The instantaneous control method - convergence analysis for finite dimensional systems

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Contents

Abstract

A new class of feedback laws for the control of dynamical systems is presented. It is obtained by applying instantaneous control, i.e. control at every time step to the underlying dynamical system. This method can be interpreted as a suboptimal closed loop controller for the dynamical system. The parameters of the controller may be adjusted so as to stabilize the dynamical system under consideration. Several numerical examples are presented.

1 Introduction

In the past control theory has been applied more and more to real world problems, such as control of robots in automobile industry [12], control of recruiting strategies [5], control of combustion phenomena [10] and of fluids [6], just to mention only a few applications. Many of these applications have in common the goal of steering the system under investigation into a desired state during a prescribed time period. Usually, the system is described by at set of differential equations, so that the control problem formally consists in steering a dynamical system from a given initial state into a desired state during a prescribed time period.

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This problem may be formulated mathematically as a constraint minimization problem for a cost function of tracking type, where the subsidiary conditions are given by the dynamical system that describes the state. The construction of optimal closed-loop controllers in the general non-linear case leads to the Hamilton-Jacobi-Bellmann equations, for which to find numerical solution approaches is an outstanding problem. Even in the case of the linear-quadratic regulator problem with quadratic costs and linear constraints the closed-loop controller is characterized by means of a Riccati matrix differential equation whose numerical solution for large scale problems necessitates an enormous amount of computational work.

The restrictions that apply to the optimal approach become even more involved when the control of partial differential equations is concerned. Especially in real world applications such as the control of three-dimensional fluid flow numerical approaches relying on todays available computing power are unworkable. This also holds true for the open-loop control approach to this complex of problems. But also academic problems such as the control of unsteady Burgers equation in one spatial dimension [14], the control of ignition and/or combustion phenomena [10], and also optimal control of fluid flow in two spatial dimensions [8] are hard to tackle numerically and require a high standard workstation equipment.

This leads to the concept of suboptimal control, where instead of an optimal control a good/-appropriate gain is to be computed/determined. The idea of suboptimal control approaches is long standing, see e.g. [5]. An overview on suboptimal control methods for fluids can be found in [6], more detailed descriptions can be found in [2, 4, 9, 13].

In this note we develop a distributed instantaneous closed-loop control approach for suboptimal control of linear and non-linear dynamical systems. Its nature is completely different from the HJB method since it does not rely on the dynamical programming principle of Bellman [1]. Rather, it successively determines approximations of the objective while marching forward in time.

The method works as follows. The dynamical system is discretized in time. Then, at every time instance a stationary control problem with an instantaneous version of the feedback gain as cost function is solved either exactly or approximately, using gradient-type methods. As is shown this discrete approach allows for the interpretation as a semi-discretization of a closed-loop control law. Provided the parameters involved in the method, such as step size in time integration, the penalty for the control cost and the descent parameter in the gradient algorithm are adjusted in a proper way the discrete control algorithms and the related continuous closed-loop control laws can be proved to be stable.

It has been shown in several numerical studies that the instantaneous control approach is implementable on workstation environments and is very effective in reducing costs, even for medium to large-scale problems such as the control of fluid flow, see [8]. Moreover, certain variants of the instantaneous method lead to practically relevant controllers for fluid flows, since they can be implemented as to determine boundary controls based on boundary information only [3].

The paper is organized as follows. In Section 2 we briefly discuss the linear quadratic regulator problem, which is chosen to be the model problem for our investigations. In Section 3 the application of the instantaneous control approach to the model problem is described. Section 4 deals with the interpretation of the instantaneous method as closed-loop control law. Stability properties and convergence of the controllers for tracking-type problems in are proven in Section 5. We close with presenting a comparison of the controller obtained by the instantaneous control method to the optimal closed-loop Riccati controller for the inverted pendulum problem in Section 6.

2 The model problem

In this section we outline the derivation of control laws at hand of the linear-quadratic regulator problem (LQR problem) in its simplest form. To make the ideas of the approach as transparent as possible we do not concern about the questions of controllability and observability. These topics in connection with the approach presented here are discussed in [15]. Thus, the model problem will be formulated as follows: Given the desired state vector $\bar{x} \in \mathbb{R}^n$ and an initial state $x_0 \in \mathbb{R}^n$, find a control vector $u(t) \in \mathbb{R}^n$ such that the objective function

$$J(u) = \frac{1}{2} \int_0^T \gamma |u(t)|^2 + |x(t) - \bar{x}|^2 \, dt \tag{1}$$

is minimal compared to the value of J at all input vectors $v(t) \in \mathbb{R}^n$ satisfying

$$\begin{aligned} \dot{x}(t) + Ax(t) &= b(t) + v(t), \\ x(0) &= x_0 \end{aligned} \tag{2}$$

Here, $\gamma > 0$ weights the costs for the control input, the matrix $A \in \mathbb{R}^{n \times n}$ is assumed to be regular and the inhomogenity is denoted by b. It is well known that this problem admits a unique solution

$$u^* = \frac{1}{\gamma} \mu,$$

where the adjoint state $\mu \in \mathbb{R}^n$ is connected to the state x by the linear forward-backward in time Hamilton equations

$$\begin{aligned} \dot{x} + Ax &= b + \tfrac{1}{\gamma}\mu, \\ -\dot{\mu} + A^*\mu &= -(x - \bar{x}), \\ x(0) &= x_0, \\ \mu(T) &= 0. \end{aligned} \tag{3}$$

Choosing for the control the Ansatz

$$u(t) = \frac{1}{\gamma} \{-P(t)(x - \bar{x}) + p(t)\} \tag{4}$$

leads to the time-variant feedback matrix P and a time-dependent vector field p satisfying the Riccati matrix differential equations

$$\begin{aligned} \dot{P} - A^*P - PA + I - \tfrac{1}{\gamma}P^2 &= 0 \\ \dot{p} - \tfrac{1}{\gamma}Pp - A^*p &= -PA\bar{x} \\ P(T) &= 0 \\ p(T) &= 0. \end{aligned} \tag{5}$$

In practical applications the use of an numerical approximation of the optimal control law (4) amounts to the storage of an $n \times n$-matrix and an n-vector at every time instance. Especially for large time intervals $T >> 0$ or/and large system dimension the application of this law is not workable, not to mention the enormous amount of computational work necessary to provide the numerical solution of (5).

For infinite time horizon $T = \infty$ the optimal control law is also given by (4), where now the feedback matrix P and the vector p are the suitable time-invariant stationary solution of Eqs. 5, i.e. P and p satisfy the quadratic system

$$\begin{array}{r} -A^*P - PA + I - \frac{1}{\gamma}P^2 = 0 \\ -\frac{1}{\gamma}Pp - A^*p = -PA\bar{x}. \end{array} \tag{6}$$

Investigations on algorithms for the numerical solution of this problem are discussed, for example, by Rauter and Sachs in [11]. However, for large system dimensions as they arise from the discretization of parabolic equations, say the effort to solve Eqs. 6 numerically is enormous. It is one goal of this paper to present an approach which provides numerically computable control laws even for systems of very large dimension.

In the next section we introduce an approach to the design of time invariant control laws that is completely different to the method sketched above as it relies on consistent time integration of the underlying dynamical system.

3 The instantaneous control strategy

As pointed out in the previous section the construction and numerical implementation of optimal control laws suffer from practical implementability, at least when the problems under consideration have large system dimension. In order to overcome these difficulties, we now suggest a different way of constructing closed-loop controllers for dynamical systems.

The approach that we take is based on a time discretization of equation (2). For this purpose let $0 = t_0 < t_1 < \cdots < t_m = T$ denote a equidistant grid on the time interval $[0, T]$ with step size $h = \frac{T}{m}$. At each discrete time level t_i a stationary control problem is solved for an approximate optimal control u_i^* and this control is used to steer the system from t_i to t_{i+1}, where a new approximate optimal control is determined. It cannot be claimed that the controls obtained in this manner approximate the optimal control for (1),(2) as the discretization parameter tends to zero. However, this procedure will be justified by the effectiveness that it exhibits for numerical examples and the interpretation that it allows for suboptimal feedback controls.

As time discretization method we choose the implicit Euler method and as cost function an instantaneous version of (1) is taken. The optimization problem in every time step then has the form

$$(P_i)\left\{\begin{array}{l} \min J(u^{j+1}) = \frac{\gamma}{2}|u^{j+1}|^2 + \frac{1}{2}|x^{j+1} - \bar{x}^j|^2 \\ \text{s.t.} \\ \quad (I + hA)x^{j+1} \quad = \quad x^j + hb^j + u^{j+1}, \end{array}\right. \tag{7}$$

which, due to the quadratic character of the cost function, admits a unique solution (x^{j+1}, u^{j+1}). This tuple together with the uniquely determined Lagrange multiplier μ^{j+1} solves the corresponding first order optimality conditions given by

$$\left.\begin{array}{rcl} (I + hA)x^{j+1} & = & x^j + hb^j + u^{j+1} \\ (I + hA^*)\mu^{j+1} & = & -(x^{j+1} - \bar{x}^j) \\ \gamma u^{j+1} - \mu^{j+1} & = & 0. \end{array}\right\} \tag{8}$$

This system has the solution

$$\left.\begin{array}{l} x^{j+1} = (I + hA)^{-1}\left(x^j + hb^j + \frac{1}{\gamma}S\left(h(A\bar{x}^j - b) - (x^j - \bar{x}^j)\right)\right) \\ \mu^{j+1} = -S(x^j - \bar{x}^j) - hS(b^j - A\bar{x}^j), \end{array}\right\} \tag{9}$$

where $S := \left((I+hA)(I+hA^*)+\frac{1}{\gamma}I\right)^{-1}$. The optimal control problem (3),(2) has thus been replaced by a sequence of stationary problems (P_i) which can be realized on workstations even for systems with large dimensions.

Having in mind to treat cost functions involving more general non-linearities a suitable method to solve for (x^{j+1}, μ^{j+1}) in (8) is the conjugate gradient method for non-linear problems.

Algorithm 3.1. Conjugate Gradient algorithm

1. Set $k = 0$ and choose u^0
2. Set $p^0 = d^0 = -\nabla J(u^0)$
3. $\rho_k = \mathrm{argmin}_{\rho\geq 0}\, H(\rho) := J(u^k + \rho d^k)$
4. $u^{k+1} = u^k + \rho_k p^k$
5. $d^{k+1} = -\nabla J(u^{k+1})$
6. $\beta_k = \begin{cases} \frac{(d^{k+1}, d^{k+1}-d^k)}{|d^k|} & \text{Polak - Rebiére} \\ \frac{|d^{k+1}|^2}{|d^k|^2} & \text{Fletcher - Reeves} \end{cases}$
7. $p^{k+1} = d^{k+1} + \beta_k p^k$
8. $k = k+1$, goto 2.

Here, the gradient of J at u in direction v is given by

$$\nabla J(u)(v) = (\gamma u - \mu, v), \tag{10}$$

where $(\cdot,\cdot)$ denotes the usual inner product in $\mathbb{R}^n$. In this algorithm the expensive part is step 3. Every evaluation $H(\rho)$ involves the solution of the first equation in system (8) with $u = u^k + \rho d^k$, so that an efficient inexact line-search should be utilized. This can be performed by exploiting the structure of problem (P_i), see [3, 7].

In order to obtain a stable control law for the dynamical system (2) we propose two possibilities. In the first one, at every time level t_i we apply exactly one step of Algorithm 3.1 with ρ_0 in step 3. replaced by some $\rho > 0$ such that $J(u^0 + \rho d^0) < J(u^0)$. The second possibility makes use of the exact solution (9) of the instantaneous control problem at every time level t_i. These two approaches for the stabilization of the dynamical system (2), here called *One-gradient-step stabilization* (ogss) and *Stationary-optimal stablization* (sos)may be written down in algorithmical form as follows.

Algorithm 3.2. (ogss)

1. Given initial values x^0, set $j = 0$, $t_0 = 0$
2. Given u_0^j, solve

$$\begin{aligned} (I+hA)x &= x^j + hb^j + u_0^j \\ (I+hA^*)\mu &= -(x - \bar{x}^j) \end{aligned}$$

3. Set $\nabla J(u_0^j) = \gamma u_0^j - \mu$

4. Given ρ, set $u^{j+1} = u_0^j - \rho \nabla J(u_0^j)$

5. Solve

$$(I + hA)x^{j+1} = x^j + hb^j + u^{j+1}$$

6. Set $t_{j+1} = t_j + h$, $j = j + 1$

7. If $t_j < T$ goto 2.

Algorithm 3.3. (sos)

1. Given initial values x^0, set $j = 0$, $t_0 = 0$

2. Compute

$$u^{j+1} = \frac{1}{\gamma}\mu^{j+1} = -\frac{1}{\gamma}S(x^j - \bar{x}^j) - \frac{h}{\gamma}S(b^j - A\bar{x}^j) \tag{11}$$

3. Solve

$$(I + hA)x^{j+1} = x^j + hb^j + u^{j+1}$$

4. Set $t_{j+1} = t_j + h$, $j = j + 1$

5. If $t_j < T$ goto 2.

In the next section we interpret Algorithms 3.2, 3.3 as semi-implicit discretization schemes of dynamical systems related to the one presented in eq. (2).

4 The discrete output control laws

The first step in proving that the instantaneous stabilization approaches presented in Algorithms 3.2 and 3.3 stabilize the dynamical system in the sense that they steer it to the desired state $\bar{x}$ relies on the interpretation of the algorithms as semi-implicit time integration scheme of the dynamical systems presented next. For this purpose we abbreviate

$$B^* := (I + hA)^{-1}, \quad B := (I + hA^*)^{-1},$$

recall

$$S := \left((I + hA)(I + hA^*) + \frac{1}{\gamma}I\right)^{-1},$$

$\bar{x} \in \mathbb{R}$, set $f := A\bar{x}$ and require $h \max\{\|A\|, \|A^*\|\} < 1$, so that the matrices B, B^* and S are well defined. In the next theorem the connection between the schemes (12), (14) and the instantaneous stabilization Algorithms 3.2 and 3.3, respectively is pointed out.

Theorem 4.1. Let $u_0^j := 0$ in Algorithms 3.2, 3.3.
Algorithm 3.2 is equivalent to the semi-implicit time discretization

$$(I + hA)x^{j+1} = x^j + hb^j - \rho BB^*(x^j - \bar{x}^j) - h\rho BB^*(b^j - f^j), \quad x^0 := x_0, \tag{12}$$

of the dynamical system

$$\dot{x} + Ax = b - \frac{\rho}{h} BB^*(x - \bar{x}) - \rho BB^*(b - f), \quad x(0) = x_0. \tag{13}$$

Algorithm 3.3 is equivalent to the semi-implicit time discretization

$$(I + hA)x^{j+1} = x^j + hb^j - \frac{1}{\gamma} S(x^j - \bar{x}^j) - \frac{h}{\gamma} S(b^j - f^j), \quad x^0 := x_0, \tag{14}$$

of the dynamical system

$$\dot{x} + Ax = b - \frac{1}{\gamma h} S(x - \bar{x}) - \frac{1}{\gamma} S(b - f), \quad x(0) = x_0. \tag{15}$$

Proof. For a proof it remains to show that for u^j in step 4. in Algorithm 3.2

$$u^j = -\rho BB^*(x^j - \bar{x}^j) - h\rho BB^*(b^j - f^j) \tag{16}$$

holds true. For this purpose note, that $f^j = A\bar{x}^j$ implies $B\bar{x}^j = BB^*\bar{x}^j + hBB^* f^j$. Using this in steps 2.-5. of Algorithm 3.2 yields the desired result for Algorithm 3.2. The proof for Algorithm 3.3 is similar and is therefore omitted. ☐

Remark 4.1. Note, that here the parameter h in Eqs. (13) and (15) serves as the time step in the discretization process.

Remark 4.2. It is worth noting that the matrices S, B, and thus B^*, also exist, and also the results of theorem 4.1 holds true in the infinite dimensional case, provided the parameter h is small enough and the operator $A : X \to X$ is linear with real-part of the spectra bounded from below, say, where X denotes a Banach space.

The next section is assigned to characterize the stability properties of the differential equations (13), (15) and their discretizations (12), (14) in terms of the parameters involved in their formulation.

5 Convergence of the control laws

This section is devoted to the study of stability properties of the semi-explicit discretizations (12), (14) of the differential equations (13) and (15), respectively, as well as to the investigation of stability for the continuous equations. It turns out that under suitable assumptions on the system parameters ρ, γ and h the discretizations mentioned above converge to a tube around $\bar{x}$. This in particular implies the convergence for large times of the iterates of the corresponding instantaneous stabilization algorithms (3.2), (3.3) to a neighbourhood of the desired state, the parameter h tending to zero. Moreover, the same conditions ensure stability of the corresponding continuous systems.

Since we are interessted in the behaviour of the difference $x(t)-\bar{x}(t)$ we instead of the difference schemes (12) and (14) from now on consider the schemes

$$
\begin{aligned}
(I+hA)\Phi^{j+1} &= \Phi^j - \rho BB^*\Phi^j - h(\rho BB^* - I)(b^j - f^j), && (17)\\
(I+hA)\Phi^{j+1} &= \Phi^j - \frac{1}{\gamma}S\Phi^j - h(\frac{1}{\gamma}S - I)(b^j - f^j), && (18)
\end{aligned}
$$

which in turn are semi-implicit time discretizations of the the differential equations

$$
\begin{aligned}
\dot{\Phi} + A\Phi &= -\frac{\rho}{h}BB^*\Phi - (\rho BB^* - I)(b - f), && (19)\\
\dot{\Phi} + A\Phi &= -\frac{1}{\gamma h}S\Phi - (\frac{1}{\gamma}S - I)(b - f) && (20)
\end{aligned}
$$

respectively, where $\Phi = x - \bar{x}$. We note that the discretization (12) of (13) differs from that for (19) given in (17) by $(I+hA)(\bar{x}^{j+1} - \bar{x}^j)$. This term is zero, provided $\bar{x}$ is a constant vector, and of size $O(h)$, if the step-size h is small and $\bar{x}$ is Lipschitz, say.

5.1 Stability of the discrete schemes

As the next step we prove that the difference schemes (12) and (14) are stable for certain configurations of the parameters involved. For this purpose it is sufficient to ensure that the matrices $C_I := (I+hA)^{-1}(I-\rho BB^*)$ in the case of (12) and $C_{II} := (I+hA)^{-1}(I-\frac{1}{\gamma}S)$ in the case of (14) have spectral radii lower than or equal to 1. Defining

$$M := \max\{\|A\|, \|A^*\|\}$$

sufficient conditions for this to hold are given in

Theorem 5.1. Let the parameters $h > 0$ and $0 < \rho < 2$ satisfy the relation

$$\frac{1}{h} \ge \frac{(14\rho + 2)\,M}{1 - |1-\rho|}. \tag{21}$$

Then the discretization (12) of (13) is stable.
Let the parameters $h > 0$ and $0 \le \gamma$ satisfy the relation

$$\frac{1}{h} \ge \left\{2\gamma + 4 + 5\frac{\gamma}{\gamma+1}(1+M)\right\} M. \tag{22}$$

Then the discretization (14) of (15) is stable.

Proof. The matrix C_I may be written as

$$C_I = I \underbrace{-\rho BB^* - hA}_{=:hC_A} + \rho hABB^* + \left(\sum_{k=2}^{\infty}(-h)^k A^k\right)(I - \rho BB^*). \tag{23}$$

For the matrix hC_A we write

$$
\begin{aligned}
hC_A = \quad & -\rho BB^* - hA = \\
& -\rho\left[I + \sum_{k=1}^{\infty}(-h)^k A^k + \sum_{k=1}^{\infty}(-h)^k A^{*k} + \right.\\
& \left. + \left(\sum_{k=1}^{\infty}(-h)^k A^k\right)\left(\sum_{k=1}^{\infty}(-h)^k A^{*k}\right)\right] - hA.
\end{aligned}
$$

Using this representation and restricting h to satisfy $2hM < 1$ straight forward estimation gives

$$\|I + hC_A\| \;\leq\; |1-\rho| + hM\,\{6\rho + 1\}\,.$$

The estimate

$$\|BB^*\| \;\leq\; \frac{1}{(1-hM)^2} \;\leq\; 4$$

together with Eq. (23) yield the desired condition.
The matrix C_{II} can be written in the form

$$C_{II} = \underbrace{I - \frac{1}{\gamma}S - hA}_{=:hC_A} + \frac{h}{\gamma}AS + \left(\sum_{k=2}^{\infty} (-h)^k A^k\right)\left(I - \frac{1}{\gamma}S\right). \tag{24}$$

Abbreviating $D := A + A^* + hAA^*$ and restricting h to satisfy $5h\gamma M < \gamma + 1$ brief estimation together with the representation

$$I - \frac{1}{\gamma}S = \frac{\gamma}{\gamma+1}I - \frac{1}{\gamma+1}\sum_{k=1}^{\infty}\left(-\frac{\gamma h}{\gamma+1}\right)^k D^k$$

leads to

$$\|I + hC_A\| \;\leq\; \frac{\gamma}{\gamma+1} + hM\left\{\frac{5\gamma}{(\gamma+1)^2} + 1\right\}.$$

Repeated application of these arguments to the remaining terms in Eq. (24) leads to the desired stability condition. □

Remark 5.1. It is worth noting that the stabilizing properties of C_{II} increase with smaller values for γ. However, the application of the operator S is related to a problem involving the operator AA^*. From this point of view the formulation given in (13) with the corresponding discretization (12) should be preferred, since the application of the operator C_I only requires the solution with B and B^*, respectively. This is of particular importance in infinite dimensions.

5.2 Stability of the continuous systems

As an immediate consequence of the proof of this theorem we obtain a conditions for exponential stability of the differential equations (13) and (15).

Theorem 5.2. Let the assumptions of Theorem 5.1 be satisfied. Furthermore, let condition (21) and (22), respectively hold with strict inequality sign. Then, the differential equation (13) and (15), respectively are exponentially stable.

Proof. We present the proof for condition (21). The matrix C_A is the system matrix of the differential equation (13). Following the lines of the proof of the previous theorem one concludes that the spectral radius r_σ of the matrix $I + hC_A$ satisfies

$$r_\sigma\,(I + hC_A) \;\leq\; \|I + hC_A\| < \;1 \quad (\text{for all } h \text{ satisfying } (21)).$$

This implies $\mathrm{Re}(\lambda(C_A)) < 0$ for all eigenvalues $\lambda(C_A)$ of C_A.
The proof for condition (22) is similar and is therefore omitted. □

5.3 Main result

In view of what follows we denote the set of m-times continuously differentiable functions with bounded derivatives up to order m on the infinite time interval $[0,\infty)$ and values in $\mathbb{R}^n$ by $C_b^m([0,\infty),\mathbb{R}^n)$.

In the next theorem we prove decay of the solution of the differential equations (13) and (15), respectively and of the iterates $\{x^j\}_{j\in\mathbb{N}}$ of the Algorithms 3.2 and 3.3 into a tube around $\bar{x}$ of size basically proportional to the parameter h.

Theorem 5.3. Let $\bar{x}$ be independent of time, let the assumptions of Theorem 4.1, 5.1 and 5.2 be satisfied. Furthermore, let the inequalities in Eqs. 21 and 22, respectively be strict and assume that $f-b\in C_b^0([0,\infty),\mathbb{R}^n)$. Then, the iterates $\{x^j\}_{j\in\mathbb{N}}$ and the controls $\{u^j\}_{j\in\mathbb{N}}$ of the Algorithms 3.2 and 3.3, respectively satisfy

$$\sup_{j\in\mathbb{N},j\geq j_0}|x^j-\bar{x}| \;\leq\; C\,h, \qquad \sup_{j\in\mathbb{N},j\geq j_0}|u^j| \;\leq\; C\,h \tag{25}$$

with some fixed $j_0(h,\rho,x^0)$ and some positive constant C independent of the parameter h.

If, in addition $\rho=1$, there exists a finite time t_0 such that for the solution of the differential equation (13)

$$\sup_{t\geq t_0}|x(t)-\bar{x}| \;\leq\; C\,h \tag{26}$$

holds with some positive constant C independent of h. Similarly, the solution of the differential equation (15) satisfies

$$\sup_{t\geq t_0}|x(t)-\bar{x}| \;\leq\; C\;(\gamma+h) \tag{27}$$

for some fixed finite time t_0, where the constant C can be chosen independent of γ and h.

Proof. We start with proving the second claim for Algorithm 3.2. To begin with, set $\Phi : x-\bar{x}$ and recall that Φ solves the stable differential equation

$$\dot{\Phi} \;=\; \underbrace{-\left(A+\frac{\rho}{h}BB^*\right)}_{=:C_A}\Phi\,\underbrace{-\,(\rho BB^*-I)}_{=:C_B}(b-f),$$

which admits the unique solution

$$\Phi(t) \;=\; e^{C_A t}\Phi(0)+\int_0^t e^{C_A(t-s)}C_B(b-f)(s)\,ds.$$

Using the expansion $BB^*=I-h\,(A+A^*)+O(h^2)$ $(h\to 0)$, the regularity requirements on the difference $b-f$, $\rho=1$ and the exponential stability of the differential equation (19), one obtains for some $t_0\geq 0$

$$|\Phi(t)| \;\leq\; Ch \quad \forall t\geq t_0,$$

where the positive constant can be chosen independent of h. A similar argument with the matrix ρBB^* replaced by the matrix $\frac{1}{\gamma}S$ establishes the estimate in (27) and thus the claim for Algorithm 3.3.

In order to prove the first claim for Algorithm 3.2 we exploit the stability properties of the semi-implicit Euler discretization (17) of (19) given by

$$(I + hA)\Phi^{j+1} \;=\; \Phi^j - \rho BB^*\Phi^j - h(\rho BB^* - I)(b^j - f^j).$$

Since strict inequality in condition (21) implies $\|C_I\| < 1$ we can conclude

$$|\Phi^j| \;\le\; \|C_I\|^j \, |x^0 - \bar{x}| + \frac{h}{1 - \|C_I\|} \|b - f\|_\infty \;\le\; C\, h \; (j \ge j_0).$$

This is the desired result for the iterates x^j. For the controls u^j the result immediately follows from the representation

$$u^{j+1} \;=\; -\rho BB^*(x^j - \bar{x}^j) - h\rho BB^*(b^j - f^j),$$

the regularity requirements on the difference $b - f$ and the the estimate for the iterates x_j.

The results for Algorithm 3.3 are derived in a similar fashion using condition (22) instead of condition (21), with the matrix C_I replaced by the the matrix C_{II}. □

Remark 5.2. Note, that if b is chosen as to satisfy $b = A\bar{x}$, the choice $\rho = 1$ in Theorem 5.3 is not necessary to obtain the estimates in (26).

As an immediate consequence of the previous theorem we obtain

Corollary 5.1. Let the assumptions of Theorems 5.1 and 5.2 be satisfied. Then, for every $\epsilon > 0$ there exists a $h > 0$ and a $j_0(h, \rho, x^0)$ such that

$$\sup_{j\in\mathbb{N}, j\ge j_0} |x^j - \bar{x}| \;\le\; \epsilon, \qquad \sup_{j\in\mathbb{N}, j\ge j_0} |u^j| \;\le\; \epsilon \tag{28}$$

holds. For the continuous controllers, similarly, for every $\epsilon > 0$ there exists a t_0 such that

$$\sup_{t \ge t_0} |x(t) - \bar{x}| \;\le \epsilon \tag{29}$$

holds true.

5.4 The nonlinear case

We now extend the results of Theorem 5.3 to nonlinear dynamical systems. For this purpose it is sufficient to suitably modify the requirements for stability in Theorem 5.1 for right-hand sides b which depend nonlinear on the state x. Since the derivation of the results for Algorithms 3.2 and 3.3 are similar we only give a detailed investigation for Algorithm 3.2. For Algorithm 3.3 we only present the results. In order to investigate Algorithm 3.2 for the nonlinear case it is sufficient to consider the differential equation

$$\dot{\Phi}(t) + A\Phi(t) \;=\; -\frac{\rho}{h} BB^*\Phi(t) - (\rho BB^* - I)(b(t, x(t)) - f(t)),\; \Phi(0) = x_0 - \bar{x}(0), \tag{30}$$

together with their semi-implicit discretization

$$(I + hA)\Phi^{j+1} \;=\; \Phi^j - \rho BB^*\Phi^j - h(\rho BB^* - I)(b^j - f^j),\; \Phi^0 = \Phi(0), \tag{31}$$

where $b^j := b(t_j, x^j)$. On the nonlinearity b and the desired $\bar{x}$ state we impose the assumptions

1. $b = b(p,q) \in C_b^1([0,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$
2. $A\bar{x}(t) = b(t, \bar{x}(t)) \; \forall \; t \in [0, \infty)$

and recall $f = A\bar{x}$. In particular, assumption 1. implies the Lipschitz-continuity of the nonlinearity b w.r.t. the 2nd component. We denote the Lipschitz constant by L_b.

We begin with deriving sufficient conditions for the stability of the discrete scheme (31). For this purpose we note that by the mean value theorem

$$b(t,x) - b(t,\bar{x}) \;=\; \underbrace{\int_0^1 b_q(t, \bar{x} + s(x-\bar{x}))ds}_{=:b_q(\xi)} \;\Phi \tag{32}$$

holds true with some $\xi \in [x, \bar{x}]$. Using this expansion, the discrete scheme (31) may be written as

$$\Phi^{j+1} \;=\; \underbrace{C_I\left(I + hb_q(\xi)\right)}_{=:\tilde{C}_I} \Phi^j, \quad \Phi^0 = \Phi(0), \tag{33}$$

where C_I denotes the matrix defined in Theorem 5.1. The same scheme is valid for Algorithm 3.3 with the matrix C_I replaced by the matrix C_{II} of Theorem 5.1. As an immediate consequence of this Theorem we obtain with $M := \max\{\|A\|, \|A^*\|\}$

Theorem 5.4. Let the parameters $h > 0$ and $0 < \rho < 2$ satisfy the relation

$$\frac{1}{h} \;\geq\; \frac{(14\rho + 2)\left(M + \frac{1}{2}L_b\right) + |1-\rho|L_b}{1 - |1-\rho|}. \tag{34}$$

Then the discretization (31) of (30) is stable.
Let the parameters $h > 0$ and $0 \leq \gamma$ satisfy the relation

$$\frac{1}{h} \;\geq\; \left\{2\gamma + 4 + \frac{5\gamma}{\gamma+1}(1+M)\right\}\left(M + \frac{1}{2}L_b\right) + \gamma L_b. \tag{35}$$

In addition, replace the matrix ρBB^* in Eqs. (30) and (31) by the matrix $\frac{1}{\gamma}S$. Then the resulting discretization is stable.

Proof. The regularity requirements on b imply $\|b_q(\xi)\| \leq L_b$. The rest of the proof is similar as the proof of Theorem 5.1 and is therefore omitted. □

As a corollary we also obtain the stability of the continuous systems. The proof of the following corollary is the same as that of Theorem 5.2.

Theorem 5.5. Let the assumptions of Theorem 5.4 be satisfied. Furthermore, let condition (34) and (35), respectively hold with strict inequality sign. Then, the differential equation (30) is exponentially stable. The same holds true for this differential equation with the matrix ρBB^* replaced by the matrix $\frac{1}{\gamma}S$.

We close this section with formulating the convergence result in the nonlinear case.

Theorem 5.6. Let the assumptions of Theorem 5.3 be satisfied. Furthermore and let the inequalities in Eqs. 34 and 35, respectively be strict. Then, the iterates $\{x^j\}_{j\in\mathbb{N}}$ and the controls $\{u^j\}_{j\in\mathbb{N}}$ of the Algorithms 3.2 and 3.3, respectively satisfy

$$\sup_{j\in\mathbb{N}, j\geq j_0} |x^j - \bar{x}| \;\leq\; C\,h, \qquad \sup_{j\in\mathbb{N}, j\geq j_0} |u^j| \;\leq\; C\,h \tag{36}$$

with some fixed $j_0(h, \rho, x^0)$ and some positive constant C independent of the parameter h. In particular, the iterates of both Algorithms 3.2 and 3.3 converge to the steady state $\bar{x}$ as h tends to zero.

There exists a finite time t_0 such that for the solution of the corresponding differential equations

$$\sup_{t\geq t_0} |x(t) - \bar{x}| \;\leq\; C\,h \tag{37}$$

holds true with some positive constant C independent of h. In particular, the solutions of both systems converge to the steady state $\bar{x}$ for t tending to infinity.

Proof. The first claim follows from the convergence of the iteration (33). The second claim is a consequence of the exponential stability of the differential equation (30). ☐

As in the linear case an immediate consequence of the previous theorem can be formulated.

Corollary 5.2. Let the assumptions of Theorems 5.5 and 5.6 be satisfied. Then, for every $\epsilon > 0$ there exists a $h > 0$ and a $j_0(h, \rho, x^0)$ such that

$$\sup_{j\in\mathbb{N}, j\geq j_0} |x^j - \bar{x}| \;\leq\; \epsilon, \qquad \sup_{j\in\mathbb{N}, j\geq j_0} |u^j| \;\leq\; \epsilon \tag{38}$$

holds. For the continuous controllers, similarly, for every $\epsilon > 0$ there exists a t_0 such that

$$\sup_{t\geq t_0} |x(t) - \bar{x}| \;\leq \epsilon \tag{39}$$

holds true.

6 An example

As example we investigate the linearization of the pendulum equation

$$\ddot{\varphi} + \sin\varphi \;=\; 0$$

around the unstable stationary solution $\varphi = \pi, \dot{\varphi} = 0$:

$$\dot{x} + \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{=:A} x = \underbrace{\begin{bmatrix} 0 \\ -\pi \end{bmatrix}}_{=:f}.$$

The state $\bar{x}$ to be observed is given by $\bar{x} = [\pi, 0]^t$. Further we set $b = [0, -\pi]$, so that $A\bar{x} = b$ holds true.

We compute the stationary Riccati solution in order to compare it to the results obtained by the control law given by Algorithm 3.2. Using the stationary Riccati solutions P and p from (6) to stabilize the linearized pendulum equations leads to the feedback law

$$\dot{x} \;=\; -\left(A + \tfrac{1}{\gamma}P\right)x + \left(f - \tfrac{1}{\gamma}p\right),$$

where the positive definite matrix P and the vector p are given by

$$P = \begin{bmatrix} \sqrt{\gamma^2+\gamma} & \gamma \\ \gamma & \sqrt{\gamma^2+\gamma} \end{bmatrix}, \quad p = \begin{bmatrix} -\sqrt{\gamma^2+\gamma}\,\pi \\ -\gamma\pi \end{bmatrix}.$$

Using this, a short calculation gives the optimal state for the infinite horizon problem,

$$x = \begin{bmatrix} \pi \\ 0 \end{bmatrix} + e^{-t\cdot\sqrt{1+\frac{1}{\gamma}}} \begin{bmatrix} a \\ b \end{bmatrix},$$

where a and b depend on the initial state of the differential equation. The control has the form $u = -\frac{1}{\gamma}(Px+p)$ and is given by

$$u = -\tfrac{1}{\gamma}\begin{bmatrix} a\sqrt{\gamma^2+\gamma}+b\gamma \\ a\gamma + b\sqrt{\gamma^2+\gamma} \end{bmatrix} e^{-t\cdot\sqrt{1+\frac{1}{\gamma}}}.$$

Two more lines of computation finally yield that the cost $J = \displaystyle\int_0^\infty \|x-\bar{x}\|^2 + \gamma\|u\|^2\,dt$ is given by

$$J = \sqrt{\gamma^2+\gamma}\left(a^2 + 2ab\sqrt{\tfrac{\gamma}{\gamma+1}} + b^2\right) \;=\; O(\gamma)\ (\gamma \to 0).$$

Now we proceed similar for the instantaneous control law. To begin with, let $\frac{1}{h} > \|A\| = 1$. Then, there holds $B = B^* = 1 - hA + O(h^2)$ $(h \to 0)$ and $BB^* = 1 - 2hA + O(h^2)$ $(h \to 0)$. For the matrix $C_A = -(A + \frac{\rho}{h}BB^*)$ (see (19)) we obtain the approximation

$$\tilde{C}_A \;\doteq\; \begin{bmatrix} -\frac{\rho}{h} & 1-2\rho \\ 1-2\rho & -\frac{\rho}{h} \end{bmatrix},$$

which is of order $O(h)$ $(h \to 0)$. Using this approximations it is sufficient to investigate the feedback control given by

$$\dot{x} = \tilde{C}_A x + \tfrac{\rho}{h}\left(I - 2hA\right)\bar{x} + b - \rho\left(I - 2hA\right)(b - f)$$

since its exact solution

$$x \;=\; ae^{\lambda_1 t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + be^{\lambda_2 t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

deviates from that of the original control law (13) only by a term of size $O(h)$ $(h \to 0)$. The eigenvalues $\lambda_{1/2}$ of the matrix $\tilde{C}_A$ are given by

$$\lambda_{1/2} \;=\; -\frac{\rho}{h} \pm \operatorname{sign}(1-2\rho)|1-2\rho|,$$

so that negative eigenvalues are guaranteed, provided the parameter h is sufficiently small. Finally, we compute the control and the costs. For the control we obtain

$$u = -\left(ae^{\lambda_1 t}\left(\tfrac{\rho}{h}+2\rho\right)\begin{bmatrix}1\\1\end{bmatrix} + be^{\lambda_2 t}\left(\tfrac{\rho}{h}+2\rho\right)\begin{bmatrix}1\\-1\end{bmatrix}\right) \longrightarrow\ 0\ (t\ \rightarrow\ \infty),$$

the value of the costs J are given by

$$J = \frac{a^2}{\lambda_1}\left(1+\gamma\left(\frac{\rho}{h}+2\rho\right)^2\right) + \frac{b^2}{\lambda_2}\left(1+\gamma\left(\frac{\rho}{h}+2\rho\right)^2\right) \leq C\left(\frac{h}{\rho}+\frac{\gamma\rho}{h}\right)$$

with some positive constant C independent of ρ, γ and h.

The results for the controller of Algorithm 3.3 are similar. For the costs one obtains in this case

$$J\ \leq\ C\left(\frac{\gamma}{h}+h\right).$$

As a result we obtain that both, the optimal Riccati controller and the controller of Algorithm 3.2 have the same qualitative behaviour.

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