Control strategies for fluid flows - optimal versus suboptimal control

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Abstract

In our talk optimal and suboptimal distributed and boundary control strategies for fluid flows governed by the Navier-Stokes equation (NSE) together with some aspects of the numerical and practical implementation of the underlying algorithms will be addressed. The objective of our numerical investigations is to compare optimal and suboptimal control strategies with regard to cost reduction and numerical expense for backward facing step flows.

To describe the approach let $\Omega \subset \mathbb{R}^2(\mathbb{R}^3)$ denote the flow region with boundary $\Gamma = \Gamma_c \cup \Gamma_{obs} \cup \Gamma_d$, where in the case of boundary control Γ_c denotes the part of the boundary where control is applied, Γ_{obs} is the observation boundary and Γ_d is the remaining part of the boundary part where fixed Dirichlet data are prescribed. For the case of boundary control and boundary observations the control problem is given by

$$(P) \quad \left\{ \begin{array}{l} \min J(\Phi) = J(\Phi, u) = \frac{l}{2} \int_0^T \int_{\Gamma_c} \Phi^2 dS dt + \int_0^T \int_{\Gamma_{obs}} F(\nabla u) dS dt \\ \text{s.t.} \\ \begin{array}{rcl} u_t - \frac{1}{Re}\Delta u + (u\nabla)u + \nabla p &=& f \text{ in } \Omega \times (0,T] \\ -div\, u &=& 0 \text{ in } \Omega \times (0,T] \\ u &=& \left\{ \begin{array}{ll} \Phi & \text{on } \Gamma_c \times (0,T] \\ 0 & \text{on } (\Gamma_d \cup \Gamma_{obs}) \times (0,T] \end{array} \right. \\ u(0) &=& u_0, \end{array} \end{array} \right. \tag{1}$$

where $F(z) = F(z_1, \ldots, z_4)$ is a sufficiently smooth, scalar valued function, u_0 denotes the initial value of the flow and Φ denotes the control. Applying a Lagrangian approach to problem (P) leads to an optimality system (OS) involving the NSE, a backward in time parabolic linear system for the adjoint variables and the optimality condition for Φ (see [1]). For the numerical solution of the OS we use the Navier-Stokes solver developed by Bänsch in [2].

Instead of looking for an optimal control for problem (P) and thus looking for an optimal cost reduction, we apply a time integration scheme to the NSE and solve at each time step an optimization problem with an instantaneous cost function related to the one of problem (1). As we shall demonstrate this allows a significant reduction of the costs with readilly available computing power. To be more precise consider (to simplify matters) the first order semi implicit time integration scheme

$$\begin{array}{rl} u_i - \frac{\delta t}{Re}\Delta u_i + \delta t \nabla p_i = \delta t(f_i - (u_{i-1}\nabla)u_{i-1}) + u_{i-1} =: R_i & \text{in } \Omega \\ -div\, u_i = 0 & \text{in } \Omega \\ u_i = \left\{ \begin{array}{ll} \Phi_i & \text{on } \Gamma_c \\ 0 & \text{on } \Gamma_d \cup \Gamma_{obs}, \end{array} \right. & \end{array} \tag{2}$$

where $u_0 = u(0)$. The minimization problem at each time step then is

$$(P_i) \quad \left\{ \begin{array}{l} \min J(\Phi_i) = J(\Phi, u_i) = \frac{l}{2} \int_{\Gamma_c} \Phi_i^2 dS + \int_{\Gamma_{obs}} F(\nabla u_i) dS \\ \text{s.t. (2).} \end{array} \right. \tag{3}$$

Having in mind the application of a gradient-type algorithm to problem (3) one needs to be able to compute the derivatives of J at Φ in directions χ. Using a Lagrange approach one obtains (subscripts are dropped)

$$\nabla J(\Phi)(\chi) = \nabla_\Phi L(\chi) = \int_{\Gamma_c} (l\Phi + c\nabla\xi\eta - \pi\eta)\chi \, dS, \tag{4}$$

where η denotes the outer normal to Ω, ξ is the adjoint velocity and π represents the adjoint pressure.

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The adjoint variables are determined by the following linear system of equations ($c = \frac{\delta t}{Re}$)

$$\begin{aligned} u - c\Delta u + \delta t\ \nabla p &= R \text{ in } \Omega \\ -div\, u &= 0 \text{ in } \Omega \\ u &= \begin{cases} \Phi & \text{on } \Gamma_c \\ 0 & \text{on } \Gamma_d \cup \Gamma_{obs}, \end{cases} \\ \xi - c\Delta\xi + \nabla\pi &= 0 \text{ in } \Omega \\ -\delta t\, div\, \xi &= 0 \text{ in } \Omega \\ \xi &= \begin{cases} \frac{1}{c} DF(\nabla u)\eta & \text{on } \Gamma_{obs} \\ 0 & \text{on } \Gamma_d \cup \Gamma_c. \end{cases} \end{aligned} \tag{5}$$

The availability of the gradient of J allows the use of gradient-type algorithms which would include a line search step solving

$$\rho^* = \text{argmin}_{\rho \geq 0} H(\rho) := J(\Phi^k - \rho \nabla J(\Phi^k)) \tag{6}$$

Every evaluation $H(\rho)$ involves the solution of system (5) with $\Phi = \Phi^k - \rho \nabla J(\Phi^k)$, so that an efficient inexact line-search should be utilized. At this stage the structure of problem (P_i) must be exploited. The optimal stepsize in the minimum problem (6) can be charactrized via a Lagrange approach. In practical applications only one (or a few) steps of the gradient-type algorithm is applied at each time step utilizing the optimal stepsize ρ^* (or a suitable estimate thereof). The numerical expense of the suboptimal approach then coincides approximately with that of one forward and one backward solve in the OS. We present numerical examples illustrating the effectiveness and scope of the approach.

Further literature concerning this class of problems can be found in [3], [4], [5], [6] [8] and [9].

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