Michael Hinze

A remark on second order methods in control of fluid flow

We present the functional analytic framework for a tracking-type control problem of the instationary Navier-Stokes equations. As solution methods for the control problem we comparatively discuss Newton's method as an example of the black box approach, and the SQP-method as one of the all-at-once methods. It is argued that black box approaches in general outperform all-at-once methods since the numerical effort necessary to numerically solve linearizations of the instationary Navier-Stokes equations compares to that of the nonlinear Navier-Stokes system. We report a numerical comparison which illustrates our argumentation. For both approaches local convergence results are cited.

1. The optimal control problem

We consider the optimal control problem

$$\left.\begin{array}{l}
\min_{(y,u)\in W\times U}\ J(y,u) := \frac{1}{2}\int\limits_{Q_o} |y-z|^2\,dxdt + \frac{\alpha}{2}\int\limits_{Q_c} |u|^2\,dxdt \\
\text{subject to} \\
\left.\begin{array}{lll}
\frac{\partial y}{\partial t} + (y\cdot\nabla)y - \nu\Delta y + \nabla p = Bu & \text{in} & Q = (0,T)\times\Omega, \\
\text{div } y = 0 & \text{in} & Q, \\
y(t,\cdot) = 0 & \text{on} & \Sigma = (0,T)\times\partial\Omega, \\
y(0,\cdot) = y_0 & \text{in} & \Omega.
\end{array}\right\} \iff e(y,u) = 0 \ \textit{in}\ Z^*.
\end{array}\right\} \tag{1}$$

Here $Q_c := \Omega_c \times (0,T)$ and $Q_o := \Omega_o \times (0,T)$, with Ω_c and Ω_o subsets of $\Omega = (0,1)^2$ denoting control and observation volumes, respectively. The first term in the cost functional values the control gain which here is to track the state z, and the second term measures the control cost, where $\alpha > 0$ denotes a weighting factor. In this form solving (1) appears at first to be a standard task. However, the formidable size of (1) and the goal of analyzing second order methods necessitate an independent analysis. One of the few contributions focusing on second order methods for optimal control of fluids are given by Ghattas *et al* [2] and Heinkenschloss [3]. These works are restricted to stationary problems, however. Among other things analytical investigations on second order methods are given by Kunisch and the author [5], where also further references can be found.

To define the spaces and operators required for the investigation of (1) we introduce the solenoidal spaces $H = \{v \in C_0^\infty(\Omega)^2 \colon \text{div } v = 0\}^{-|\cdot|_{L^2}}$, $V = \{v \in C_0^\infty(\Omega)^2 \colon \text{div } v = 0\}^{-|\cdot|_{H^1}}$, with the superscripts denoting closures in the respective norms. Further we define $W = \{v \in L^2(V) \colon v_t \in L^2(V^*)\}$ and $Z := L^2(V) \times H$, where W is endowed with the norm $|v|_W = (|v|^2_{L^2(V)} + |v_t|^2_{L^2(V^*)})^{1/2}$, and set $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2(V^*),L^2(V)}$, with V^* denoting the dual space of V. Here $L^2(V)$ is an abbreviation for $L^2(0,T;V)$ and similarly $L^2(V^*) = L^2(0,T;V^*)$. In (1) further $U = L^2(Q_c)$ denotes the Hilbert space of controls which is identified with its dual U^*. The state z is assumed to an element of $L^2(H)$. It is not hard to show that the cost functional $J \colon L^2(H) \times U \to \mathbb{R}$ is bounded from below, weakly lower semi-continuous, twice Fréchet differentiable with locally Lipschitzean second derivative, and radially unbounded in u, i.e. $J(y,u) \to \infty$ as $|u|_U \to \infty$, for every $y \in W$. It is well known that the nonlinear mapping $e \colon W \times U \to Z^*$, $e(y,u) = (\frac{\partial y}{\partial t} + (y\cdot\nabla)y - \nu\Delta y - Bu, y(0) - y_0)$ is onto [1]. Here, $Bu(t,\cdot)$ denotes the Leray-projection of the extension by zero of $u(t,\cdot)$ to the whole of Ω and $y_0 \in H$. Therefore, with respect to existence (1) can equivalently be rewritten as $\min \hat{J}(u) = J(y(u),u)$ subject to $u \in U$, where $y(u) \in W$ satisfies $e(y(u),u) = 0$. It is proved in [1] that this problem admits a solution $x^* = (y^*,u^*) \in W \times U =: X$.

We distinguish two numerical approaches for the numerical solution of (1): In the *black-box* approach Newton's method is applied to the equation $\nabla\hat{J}(u) = 0$ in U. For a given control iterate $u = u^0$ the Newton step amounts to the solution of one Navier-Stokes system for $y(u)$, the solution of one adjoint system for $\lambda(u)$, and the solution of the Newton system $H((y(u),u),\lambda(u))\delta u = -\nabla\hat{J}(u)$, see [5]. In the *all-at-once* approach the SQP-method is applied to solve (1). This is equivalent to apply Newton's method to solve $\nabla L(x,\lambda) = 0$, where $L \colon X \times Z \to \mathbb{R}$, $L(x,\lambda) = J(x) + \langle e(x),\lambda\rangle_{Z^*,Z}$ denotes the Lagrangian corresponding to problem (1). For a given iterate (x,λ) the numerical amount in every iteration step of this approach consists in solving one linearized Navier-Stokes system (with solution b1), and two adjoint-like linear systems (with solutions b2 and b3, respectively) to provide the righthand-side r in the reduced system $H(y,u,\lambda)\delta u = r$, and of the solution of the reduced system for δu. The increments δy and $\delta\lambda$ can than be obtained from δu, b1, b2 and b3 [5]. In both approaches $H(x,\lambda) = T^*(x)L_{xx}(x,\lambda)T(x)$ denotes the reduced Hessian, where $T(x) := \left[-e_y^{-1}(x)e_u(x), Id_U\right]^t$ with $x = (y,u)$. It is worth noting that due to the structure of the reduced Hessian these systems themselves have to be solved iteratively. For the numerical example presented in Tab. 1 we utilized the cg-algorithm. Proofs of local convergence with expected rates for both methods are given in [5] for a wide range of control problems. In [3] it is claimed that *all-at-once* approaches in general numerically outperform *black-box* approaches, since in the latter ones nonlinear systems have to be solved iteratively, whereas in the *all-at-once* approaches only linear system solves are required. Arguing in this way seems not to be correct for control problems for fluid flow since the numerical effort necessary to numerically solve linearizations of the instationary Navier-Stokes equations compares to that of the nonlinear Navier-Stokes system [5].

In Tab. 1 we present a numerical comparison of Newton's method and the SQP-approach to the control of a cavity flow at Reynolds number $1/\nu$ = Re=400. We chose $\Omega_c = \Omega_o = (0,1)^2$ and $\alpha = 10^{-2}$. As initial guess for the SQP-algorithm we utilized the tuple (x^0, λ^0) where $x^0 = (y^0, u^0)$ is obtained from applying one step of Newton's method with initial control equal to zero to the numerical solution of $\nabla\hat{J}(u) = 0$, i.e. with u^0 computed by Newton's method y^0 denotes the solution of the Navier-Stokes equations in (1) with $u = u^0$. Finally, the initial guess for the Lagrange multiplier is set to $\lambda^0 = -e_y(x^0)J_y(x^0)$. Newton's method is initialized with $u^0 = 0$. The numerical details can be found in [4,6]. The SQP-method takes 10 iterations and approximately 3 hours cpu to calculate a numerical solution, whereas Newton's method takes only 4 iterations at 1.3 hours cpu. As is expected due the inexact solution of the reduced systems both methods only converge super-linearly. All computations were performed on a ORIGIN^{TM} 200.

Table 1: Performance of the SQP- method (left) versus Newton's Method (right)

It	CG-It	$\frac{\lvert L'(x,\lambda)\rvert}{\lvert L'(x^0,\lambda^0)\rvert}$	$\frac{\lvert\delta(x^k,\lambda^k)\rvert}{\lvert\delta(x^{k+1},\lambda^{k+1}\rvert}$	$J(x^k)$	CG-It	$\frac{\lvert\hat{J}'(u^k)\rvert}{\lvert\hat{J}'(u^0)\rvert}$	$\frac{\lvert\delta u^k\rvert}{\lvert\delta u^{k-1}\rvert}$	$\hat{J}(u^k)$
0	-	-	-	1.188772e-2	-	1.e0	-	1.188772e-2
1	11	1.e0	1.	3.216904e-3	13	3.343476e-1	1.	3.216904e-3
2	3	2.342777e-1	0.456	1.661840e-3	11	5.032540e-2	0.489	1.618184e-3
3	16	5.846246e-1	1.110	2.041436e-3	18	8.839902e-3	0.428	1.479364e-3
4	5	1.574504e-1	0.959	1.548152e-3	16	1.126838e-4	0.079	1.477881e-3
5	14	2.718657e-2	0.554	1.683024e-3	-	-	-	-
6	23	6.744024e-2	0.914	1.485434e-3	-	-	-	-
7	18	8.005254e-2	0.874	1.521882e-3	-	-	-	-
8	18	1.852064e-2	0.197	1.480751e-3	-	-	-	-
9	23	1.343532e-3	0.146	1.479234e-3	-	-	-	-
10	26	1.698641e-4	0.127	1.479219e-3	-	-	-	-

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2. References

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Addresses: Prof. Dr. Michael Hinze, TU Dresden, Fakultät für Mathematik, Zellescher Weg 12-14, D-01069 Dresden.