang (2002)

M. Hintermüller/H. et al. (2006) A. Gaevskaya/H. et al. (2006/07)

A. Gaevskaya/H. and S. Repin (2006/07) M. Hintermüller/H. (2007)

B. Vexler and W. Wollner (2007)

Model Problem (Distributed Elliptic Control Problem with Control Constraints)

Given a bounded domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\Gamma = \partial\Omega$, a function $y^d, \psi \in L^2(\Omega)$, and $\alpha > 0$, consider the distributed optimal control problem

$$\begin{aligned}
&\text{Minimize} && J(y,u) := \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u\|^2_{0,\Omega}\,,\\
&\text{over} && (y,u) \in H^1_0(\Omega) \times L^2(\Omega)\,,\\
&\text{subject to} && -\Delta\, y = u\,,\\
& && u \in K := \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}\,.
\end{aligned}$$

Optimality Conditions for the Distributed Control Problem

There exists an adjoint state $p \in H_0^1(\Omega)$ and an adjoint control $\boldsymbol{\sigma} \in L^2(\Omega)$ such that the quadruple $(y, p, u, \boldsymbol{\sigma})$ satisfies

$$\begin{aligned} a(y,v) &= (u,v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) \; , \\ a(p,v) &= (y^d - y, v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) \; , \\ \alpha\, u &= p - \boldsymbol{\sigma} \; , \\ \boldsymbol{\sigma} \;\geq\; 0 \; , \; u \;\leq\; \psi \quad &, \quad (\boldsymbol{\sigma}; u - \psi)_{0,\Omega} \;=\; 0 \; , \end{aligned}$$

where $a(\cdot,\cdot)$ stands for the bilinear form

$$a(w,z) \;=\; \int_\Omega \nabla w \cdot \nabla z \; dx \quad , \quad w,z \in H_0^1(\Omega) \; .$$

Finite Element Approximation of the Distributed Control Problem

Let $\mathcal{T}_\ell(\Omega)$ be a shape regular, simplicial triangulation of Ω and let

$$V_\ell \ := \ \{ \ v_\ell \in C(\Omega) \mid v_\ell|_T \in P_{k_1}(T) \ , \ T \in \mathcal{T}_\ell(\Omega) \ , \ k_1 \in \mathbb{N} \ , \ v_H|_{\partial\Omega} = 0 \ \}$$

be the FE space of continuous, piecewise polynomial functions (of degree k_1) and

$$W_\ell \ := \ \{ \ w_\ell \in L^2(\Omega) \mid w_\ell|_T \in P_{k_2}(T) \ , \ T \in \mathcal{T}_\ell(\Omega) \ , \ k_2 \in \mathbb{N} \cup \{0\} \ \}$$

the linear space of elementwise polynomial functions (of degree k_2).

Consider the following FE Approximation of the distributed control problem

$$\begin{aligned}
&\text{Minimize} && J_\ell(y_\ell, u_\ell) \ := \ \frac{1}{2} \, \|y_\ell - y_\ell^d\|_{0,\Omega}^2 \ + \ \frac{\alpha}{2} \, \|u_\ell\|_{0,\Omega}^2 \ , \\
&\text{over} && (y_\ell, u_\ell) \in V_\ell \times W_\ell \ , \\
&\text{subject to} && a(y_\ell, v_\ell) \ = \ (u_\ell, v_\ell)_{0,\Omega} \ , \ v_\ell \in V_\ell \ , \\
& && u_\ell \in K_\ell \ := \ \{ w_\ell \in W_\ell \mid w_\ell|_T \le \psi_\ell|_T \ , \ T \in \mathcal{T}_\ell(\Omega) \} \ .
\end{aligned}$$

where $\psi_\ell \in W_\ell$ is the discrete control constraint.

Optimality Conditions for the FE Discretized Control Problem

There exists an adjoint state $p_\ell \in V_\ell$ and an adjoint control $\boldsymbol{\sigma}_\ell \in W_\ell$ such that the quadruple $(y_\ell, u_\ell, p_\ell, \boldsymbol{\sigma}_\ell)$ satisfies

$$\begin{aligned} a(y_\ell, v_\ell) &= (u_\ell, v_\ell)_{0,\Omega} \quad , \quad v_\ell \in V_\ell \ , \\ a(p_\ell, v_\ell) &= (y_\ell^d - y, v_\ell)_{0,\Omega} \quad , \quad v_\ell \in V_\ell \ , \\ \alpha \ u_\ell &= M_\ell \ p_\ell - \boldsymbol{\sigma}_\ell \ , \\ \boldsymbol{\sigma}_\ell \ \geq \ 0 \ , \quad u_\ell \ &\leq \ \psi_\ell \quad , \quad (\boldsymbol{\sigma}_\ell, u_\ell - \psi_\ell)_{0,\Omega} \ = \ 0 \ , \end{aligned}$$

where $y_\ell^d \in V_\ell$ and $M_\ell : V_\ell \to W_\ell$, e.g., for $k_2 = 0$:

$$(M_\ell v_\ell)|_T \ := \ |T|^{-1} \int_T v_\ell \ dx \quad , \quad T \in \mathcal{T}_\ell(\Omega) \ .$$

Primal-Dual Weighted Error Representation (Control Constraints)

Theorem. Let $(x,\sigma) \in X \times L^2(\Omega)$ and $(x_\ell,\sigma_\ell) \in X_\ell \times W_\ell$ be the solutions of the continuous and discrete optimality systems, respectively. Then, there holds

$$J(y,u) - J_\ell(y_\ell,u_\ell) \;=\; -\frac{1}{2}\nabla_{xx}\mathcal{L}(x_\ell - x, x_\ell - x) \;+\; (\sigma, u_\ell - \psi)_{0,\Omega} \;+\; \mathrm{osc}_\ell^{(1)} \;.$$

Remark: In the unconstrained case, i.e., $\sigma = \sigma_\ell = 0$, the above result reduces to the error representation in [Becker, Kapp, and Rannacher (2000)].

Primal-Dual Weighted Error Representation (Control Constraints)

Theorem. Under the assumptions of the previous Theorem let $i_\ell^z, z \in \{y,u,p\}$, be the interpolation operators introduced before. Then, there holds

$$J(y,u) - J_\ell(y_\ell,u_\ell) \;=\; -\Big(r(i_\ell^y y - y) \;+\; r(i_\ell^p p - p) \;+\; r(i_\ell^u u - u) \Big) \;+\; \mu_\ell(x,\sigma) \;+\; \mathrm{osc}_\ell^{(1)} \;+\; \mathrm{osc}_\ell^{(2)} \;,$$

where $r(i_\ell^y y - y)$, $r(i_\ell^p p - p)$ and $r(i_\ell^u u - u)$ stand for the primal-dual weighted residuals

$$r(i_\ell^y y - y) \;:=\; \frac{1}{2}\Big((y_\ell - y_\ell^d, i_\ell^y y - y)_{0,\Omega} \;+\; (\nabla(i_\ell^y y - y), \nabla p_\ell)_{0,\Omega} \Big) \;,$$

$$r(i_\ell^p p - p) \;:=\; \frac{1}{2}\Big((\nabla(i_\ell^p p - p), \nabla y_\ell)_{0,\Omega} \;-\; (u_\ell, i_\ell^p p - p)_{0,\Omega} \Big) \;, \quad r(i_\ell^u u - u) \;:=\; \frac{1}{2}\,(M_\ell p_\ell - p_\ell, i_\ell^u u - u)_{0,\Omega} \;.$$

Moreover, $\mu_\ell(x,\sigma)$ represents the primal-dual mismatch in complementarity

$$\mu_\ell(x,\sigma) \;:=\; \frac{1}{2}\Big((\sigma, u_\ell - \psi)_{0,\Omega} \;+\; (\sigma_\ell, \psi_\ell - u)_{0,\Omega} \Big) \;,$$

and $\mathrm{osc}_\ell^{(2)}$ is a further oscillation term

$$\mathrm{osc}_\ell^{(2)} \;:=\; \sum_{T \in \mathcal{T}_\ell(\Omega)} \mathrm{osc}_T^{(2)} \quad , \quad \mathrm{osc}_T^{(2)} \;:=\; \frac{1}{2}\,(y^d - y_\ell^d, y_\ell - y)_{0,T} \;.$$

Primal-Dual Weighted Residuals (Control Constraints)

Theorem. The primal-dual residuals can be estimated according to

$$|r(i_\ell^y y - y)| \;\leq\; C \sum_{T \in \mathcal{T}_\ell(\Omega)} \omega_T^y \rho_T^y \quad , \quad |r(i_\ell^p p - p)| \;\leq\; C \sum_{T \in \mathcal{T}_\ell(\Omega)} \Big(\omega_T^p \rho_T^{p,1} \;+\; \omega_T^u \rho_T^{p,2} \Big) \; .$$

Here, ρ_T^y and $\rho_T^{p,1}$ are L^2-norms of the residuals associated with the state and the adjoint state

$$\rho_T^y \;:=\; \Big(\, \|u_\ell\|_{0,T}^2 \;+\; h_T^{-1} \, \|\frac{1}{2}\nu \cdot [\nabla y_\ell]\|_{0,\partial T}^2 \, \Big)^{1/2} \; ,$$

$$\rho_T^{p,1} \;:=\; \Big(\, \|y_\ell - y_\ell^d\|_{0,T}^2 \;+\; h_T^{-1} \, \|\frac{1}{2}\nu \cdot [\nabla p_\ell]\|_{0,\partial T}^2 \, \Big)^{1/2} \; .$$

The corresponding dual weights ω_T^u and ω_T^p are given by

$$\omega_T^y \;:=\; \Big(\, \|i_\ell^p p - p\|_{0,T}^2 \;+\; h_T \, \|i_\ell^p p - p\|_{0,\partial T}^2 \, \Big)^{1/2} \; ,$$

$$\omega_T^p \;:=\; \Big(\, \|i_\ell^y y - y\|_{0,T}^2 \;+\; h_T \, \|i_\ell^y y - y\|_{0,\partial T}^2 \, \Big)^{1/2} \; .$$

The residual $\rho_T^{p,2}$ and its dual weight ω_T^u are given by

$$\rho_T^{p,2} \;:=\; \|M_\ell p_\ell - p_\ell\|_{0,T} \quad , \quad \omega_T^u \;:=\; \|i_\ell^u u - u\|_{0,T} \; .$$

Primal-Dual Mismatch in Complementarity (Control Constraints)

Using the complementarity conditions

$$\mathrm{u} \le \psi \quad , \quad \sigma \ge 0 \quad , \quad (\sigma, \mathrm{u} - \psi)_{0,\Omega} = 0 \quad , \quad \alpha \mathrm{u} - \mathrm{p} + \sigma = 0 \, ,$$

$$\mathrm{u}_\ell \le \psi_\ell \quad , \quad \sigma_\ell \ge 0 \quad , \quad (\sigma_\ell, \mathrm{u}_\ell - \psi_\ell)_{0,\Omega} = 0 \quad , \quad \alpha \mathrm{u}_\ell - \mathrm{M}_\ell \, \mathrm{p}_\ell + \sigma_\ell = 0 \, ,$$

the primal-dual mismatch $\mu_\ell := \mu_\ell(\mathrm{x}, \sigma)$ can be further assessed according to

$$\begin{aligned}
\mu_\ell(\mathcal{I} \cap \mathcal{I}_\ell) &= 0 \, , \\
\mu_\ell(\mathcal{A} \cap \mathcal{A}_\ell) &= \frac{1}{2} \, (\sigma + \sigma_\ell, \psi_\ell - \psi)_{0,\mathcal{A} \cap \mathcal{A}_\ell} \, , \\
\mu_\ell(\mathcal{I} \cap \mathcal{A}_\ell) &= \frac{1}{2} \, (\sigma_\ell, \psi_\ell - \alpha^{-1} \mathrm{p})_{0,\mathcal{I} \cap \mathcal{A}_\ell}, \\
\mu_\ell(\mathcal{A} \cap \mathcal{I}_\ell) &= \frac{\alpha}{2} \, \|\mathrm{u} - \mathrm{u}_\ell\|^2_{0,\mathcal{I} \cap \mathcal{A}_\ell} \; + \; \frac{1}{2} \, (\mathrm{p} - \mathrm{M}_\ell \, \mathrm{p}_\ell, \mathrm{u}_\ell - \mathrm{u})_{0,\mathcal{I} \cap \mathcal{A}_\ell}.
\end{aligned}$$

and we finally obtain

$$|\mu_\ell(\mathcal{I} \cap \mathcal{A}_\ell) \; + \; \mu_\ell(\mathcal{A} \cap \mathcal{I}_\ell)| \; \le \; \nu_\ell$$

with a fully computable a posteriori term ν_ℓ (consistency error).

Numerical Results: Distributed Control Problem with Control Constraints I

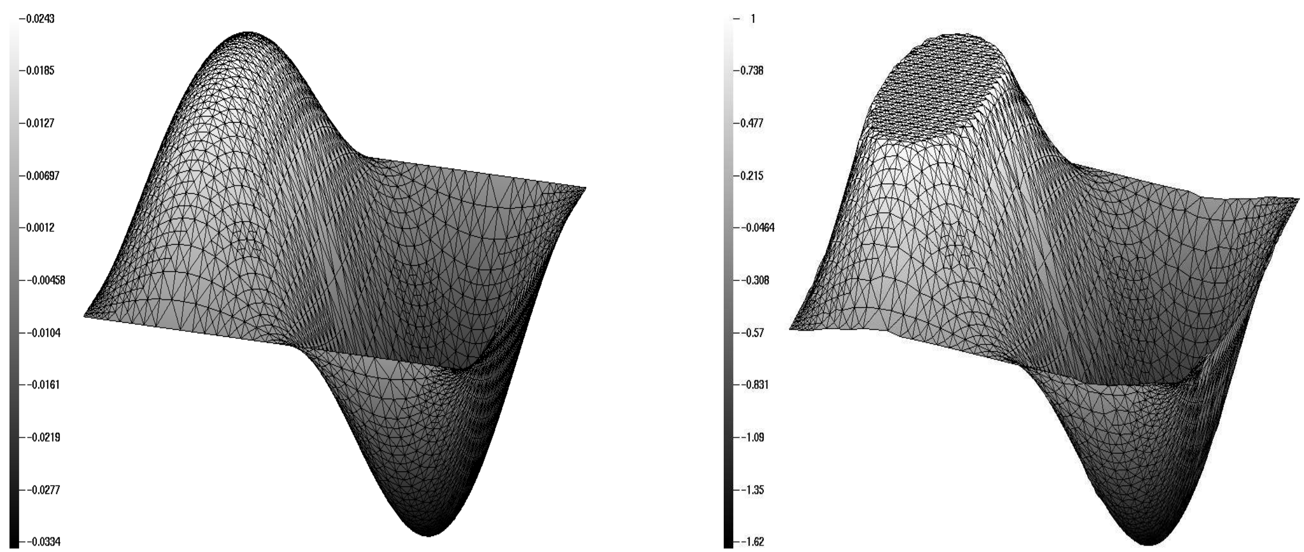
$$
\begin{array}{ll}
\text{Minimize} & J(y,u) := \frac{1}{2} \|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2} \|u\|_{0,\Omega}^2 \\
\text{over} & (y,u) \in H_0^1(\Omega) \times L^2(\Omega) \\
\text{subject to} & -\Delta y = u \quad \text{in } \Omega , \\
 & u \in K := \{v \in L^2(\Omega) \mid v \leq \psi \text{ a.e. in } \Omega\}
\end{array}
$$

Data: $\Omega := (0,1)^2$,

$$
y^d := \begin{cases} 200\; x_1\; x_2\; (x_1 - 0.5)^2\; (1 - x_2) \;,\; 0 \leq x_1 \leq 0.5 \\ 200\; (x_1 - 1)\; (x_2\; (x_1 - 0.5)^2\; (1 - x_2) \;,\; 0.5 < x_1 \leq 1 \end{cases} ,
$$

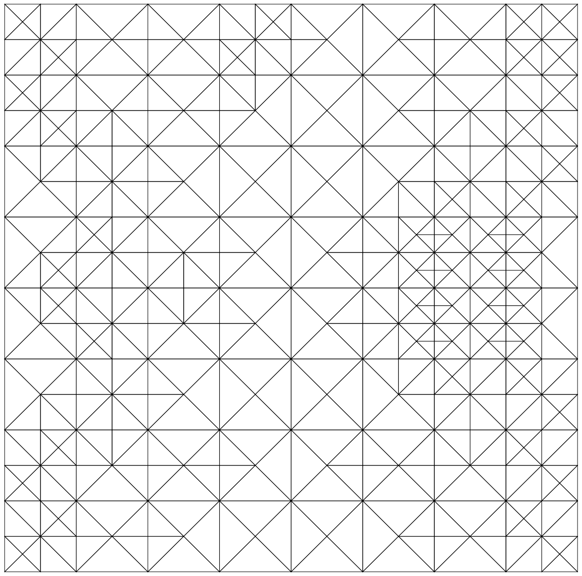
$\alpha = 0.01$, $\psi = 1$.

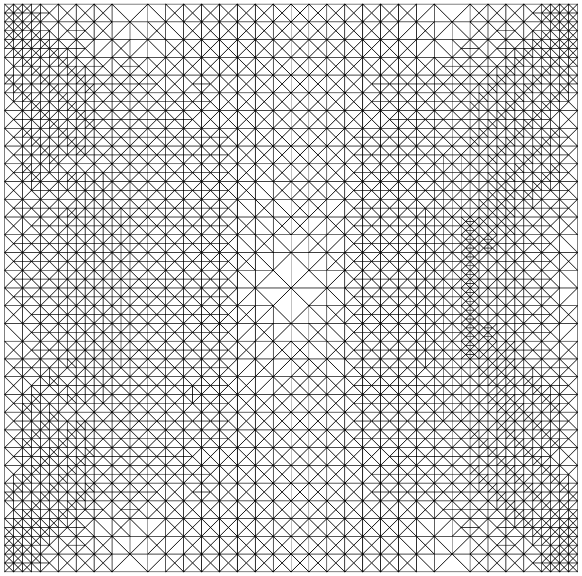
Numerical Results: Distributed Control Problem with Control Constraints I



Optimal state (left) and optimal control (right)

Numerical Results: Distributed Control Problem with Control Constraints I





Grid after 6 (left) and 10 (right) refinement steps

Numerical Results: Distributed Control Problem with Control Constraints I

l	N_{dof}	δ_h	η_h	osc_h	ν_h
0	12	2.73E-03	1.47E-02	1.17E-01	0.00E+00
1	25	8.57E-04	2.03E-02	6.23E-02	2.04E-03
2	42	5.09E-04	1.42E-02	3.44E-02	4.86E-03
4	138	1.52E-04	4.61E-03	1.27E-02	1.66E-04
6	478	4.24E-05	1.35E-03	4.20E-03	3.67E-05
8	1706	9.91E-06	3.67E-04	2.08E-03	4.27E-06
10	6237	2.52E-06	9.95E-05	6.60E-04	3.82E-07
12	22639	5.92E-07	2.74E-05	1.63E-04	1.63E-07
14	81325	1.57E-07	7.57E-06	5.05E-05	7.60E-09
16	299028	4.65E-08	2.05E-06	1.58E-05	1.32E-09

Error (quantity of interest), estimator, oscillations, and consistency error

Numerical Results: Distributed Control Problem with Control Constraints II

$$\begin{aligned}
&\text{Minimize} && J(y,u) := \frac{1}{2}\,\|y-y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u-u^d\|^2_{0,\Omega} \\
&\text{over} && (y,u)\in H^1_0(\Omega)\times L^2(\Omega) \\
&\text{subject to} && -\Delta y = f + u \quad \text{in } \Omega\,, \\
& && u\in K := \{v\in L^2(\Omega)\ |\ v\le\psi \text{ a.e. in } \Omega\}
\end{aligned}$$

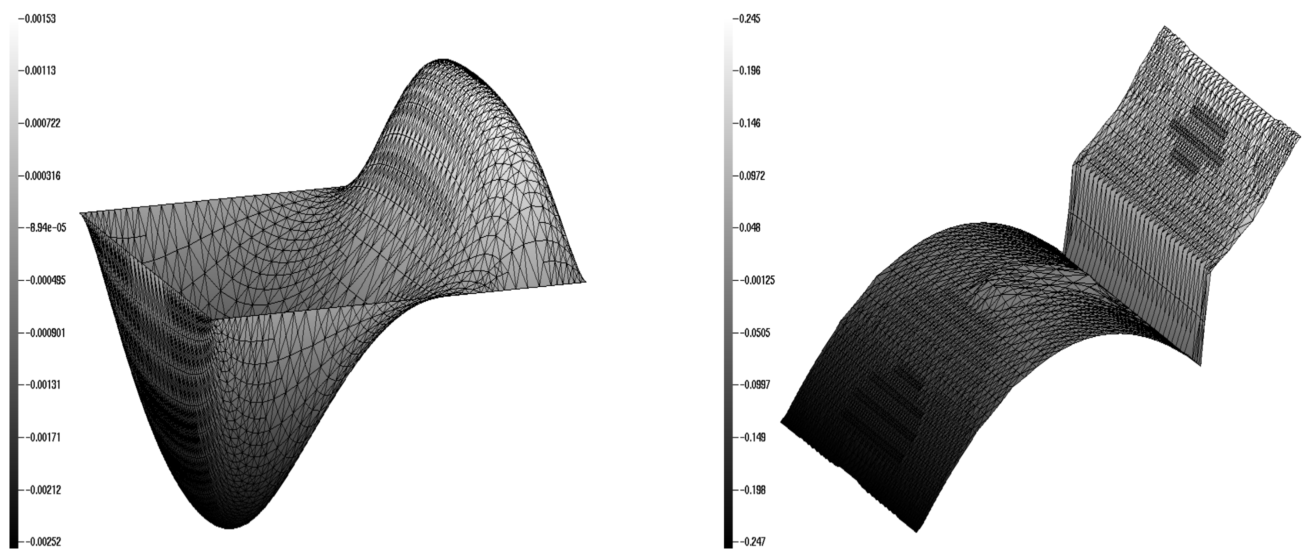
Data: $\Omega := (0,1)^2 \quad , \quad y^d := 0 \quad , \quad u^d := \hat{u} + \alpha^{-1}(\hat{\sigma} - \Delta^{-2}\hat{u})\,,$

$$\psi := \begin{cases} (x_1-0.5)^8 & , \quad (x_1,x_2)\in\Omega_1, \\ (x_1-0.5)^2 & , \quad \text{otherwise} \end{cases} \quad , \quad \alpha := 0.1 \quad , \quad f := 0$$

$$\hat{u} := \begin{cases} \psi & , \quad (x_1,x_2)\in\Omega_1\cup\Omega_2, \\ -1.01\,\psi & , \quad \text{otherwise} \end{cases} \quad , \quad \hat{\sigma} := \begin{cases} 2.25\,(x_1-0.75)\cdot 10^{-4} & , \quad (x_1,x_2)\in\Omega_2, \\ 0 & , \quad \text{otherwise} \end{cases} \quad ,$$

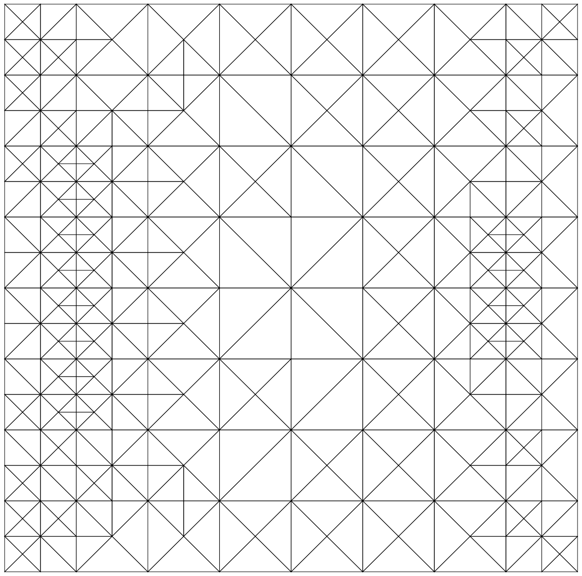
$$\Omega_1 := \{(x_1,x_2)\in\Omega\ |\ \Big((x_1-0.5)^2+(x_2-0.5)^2\Big)^{1/2}\le 0.15\} \quad , \quad \Omega_2 := \{(x_1,x_2)\in\Omega\ |\ x_1\ge 0.75\}\,.$$

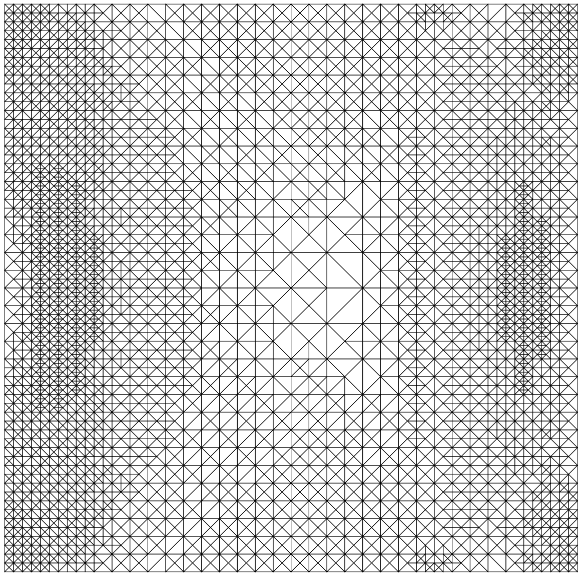
Numerical Results: Distributed Control Problem with Control Constraints II



Optimal state (left) and optimal control (right)

Numerical Results: Distributed Control Problem with Control Constraints II





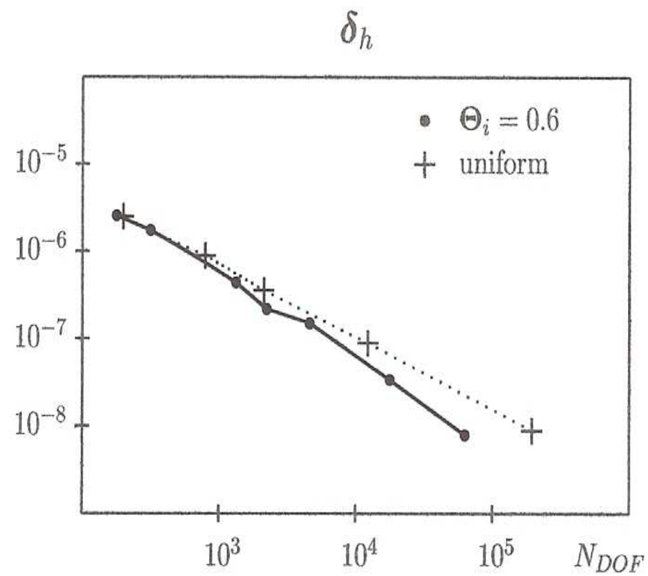
Grid after 6 (left) and 10 (right) refinement steps

Numerical Results: Distributed Control Problem with Control Constraints I

l	N_{dof}	δ_h	η_h	osc_h	ν_h
0	5	2.41E-04	2.58E-06	1.07E-01	0.00E+00
1	12	1.61E-04	5.26E-06	8.11E-02	2.71E-07
2	26	7.62E-05	4.78E-06	5.25E-02	4.19E-07
4	73	1.54E-05	2.08E-06	2.89E-02	0.00E+00
6	253	4.09E-06	6.45E-07	1.59E-02	0.00E+00
8	953	1.16E-06	1.79E-07	8.39E-03	9.86E-12
10	3507	3.41E-07	4.87E-08	4.70E-03	2.66E-13
12	12684	1.03E-07	1.33E-08	2.59E-03	3.08E-14
14	45486	2.99E-08	3.71E-09	1.52E-03	2.23E-15
16	165366	8.12E-09	1.05E-09	9.06E-04	2.65E-16

Error (quantity of interest), estimator, oscillations, and consistency error

Numerical Results: Distributed Control Problem with Control Constraints II



Decrease in the quantity of interest versus total number of DOFs

Elliptic Optimal Control Problems

Constraints on the Gradient of the State

Elliptic Optimal Control with Pointwise Gradient-State Constraints

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary Γ, $y^d \in L^2(\Omega)$ a desired state, f a forcing term, $\psi \in L^2(\Omega)$ s.th. $\psi \geq \psi_{\min} > 0$ a.e. in Ω, and $\alpha > 0$, find $(y,u) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$(P) \qquad \inf_{(y,u)} J(y,u) := \frac{1}{2} \int_\Omega |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx,$$

$$\text{subject to} \qquad \begin{aligned} Ly := -\nabla \cdot a \nabla y + cy &= f + u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \Gamma, \\ \nabla y \in K := \{v \in L^2(\Omega)^2 \mid & \; |v| \leq \psi \text{ a.e. in } \Omega\}. \end{aligned}$$

Pointwise Gradient-State Constraints: State-Reduced Formulation

Let $\hat{V} \subset H_0^1(\Omega)$ be a reflexive Banach space and let $\hat{G} : L^2(\Omega) \to \hat{V}$ be the map that assigns to the rhs $f + u$ the solution $y = \hat{G}(f + u)$ of the state equation. Assume that $\hat{G}$ is a bounded linear operator which is invertible such that $u = \hat{G}^{-1} y - f$.
This leads to the state-reduced formulation:
Find $y \in \hat{K} := \{v \in \hat{V} \mid |\nabla v| \leq \psi \text{ bf a.e. } \text{ in } \Omega\}$ such that

$$\inf_{y \in \hat{K}} J_{red}(y) := \frac{1}{2} \int_\Omega |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |\hat{G}^{-1} y - f|^2 \, dx.$$

Unconstrained formulation:

$$\inf_{y \in \hat{V}} J_{red}(y) + I_{\hat{K}}(y)$$

where $I_{\hat{K}}$ stands for the indicator function of the set $\hat{K}$.

State-Reduced Formulation: Optimality Conditions

Theorem. The gradient-state constrained optimal control problem admits a unique solution $(y,u) \in \hat{K} \times L^2(\Omega)$ which is characterized by the existence of a unique pair $(p,w) \in L^2(\Omega) \times \hat{V}^*$ satisfying

$$Lp = -\nabla \cdot (a\nabla p) + cp \;=\; y^d - y - w \quad \text{in } \hat{V}^*,$$
$$p \;=\; \alpha u \quad \text{in } L^2(\Omega),$$
$$w \in N_{\hat{K}}(y) := \{\xi \in \hat{V}^* \;\mid\; \langle \xi, z - y\rangle_{\hat{V}^*,\hat{V}} \le 0,\; z \in \hat{K}\}.$$

Remark. If $\hat{V} = W^{2,r}(\Omega) \cap H_0^1(\Omega), r > 2$, there exists a Slater point, i.e., $y_0 \in \text{int } \hat{K}$ and $|\nabla(y_0 + v)| \le \psi$ in Ω for all $v \in C^1(\bar{\Omega})$ s.th. $\|v\|_{C^1(\bar{\Omega})} \le \delta$ for sufficiently small $\delta > 0$.

$$0 \in J'_{red}(y) + \partial(I_{\hat{K}} \circ \nabla)(y) = J'_{red}(y) - \nabla \cdot \partial I_{\hat{K}}(\nabla y),$$

i.e., there exists $\mu \in \partial I_{\hat{K}}(\nabla y) \subset M(\bar{\Omega})^2$ such that $w = -\nabla \cdot \mu$.

Control-Reduced Formulation and Dual Problem

Denoting by $G : H^{-1}(\Omega) \to H_0^1(\Omega)$ the solution operator associated with the state equation, the optimal control problem can be written according to

$$\inf_{u \in L^2(\Omega)} \mathcal{F}(u) + \mathcal{G}(\Lambda u)$$

where

$$\mathcal{F}(u) := J(G(f+u),u), \quad \mathcal{G}(q) := I_K(q), \quad \Lambda := \nabla G.$$

Denoting by $\mathcal{F}^*$ and $\mathcal{G}^*$ the Fenchel conjugates of $\mathcal{F}$ and $\mathcal{G}$

$$\mathcal{F}^*(u^*) = \frac{1}{2}\, \|u^* + G^* y^d + \alpha f\|^2_{M^{-1}}, \quad \mathcal{G}^*(q^*) = \int_\Omega \psi |q^*| dx,$$

where $M := G^*G + \alpha I$ and $\|\cdot\|^2_{M^{-1}} := (M^{-1}\cdot,\cdot)_{0,\Omega}$, the dual problem reads as follows:

$$(D) \quad \sup_{q^* \in L^2(\Omega)} -\mathcal{F}^*(\Lambda^* q^*) - \mathcal{G}^*(-q^*) \iff \inf_{\mu \in L^2(\Omega)} \frac{1}{2}\, \|G^*(\nabla^* \mu + y^d) + \alpha f\|^2_{M^{-1}} + \int_\Omega \psi |\mu| dx.$$

Tightened Formulation of the Primal Problem

Consider the following tightened formulation of the primal problem

$$(\hat{\mathrm{P}}) \quad \inf_{(y,u) \in \hat{V} \times L^2(\Omega)} J(y,u) := \frac{1}{2} \int_\Omega |y - y^d|^2 dx + \frac{\alpha}{2} \int_\Omega |u|^2 dx,$$

subject to

$$Ly = f + u \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \Gamma, \quad |\nabla y| \le \psi \quad \text{a.e. in } \Omega.$$

Theorem. Let $\{\mu_n\}_{\mathbb{N}} \subset L^2(\Omega)^2$ be a minimizing sequence for the dual $(\hat{D})$ to $(\hat{P})$. Then, there exist a subsequence $\{\mu_n\}_{\mathbb{N}'}$ and $\mu \in M(\bar{\Omega})^2$ such that

$$w^* - \lim \mu_n = \mu \quad \text{in } M(\bar{\Omega})^2 \qquad \text{and} \qquad w - \lim \nabla \cdot \mu_n = -w \quad \text{in } \hat{V}^*.$$

Moreover, the limit $w \in \hat{V}^*$ satisfies

$$(*) \qquad Ly = f + u \quad \text{in } L^2(\Omega), \quad Lp = y^d - y - w \quad \text{in } \hat{V}^*, \quad p = \alpha u \quad \text{in } L^2(\Omega).$$

Remark. A quadruple $(y,u,p,w) \in V \times L^2(\Omega) \times L^2(\Omega) \times \hat{V}^*$ such that $(*)$ holds true and $\nabla y \in (M(\bar{\Omega})^2)^* \setminus C(\bar{\Omega})^2$, is called a weak solution of (P).

Finite Element Discretization of (P) and $(\hat{\mathrm{P}})$

Let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation of Ω and denote by $\mathcal{E}_h(D)$ the set of edges of $\mathcal{T}_h(\Omega)$ in $D \subset \Omega$. We refer to $V_h := \{v_h \in C_0(\Omega) \mid v_h|_T \in P_1(T), T \in \mathcal{T}_h(\Omega)\}$ as the finite element space of P1 conforming FEs w.r.t. $\mathcal{T}_h(\Omega)$ and set $W_h := \{w_h : \bar{\Omega} \to \mathbb{R}^2 \mid w_h|_T \in P_0(T)^2, T \in \mathcal{T}_h(\Omega)\}$. We define ψ_h according to $\psi_h|_T := |T|^{-1} \int_T \psi dx, T \in \mathcal{T}_h(\Omega)$ and set

$$K_h := \{z_h \in W_h \mid |z_h|_T| \le \psi_h|_T, \; T \in \mathcal{T}_h(\Omega)\}.$$

The discrete optimal control problems reads:

$$(\hat{P}_h) \qquad \inf_{(y_h, u_h)} \; J(y_h, u_h) \; := \; \frac{1}{2} \int_\Omega |y_h - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u_h|^2 \, dx,$$

$$\text{subject to} \quad a(y_h, v_h) \; = \; (f + u_h, v_h)_{0,\Omega}, \quad v_h \in V_h,$$

$$\nabla y_h \; \in \; K_h.$$

Discrete Optimal Control Problem: Optimality Conditions

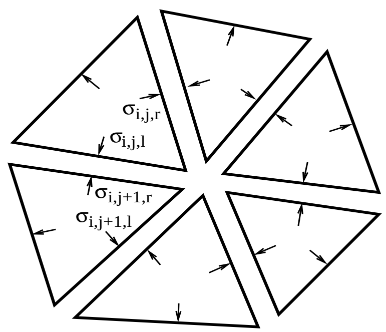
Theorem. The discrete optimal control problem $(\hat{P}_h)$ admits a unique solution $(y_h, u_h) \in V_h \times V_h$ which is characterized by the existence of an adjoint state $p_h \in V_h$ and a multiplier $\mu_h \in W_h$ such that

$$\begin{aligned} a(p_h, v_h) - (y^d - y_h, v_h)_{0,\Omega} + \sum_{T \in \mathcal{T}_h(\Omega)} (\mu_h|_T, \nabla v_h|_T)_{0,T} &= 0, \quad v_h \in V_h, \\ p_h - \alpha u_h &= 0, \\ \sum_{T \in \mathcal{T}_h(\Omega)} (\mu_h|_T, q_h|_T - \nabla y_h|_T)_{0,T} &\le 0, \quad q_h \in K_h. \end{aligned}$$

Remark. The Fenchel dual associated with $(\hat{P}_h)$ reads

$$(\hat{D}_h) \qquad \inf_{\mu_h \in W_h} \ \frac{1}{2} \, \| G_h^*(\nabla^* \mu_h + y^d) + \alpha f \|^2_{M_h^{-1}} + \int_\Omega \psi_h |\mu_h| dx.$$

Prager-Synge Equilibration (cf. Braess/Schöberl (08), Braess/H./Schöberl (09))



Construct $\tilde{\mu}_h \in RT_0(\Omega; \mathcal{T}_h(\Omega))$ such that

$$\sum_{T \in \mathcal{T}_h(\Omega)} (\mu_h|_T, \nabla v_h|_T)_{0,T} = \sum_{E \in \mathcal{E}_h(\Omega)} (n_E \cdot [\mu_h]_E, v_h)_{0,E}$$

$$= - \sum_{T \in \mathcal{T}_h(\Omega)} (\nabla \cdot \tilde{\mu}_h, v_h)_{0,T}, \quad v_h \in V_h.$$

Then the discrete optimality system can be written according to

$$a(p_h, v_h) - (y^d - y_h, v_h)_{0,\Omega} - \sum_{T \in \mathcal{T}_h(\Omega)} (\nabla \cdot \tilde{\mu}_h, \nabla v_h)_{0,T} \; = \; 0, \quad v_h \in V_h,$$

$$p_h - \alpha u_h \; = \; 0,$$

$$\sum_{T \in \mathcal{T}_h(\Omega)} (\mu_h|_T, q_h|_T - \nabla y_h|_T)_{0,T} \; \leq \; 0, \quad q_h \in K_h.$$

Residual-Type A Posteriori Error Estimator

We choose $\hat{V} = W_0^{1,r}(\Omega), r > 2,$ such that $\hat{V}^* = W^{-1,s}(\Omega), 1/r + 1/s = 1.$
The associated residual-type a posteriori error estimator reads

$$\eta_h := \Big(\sum_{T \in \mathcal{T}_h(\Omega)} \eta_{y,T}^r \Big)^{1/r} + \Big(\sum_{E \in \mathcal{E}_h(\Omega)} \eta_{y,E}^r \Big)^{1/r} + \Big(\sum_{T \in \mathcal{T}_h(\Omega)} \eta_{p,T}^s \Big)^{1/s} + \Big(\sum_{E \in \mathcal{E}_h(\Omega)} \eta_{p,E}^s \Big)^{1/s},$$

where the element residuals $\eta_{y,T}, \eta_{p,T}$ and the edge residuals $\eta_{y,E}, \eta_{p,E}$ are given by

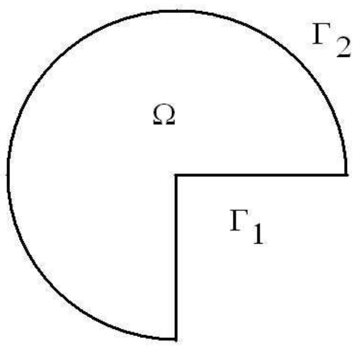
$$\begin{aligned}
\eta_{y,T}^r &:= h_T^r \, \|f + u_h + \nabla\cdot(a\nabla y_h) - cy_h\|_{0,T}^r, \\
\eta_{p,T}^s &:= h_T^s \, \|y^d - y_h + \nabla \cdot (a\nabla p_h) - cp_h + \nabla \cdot \tilde{\mu}_h\|_{0,T}^s, \\
\eta_{y,E}^r &:= h_E^{r/2} \, \|n_E \cdot [a\nabla y_h]_E\|_{0,E}^r, \quad \eta_{p,E}^s := h_E^{s/2} \, \|n_E\cdot[a\nabla p_h]_E\|_{0,E}^s.
\end{aligned}$$

Reliability of the A Posteriori Error Estimator

Theorem. Let $(y, u, p, w) \in W_0^{1,r} \times L^2(\Omega) \times W_0^{1,s}(\Omega) \times W^{-1,s}(\Omega)$ and $(y_h, u_h, p_h, w_h) \in V_h \times V_h \times V_h \times W_h$ be the solution of $(\hat{P})$ and $(\hat{P}_h)$, respectively. Let further η be the residual error estimator. Then, there holds

$$\|y - y_h\|^2_{W_0^{1,r}} + \|u - u_h\|^2_{0,\Omega} \lesssim \eta^2 + |\langle w, y - y_h \rangle_{W^{-1,s}(\Omega), W_0^{1,r}(\Omega)}|.$$

Gradient-State Constraints: Numerical Examples



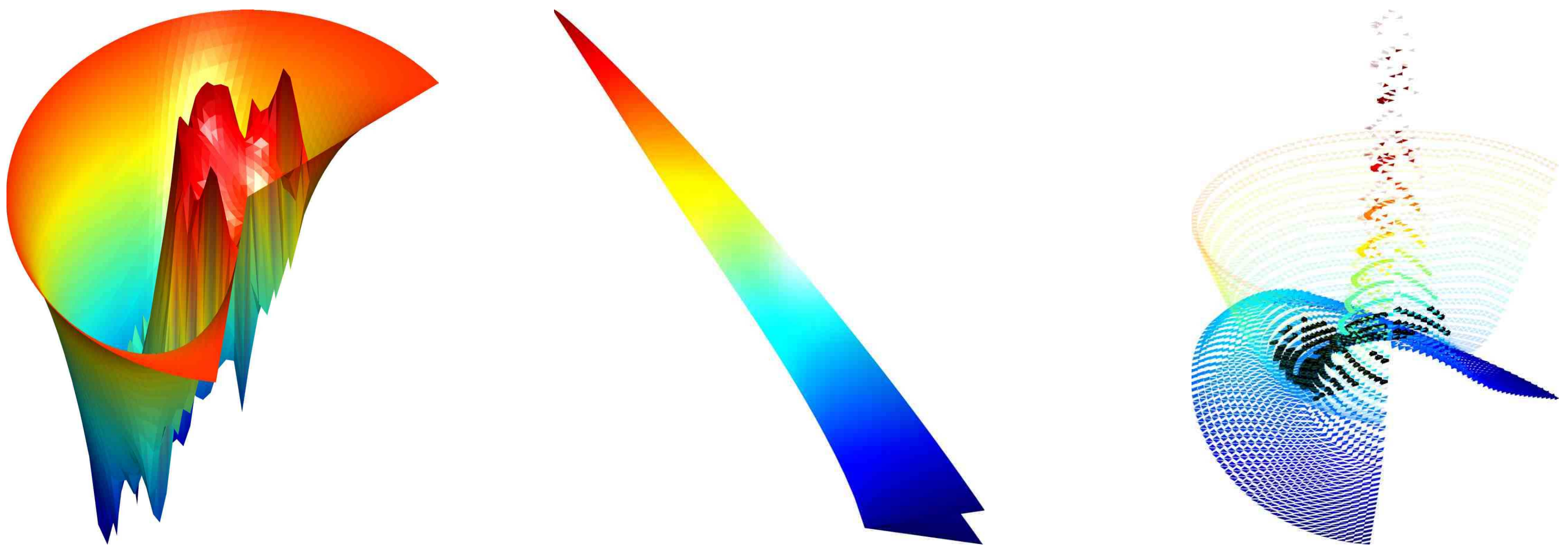
We choose $\Omega := \{(r,\varphi) \mid r \in (0,1), \varphi \in (0,\omega)\}$ with boundaries $\Gamma_1 := [0,1] \times \{0\} \cup \{(r\cos\omega, r\sin\omega) \mid r \in [0,1]\}$. and $\Gamma_2 := \{(\cos\varphi, \sin\varphi) \mid \varphi \in (0,\omega)\}$. We further choose $y^d := r^{\pi/\omega}\sin(\pi\varphi/\omega),\ \psi \in L^q(\Omega)$ for some $q > 2$ and $\alpha = 1$ as well as $a = 1, c = 0$ and $f = 0$.

Remark. The state satisfies $y \in W^{1,r}(\Omega)$ with $r := \frac{2\omega}{\omega - \pi}$.

Ex. 1: $\omega = \frac{5}{4}\pi,\ r = 10,\ \psi(x) := 2|x|^{-1/5} + |x| - 1.9\ (\psi \in L^{10}(\Omega))$.

Ex. 2: $\omega = \frac{3}{2}\pi,\ r = 6,\ \psi(x) := 0.1|x|^{-1/3} + 0.9\ (\psi \in L^{6}(\Omega))$..

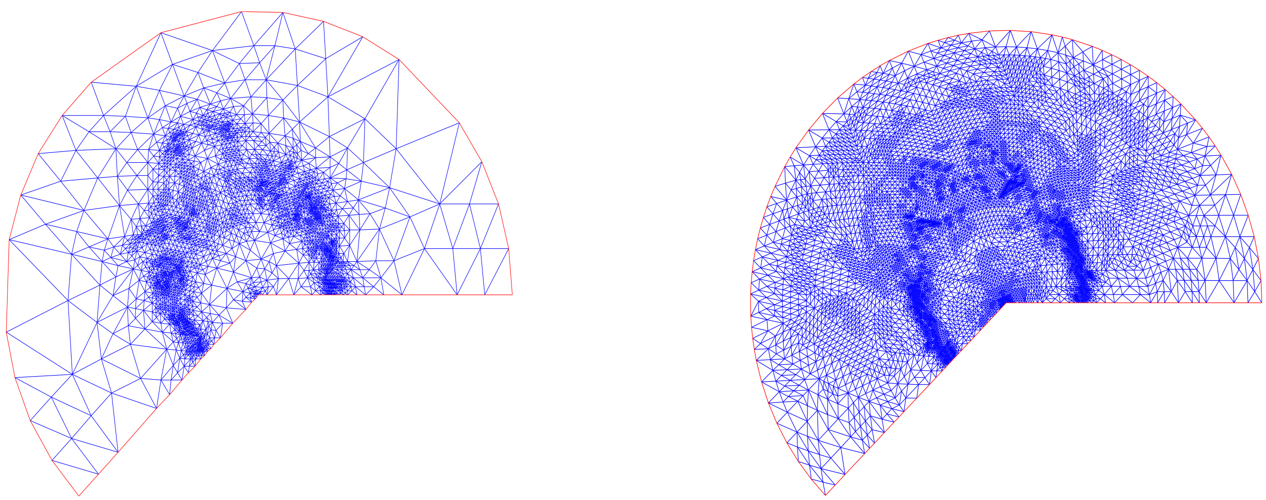
Gradient-State Constraints: Numerical Example I



Computed optimal control u_h (l.), state y_h (m.), and $|\nabla y_h|_T|$ (r.)

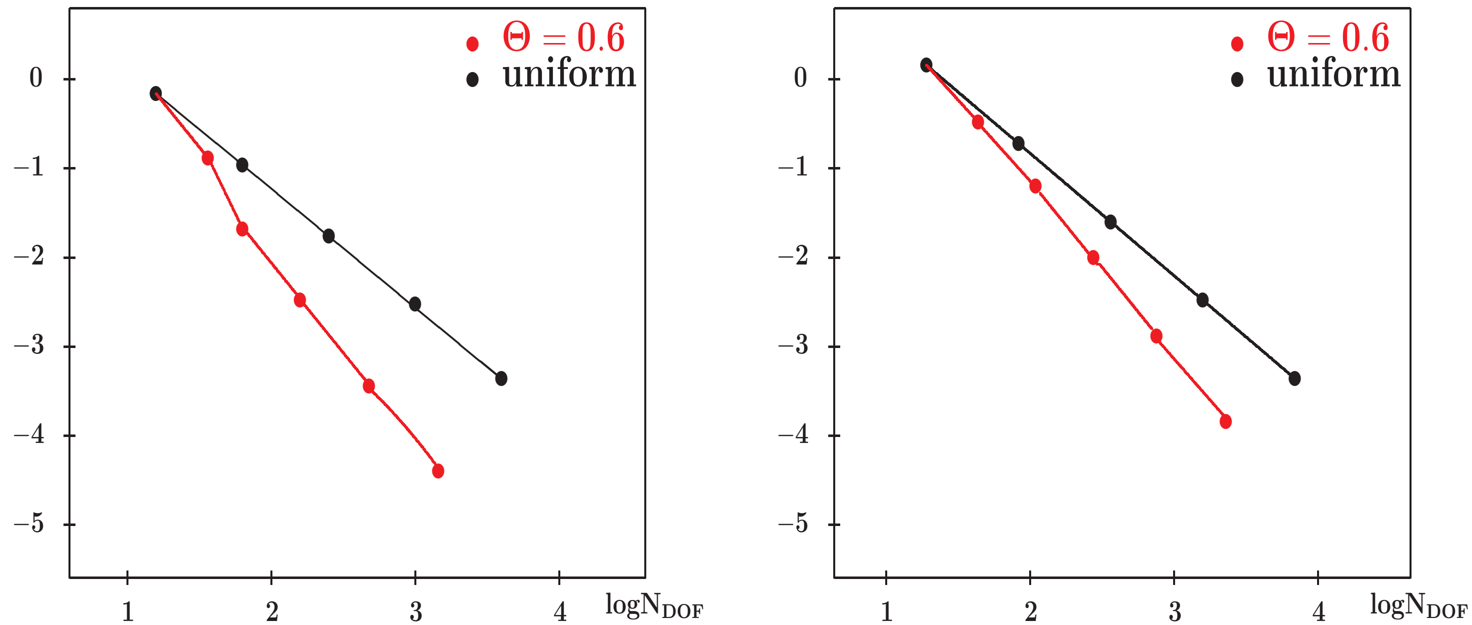
Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Gradient-State Constraints: Numerical Example I



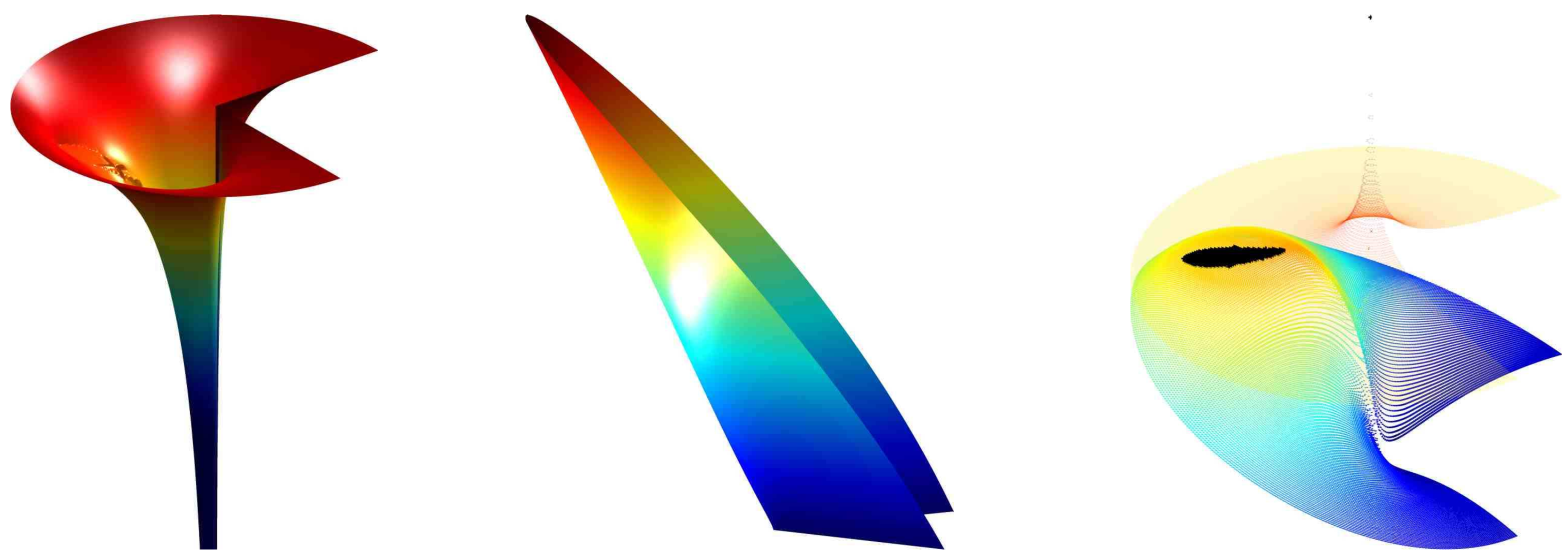
Adaptively refined meshes with $\#\mathcal{N}_h(\Omega) = 4020$ (l.) and $\#\mathcal{N}_h(\Omega) = 9088$ (r.)

Gradient-State Constraints: Numerical Examples I/II



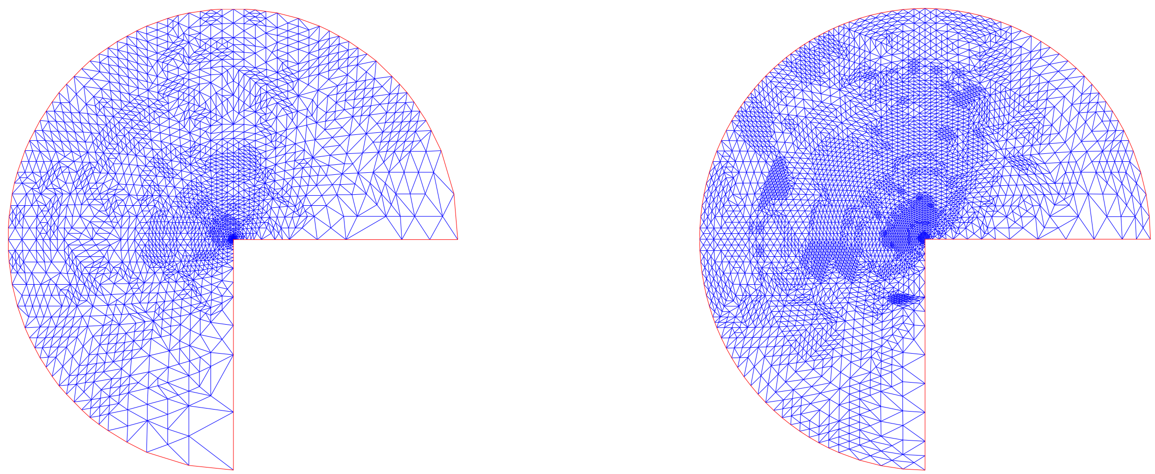
Convergence history: Example 1 (l.) and Example 2 (r.)

Gradient-State Constraints: Numerical Example II



Computed optimal control u_h (l.), state y_h (m.), and $|\nabla y_h|_T|$ (r.)

Gradient-State Constraints: Numerical Example II



Adaptively refined meshes with $\#\mathcal{N}_h(\Omega) = 2345$ (l.) and $\#\mathcal{N}_h(\Omega) = 4289$ (r.)