Numerical analysis and algorithms in control and state constrained optimization with pdes

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We consider an elliptic optimal control problem with control and pointwise state constraints. The cost functional is approximated by a sequence of functionals which are obtained by discretizing the state equation with the help of linear finite elements and enforcing the state constraints in the nodes of the triangulation. The control variable is not discretized. A general error bound for control and state is obtained which forms the starting point for optimal error estimates in both in two and three space dimensions. For the numerical implementation of the discrete concept fix-point iterations or generalized Newton methods are proposed.

1 Optimization problem

Let \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) be a bounded domain with a smooth boundary \( \partial \Omega \) and consider an uniformly elliptic, coercive differential operator \( A_y := -\sum_{i,j=1}^{d} \partial_j \left( a_{ij} y_{ij} \right) + \sum_{i=1}^{d} b_i y_{ix} + c y \), with associated bilinear form \( a \) defined on \( H^1(\Omega) \). We are interested in finite element analysis of the following control problem

\[
\min_{u \in U_{ad}} J(u) = \frac{1}{2} \int_{\Omega} |y - y_0|^2 + \frac{\alpha}{2} \| u - u_0 \|^2_{L^2} \quad \text{subject to } y = G(Bu) \text{ and } y(x) \leq b(x) \text{ in } \Omega. \tag{1}
\]

Here, \( G \) denotes the solution operator associated to \( A \), and \( U_{ad} \subseteq U \) denotes the set of admissible controls which is assumed to be a closed and convex subset of the Hilbert space \( U \). Furthermore, we suppose that \( \alpha > 0 \) and that \( y_0 \in H^1(\Omega) \), \( u_0 \in U \), \( b \in W^{2,\infty}(\Omega) \) are given, and that \( B : U \to (H^1(\Omega))^* \) is linear and bounded. Clearly, (1) admits a unique solution \( u \in U_{ad} \).

We suppose a so called Slater condition, i.e. there exists some \( \bar{u} \in U_{ad} \) such that \( G(\bar{u}) < b \) in \( \Omega \).

Finite element analysis for elliptic control problems in the presence of control and state constraints is presented by Casas in [1] who proves convergence of finite element approximations for finitely many state constraints. Casas and Mateos extend these results in [2] to a less regular setting for the states and prove convergence of finite element approximations to semilinear distributed and boundary control problems. In [9] Meyer considers a fully discrete strategy to approximate an elliptic control problem with pointwise state and control constraints. His results are similar to those presented by the authors in [3,4]. Constraints on the gradient of the state are considered by the authors in [5]. The discretization concept introduced in the next section is developed in [6], where also tailored fixed point iterations and generalized Newton methods are proposed for the numerical solution of the corresponding discrete problem (2), compare also [7].

2 Finite element discretization

Let \( \mathcal{T}_h \) be a quasi–uniform triangulation of \( \Omega \) with maximum mesh size \( h := \max_{T \in \mathcal{T}_h} \text{diam}(T) \) and vertices \( x_1, \ldots, x_m \). Furthermore, let \( X_h \subset H^1(\Omega) \) denote some associated finite element space consisting of continuous, piecewise polynomial functions. We suppose that \( \Omega \) is the union of the elements of \( \mathcal{T}_h \) so that element edges lying on the boundary are curved. Problem (1) is now approximated by the following sequence of control problems depending on the mesh parameter \( h \):

\[
\min_{u \in U_{ad}} J_h(u) := \frac{1}{2} \int_{\Omega} |y_h - y_0|^2 + \frac{\alpha}{2} \| u - u_{0,h} \|^2_{L^2} \quad \text{subject to } y_h = G_h(Bu) \text{ and } y_h(x_j) \leq b(x_j) \text{ for } j = 1, \ldots, m. \tag{2}
\]

Here, \( G_h \) denotes the finite element solution operator associated to \( G \), and \( u_{0,h} \) denotes an approximation to \( u_0 \) which is assumed to satisfy \( \| u_0 - u_{0,h} \| \leq Ch \). Clearly, problem (2) admits a unique solution \( u_h \in U_{ad} \). We note that the set \( U_{ad} \) is not discretized so that \( u_h \) in general does not represent a finite element function. This is different to the common approaches considered in the literature to approximate (1). There holds

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Theorem 2.1 Let $u, u_h$ denote the unique solutions to (1), (2) with associated states $y, y_h$. Then
\begin{align*}
\alpha \|u - u_h\|^2 + \|y - y_h\|^2 & \leq \langle (u_h - u), (p - p^h) \rangle_{(H^1)' \times H^1} + \int_\Omega (y - y_h)(y - y^h) - \\
& - \alpha (u_0 - u_h, u_h - u) + \int_\Omega (y - y^h) d\mu + \int_\Omega (I_h b - b) d\mu + \int_\Omega (y^h - y) d\mu_h + \int (b - I_h b) d\mu_h.
\end{align*}

Here, $y_h = G_h(Bu_h), y^h = G_h(Bu)$, $I_h$ denotes the Lagrange interpolation operator, and $p^h \in X_h$ denotes the unique solution of $a(w_h, p^h) = \int_\Omega (y - y_0) w_h + \int_\Omega w_h d\mu$ for all $w_h \in X_h$. Furthermore, $p, p_h$ denote adjoint and discrete adjoint states associated to the state and discrete state equation, respectively, and $\mu, \mu_h$ denote multipliers associated to the constraints on the state and its finite element discretization, respectively. A proof of this theorem can be deduced from the proof of [4, Theorem 3.2]. Estimate (3) is optimal, since it allows to estimate the error $\|u - u_h\|$ in the controls using the weakest possible norms of $p - p^h$ and $y - y^h$, respectively. Using standard finite element analysis together with the fact that $\mu_h$ is uniformly bounded w.r.t. $h$ in the space of regular Borel measures ( [4, Lemma 2.4]) it is possible to deduce from (3) for continuous, linear finite elements
\begin{equation}
\|u - u_h\| \leq \begin{cases}
C h^{1 - \frac{4}{\pi}}, & \text{if } Bu \in W^{1,s}(\Omega) \text{ for } s \in \left[1, \frac{4}{\pi - 1}\right].
\end{cases}
\end{equation}

2.1 Numerical example

The following test problem is taken - in a slightly modified form - from [8, Example 6.2]. Let $\Omega := B_1(0) \subset \mathbb{R}^2$, $\alpha > 0$, $U_{ad} \equiv U := L^2(\Omega) \times H^1(\Omega)^\ast$ denote the injection,
\begin{align*}
y_0(x) := 4 + \frac{1}{\pi} - \frac{1}{4\pi} |x|^2 + \frac{1}{2\pi} \log |x|, \quad u_0(x) := 4 + \frac{1}{4\alpha\pi} |x|^2 - \frac{1}{2\alpha\pi} \log |x|
\end{align*}
and $b(x) := |x|^2 + 4$. We consider problem (2) for this setting, where we use linear, continuous finite elements for the approximation of $y_h$. By checking the optimality conditions of first order one verifies that $u \equiv 4$ is the unique solution of (1) with corresponding state $y \equiv 4$, see [3]. The experimental order of convergence for a sequence of conformi uniform refinements of the unit disc up to $RL = 5$ refinement levels is reported in Table 1 for the error functionals $E(h) := \|u - u_h\|, \|y - y_h\|$. It confirms the analytical findings of (4) for the controls $u, u_h$. The order of convergence for $y, y_h$ is better than expected which may be explained by the fact that $y \in X_h$.

The experimental order of convergence for an error functional $E(h) > 0$ is defined by
\begin{align*}
EOC := \frac{\log E(h_1) - \log E(h_2)}{\log h_1 - \log h_2}.
\end{align*}

Its value is equal to $\kappa$ if $E(h) \leq C h^\kappa$, with some constant $C > 0$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$RL$ & $\|u - u_h\|$ & $\|y - y_h\|$ \\
\hline
1 & 0.788985 & 0.536461 \\
2 & 0.759556 & 1.147861 \\
3 & 0.919917 & 1.389378 \\
4 & 0.966078 & 1.518381 \\
5 & 0.988686 & 1.598421 \\
\hline
\end{tabular}
\caption{Experimental order of convergence}
\end{table}

References