An Introduction to the A Posteriori Error Analysis of Elliptic Optimal Control Problems

Ronald H.W. Hoppe[1,2]

[1] Department of Mathematics, University of Houston
[2] Institute of Mathematics, University of Augsburg

Mathematisches Forschungsinstitut Oberwolfach

Oberwolfach, November 2010

Adaptive Finite Element Methods for Optimal Control Problems

M. Hintermüller, R.H.W. Hoppe, Y. Iliash, and M. Kieweg; An a posteriori error analysis of adaptive finite element methods for distributed elliptic control problems with control constraints.
ESAIM: Control, Optimisation and Calculus of Variations 14, 540–560, 2008.

M. Hintermüller and R.H.W. Hoppe; Goal-oriented adaptivity in control constrained optimal control of partial differential equations.
SIAM J. Control Optim. 47, 1721–1743, 2008.

M. Hintermüller and R.H.W. Hoppe; Goal-oriented adaptivity in pointwise state constrained optimal control of partial differential equations.
SIAM J. Control Optim., 2010 (in press).

Adaptive Finite Element Methods for Optimal Control Problems

M. Hintermüller, M. Hinze, and R.H.W. Hoppe; Weak-duality based adaptive finite element methods for PDE constrained optimization with pointwise gradient state constraints. submitted to JCM, 2010.

Recent Books on Optimal Control of PDEs

R. Glowinski, J.L. Lions, and J. He; Exact and Approximate Controllability for Distributed Parameter Systems: A Numerical Approach. Cambridge University Press, Cambridge, 2008.

M. Hinze, R. Pinnau, and M. Ulbrich; Optimization with PDE Constraints. Springer, Berlin-Heidelberg-New York, 2008.

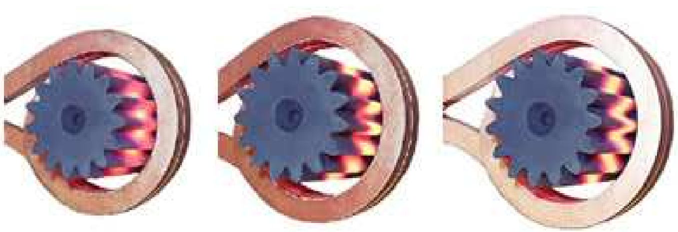
F. Tröltzsch; Optimal Control of Partial Differential Equations. Theory, Methods, and Applications. American Mathematical Society, Providence, 2010.

Contents

- The optimality systems for unconstrained, control, state, and gradient-state constrained elliptic optimal control problems
- Review of the a posteriori error analysis of adaptive finite element methods
- A posteriori error analysis of unconstrained elliptic optimal control problems
- Residual-type a posteriori error analysis of control constrained problems
- The goal-oriented dual weighted approach for state constrained problems

Modeling Our Complex World ...

Application I: Optimal Control of Induction Hardening



Optimal hardening of a steel workpiece by electromagnetic induction and quenching: The creation of martensitic structures increases the durability of the steel.

$$\inf_{u \in K, y} \quad \frac{1}{2}\|\theta(\cdot, T) - \theta^d\|^2_{0,\hat{\Omega}_2} + \frac{\alpha}{2}\int_0^T \|u\|^2_{0,\Gamma_S} dt,$$

where the control $u \in K := \{v \in L^2((0,T), L^2(\Gamma_2)) \mid u_{min} \le u \le u_{max} \text{ a.e.}\}$ is a current density applied at $\Gamma_S \subset \Omega_1$ and the state $y = (\varphi, A, \theta, z)$ consists of the electric potential φ, the magnetic vector potential A, the temperature θ, and the phase function z.

The state $y = (\varphi, A, \theta, z)$ satisfies the state equations

$$\sigma \frac{\partial A}{\partial t} + \text{curl}(\mu^{-1}\text{curl } A) + \sigma \nabla\varphi = 0 \quad \text{in } Q := D \times (0,T),$$

$$A \wedge n_{\partial D} = 0 \quad \text{on } \Sigma := \partial D \times (0,T), \quad A(\cdot,0) = A_0 \quad \text{in } D,$$

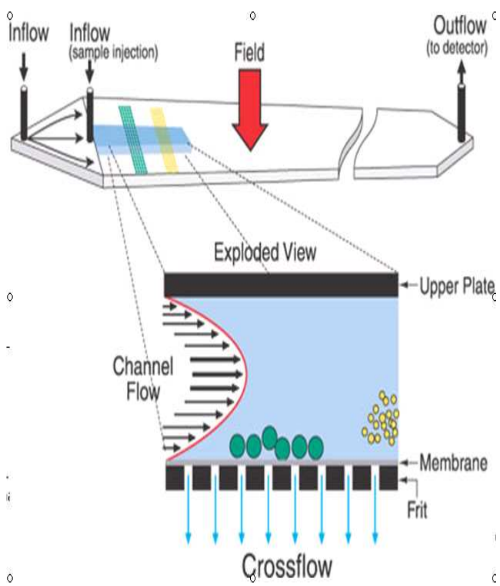
$$-\sigma\Delta\varphi = 0 \quad \text{in } \Omega_1 \times (0,T), \quad n_{\partial\Omega_1}\cdot\nabla\varphi = \begin{cases} u \text{ on } \Gamma_S \times (0,T) \\ 0 \text{ on } (\partial\Omega_1 \setminus \Gamma_s) \times (0,T) \end{cases},$$

$$\rho c \frac{\partial \theta}{\partial t} - \nabla\cdot(\kappa\nabla\theta) = -\rho L \frac{\partial z}{\partial t} + \sigma \left|\frac{\partial A}{\partial t}\right|^2 \quad \text{in } Q_2 := \Omega_2 \times (0,T),$$

$$n_{\partial\Omega_2} \cdot \kappa\nabla\theta = 0 \quad \text{on } \Sigma_2 := \partial\Omega_2 \times (0,T), \quad \theta(\cdot,0) = \theta_0 \quad \text{in } \Omega_2,$$

$$\tau \frac{dz}{dt} = g(\theta, z) \quad \text{in } (0,T), \quad z(0) = z_0.$$

Application II: Optimal Control of AF^4

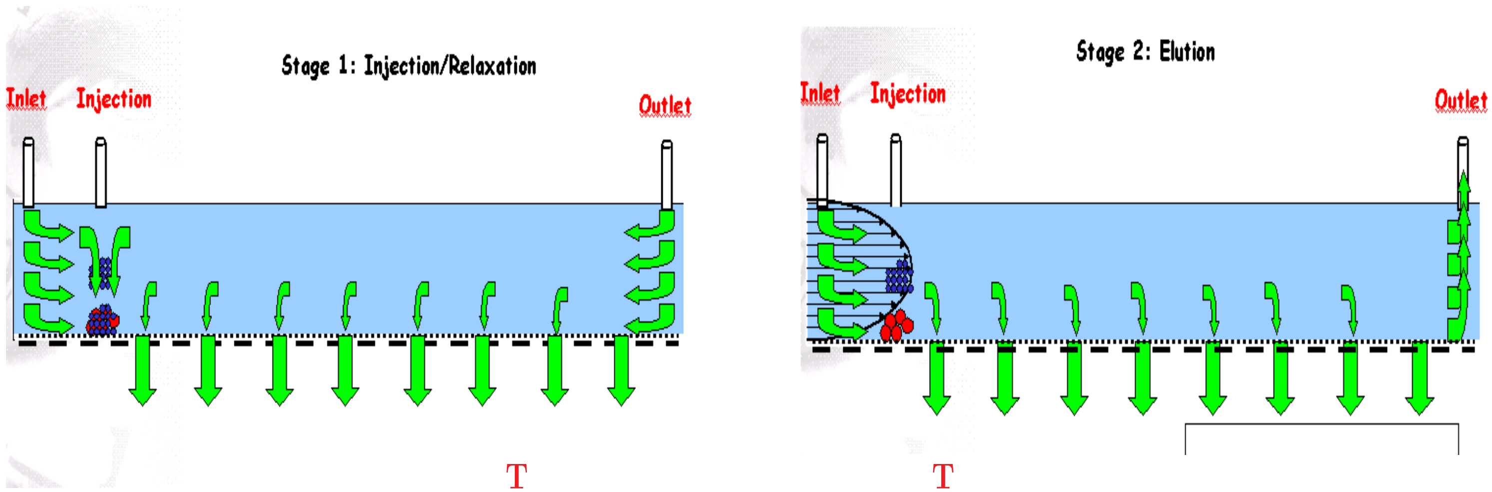


AF^4 (Asymmetric Flow Field Flow Fractionation) is a process for the efficient separation of particles of different size (μm - nm) in microfluidic flows.

AF^4 is used in chemical analytics, hematology, pharmacology, proteomics, and cytometry.

The principle of AF^4 relies on the separation in a microchannel due to a force induced by a cross flow through a porous membrane permeable for the carrier fluid, but impermeable for the particles.

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg



$$\inf \quad \frac{1}{2}\int_{T^*}^{T} \|c - c^d\|^2_{0,\Gamma_{out}} dt + \frac{\alpha}{2}\int_0^T \|u\|^2_{0,\Gamma_{in}} dt,$$

where the control u is the inflow velocity at the inlet and the state $y = (v, p, c)$ satisfies the Navier-Stokes Brinkman equations for (v, p) and advection-diffusion equations for the analytes $c = (c_1, \cdots, c_M)^T$.

The Adaptive Cycle

The Loop in Adaptive Finite Element Methods (AFEM)

Adaptive Finite Element Methods (AFEM) consist of successive loops of the cycle

SOLVE $\Longrightarrow$ ESTIMATE $\Longrightarrow$ MARK $\Longrightarrow$ REFINE

SOLVE:	Numerical solution of the FE discretized problem
ESTIMATE:	Residual and hierarchical a posteriori error estimators Error estimators based on local averaging Goal oriented weighted dual approach Functional type a posteriori error bounds
MARK:	Strategies based on the max. error or the averaged error Bulk criterion for AFEMs
REFINE:	Bisection or 'red/green' refinement or combinations thereof

Elliptic Optimal Control Problems

Unconstrained Case

Optimize first, then discretize

Elliptic Optimal Control Problems: Unconstrained Case

Given $y^d \in L^2(\Omega)$ and $\alpha > 0$, find $(y,u) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\inf_{(y,u)} J(y,u) \;:=\; \frac{1}{2} \int_\Omega |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx,$$

$$\text{subject to} \qquad -\Delta y = u \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \Gamma.$$

Reduced formulation: Denoting by $G : H^{-1}(\Omega) \to H_0^1(\Omega)$ the control-to-state map, which assigns to a control $u \in L^2(\Omega)$ the solution $y = G(u) \in H_0^1(\Omega)$ of the state equation, the reduced formulation reads:

$$\inf_{u} J_{red}(u) \;:=\; \frac{1}{2} \int_\Omega |G(u) - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx.$$

Unconstrained Minimization in Function Space

Theorem. Let V be a reflexive Banach space and assume that $J : V \to (-\infty, +\infty]$ is a proper convex, lower semicontinuous (lsc), and coercive functional. Then, the unconstrained minimization problem

$$\inf_{v \in V} J(v)$$

has a solution $u \in V$.

If J is strictly convex, the solution is unique.

Elliptic Optimal Control Problems: Unconstrained Case

Theorem. The unconstrained optimal control problems admits a unique solution.
Proof. Minimizing sequence argument.

Theorem. If (y, u) is the optimal solution, then there exists $p \in H_0^1(\Omega)$ such that

$$\begin{aligned} -\Delta p &= y^d - y \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \Gamma, \end{aligned}$$

and

$$p = \alpha u \quad \text{in } \Omega.$$

Proof. Let $u \in L^2(\Omega)$ be the unique solution of the optimal control problem. The necessary (and here also sufficient) optimality condition for

$$\inf_u J_{red}(u) \;:=\; \frac{1}{2}\int_\Omega |G(u) - y^d|^2 \, dx + \frac{\alpha}{2}\int_\Omega |u|^2 \, dx.$$

reads

$$(J'_{red}(u), v)_{0,\Omega} = (G(u) - y^d, G(v))_{0,\Omega} + \alpha(u, v)_{0,\Omega} = 0, \quad v \in L^2(\Omega),$$

where $J'_{red}(u)$ is the Gâteaux derivative of J_{red} at u.
Straightforward computation yields

$$(J'_{red}(u), v)_{0,\Omega} = (\underbrace{G^*(\underbrace{G(u)}_{=\,y} - y^d)}_{=\,-p} + \alpha u, v)_{0,\Omega} = 0, \quad v \in L^2(\Omega),$$

and hence, $p = G^*(y^d - y)$ and $p - \alpha u = 0$.

Optimality Conditions: Lagrange Multiplier Approach

Let $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the operator associated with the bilinear form $a(y,v) := (\nabla y, \nabla v)_{0,\Omega}$. Couple the PDE constraint $Ay = u$ by a Lagrange multiplier $p \in H_0^1(\Omega)$:

$$\inf_{y,u} \sup_{p} \mathcal{L}(y,u.p), \qquad \mathcal{L}(y,u,p) := J(y,u) + \langle Ay - u, p \rangle_{H^{-1},H_0^1}.$$

Optimality Conditions:

$$\mathcal{L}_p(y,u,p) = Ay - u = 0 \quad \Longrightarrow \quad Ay = 0,$$

$$\mathcal{L}_y(y,u,p) = y - y^d + A^* p = 0 \quad \Longrightarrow \quad A^* p = y^d - y,$$

$$\mathcal{L}_u(y,u,p) = \alpha u - p = 0 \quad \Longrightarrow \quad p = \alpha u.$$

Finite Element Approximation of the Distributed Control Problem

Let $\mathcal{T}_h(\Omega)$ be a shape regular, simplicial triangulation of Ω and let

$$V_h \ := \ \{ \ v_h \in C(\Omega) \mid v_h|_T \in P_1(T) \ , \ T \in \mathcal{T}_h(\Omega) \ , \ v_h|_{\partial\Omega} = 0 \ \}$$

be the FE space of continuous, piecewise linear finite elements.

Consider the following FE Approximation of the distributed control problem

$$\begin{array}{lll} \text{Minimize} & J(y_h, u_h) \ := \ \dfrac{1}{2} \, \|y_h - y^d\|^2_{L^2(\Omega)} \ + \ \dfrac{\alpha}{2} \, \|u_h\|^2_{L^2(\Omega)} \ , \\ \text{over} & (y_h, u_h) \in V_h \times V_h \ , \\ \text{subject to} & a(y_h, v_h) \ = \ (u_h, v_h)_{L^2(\Omega)} \ , \ v_h \in V_h \ . \end{array}$$

Optimality Conditions for the FE Discretized Control Problem

There exists an adjoint state $p_h \in V_h$ such that the triple (y_h, p_h, u_h) satisfies

$$\begin{aligned} a(y_h, v_h) &= (u_h, v_h)_{L^2(\Omega)} \quad , \quad v_h \in V_h \; , \\ a(p_h, v_h) &= -\, (y_h - y^d, v_h)_{L^2(\Omega)} \quad , \quad v_h \in V_h \; , \\ p_h - \alpha u_h &= 0 \; . \end{aligned}$$

Algebraic Formulation:

$$\begin{pmatrix} A_h & 0 \\ M_h & A_h \end{pmatrix} \begin{pmatrix} y_h \\ p_h \end{pmatrix} = \begin{pmatrix} M_h u_h \\ y_h^d \end{pmatrix} \quad \underset{u_h = \alpha^{-1} p_h}{\Longrightarrow} \quad \begin{pmatrix} A_h & -\alpha^{-1} M_h \\ M_h & A_h \end{pmatrix} \begin{pmatrix} y_h \\ p_h \end{pmatrix} = \begin{pmatrix} 0 \\ y_h^d \end{pmatrix} .$$

Solver: Multigrid with preconditioned Uzawa as a smoother

Multigrid Solvers for Elliptic Optimal Control Problems

A. Borzi, K. Kunisch, and D. Y. Kwak; Accuracy and convergence properties of the finite difference multigrid solution of an optimal control optimality system.
SIAM J. Control Optimization 41, 1477-1497, 2003.

A. Borzi and V. Schulz; Multigrid methods for PDE optimization.
SIAM Rev. 51, 361-395, 2009.

J. Schöberl, R. Simon, and W. Zulehner; A robust multigrid method for elliptic optimal control problems.
Preprint, Inst. of Comput. Math., University of Linz, 2010.

Elliptic Optimal Control Problems

Control Constraints

Elliptic Optimal Control Problems: Control Constrained Case

Given $y^d \in L^2(\Omega)$, $\alpha > 0$, and the closed convex set

$$K := \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\},$$

where ψ is an affine function, find $(y,u) \in H_0^1(\Omega) \times K$ such that

$$\inf_{(y,u) \in H_0^1(\Omega) \times K} J(y,u) := \frac{1}{2} \int_\Omega |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx,$$

$$\text{subject to} \quad \begin{aligned} -\Delta y &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Elliptic Optimal Control Problems: Control Constrained Case

Theorem. The control constrained optimal control problem has a unique solution.
Proof. Minimizing sequence argument.

Theorem. If $(y,u) \in H_0^1(\Omega) \times K$ is the optimal solution, then there exists an adjoint state $p \in H_0^1(\Omega)$ and an adjoint control $\boldsymbol{\lambda} \in L_2(\Omega)$ such that

$$\begin{aligned} -\Delta p &= y^d - y \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \Gamma, \end{aligned}$$

$$p = \alpha u + \boldsymbol{\lambda} \quad \text{in } \Omega,$$

$$\boldsymbol{\lambda} \in L_+^2(\Omega), \ \psi - u \geq 0, \ (\boldsymbol{\lambda}, \psi - u)_{0,\Omega} = 0.$$

Elliptic Optimal Control Problems: Control Constrained Case

Reduced formulation: Denoting by $G : H^{-1}(\Omega) \to H_0^1(\Omega)$ the control-to-state map, which assigns to $u \in H^{-1}(\Omega)$ the solution $y = G(u) \in H_0^1(\Omega)$ of the state equation, the reduced formulation reads:

$$\inf_{u \in K} J_{red}(u) \; := \; \frac{1}{2} \int_\Omega |G(u) - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx.$$

Unconstrained formulation: Let I_K be the indicator function of the constraint set K. Then, the unconstrained formulation of the control constrained optimal control problem is given by

$$\inf_{u \in L^2(\Omega)} \hat{J}(u) \; := \; J_{red}(u) \; + \; I_K(u).$$

Proof. The necessary and sufficient optimality condition is given by

$$0 \in \partial \hat{J}(u) = J'_{red}(u) + \partial I_K(u),$$

where $\partial I_K(u)$ is the subdifferential of I_K at u. Hence, there exists $\lambda \in \partial I_K(u)$ such that

$$\underbrace{G^*(\underbrace{G(u)}_{=\, y} - y^d)}_{=\, -p} + \alpha u + \lambda = 0 \quad \Longrightarrow \quad p = \alpha u + \lambda.$$

Since $\partial I_K(u) = \{\mu \in L^2(\Omega) \mid (\mu, u - v)_{0,\Omega} \geq 0,\ v \in I_K(u)\}$, choosing $v = u - w_+,\ w_+ \in L^2_+(\Omega)$, it follows that $\lambda \in L^2_+(\Omega)$. On the other hand, choosing $v = \psi$ allows to deduce

$$(\lambda, \psi - u)_{0,\Omega} = 0.$$

Moreau-Yosida Approximation of Multivalued Maps I

Weighted Duality Mapping: Assume that V is a Banach space with dual V^* and let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and non-decreasing function such that $h(0) = 0$ and $h(t) \to \infty$ as $t \to \infty$. Then the mapping $J_h : V \to 2^{V^*}$

$$J_h(u) := \{u^* \in V^* \mid \langle u^*, u\rangle = \|u\|\|u^*\| \ , \ \|u^*\| = h(\|u\|)\}$$

is called the duality mapping with weight h.

Example: For $V = L^p(\Omega), V^* = L^q(\Omega), 1 < p, q < \infty, 1/p + 1/q = 1$, and $h(t) = t^{p-1}$ we have

$$J_h(u)(x) = \begin{cases} |u(x)|^{p-1} \operatorname{sgn}(u(x)) \ , & u(x) \neq 0 \\ 0 \ , & u(x) = 0 \end{cases} .$$

Moreau-Yosida Approximation of Multivalued Maps II

Moreau-Yosida proximal map: Let $f : V \to \bar{\mathbb{R}}$ be a lower semi-continuous proper convex function with subdifferential ∂f. For $c > 0$, the Moreau-Yosida proximal map $P_c^{\partial f} : V \to 2^V$ is defined such that $P_c^{\partial f}(w)$, $w \in V$, is the set of minimizers of

$$\inf_{v \in V} f(v) + c j_h\left(\frac{v - w}{c}\right),$$

where $\partial j_h = J_h$.

Moreau-Yosida approximation: If J_h is single-valued, then for $c > 0$ the Moreau-Yosida approximation $(\partial f)_c$ of ∂f is given by

$$(\partial f)_c(w) := J_h(c^{-1} w - c^{-1} P_c^{\partial f}(w)).$$

Moreau-Yosida Approximation of ∂I_{K_C}

Idea: Approximate ∂I_{K_C} by its Moreau-Yosida approximation $(\partial I_{K_C})_c$.

Theorem. For any $c > 0$, we have

$$\lambda \in (\partial I_{K_C})_c,$$

if and only if there holds

$$\lambda = c\Big(u + c^{-1}\lambda - \Pi_{K_C}(u + c^{-1}\lambda)\Big) = c\ \max(0, u + c^{-1}\lambda - \psi),$$

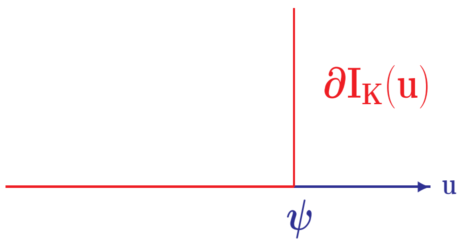
and this is equivalent to

$$u = \Pi_{K_C}(u + c^{-1}\lambda),$$

where Π_{K_C} denotes the L^2-projection onto K_C.

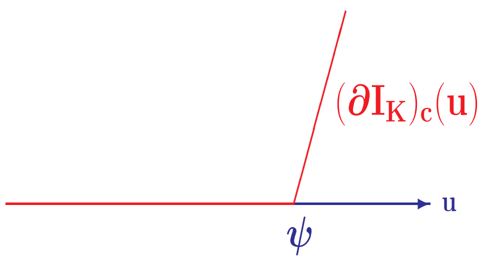
Elliptic Optimal Control Problems: Control Constrained Case

Problem: The subdifferential $\partial I_K(u)$ is a multivalued function.



$$\partial I_K(u) = \begin{cases} 0 \ , & u < \psi \\ [0,+\infty) \ , & u = \psi \end{cases} .$$

Remedy: Moreau-Yosida approximation of multivalued functions.



$$(\partial I_K)_c(u) = \begin{cases} 0 \ , & u < \psi \\ c(u-\psi) \ , & u \geq \psi \end{cases} .$$

$$\lambda \in (\partial I_K)_c(u) \quad \Longleftrightarrow \quad \lambda = \max(0, \lambda + c(u-\psi)).$$

Primal-Dual Active Set Strategy I

Step 1 (Initialization):

Choose $\mathbf{c} > \mathbf{0}$, start-iterates $\mathbf{y}_h^{(0)}, \mathbf{u}_h^{(0)}, \boldsymbol{\lambda}_h^{(0)}$ and set $\mathbf{n} = \mathbf{1}$.

Step 2 (Specification of active/inactive sets):

Compute the active/inactive sets $\mathcal{A}_n$ and $\mathcal{I}_n$ according to

$$\mathcal{A}_n := \{\mathbf{1} \le \mathbf{i} \le \mathbf{N} \mid (\mathbf{u}_h^{(n-1)} + \mathbf{c}^{-1}\boldsymbol{\lambda}_h^{(n-1)})_i > (\boldsymbol{\psi}_h)_i\} \quad , \quad \mathcal{I}_n := \{\mathbf{1}, \cdots, \mathbf{N}\} \setminus \mathcal{A}_n .$$

Step 3 (Termination criterion):

If $\mathbf{n} \ge \mathbf{2}$ and $\mathcal{A}_n = \mathcal{A}_{n-1}$, stop the algorithm. Otherwise, go to Step 4.

Primal-Dual Active Set Strategy II

Step 4 (Update of the state, adjoint state, and control):

Compute $y_h^{(n)}, p_h^{(n)}$ as the solution of

$$(A_h y_h^{(n)})_i = \begin{cases} (\psi_h)_i \ , & \text{if } i \in \mathcal{A}_n \\ \alpha^{-1}(M_h^{-1} p_h^{(n)})_i \ , & \text{if } i \in \mathcal{I}_n \end{cases} \quad , \quad A_h p_h^{(n)} = -M_h y_h^{(n)} + y_h^d ,$$

and set

$$(u_h^{(n)})_i := \begin{cases} (\psi_h)_i \ , & \text{if } i \in \mathcal{A}_n \\ \alpha^{-1}(M_h^{-1} p_h^{(n)})_i \ , & \text{if } i \in \mathcal{I}_n \end{cases} .$$

Step 5 (Update of the multiplier):

Set $\lambda_h^{(n)} := p_h^{(n)} - \alpha M_h u_h^{(n)}$, $n := n+1$, and go to Step 2.

Elliptic Optimal Control Problems

State Constraints

Elliptic Optimal Control Problems: State Constrained Case

Given $y^d \in L^2(\Omega)$, $\alpha > 0$, and the closed convex set

$$K := \{v \in W^{1,r}(\Omega) \cap H_0^1(\Omega),\; r > d \mid v \le \psi \text{ in } \Omega\},$$

where $\psi \in W^{1,\infty}(\Omega),\; \psi|_\Gamma > 0$, find $(y,u) \in K \times L^2(\Omega)$ such that

$$\inf_{(y,u)\in K\times L^2(\Omega)} J(y,u) := \frac{1}{2}\int_\Omega |y - y^d|^2\, dx + \frac{\alpha}{2}\int_\Omega |u|^2\, dx,$$

$$\text{subject to} \quad \begin{aligned} -\Delta y &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Elliptic Optimal Control Problems: State Constrained Case

Reduced formulation: Denoting by $G : W^{-1,s}(\Omega) \to W_0^{1,r}(\Omega)$ the control-to-state map, which assigns to $u \in W^{-1,s}(\Omega)$ the solution $y = G(u) \in W_0^{1,r}(\Omega)$ of the state equation, the reduced formulation reads:

$$\inf_{G(u) \in K} J_{red}(u) \;:=\; \frac{1}{2} \int_\Omega |G(u) - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx.$$

Unconstrained formulation: Let I_K be the indicator function of the constraint set K. Then, the unconstrained formulation of the control constrained optimal control problem is given by

$$\inf_{u \in L^2(\Omega)} \hat{J}(u) \;:=\; J_{red}(u) \;+\; (I_K \circ G)(u).$$

Elliptic Optimal Control Problems: State Constrained Case

Theorem. The state constrained optimal control problem has a unique solution.
Proof. Minimizing sequence argument.

Theorem. Assume that the following **Slater condition** holds true:
There exists $u_0 \in L^2(\Omega)$ such that the associated solution $y_0 = G(u_0) \in W_0^{1,r}(\Omega)$ satisfies $y_0 \in \mathrm{int}(K)$. If $(y,u) \in K \cap L^2(\Omega)$ is the unique solution of the state constrained optimal control problem, there exist

$$p \in W_0^{1,s}(\Omega),\ \frac{1}{r} + \frac{1}{s} = 1 \quad \text{and} \quad \boldsymbol{\lambda} \in M(\bar{\Omega}) \quad \text{s.th.}$$

$$\langle \nabla p, \nabla v \rangle_{L^s,L^r} = (y^d - y, v)_{0,\Omega} - \langle \boldsymbol{\lambda}, v \rangle_{M(\bar{\Omega}),C(\bar{\Omega})},\ v \in W_0^{1,r}(\Omega),$$

$$p = \alpha u,$$

$$\boldsymbol{\lambda} \in M_+(\bar{\Omega}), \quad \psi - y \geq 0, \quad \langle \boldsymbol{\lambda}, \psi - y \rangle_{M(\bar{\Omega}),C(\bar{\Omega})} = 0.$$

Proof. The necessary and sufficient optimality condition reads

$$0 \in \partial \hat{J}(u) = J'_{red}(u) + \partial(I_K \circ G)(u).$$

What do we know about $\partial(I_K \circ G)(u)$?

Subdifferential Calculus: Subdifferential of Composite Maps

Theorem. Let X, Y be Banach spaces with duals X^*, Y^*. Let $f : X \to (-\infty, +\infty]$ be proper convex and lsc, and let $A : Y \to X$ be a bounded linear operator. Assume that there exists $\tilde{u} \in Y$ such that f is continuous and finite at $A\tilde{u}$. Then there holds

$$\partial(f \circ A)(u) = A^* \partial f(A(u)).$$

Proof. The necessary and sufficient optimality condition reads

$$0 \in \partial \hat{J}(u) = J'_{red}(u) + \partial(I_K \circ G)(u).$$

Due to the Slater condition there holds

$$\partial(I_K \circ G)(u) = G^*(\partial I_K(G(u))).$$

Hence, there exists $\lambda \in \partial I_K(y)$ such that

$$\underbrace{G^*(\underbrace{G(u)}_{=\ y} + \lambda)}_{=\ -p} + \alpha u = 0 \quad \Longrightarrow \quad p = \alpha u \quad \text{and} \quad p = G^*(y^d - y - \lambda).$$

PDE theory tells us that $p \in W_0^{1,s}(\Omega)$.

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Elliptic Optimal Control Problems

Constraints on the Gradient of the State

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Elliptic Optimal Control with Pointwise Gradient-State Constraints

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary Γ, $y^d \in L^2(\Omega)$ a desired state, f a forcing term, $\psi \in L^2(\Omega)$ s.th. $\psi \geq \psi_{\min} > 0$ a.e. in Ω, and $\alpha > 0$, find $(y, u) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$(P) \qquad \inf_{(y,u)} J(y,u) := \frac{1}{2} \int_\Omega |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx,$$

$$\text{subject to} \qquad \begin{aligned} Ly := -\nabla \cdot a \nabla y + cy &= f + u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \Gamma, \\ \nabla y \in K := \{v \in L^2(\Omega)^2 \;|\; &|v| \leq \psi \text{ a.e. in } \Omega\}. \end{aligned}$$

Pointwise Gradient-State Constraints: State-Reduced Formulation

Let $\hat{V} \subset H_0^1(\Omega)$ be a reflexive Banach space and let $\hat{G} : L^2(\Omega) \to \hat{V}$ be the map that assigns to the rhs $f + u$ the solution $y = \hat{G}(f + u)$ of the state equation. Assume that $\hat{G}$ is a bounded linear operator which is invertible such that $u = \hat{G}^{-1} y - f$.
This leads to the state-reduced formulation:
Find $y \in \hat{K} := \{v \in \hat{V} \mid |\nabla v| \leq \psi \text{ bf a.e. in } \Omega\}$ such that

$$\inf_{y \in \hat{K}} J_{red}(y) \; := \; \frac{1}{2} \int_\Omega |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |\hat{G}^{-1} y - f|^2 \, dx.$$

Unconstrained formulation:

$$\inf_{y \in \hat{V}} J_{red}(y) + I_{\hat{K}}(y)$$

where $I_{\hat{K}}$ stands for the indicator function of the set $\hat{K}$.

State-Reduced Formulation: Optimality Conditions

Theorem. The gradient-state constrained optimal control problem admits a unique solution $(y, u) \in \hat{K} \times L^2(\Omega)$ which is characterized by the existence of a unique pair $(p, w) \in L^2(\Omega) \times \hat{V}^*$ satisfying

$$Lp = -\nabla \cdot (a \nabla p) + cp \; = \; y^d - y - w \quad \text{in } \hat{V}^*,$$
$$p \; = \; \alpha u \quad \text{in } L^2(\Omega),$$
$$w \in N_{\hat{K}}(y) := \{\xi \in \hat{V}^* \;\; | \;\; \langle \xi, z - y \rangle_{\hat{V}^*, \hat{V}} \le 0, \; z \in \hat{K}\}.$$

Remark. If $\hat{V} = W^{2,r}(\Omega) \cap H_0^1(\Omega), r > 2$, there exists a Slater point, i.e., $y_0 \in \text{int } \hat{K}$ and $|\nabla(y_0 + v)| \le \psi$ in Ω for all $v \in C^1(\bar{\Omega})$ s.th. $\|v\|_{C^1(\bar{\Omega})} \le \delta$ for sufficiently small $\delta > 0$.

$$0 \in J'_{red}(y) + \partial(I_{\hat{K}} \circ \nabla)(y) = J'_{red}(y) - \nabla \cdot \partial I_{\hat{K}}(\nabla y),$$

i.e., there exists $\mu \in \partial I_{\hat{K}}(\nabla y) \subset M(\bar{\Omega})^2$ such that $w = -\nabla \cdot \mu$.

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Control-Reduced Formulation and Dual Problem

Denoting by $G : H^{-1}(\Omega) \to H_0^1(\Omega)$ the solution operator associated with the state equation, the optimal control problem can be written according to

$$\inf_{u \in L^2(\Omega)} \mathcal{F}(u) + \mathcal{G}(\Lambda u)$$

where

$$\mathcal{F}(u) := J(G(f+u), u), \quad \mathcal{G}(q) := I_K(q), \quad \Lambda := \nabla G.$$

Denoting by $\mathcal{F}^*$ and $\mathcal{G}^*$ the Fenchel conjugates of $\mathcal{F}$ and $\mathcal{G}$

$$\mathcal{F}^*(u^*) = \frac{1}{2} \|u^* + G^* y^d + \alpha f\|^2_{M^{-1}}, \quad \mathcal{G}^*(q^*) = \int_\Omega \psi |q^*| dx,$$

where $M := G^*G + \alpha I$ and $\|\cdot\|^2_{M^{-1}} := (M^{-1}\cdot, \cdot)_{0,\Omega}$, the dual problem reads as follows:

$$\text{(D)} \quad \sup_{q^* \in L^2(\Omega)} -\mathcal{F}^*(\Lambda^* q^*) - \mathcal{G}^*(-q^*) \iff \inf_{\mu \in L^2(\Omega)} \frac{1}{2} \|G^*(\nabla^* \mu + y^d) + \alpha f\|^2_{M^{-1}} + \int_\Omega \psi |\mu| dx.$$

The Fenchel Conjugate (Polar Function)

Let $f : V \to (-\infty, +\infty]$ be a proper convex function.
The Fenchel conjugate $J^* : V^* \to (-\infty, +\infty]$ is defined by means of

$$J^*(u^*) := \sup_{u \in V} \Big(\langle u^*, u \rangle - J(u) \Big).$$

Example. Let $K \subset V$ be a closed convex set with indicator function I_K.
The Fenchel conjugate I_K^* is given by

$$I_K^*(u^*) = \sup_{u \in K} \langle u^*, u \rangle.$$

The Fenchel Conjugate of $\mathcal{G} : \mathbf{L}^2(\Omega)^2 \to \mathbb{R}, \ \mathcal{G}(\mathbf{q}) := \mathbf{I}_K(\mathbf{q})$

We claim $\mathcal{G}^*(\mathbf{q}^*) = \int_\Omega \psi |\mathbf{q}^*| dx$

Proof. We have

$$\mathcal{G}^*(\mathbf{q}^*) = \sup_{\mathbf{q} \in K} (\mathbf{q}^*, \mathbf{q})_{0,\Omega}.$$

Since $|\mathbf{q}| \leq \psi$, there obviously holds

$$(\mathbf{q}^*, \mathbf{q})_{0,\Omega} \leq \int_\Omega \psi |\mathbf{q}^*| dx.$$

On the other hand, the special choice $\mathbf{q} := \psi \mathbf{q}^* |\mathbf{q}^*|^{-1}$ implies

$$(\mathbf{q}^*, \mathbf{q})_{0,\Omega} = (\mathbf{q}^*, \psi \mathbf{q}^* |\mathbf{q}^*|^{-1})_{0,\Omega} = \int_\Omega \psi |\mathbf{q}^*| dx.$$

Tightened Formulation of the Primal Problem

Consider the following tightened formulation of the primal problem

$$(\hat{P}) \qquad \inf_{(y,u) \in \hat{V} \times L^2(\Omega)} J(y,u) := \frac{1}{2} \int_\Omega |y - y^d|^2 dx + \frac{\alpha}{2} \int_\Omega |u|^2 dx,$$

subject to

$$Ly = f + u \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \Gamma, \quad |\nabla y| \le \psi \quad \text{a.e. in } \Omega.$$

Theorem. Let $\{\mu_n\}_{\mathbb{N}} \subset L^2(\Omega)^2$ be a minimizing sequence for the dual $(\hat{D})$ to $(\hat{P})$. Then, there exist a subsequence $\{\mu_n\}_{\mathbb{N}'}$ and $\mu \in M(\bar{\Omega})^2$ such that

$$w^* - \lim \mu_n = \mu \quad \text{in } M(\bar{\Omega})^2 \qquad \text{and} \qquad w - \lim \nabla \cdot \mu_n = -w \quad \text{in } \hat{V}^*.$$

Moreover, the limit $w \in \hat{V}^*$ satisfies

$$(*) \qquad Ly = f + u \quad \text{in } L^2(\Omega), \quad Lp = y^d - y - w \quad \text{in } \hat{V}^*, \quad p = \alpha u \quad \text{in } L^2(\Omega).$$

Remark. A quadruple $(y,u,p,w) \in V \times L^2(\Omega) \times L^2(\Omega) \times \hat{V}^*$ such that $(*)$ holds true and $\nabla y \in (M(\bar{\Omega})^2)^* \setminus C(\bar{\Omega})^2$, is called a weak solution of (P).

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Basic Concepts of Adaptive Finite Element Methods for Elliptic Boundary Value Problems

Ronald H.W. Hoppe[1,2]

[1] Department of Mathematics, University of Houston
[2] Institute of Mathematics, University of Augsburg

Mathematisches Forschungsinstitut Oberwolfach

Oberwolfach, November, 2010

Foundations of AFEM I

For a closed subspace $V \subset H^1(\Omega)$ we assume

$$a(\cdot,\cdot) : V \times V \to \mathbb{R}$$

to be a bounded, V-elliptic bilinear form, i.e.,

$$|a(v,w)| \leq C\|v\|_{k,\Omega}\|w\|_{k,\Omega}, \quad v,w \in V, \quad a(v,v) \geq \gamma\|v\|^2_{k,\Omega}, \quad v \in V,$$

for some constants $C > 0$ and $\gamma > 0$. We further assume $\ell \in V^*$ where V^* denotes the algebraic and topological dual of V and consider the variational equation: Find $u \in V$ such that

$$a(u,v) = \ell(v) \quad , \quad v \in V.$$

It is well-known by the Lax-Milgram Lemma that under the above assumptions the variational problem admits a unique solution.

Foundations of AFEM II

Finite element approximations are based on the Ritz-Galerkin approach: Given a finite dimensional subspace $V_h \subset V$ of test/trial functions, find $u_h \in V_h$ such that

$$a(u_h, v_h) \; = \; \ell(v_h), \quad v_h \in V_h.$$

Since $V_h \subset V$, the existence and uniqueness of a discrete solution $u_h \in V_h$ follows readily from the Lax-Milgram Lemma. Moreover, we deduce that the error $e_u := u - u_h$ satisfies the Galerkin orthogonality

$$a(u - u_h, v_h) \; = \; 0, \quad v_h \in V_h,$$

i.e., the approximate solution $u_h \in V_h$ is the projection of the solution $u \in V$ onto V_h with respect to the inner product $a(\cdot,\cdot)$ on V (elliptic projection). Using the Galerkin orthogonality, it is easy to derive the a priori error estimate

$$\|u - u_h\|_{1,\Omega} \; \leq \; M \inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega},$$

where $M := C/\gamma$. This result tells us that the error is of the same order as the best approximation of the solution $u \in V$ by functions from the finite dimensional subspace V_h. It is known as Céa's Lemma.

Foundations of AFEM III

The Ritz-Galerkin method also gives rise to an a posteriori error estimate in terms of the residual $r : V \to \mathbb{R}$

$$r(v) \;:=\; \ell(v) \;-\; a(u_h, v), \quad v \in V.$$

In fact, it follows that for any $v \in V$

$$\gamma \|u - u_h\|_{1,\Omega}^2 \le a(u - u_h, u - u_h) = r(u - u_h) \le \|r\|_{-1,\Omega} \; \|u - u_h\|_{1,\Omega},$$

whence

$$\|u - u_h\|_{1,\Omega} \;\le\; \frac{1}{\gamma} \; \sup_{v \in V} \frac{|r(v)|}{\|v\|_{1,\Omega}}.$$

Foundations of AFEM IV

Definition. An error estimator η_h is called reliable, if it provides an upper bound for the error up to data oscillations osc_h^{rel}, i.e., if there exists a constant $C_{rel} > 0$, independent of the mesh size h of the underlying triangulation, such that

$$\|e_u\|_a \leq C_{rel}\, \eta_h + \mathrm{osc}_h^{rel} .$$

On the other hand, an estimator η_h is said to be efficient, if up to data oscillations osc_h^{eff} it gives rise to a lower bound for the error, i.e., if there exists a constant $C_{eff} > 0$, independent of the mesh size h of the underlying triangulation, such that

$$\eta_h \leq C_{eff}\, \|e_u\|_a + \mathrm{osc}_h^{eff}.$$

Finally, an estimator η_h is called asymptotically exact, if it is both reliable and efficient with $C_{rel} = C_{eff}^{-1}$.

Reliability and Efficiency of Error Estimators II

Remark. The notion 'reliability' is motivated by the use of the error estimator in error control. Given a tolerance tol, an idealized termination criterion would be

$$\|e_u\|_a \;\le\; \mathrm{tol}.$$

Since the error $\|e_u\|_a$ is unknown, we replace it with the upper bound, i.e.,

$$C_{rel}\,\eta_h \;+\; \mathrm{osc}_h^{rel} \;\le\; \mathrm{tol}.$$

We note that the termination criterion both requires the knowledge of C_{rel} and the incorporation of the data oscillation term osc_h^{rel}. In the special case $C_{rel} = 1$ and $\mathrm{osc}_h^{rel} \equiv 0$, it reduces to

$$\eta_h \;\le\; \mathrm{tol}.$$

An alternative, but less used termination criterion is based on the lower bound, i.e., we require

$$\frac{1}{C_{eff}}\left(\eta_h \;-\; \mathrm{osc}_h^{eff}\right) \;\le\; \mathrm{tol}.$$

Typically, this criterion leads to less refinement and thus requires less computational time which motivates to call the estimator efficient.

The Role of the Residual

The error estimate

$$\|u - u_h\|_{1,\Omega} \;\le\; \frac{1}{\gamma} \sup_{v \in V} \frac{|r(v)|}{\|v\|_{1,\Omega}}$$

shows that in order to assess the error $\|e_u\|_a$ we are supposed to evaluate the norm of the residual with respect to the dual space V^*, i.e.,

$$\|r\|_{V^*} \;:=\; \sup_{v \in V \setminus \{0\}} \frac{|r(v)|}{\|v\|_a}.$$

In particular, we have the equality

$$\|r\|_{V^*} \;=\; \|e_u\|_a,$$

whereas for the relative error of $r(v), v \in V$, as an approximation of $\|e_u\|_a$ we obtain

$$\frac{(\|e_u\|_a - r(v))}{\|e_u\|_a} \;=\; \frac{1}{2}\,\|v - \frac{e_u}{\|e_u\|_a}\|_a^2, \quad v \in V \text{ with } \|v\|_a = 1.$$

The goal is to obtain lower and upper bounds for $\|r\|_{V^*}$ at relatively low computational expense.

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Model problem: Let Ω be a bounded simply-connected polygonal domain in Euclidean space $\mathbb{R}^2$ with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and consider the elliptic boundary value problem

$$Lu := -\nabla \cdot (a \nabla u) = f \quad \text{in } \Omega ,$$

$$u = 0 \quad \text{on } \Gamma_D , \quad n \cdot a \nabla u = g \quad \text{on } \Gamma_N,$$

where $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$ and $a = (a_{ij})_{i,j=1}^2$ is supposed to be a matrix-valued function with entries $a_{ij} \in L^\infty(\Omega)$, that is symmetric and uniformly positive definite. The vector n denotes the exterior unit normal vector on Γ_N. Setting

$$H^1_{0,\Gamma_D}(\Omega) := \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0 \},$$

the weak formulation is as follows: Find $u \in H^1_{0,\Gamma_D}(\Omega)$ such that

$$a(u,v) = \ell(v) \quad , \quad v \in H^1_{0,\Gamma_D}(\Omega),$$

where

$$a(v,w) := \int_\Omega a \nabla v \cdot \nabla w \, dx, \quad \ell(v) := \int_\Omega f \, v \, dx + \int_{\Gamma_N} g \, v \, d\sigma \quad , \quad v \in H^1_{0,\Gamma_D}(\Omega).$$

FE Approximation: Given a geometrically conforming simplicial triangulation $\mathcal{T}_h$ of Ω, we denote by

$$S_{1,\Gamma_D}(\Omega;\mathcal{T}_h) := \{ v_h \in H^1_{0,\Gamma_D}(\Omega) \mid v_h|_T \in P_1(K) , T \in \mathcal{T}_h \}$$

the trial space of continuous, piecewise linear finite elements with respect to $\mathcal{T}_h$. Note that $P_k(T)$, $k \geq 0$, denotes the linear space of polynomials of degree $\leq k$ on T. In the sequel we will refer to $\mathcal{N}_h(D)$ and $\mathcal{E}_h(D)$, $D \subseteq \bar{\Omega}$ as the sets of vertices and edges of $\mathcal{T}_h$ on D. We further denote by $|T|$ the area, by h_T the diameter of an element $T \in \mathcal{T}_h$, and by $h_E = |E|$ the length of an edge $E \in \mathcal{E}_h(\Omega \cup \Gamma_N)$. We refer to $f_T := |T|^{-1} \int_T f dx$ the integral mean of f with respect to an element $T \in \mathcal{T}_h$ and to $g_E := |E|^{-1} \int_E g ds$ the mean of g with respect to the edge $E \in \mathcal{E}_h(\Gamma_N)$.
The conforming P1 approximation reads as follows: Find $u_h \in S_{1,\Gamma_D}(\Omega;\mathcal{T}_h)$ such that

$$a(u_h, v_h) = \ell(v_h), \quad v_h \in S_{1,\Gamma_D}(\Omega;\mathcal{T}_h).$$

Representation of the Residual I

The residual r is given by

$$r(v) := \int_\Omega f\, v\, dx + \int_{\Gamma_N} g\, v ds - a(u_h, v) \quad , \quad v \in V.$$

Applying Green's formula elementwise yields

$$a(u_h, v) = \sum_{T \in \mathcal{T}_h} \int_T a\, \nabla u_h \cdot \nabla v\, dx = \sum_{E \in \mathcal{E}_h(\Omega)} \int_E [n \cdot a\, \nabla u_h]\, v\, ds + \sum_{E \in \mathcal{E}_h(\Gamma_N)} \int_E n \cdot a\, \nabla u_h\, v\, ds,$$

where $[n \cdot a\, \nabla u_h]$ denotes the jump of the normal derivative of u_h across $E \in \mathcal{E}_h(\Omega)$ and where we have used that $\Delta u_h \equiv 0$ on $T \in \mathcal{T}_h$, since $u_h|_T \in P_1(T)$. We thus obtain

$$r(v) := \sum_{T \in \mathcal{T}_h} r_T(v) + \sum_{E \in \mathcal{E}_h(\Omega \cup \Gamma_N)} r_E(v).$$

Representation of the Residual II

Here, the local residuals $r_T(v), T \in \mathcal{T}_h$, are given by

$$r_T(v) \ := \int\limits_T (f - Lu_h)v \ dx,$$

whereas for $r_E(v)$ we have

$$r_E(v) \ := -\int\limits_E [n \cdot a \ \nabla u_h] v \ ds, \ \ E \in \mathcal{E}_h(\Omega),$$

$$r_E(v) \ := \int\limits_E \Big(g - n \cdot a \ \nabla u_h\Big) v \ ds, \ \ E \in \mathcal{E}_h(\Gamma_N).$$

A Posteriori Error Estimator and Data Oscillations

The error estimator η_h consists of element residuals $\eta_T, T \in \mathcal{T}_h$, and edge residuals $\eta_E, E \in \mathcal{E}_H(\Omega \cup \Gamma_N)$, according to

$$\eta_h \ := \ \Big(\sum_{T \in \mathcal{T}_h} \eta_T^2 \ + \sum_{E \in \mathcal{E}_H(\Omega \cup \Gamma_N)} \eta_E^2 \Big)^{1/2},$$

where η_T and η_E are given by

$$\eta_T \ := \ h_T \ \|f_T - Lu_h\|_{0,T} \ , \ T \in \mathcal{T}_h,$$

$$\eta_E \ := \ \begin{cases} h_E^{1/2} \| [n \cdot a \ \nabla u_h] \|_{0,E} \ , \ E \in \mathcal{E}_h(\Omega), \\ h_E^{1/2} \|g_E - n \cdot a \ \nabla u_h\|_{0,E} \ , \ E \in \mathcal{E}_h(\Gamma_N) \end{cases} .$$

The a posteriori error analysis further invokes the data oscillations

$$\mathrm{osc}_h \ := \ \Big(\sum_{T \in \mathcal{T}_h} \mathrm{osc}_T^2(f) \ + \sum_{E \in \mathcal{E}_h(\Gamma_N)} \mathrm{osc}_E^2(g) \Big)^{1/2},$$

where $\mathrm{osc}_T(f)$ and $\mathrm{osc}_E(g)$ are given by

$$\mathrm{osc}_T(f) := h_T \ \|f - f_T\|_{0,T}, \quad \mathrm{osc}_E(g) := h_E^{1/2} \ \|g - g_E\|_{0,E}.$$

Clément's Quasi-Interpolation Operator I

For $p \in \mathcal{N}_h(\Omega) \cup \mathcal{N}_h(\Gamma_N)$ we denote by φ_p the basis function in $S_{1,\Gamma_D}(\Omega; \mathcal{T}_h)$ with supporting point p, and we refer to D_p as the set

$$D_p \ := \bigcup \{ \ T \in \mathcal{T}_h \ | \ p \in \mathcal{N}_h(T) \ \}.$$

We refer to π_p as the L^2-projection onto $P_1(D_p)$, i.e.,

$$(\pi_p(v), w)_{0,D_p} \ = \ (v, w)_{0,D_p} \quad , \quad w \in P_1(D_p),$$

where $(\cdot, \cdot)_{0,D_p}$ stands for the L^2-inner product on $L^2(D_p) \times L^2(D_p)$. Then, Clément's interpolation operator P_C is defined as follows

$$P_C \ : \ L^2(\Omega) \longrightarrow S_{1,\Gamma_D}(\Omega, \mathcal{T}_h), \quad P_C v := \sum_{p \in \mathcal{N}_h(\Omega) \cup \mathcal{N}_h(\Gamma_N)} \pi_P(v) \ \varphi_P.$$

Clément's Quasi-Interpolation Operator II

Theorem. Let $v \in H^1_{0,\Gamma_D}(\Omega)$. Then, for Clément's interpolation operator there holds

$$\|P_C\, v\|_{0,T} \le C\, \|v\|_{0,D_T^{(1)}}, \quad \|P_C\, v\|_{0,E} \le C\, \|v\|_{0,D_E^{(1)}}, \quad \|\nabla P_C v\|_{0,T} \le C\, \|\nabla v\|_{0,D_T^{(1)}},$$

$$\|v - P_C\, v\|_{0,T} \le C\, h_T\, \|v\|_{1,D_T^{(1)}}, \quad \|v - P_C\, v\|_{0,E} \le C\, h_E^{1/2}\, \|v\|_{1,D_E^{(1)}}.$$

Further, we have

$$\Big(\sum_{T \in \mathcal{T}_h} \|v\|^2_{\mu,D_K^{(1)}} \Big)^{1/2} \le C\, \|v\|_{\mu,\Omega}, \quad 0 \le \mu \le 1,$$

$$\Big(\sum_{E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)} \|v\|^2_{\mu,D_E^{(1)}} \Big)^{1/2} \le C\, \|v\|_{\mu,\Omega}, \quad 0 \le \mu \le 1.$$

where $D_T^{(1)} := \bigcup \{ T' \in \mathcal{T}_h \mid \mathcal{N}_h(T') \cap \mathcal{N}_h(T) \neq \emptyset \}$, $D_E^{(1)} := \bigcup \{ T' \in \mathcal{T}_h \mid \mathcal{N}_h(E) \cap \mathcal{N}_h(T') \neq \emptyset \}$.

Element and Edge Bubble Functions I

The element bubble function ψ_T is defined by means of the barycentric coordinates $\lambda_i^T, 1 \leq i \leq 3$, according to

$$\psi_T \; := \; 27 \, \lambda_1^T \, \lambda_2^T \, \lambda_3^T .$$

Note that $\text{supp } \psi_T = T_{int}$, i.e., $\psi_T \mid_{\partial T} = 0 \; , \; T \in \mathcal{T}_h$. On the other hand, for $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ and $T \in \mathcal{T}_h$ such that $E \subset \partial T$ and $p_i^E \in \mathcal{N}_h(E) \; , \; 1 \leq i \leq 2$, we introduce the edge-bubble functions ψ_E

$$\psi_E \; := \; 4 \, \lambda_1^T \, \lambda_2^T .$$

Note that $\psi_E \mid_{E'} = 0$ for $E' \in \mathcal{E}_h(T), E' \neq E$.

Element and Edge Bubble Functions II

The bubble functions ψ_T and ψ_E have the following important properties that can be easily verified taking advantage of the affine equivalence of the finite elements:

Lemma. There holds

$$\|p_h\|_{0,T}^2 \le C \int_T p_h^2 \, \psi_T \, dx, \quad p_h \in P_1(T),$$

$$\|p_h\|_{0,E}^2 \le C \int_E p_h^2 \, \psi_E \, d\sigma, \quad p_h \in P_1(E),$$

$$|\, p_h \, \psi_T \,|_{1,T} \le C \, h_T^{-1} \, \|p_h\|_{0,T}, \quad p_h \in P_1(T),$$

$$\|p_h \, \psi_T\|_{0,T} \le C \, \|p_h\|_{0,T}, \quad p_h \in P_1(T),$$

$$\|p_h \, \psi_E\|_{0,E} \le C \, \|p_h\|_{0,E} \quad , \quad p_h \in P_1(E).$$

Element and Edge Bubble Functions III

For functions $p_h \in P_1(E)$, $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ we further need an extension $p_h^E \in L^2(T)$ where $T \in \mathcal{T}_h$ such that $E \subset \partial T$. For this purpose we fix some $E' \subset \partial T$, $E' \neq E$, and for $x \in T$ denote by x_E that point on E such that $(x - x_E) \parallel E'$. For $p_h \in P_1(E)$ we then set

$$p_h^E \;:=\; p_h(x_E).$$

Further, for $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ we define $D_E^{(2)}$ as the union of elements $T \in \mathcal{T}_h$ containing E as a common edge

$$D_E^{(2)} \;:=\; \bigcup \{\, K \in \mathcal{T}_h \;\mid\; E \in \mathcal{E}_h(T) \,\}.$$

Element and Edge Bubble Functions IV

Lemma. There holds

$$| p_h^E \, \psi_E |_{1,D_E^{(2)}} \le C \, h_E^{-1/2} \, \|p_h\|_{0,e}, \quad p_h \in P_1(E),$$

$$\|p_h^E \, \psi_E\|_{0,D_E^{(2)}} \le C \, h_E^{1/2} \, \|p_h\|_{0,E}, \quad p_h \in P_1(E).$$

Further, for all $v \in V$ and $\mu = 0, 1$ there holds

$$\Big(\sum_{E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)} h_E^{1-\mu} \, \|v\|^2_{\mu, D_E^{(2)}} \Big)^{1/2} \le C \, \Big(\sum_{T \in \mathcal{T}_h} h_T^{1-\mu} \, \|v\|^2_{\mu,T} \Big)^{1/2}.$$

Step MARK of the Adaptive Cycle: Bulk Criterion

Given a universal constant $0 < \Theta < 1$, specify a set $\mathcal{M}_T$ of elements and a set $\mathcal{M}_E$ of edges such that (bulk criterion, Dörfler marking)

$$\Theta \left(\sum_{T \in \mathcal{T}_H(\Omega)} \eta_T^2 + \sum_{E \in \mathcal{E}_H(\Omega)} \eta_E^2 \right) \leq \sum_{T \in \mathcal{M}_T} \eta_T^2 + \sum_{E \in \mathcal{M}_E} \eta_E^2 \, .$$

Step REFINE of the Adaptive Cycle: Refinement Rules

- Any $T \in \mathcal{M}_T, E \in \mathcal{M}_E$ is refined by bisection.
- Further bisection is used to create a geometrically conforming triangulation $\mathcal{T}_h(\Omega)$.

Adaptive Finite Element Methods for Unconstrained Optimal Elliptic Control Problems

Ronald H.W. Hoppe[1,2]

[1] Department of Mathematics, University of Houston
[2] Institute of Mathematics, University of Augsburg

Institute for Mathematics and its Applications

Minneapolis, November, 2010

Elliptic Optimal Control Problems: Unconstrained Case

Let Ω be a bounded polygonal domain with boundary $\Gamma = \partial\Omega$. Given a desired state $y^d \in L^2(\Omega)$, $f \in L^2\Omega)$, and $\alpha > 0$, find $(y,u) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\inf_{(y,u)} J(y,u) \;:=\; \frac{1}{2}\int_\Omega |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx,$$

$$\begin{aligned} \text{subject to} \quad -\Delta y &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Reduced Optimality Conditions in y and p

Substituting u in the state equation by $p = \alpha u$, we arrive at the following system of two variational equations:

$$a(y,v) - \alpha^{-1}(p,v)_{0,\Omega} = \ell_1(v), \quad v \in V := H_0^1(\Omega) \;,$$
$$a(p,w) + (y,w)_{0,\Omega} = \ell_2(w), \quad w \in V,$$

where the functionals $\ell_\nu : V \to \mathbb{R}, 1 \le \nu \le 2,$ are given by

$$\ell_1(v) := 0, \; v \in V, \quad \ell_2(w) := (y^d, w)_{0,\Omega}, \; w \in V.$$

The operator-theoretic formulation reads

$$\mathcal{L}(y,p) = (\ell_1, \ell_2)^T,$$

where the operator $\mathcal{L} : V \times V \to V^* \times V^*$ is defined according to

$$(\mathcal{L}(y,p))(v,w) := a(y,v) - \alpha^{-1}(p,v)_{0,\Omega} + a(p,w) + (y,w)_{0,\Omega}.$$

Operator Theoretic Formulation of the Optimality System I

Theorem. The operator $\mathcal{L}$ is a continuous, bijective linear operator. Hence, for any $(\ell_1, \ell_2) \in V^* \times V^*$ the system admits a unique solution $(y, p) \in V \times V$. The solution depends continuously on the data according to

$$\|(y, p)\|_{V \times V} \leq \ C \ \|(\ell_1, \ell_2)\|_{V^* \times V^*}.$$

Proof. The linearity and continuity are straightforward. For the proof of the inf-sup condition, we choose $v = \alpha y - p$ and $w = p + y$. It follows that

$$(\mathcal{L}(y, p))(\alpha y - p, y + p) = \alpha \ a(y, y) + a(p, p) + (y, y)_{0,\Omega} + \alpha^{-1} \ (p, p)_{0,\Omega},$$

which allows to conclude.

Operator Theoretic Formulation of the Optimality System II

Corollary. Let $(y_h, p_h) \in V_h \times V_h$, $V_h \subset V$, be an approximate solution of $(y, p) \in V \times V$.

Then, there holds

$$\|(y - y_h, p - p_h)\|_{V \times V} \leq C \; \|(\mathcal{R}es_1, \mathcal{R}es_2)\|_{V^* \times V^*},$$

where the residuals $\mathcal{R}es_1 \in V^*, \mathcal{R}es_2 \in V^*$ are given by

$$\begin{aligned} \mathcal{R}es_1(v) &:= \ell_1(v) - a(y_h, v) + \alpha^{-1}(p_h, v)_{0,\Omega}, \;\; v \in V, \\ \mathcal{R}es_2(w) &:= \ell_2(w) - a(p_h, w) - (y_h, w)_{0,\Omega}, \;\; w \in W. \end{aligned}$$

Proof. The assertion is an immediate consequence of the previous theorem.

Using Galerkin orthogonality and Clément's quasi-interpolation operator P_C, for the first residual Res_1 we find

$$\mathrm{Res}_1(v) = \sum_{T \in \mathcal{T}_h(\Omega)} (f, v - P_C v)_{0,T} - \sum_{T \in \mathcal{T}_h(\Omega)} \Big(a(u_h, v - P_C v) + \alpha^{-1} (p_h, v - P_C v)_{0,T} \Big).$$

By an elementwise application of Green's formula and the local approximation properties of P_C it follows that

$$\|\mathrm{Res}_1\|_{V^*} \le C \Big(\sum_{T \in \mathcal{T}_h(\Omega)} \eta_{T,1}^2 + \sum_{E \in \mathcal{E}_h(\Omega)} \eta_{E,1}^2 \Big)^{1/2},$$

The local residuals are given by

$$\eta_{T,1} := h_T \, \|\Delta y_h + u_h\|_{0,T},$$
$$\eta_{E,1} := h_E^{1/2} \, \|n \cdot [\nabla y_h]\|_{0,E}.$$

Likewise, for the second residual Res_2 we obtain

$$\|\mathrm{Res}_2\|_{V^*} \leq C\Big(\sum_{T \in \mathcal{T}_h(\Omega)} \eta_{T,2}^2 + \sum_{E \in \mathcal{E}_h(\Omega)} \eta_{E,2}^2 \Big)^{1/2},$$

where the local residuals are given by

$$\begin{aligned} \eta_{T,2} &:= h_T \, \|y^d + \Delta p_h - y_h\|_{0,T}, \;\; T \in \mathcal{T}_h(\Omega), \\ \eta_{E,2} &:= h_E^{1/2} \, \|n \cdot [\nabla p_h]\|_{0,E}, \;\; E \in \mathcal{E}_h(\Omega). \end{aligned}$$

Reliability of the Residual-Type A Posteriori Error Estimator

Theorem. Let $(y, p) \in V \times V$ and $(y_h, p_h) \in V_h \times V_h$ be the solutions of the continuous and discrete optimality system, respectively. Then, there holds

$$\|(y - y_h, p - p_h)\|_{V \times V} \leq C\eta_h,$$

where the estimator η_h is given by

$$\eta_h := \Big(\sum_{T \in \mathcal{T}_h(\Omega)} (\eta_{T,1}^2 + \eta_{T,2}^2) + \sum_{E \in \mathcal{E}_h(\Omega)} (\eta_{E,1}^2 + \eta_{E,2}^2) \Big)^{1/2}.$$

Efficiency of the Residual-Type A Posteriori Error Estimator I

Lemma. Let $(y,p) \in V \times V$ and $(y_h, p_h) \in V_h \times V_h$ be the solutions of the continuous and discrete optimality system, respectively. Then, there exists a positive constant c depending only on the shape regularity of $\{\mathcal{T}_h(\Omega)\}$ such that for $T \in \mathcal{T}_h(\Omega)$

$$\eta_{T,1}^2 \;\le\; c\,\left(|y - y_h|_{1,T}^2 + h_T^2\;\|u - u_h\|_{0,T}^2\right).$$

Proof. Setting $z_h := u_h|_T \psi_T$ and observing that $\Delta y_h|_T = 0$, Green's formula and the fact that z_h is an admissible test function imply

$$\begin{aligned}
&\eta_{T,1}^2 = h_T^2\;\|u_h\|_{0,T}^2 \le c\;h_T^2\;\;(u_h + \Delta y_h, z_h)_{0,T} = c\;h_T^2\;\left(-a(y_h, z_h) + (u, z_h)_{0,T}\right.\\
&\left.+(u_h - u, z_h)_{0,T}\right) = c\;h_T^2\;\left(a(y - y_h, z_h) + (u_h - u, z_h)_{0,T}\right)\\
&\le\; c(\;h_T^2\;|y - y_h|_{1,T}|z_h|_{1,T} + h_T^2\;\|u - u_h\|_{0,T}\;\|z_h\|_{0,T}).
\end{aligned}$$

Proof cont'd. By the property of the element bubble function

$$| p_h \ \psi_T |_{1,T} \leq c \ h_T^{-1} \ \|p_h\|_{0,T} \quad , \quad p_h \in P_1(T),$$

and Young's inequality we obtain

$$h_T^2 \ \|u_h\|_{0,T}^2 \leq c(|y - y_h|_{1,T}^2 + h_T^2 \|u - u_h\|_{0,T}^2) + \frac{1}{2} \ h_T^2 \ \|u_h\|_{0,T}^2,$$

which gives the assertion.

Efficiency of the Residual-Type A Posteriori Error Estimator II

Lemma. Let $(y, p) \in V \times V$ and $(y_h, p_h) \in V_h \times V_h$ be the solutions of the continuous and discrete optimality system, respectively. Then, there exists a positive constant c depending only on the shape regularity of $\{\mathcal{T}_h(\Omega)\}$ such that for $T \in \mathcal{T}_h(\Omega)$

$$\eta_{T,2}^2 \;\le\; c\;(|p - p_h|_{1,T}^2 + h_T^2\;\|y - y_h\|_{0,T}^2 + \mathrm{osc}_T^2),$$

where

$$\mathrm{osc}_T := h_T\;\|y^d - y_h^d\|_{0,T}, \quad T \in \mathcal{T}_h(\Omega).$$

Proof. The assertion can be proved along the same lines as in the previous lemma.

Efficiency of the Residual-Type A Posteriori Error Estimator III

Lemma. Let $(y,p) \in V \times V$ and $(y_h, p_h) \in V_h \times V_h$ be the solutions of the continuous and discrete optimality system, respectively. Then, there exists a positive constant c depending only on the shape regularity of $\{\mathcal{T}_h(\Omega)\}$ such that for $E \in \mathcal{E}_h(\Omega)$

$$\eta_{E,1}^2 \ \leq \ c(|y - y_h|_{1,\omega_E}^2 + h_E^2 \ \|u - u_h\|_{0,\omega_E}^2 + \sum_{\nu=1}^{2} \eta_{T_\nu,1}^2) \ .$$

Proof. We set $\zeta_E := (n_E \cdot [\nabla y_h])|_E$ and $z_h := \tilde{\zeta}_E \psi_E$. Then, applying Green's formula and observing that z_h is an admissible test function, we find

$$\eta_{E,1}^2 = h_E \|n_E \cdot [\nabla y_h]\|_{0,E}^2 \leq c \ h_E \ (n_E \cdot [\nabla y_h], \zeta_E \psi_E)_{0,E} = c \ h_E \ \sum_{\nu=1}^{2} (n_{\partial T_\nu} \cdot [\nabla y_h], z_h)_{0,\partial T_\nu}$$

$$= \ c \ h_E \ (a(y_h - y, z_h) + (u - u_h, z_h)_{0,\omega_E} + (f + u_h, z_h)_{0,\omega_E})$$

$$\leq \ c \ h_E^{1/2} \|\nu_E \cdot [\nabla y_h]\|_{0,E} (|y - y_h|_{1,\omega_E} (h_E \ \|u - u_h\|_{0,\omega_E} + (\sum_{\nu=1}^{2} \eta_{T_\nu,1}^2)^{1/2})),$$

which allows to conclude.

Efficiency of the Residual-Type A Posteriori Error Estimator IV

Lemma. Let $(y, p) \in V \times V$ and $(y_h, p_h) \in V_h \times V_h$ be the solutions of the continuous and discrete optimality system, respectively. Then, there exists a positive constant c depending only on the shape regularity of $\{\mathcal{T}_h(\Omega)\}$ such that for $E \in \mathcal{E}_h(\Omega)$

$$\eta_{E,2}^2 \;\le\; c\big(|p - p_h|_{1,\omega_E}^2 + h_E^2\, \|y - y_h\|_{0,\omega_E}^2 + \sum_{\nu=1}^{2} \eta_{T_\nu,2}^2\big) \;.$$

Proof. The proof is similar to the one in the previous lemma.

Efficiency of the Residual-Type A Posteriori Error Estimator V

Theorem. Let $(y, p) \in V \times V$ and $(y_h, p_h) \in V_h \times V_h$ be the solutions of the continuous and discrete optimality system, respectively. Then, there exist positive constants C and c depending only on Ω and the shape regularity of the triangulations such that

$$\|(y - y_h, p - p_h)\|^2_{V \times V} + \|u - u_h\|^2_{0,\Omega} \geq C\ \eta^2_h - c\ \mathrm{osc}^2_h.$$

where

$$\mathrm{osc}^2_h := \sum_{T \in \mathcal{T}_h(\Omega)} \mathrm{osc}^2_T.$$

Proof. Combining the results of the previous four lemmas gives the assertion.