INSTANTANEOUS CONTROL FOR THE INSTATIONARY BURGERS EQUATION – CONVERGENCE ANALYSIS & NUMERICAL IMPLEMENTATION –

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ABSTRACT. Instantaneous control is applied to the control of the Burgers equation. This control technique is closely related to receding horizon control and allows for an interpretation as suboptimal closed loop controller, whose parameters may be adjusted so as to stabilize the the nonlinear equation under consideration. Besides convergence analysis for the distributed control case several numerical examples for the continuous and discrete-in-time control laws are presented, including boundary control.

1. INTRODUCTION

In this research we are concerned with optimal and instantaneous control applied to the Burgers equation. As a model we take the distributed control problem

$$\textbf{(P)} \quad \begin{cases} \min J(y,u) \;=\; \frac{1}{2}\int_0^T\int_\Omega |y-z|^2\,dxdt \;+\; \frac{\alpha}{2}\int_0^T\int_{\Omega_c} |u|^2\,dxdt \\ \text{s.t.} \\ \quad y_t - \nu y'' + yy' = u\chi_{\Omega_c} \quad \text{in } (0,T)\times\Omega, \\ \quad\qquad\qquad y = 0 \qquad\quad \text{on } (0,T)\times\partial\Omega, \\ \quad\qquad\qquad y(0) = \phi \qquad\quad \text{in } \Omega. \end{cases}$$

Here, the control target is to match the given desired state z in the L^2-sense by adjusting the body force u in a control volume $\Omega_c \subseteq \Omega = (0,1)$ in the L^2-sense, i.e. with minimal energy and work. The first term in the cost functional measures the physical objective, the second one the size of the control, where the parameter $\alpha > 0$ plays the role of a weight.

If the control goal is primarily to match the desired state without considering minimizing the energy and work linear body force feedback control can be used instead of optimal control [8, 12, 13]. The idea behind instantaneous control in this context is two-fold - to construct a feedback law which matches the desired state, but at considerable control costs. The method works as follows. The uncontrolled Burgers equation is discretized with respect to time. Then, at selected time slices an instantaneous version of the cost functional is approximately minimized subject to a stationary quasi-Burgers equation, whose structure depends on the chosen discretization method. The control obtained is used to steer the system to the next time slice, where the procedure is repeated.

Instantaneous control is closely related to receding horizon control (rhc) or model predictive control (mpc) with finite time horizon [7, 16, 17]. Its difference to (rhc) and (mpc) can best be explained in terms of a chess analogy: (rhc) and (mpc) compute the next n, say, optimal moves and apply the first one to steer the system to the next time slice. Instantaneous control computes an approximation to the next

optimal move by applying exactly one step of the gradient method to the solution of the instantaneous control problem and applies the approximate control to advance to the next time slice. Clearly, it can not be expected that the instantaneous control technique is a candidate to stabilize arbitrary dynamical systems, compare the citations above. However, as we shall show at hand of the Burgers equation, it is capable of enhancing the stability properties of dissipative systems. In its nature instantaneous control is a discrete-in-time feedback control approach and, as shall be shown, can be interpreted as the stable time discretization of a closed-loop control law [10].

The optimal control problem **(P)** was studied by the second author in [21], where the augmented Lagrange SQP technique was used to solve **(P)**. Related contributions were presented in [1], where optimality systems and existence of optimal controls for the control of the instationary Navier-Stokes equations in two space dimensions are discussed. A gradient technique was proposed in [9] for the solution of **(P)**, similar approaches to the numerical solution of optimal control problem can be found in [2].

The instantaneous control technique was successfully applied to a great variety of fluid mechanical control problems [3, 4, 5, 11, 14, 15] and in the recent past also as real-time control approach to the cooling of steel [19]. However, as far as the authors know stability investigations for its application to the control of infinite-dimensional systems have not yet been provided. Stability analysis of the approach for finite-dimensional systems with distributed controls can be found in [10]. In the present work we extend these results to the control of the instationary Burgers equation.

The paper is organized as follows. In Section 2 we briefly discuss existence and second order sufficiency for solutions of problem **(P)**. Section 3 is devoted to the derivation of the instantaneous control approach and the formulation of the algorithm. In Section 4 we relate the discrete-in-time control approach to a continuous closed-loop control law and prove both stability of the continuous and the discrete controller. Finally, in Section 5 we present a variety of numerical examples, which complete the theoretical results.

Throughout this work c and C denote global generic constants whose dependencies are mentioned when necessary.

2. Optimal control

For given $T > 0$ let $Q = (0,T) \times \Omega$, where $\Omega = (0,1) \subset \mathbb{R}$. We set $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and identify the Hilbert space H with its dual H'. On H we use the common inner product, and endow the Hilbert space V with the inner product

$$(\varphi, \psi)_V = (\varphi', \psi')_H \text{ for } \varphi, \psi \in V.$$

Moreover, by $L^2(0,T;V)$ we denote the space of measurable functions from $[0,T]$ to V, which are square integrable in the sense of Bochner. We denote for instance by $u(t)$ the function $u(t,\cdot)$ considered as a function of x only when t is fixed.

The control acts on a subset $Q_c \subseteq Q$ with positive measure. Here, we restrict ourselves to the case $Q_c = (0,T) \times \Omega_c$ for some $\Omega_c \subseteq \Omega$. In this context we introduce the bounded linear extension operator $\mathcal{B} : L^2(\Omega_c) \to V'$ by

$$\langle \mathcal{B}u, \varphi \rangle_{V',V} = (u, \varphi)_{L^2(\Omega_c)} \text{ for } u \in L^2(\Omega_c) \text{ and } \varphi \in V. \tag{1}$$

In order to formulate the weak form of the Burgers equation we introduce the Hilbert space

$$W(0,T;V) = \{\varphi \in L^2(0,T;V) : \varphi_t \in L^2(0,T;V')\}$$

supplied with the common inner product.

Now, for controls $u \in L^2(0,T;L^2(\Omega_c))$ the state $y \in W(0,T;V)$ is given by the weak solution of the Burgers equation, i.e.

$$\begin{aligned} &\frac{d}{dt}(y(t),\varphi)_H + \nu(y(t),\varphi)_V + (y(t)y_x(t),\varphi)_H \\ &\qquad = \langle \mathcal{B}u(t),\varphi\rangle_{V',V} \text{ for all } \varphi \in V \text{ and a.e. } t \in [0,T] \end{aligned} \tag{2a}$$

and

$$(y(0),\chi)_H = (\phi,\chi)_H \text{ for all } \chi \in H. \tag{2b}$$

Here, $\nu > 0$ is a viscosity parameter.

Theorem 1. For any $\phi \in H$ and for every control $u \in L^2(0,T;L^2(\Omega_c))$ equation (2) admits a unique weak solution $y \in W(0,T;V)$.

The proof of this theorem is similar to that for the unsteady Navier-Stokes equations, see [18]. We recall that every element $y \in W(0,T;V)$ has a continuous representation with respect to time with values in H, i.e. $y \in C(0,T;H)$, see e.g. [6].

The cost function in **(P)** can be rewritten as

$$J(y,u) = \frac{1}{2}\|y-z\|^2_{L^2(0,T;H)} + \frac{\alpha}{2}\|u\|^2_{L^2(0,T;L^2(\Omega_c))},$$

where $z \in L^2(0,T;H)$ is a desired state and $\alpha > 0$ is fixed. The minimization of the first term involving $y - z$ is the real goal. The other term has been introduced in order to bound the control function and to prove existence of an optimal control.

To shorten notation we set

$$X = W(0,T;V) \times L^2(0,T;L^2(\Omega_c)) \text{ and } Y = L^2(0,T;V') \times H.$$

Introducing the operator $e : X \to Y$ by

$$e(y,u) = (e_1(y,u), e_2(y,u)) = \big(y_t - \nu y_{xx} + yy_x - \mathcal{B}u, y(0) - \phi\big)$$

the optimal control problem **(P)** can be understood as a minimization problem with equality constraints:

$$\text{minimize } J(y,u) \quad \text{s.t.} \quad e(y,u) = 0.$$

Both J and e are twice continuously Fréchet-differentiable and their second Fréchet-derivatives are Lipschitz continuous. In the following the Fréchet-derivatives with respect to the variable $x = (y,u)$ are denoted by primes.

For every $c \geq 0$ let us denote the augmented Lagrangian by

$$\begin{aligned} L_c(y,u,\lambda,\mu) &= J(y,u,\lambda,\mu) + \langle e_1(y,u),\lambda\rangle_{L^2(0,T;V'),L^2(0,T;V)} \\ &\quad + (e_2(y,u),\mu)_H + \frac{c}{2}\|e(y,u)\|^2_Y. \end{aligned}$$

For problem **(P)** the following results were proved in [21].

Theorem 2. There exists a solution $(y_*, u_*, \lambda_*, \mu_*)$ to the first-order necessary optimality conditions

$$L_c'(y_*, u_*, \lambda_*, \mu_*) = 0, \quad e(y_*, u_*) = 0 \text{ for } c \geq 0.$$

If in addition $\|y_* - z\|_{L^2(0,T;H)}$ is sufficiently small, then the second-order sufficient optimality condition holds, i.e. there exists a constant $\gamma > 0$ such that

$$\langle L_c''(y_*, u_*, \lambda_*, \mu_*)v, v\rangle_{X',X} \geq \gamma \, \|v\|_X^2 \quad \text{for all } v \in X \text{ satisfying } e'(y_*, u_*)v = 0.$$

Hence, (y_*, u_*) is a local solution to **(P)**.

3. Instantaneous control strategy

We now describe the moving horizon approach which is the main subject of the investigations.

The uncontrolled Burgers equation is discretized with respect to time. At every time step a now stationary tracking type control problem is approximately solved by applying exactly one gradient-step with step-size $\rho \geq 0$. The control obtained is used to steer state to the next time slice and, in the long run, hopefully towards the desired state. By patching together the controls one obtains a piece-wise constant control function defined over the time interval $(0, T)$. Clearly, it can not be expected that the control function obtained by the instantaneous control strategy is optimal, i.e. a solution optimal to problem **(P)**. However, as several applications of the method to large control problems show, it provides good controls at low numerical costs [4, 5, 19] and, which is even more important, allows for an interpretation as closed-loop control law [4, 10]. Clearly, the instantaneous control approach depends on the time integration scheme used. We discuss both, the fully implicit and the semi-implicit (implicit with respect to the linear, explicit with respect to the nonlinear part) Euler scheme and anticipate that from the applications point of view semi-implicit time integration is advantageous.

For notational purposes we set $A = -\nu \frac{d^2}{dx^2}$ with domain $D(A) = H^2(\Omega) \cap V$. Then A is a linear, self-adjoint, positive definite unbounded operator which maps $D(A)$ into H. Moreover, $A^{-1} : H \to H$ exists and is self-adjoint, positive definite and compact. Furthermore, we set

$$b(y) = -yy'.$$

For $m \in \mathbb{N}$ an equidistant discretization of the time interval $(0, T)$ is defined by $h = \frac{T}{m}$ and $t_k = kh$, $k = 0, 1, \ldots, m$. As instantaneous versions of the cost functional J we take

$$J^k : V \times L^2(\Omega_c) \to \mathbb{R}, \quad (y, u) \mapsto \frac{1}{2} \, \|y - z^k\|_H^2 + \frac{\alpha}{2} \, \|u\|_{L^2(\Omega_c)}^2,$$

where

$$z^k = \frac{1}{h} \int_{t_k - \frac{h}{2}}^{t_k + \frac{h}{2}} z(s, \cdot) \, ds \tag{3}$$

and $z(t, \cdot) = 0$ for $t > T$. Finally, for $k = 1, \ldots, m$ and $i = 1, 2$ we introduce the operators $e_i^k : V \times L^2(\Omega_c) \to V'$ by

$$e_1^k(y, u) = (I + hA)y \, - \, hb(y) - y^{k-1} - \mathcal{B}u$$

and

$$e_2^k(y, u) = (I + hA)y \, - \, hb(y^{k-1}) - y^{k-1} - \mathcal{B}u.$$

3.1. **Fully implicit time integration.** As time discretization scheme for the uncontrolled Burgers equation we use the fully implicit Euler scheme and take at time slice t_k the resulting stationary equation as state equation for the instantaneous control problem. This control problem then is given by

$$(\mathbf{P}_1^k) \quad \text{minimize } J^k(y,u) \text{ subject to } e_1^k(y,u) = 0 \text{ in } V',$$

where $y^0 = \phi$. We recall that for given y^{k-1} a pair (y^k, u^k) satisfies the subsidiary condition $e_1^k(y,u) = 0$ in V' if and only if

$$\nu h\,(y^k,\varphi)_V + (y^k - hb(y^k),\varphi)_H = (y^{k-1},\varphi)_H + (\mathcal{B}u^k,\varphi)_{V',V} \quad \forall\,\varphi \in V. \tag{4}$$

The right-hand side in this equation defines a bounded linear functional on V. Thus, for every $u^k \in L^2(\Omega_c)$ Eq. (4) admits a unique solution $y^k \in V$, which satisfies the a-priori estimate

$$\|y^k\|_V \le \frac{C}{\nu h}\left(\|y^{k-1}\|_H + \|u^k\|_{L^2(\Omega_c)}\right).$$

A proof can be found in [20], where control problems for the stationary Burgers equation were studied.

For every $k = 1,\dots,m$ problem $(\mathbf{P}_1^k)$ admits a solution $(y_*^k, u_*^k) \in V \times L^2(\Omega_c)$. Furthermore, since $(e_1^k)'(y_*,u_*)$ is surjective, there exists a Lagrange multiplier $\lambda_*^k \in V$, which is even unique, such that the triple $(y_*^k, u_*^k, \lambda_*^k)$ satisfies the first-order necessary optimality conditions

$$(I + hA)y - b(y) = \mathcal{B}u + y^{k-1}, \tag{5a}$$

$$(I + hA)\lambda - b'(y)\lambda = -(y - z^k), \tag{5b}$$

$$\alpha u - \mathcal{B}^\star\lambda = 0, \tag{5c}$$

where the adjoint operator $\mathcal{B}^\star : V \to L^2(\Omega_c)$ of $\mathcal{B}$ is defined by

$$\langle \mathcal{B}v, \psi\rangle_{V',V} = (v, \mathcal{B}^\star\psi)_{L^2(\Omega_c)} \text{ for all } v \in L^2(\Omega_c) \text{ and } \psi \in V.$$

Moreover, if $\|y_*^k - z^k\|_H$ is sufficiently small, the Lagrangian associated to problem $(\mathbf{P}_1^k)$ satisfies a second-order sufficient optimality condition at $(y_*^k, u_*^k, \lambda_*^k)$ so that (y_*^k, u_*^k) locally solves $(\mathbf{P}_1^k)$.

The optimal control problem $(\mathbf{P}_1^k)$ with respect to existence is equivalent to the unconstrained minimization of the functional

$$\hat{J}_1^k(u) = J^k(y(u),u)$$

over $L^2(\Omega_c)$, where for a control $u \in L^2(\Omega_c)$ the state $y(u) \in V$ is given as the unique solution to (4). The gradient of $\hat{J}_1^k$ at u is given by

$$\nabla\hat{J}_1^k(u) = \alpha u - \mathcal{B}^\star\lambda,$$

where for given u the function λ is obtained by first solving the nonlinear problem (5a) for the state y, and then solving the linearized Burgers equation (5b) for λ.

Remark 1. Using implicit time integration the gradient $\nabla\hat{J}_1^k(u)$ at a control u depends on the observation $y^k - z^k$, which occurs as right-hand side in the adjoint equation for the computation of the auxiliary variable λ, and on the whole state y^k, which enters as a coefficient into the adjoint equation for λ. This structure remains valid even in the case of boundary observation, where the observation enters as boundary condition into the adjoint equation, but the whole state y^k again as a coefficient function. As a consequence, computation of gradient information for $\hat{J}_1^k$ can not be based on observations alone.

3.2. **Semi-implicit time integration.** We proceed as in Section 3.1, except that instead of the fully implicit Euler scheme we utilize a semi-implicit scheme for time discretization of the uncontrolled Burgers equation. The instantaneous control problem now is given by

$$(\mathbf{P}_2^k) \quad \text{minimize } J^k(y,u) \text{ subject to } e_2^k(y,u) = 0 \text{ in } V',$$

where $y^0 = \phi$. The initial value ϕ now is required to be an element of the space V. For given y^{k-1} a pair (y^k, u^k) satisfies the subsidiary condition $e_2^k(y,u) = 0$ in V' if and only if

$$\nu h\ (y^k, \varphi)_V + (y^k, \varphi)_H = (y^{k-1} + hb(y^{k-1}), \varphi)_H + (\mathcal{B}u^k, \varphi)_{V',V} \quad \forall\, \varphi \in V. \tag{6}$$

Since $\phi \in V$ holds, the right-hand side in this linear equation defines a bounded linear functional on V. Thus, for every $u^k \in L^2(\Omega_c)$ Eq. (6) admits a unique solution $y^k \in V$ which satisfies the a-priori estimate

$$\|y^k\|_V \le \frac{C}{\nu h} \left(\|y^{k-1}\|_H + h\|y^{k-1}\|_V + \|u^k\|_{L^2(\Omega_c)} \right).$$

Arguing as in the previous section problem $(\mathbf{P}_2^k)$, $k = 1, \ldots, m$, admits a solution $(y_*^k, u_*^k) \in V \times L^2(\Omega_c)$, which due to the quadratic cost functional and the linear constraints is unique. Furthermore, the unique Lagrange multiplier $\lambda_*^k \in V$ together with the solution (y_*^k, u_*^k) satisfies the first-order necessary optimality conditions

$$(I + hA)y = \mathcal{B}u + y^{k-1} + b(y^{k-1}), \tag{7a}$$

$$(I + hA)\lambda = -(y - z^k), \tag{7b}$$

$$\alpha u - \mathcal{B}^\star \lambda = 0. \tag{7c}$$

Furthermore, the second-order sufficient optimality condition holds on the whole space $V \times L^2(\Omega_c) \times V$. Hence, the solution (y_*^k, u_*^k) of (7) is the minimum of $(\mathbf{P}_2^k)$.

The optimal control problem $(\mathbf{P}_2^k)$ with respect to existence is equivalent to the unconstrained minimization of the functional

$$\hat{J}_2^k(u) = J^k(y(u), u)$$

over $L^2(\Omega_c)$, where for a control $u \in L^2(\Omega_c)$ the state $y(u) \in V$ is given as the unique solution to (6). The gradient of $\hat{J}_2^k$ at u is given by

$$\nabla \hat{J}_2^k(u) = \alpha u - \mathcal{B}^\star \lambda,$$

where, similar as the previous section for given u the function λ is obtained by first solving the nonlinear problem (7a) for the state y, and then solving the linearized Burgers equation (7b) for λ.

Remark 2. The adjoint equation (7b) only depends on the observation $y^k - z^k$ and not, as it is the case for fully implicit time integration, on the whole state y^k in terms of a coefficient. Therefore, gradient information for the functional $\hat{J}_2^k$ is available utilizing the observations only. In the particular case of boundary observation no information of the state in the computational domain is needed at all.

3.3. **The algorithm.** In order to stabilize Burgers equation at every time slice we apply one gradient step to the solution of the control problem $(\mathbf{P_2^k})$. The control obtained in this way is used to steer the system to the next time slice. With the gradient of the functional $\hat{J}_2^k$ available this procedure in algorithmical form can be formulated as follows.

Algorithm 1 (Instantaneous control)**.**

1.) Set $y^0 = \phi$, $k = 0$ and $t^0 = 0$.
2.) Given an initial control u_0^k, solve

$$\begin{aligned}(I + hA)y &= y^k + hb(y^k) + \mathcal{B}u_0^k,\\ (I + hA)\lambda &= -(y - z^k).\end{aligned}$$

3.) Set $\nabla\hat{J}(u_0^k) = \alpha u_0^k - \mathcal{B}^\star\lambda$.
4.) Given $\rho > 0$, set $u^{k+1} = u_0^k - \rho\nabla\hat{J}(u_0^k)$.
5.) Solve

$$(I + hA)y^{k+1} = y^k + hb(y^k) + \mathcal{B}u^{k+1}.$$

6.) Set $t_{k+1} = t_k + h$, $k = k + 1$. If $t_k < T$ goto 2.

Here, $\hat{J} = \hat{J}_2^k$. The choice of the stepsize ρ in step 4.) of the algorithm is crucial. Since $(\mathbf{P_2^k})$ is quadratic with linear constraints, the optimal choice ρ^* can be computed exactly and is given by

$$\rho^* = \frac{\|\nabla\hat{J}(u)\|^2_{L^2(\Omega_c)}}{(D^2\hat{J}(u)\nabla\hat{J}(u), \nabla\hat{J}(u))_{L^2(\Omega_c)}}, \tag{8}$$

where

$$\begin{aligned}(D^2\hat{J}(u)\nabla\hat{J}(u), \nabla\hat{J}(u)))_{L^2(\Omega_c)} &= \alpha\ \|\nabla\hat{J}(u)\|^2_{L^2(\Omega_c)}\\ &\quad + \mathcal{B}^\star(D_y e_2(y(u), u))^{-\star} I_H (D_y e_2(y(u), u))^{-1}\mathcal{B}\nabla\hat{J}(u).\end{aligned}$$

Since $D^2\hat{J}(u)$ is positive definite close, the denominator in (8) never vanishes. Note, that the computation of ρ^* requires the solution of two auxiliary linear problems and is bounded from above by $1/\alpha$.

Remark 3. An approximation of ρ^* utilizing only the solution of one additional auxiliary problem can be obtained as follows. Set $d = -\nabla\hat{J}(u)$. Decompose the function $y(u+\rho d)$ into its affine part $y(u)$ and its homogeneous part $y(d)$, i.e. write $y(u + \rho d) = y(u) + \rho y(d)$ and set

$$h(\rho) = J(y(u + \rho d), u + \rho d).$$

Then h is a quadratic polynomial in ρ which takes its minimum value at

$$\rho_1^* = -\frac{(y(u) - z, y(d))_H + \alpha(u, d)_{L^2(\Omega_c)}}{\|y(d)\|^2_H + \alpha\ \|d\|^2_{L^2(\Omega_c)}}. \tag{9}$$

The computation of ρ_1^* requires only the computation of the auxiliary function $y(d)$.

4. The control laws

Instantaneous control allows for an interpretation as a non-linear discrete-in-time suboptimal closed loop control method, which turns out to be the stable time discretization of some continuous closed loop controller. We now outline the reconstruction of the continuous controller from the discrete-in-time control approach given by Algorithm 1. Furthermore, we present existence and uniqueness results for the continuous controller, and investigate the asymptotic behavior of both, the continuous and the discrete control law.

4.1. **Existence and uniqueness of the continuous controller.** To further simplify notations we denote the solution operator in (6) by B_h, i.e. for given $f \in V'$ the function $v_h = B_h f$ solves

$$v_h - \nu h v_h'' = f \text{ in } V' \quad \text{and} \quad v_h(0) = v_h(1) = 0. \tag{10}$$

The operator B_h is linear, bounded and self-adjoint and we have $B_h^{-1} = I + hA$ holds.

In many applications of distributed control applied to systems governed by parabolic equations the operator $\mathcal{B}$ defined in (1) plays the role of an extension operator. For this reason and in order to simplify the exposition we from now on assume that $\mathcal{B} = I$.

Algorithm 1 allows the following interpretation.

Theorem 3. For $u_0^k = 0$ Algorithm 1 is equivalent to the semi-implicit time discretization with discretization step size h

$$\begin{aligned}(I + hA)y^{k+1} = y^k + hb(y^k) - \rho B_h B_h(y^k - z^k) \\ - h\rho B_h B_h(b(y^k) - Az^k), \quad y^k = \phi,\end{aligned} \tag{11}$$

of the dynamical system

$$\dot{y} + Ay = b(y) - \frac{\rho}{h} B_h B_h(y - z) - \rho B_h B_h(b(y) - Az), \quad y(0) = \phi. \tag{12}$$

Proof. For a proof it remains to show that for u^k in step 4.) of Algorithm 1

$$u^k = -\rho B_h B_h(y^k - z^k) - h\rho B_h B_h(b(y^k) - Az^k) \tag{13}$$

holds true. Note that

$$B_h z^k = B_h B_h(I + hA)z^k = B_h B_h z^k + hB_h B_h Az^k.$$

Using this in steps 2.)-5.) of the algorithm yields the desired result. □

Due to Theorem 3 the term

$$K_h y = -\frac{\rho}{h} B_h B_h(y - z) - \rho B_h B_h(b(y) - Az) \tag{14}$$

in (12) can be interpreted as a non-linear closed-loop control policy for the Burgers equation. It is important to note that the discretization step-size h and the descent parameter ρ in the gradient step of Algorithm 1 in the continuous case may now be regarded as parameters defining the controller.

First, we investigate existence and regularity questions for the dynamical system (12). Since in this context the desired state plays the role of an inhomogeneity, whose regularity properties could be adapted to the necessities of the proof, we for

the moment set $z = 0$, which in turn implies $Az = 0$. Furthermore, for $h > 0$ let $\rho_0 = \rho_0(h)$ be given by

$$\rho_0 = \min\Big(\frac{\sqrt{2}\nu^{3/2}}{4\sqrt{2}\nu^{3/2} + 4C_A\sqrt{h}\,\|\phi\|_H^2}, \frac{6\nu + 3d_H}{2\nu^2}\Big). \tag{15}$$

and set

$$d_H = 2\,\|\phi\|_H^2 \text{ and } d_V = 2\,\|\phi\|_V^2.$$

The next two theorems will be proved in the appendix.

Theorem 4. Let $\phi \in H$ be a given initial state, let $h > 0$ be fix and let ρ_0 be given by (15). Then, for every $0 < \rho \le \rho_0$ the equation

$$\begin{cases} y_t + Ay - b(y) = -\frac{\rho}{h}B_hB_hy - \rho B_hB_hb(y) \\ y(0) = \phi \end{cases} \tag{16}$$

for every $T > 0$ admits a unique weak solution $y \in W(0,T;V)$, which satisfies the a-priori estimates

$$\begin{aligned} \|y(t)\|_H^2 &\le \exp\big(-\tfrac{\rho t}{h}\big)\, d_H && \text{for all } t \in [0,T] \\ \|y\|^2_{L^\infty(0,T;H)} &\le d_H, \\ \|y\|^2_{L^2(0,T;V)} &\le \tfrac{2}{\nu}\, d_H && \text{and} \\ \|y\|^2_{L^2(0,T;V')} &\le C(\nu)\,\big(1 + \tfrac{\rho}{h}\big)\, d_H. \end{aligned} \tag{17}$$

For more regular initial values we can say even more.

Theorem 5. Let ρ_0 be given by (15), let $0 < \rho \le \rho_0$ and let $\phi \in V$. Then the solution of problem (16) satisfies $y \in H^{2,1}(Q)$ and the a-priori estimates

$$\begin{aligned} \|y\|_V^2 &\le C\exp\big(-\tfrac{\rho t}{h}\big)\, d_V && \text{for all } t \in [0,T], \\ \|y\|^2_{L^\infty(0,T;V)} &\le C(\nu)\, d_V, \\ \|y\|^2_{L^2(0,T;V)} &\le C(\nu)\, d_V && \text{and} \\ \|y\|^2_{L^2(0,T;V')} &\le C(\nu)\,\big(1 + \tfrac{\rho}{h}\big)\, d_V. \end{aligned} \tag{18}$$

Since the desired state z need not satisfy the homogeneous equation

$$z_t + Az - b(z) = -\rho B_hB_h(b(z) - Az), \quad z(0) = \phi$$

we generalize the control law (14) to

$$K_hy = -\frac{\rho}{h}B_hB_h(y - z) - \rho B_hB_h(b(y) - b(z)) + z_t + Az - b(z) \tag{19}$$

so that the controlled Burgers equation becomes the form

$$y_t + Ay - b(y) = K_hy \quad \text{in } L^2(0,T;V') \text{ and } \quad y(0) = \phi. \tag{20}$$

Remark 1. Choosing the initial control u_0^k in Algorithm 1 as solution of

$$\Big(I - \frac{\rho}{1-\rho\alpha}B_hB_h\Big)\,u_0^k = \frac{1}{1-\rho\alpha}\,\big(z^{k+1} - z^k + Az^{k+1} - b(z^k) + \rho B_hB_h(b(z^k) - Az^k)\big)$$

in Theorem 3 we would end up with control law (19) instead of control law (14).

4.2. **Stability of the continuous controller.**

Assumption 1. Let z be given such that $z \in W(0,T;V)$, and $\phi - z(0) \in V$ hold.

In this section the stability properties of the control law (20) are investigated. It follows from the a-priori estimates to be derived that this feedback law admits a unique solution provided Assumption 1 is satisfied. Moreover, the H and V norm of difference $w = y - z$ decay exponentially with rate $-\frac{\rho}{h}$, which coincides with the results for $z = 0$. To achieve these results the value of ρ_0 in (15) has to be decreased depending on the size of $\|z\|^2_{L^\infty(0,T;H)}$.

Using w as test function in the variational formulation of (20) leads to

$$\begin{aligned}(21)\quad \frac{d}{dt}\,\|w_t\|_H^2 + \nu\,\|w\|_V^2 + \frac{\rho}{h}\,\|B_h w\|_H^2 = \\ -\int_\Omega (ww' + zw' + z'w)\,w\,dx \\ -\rho\int_\Omega B_h(ww' + zw' + z'w)B_h w\,dx = I_1 + \ldots + I_6.\end{aligned}$$

Recall that $I_1 = 0$ holds. Using integration by parts, Hölder's and Young's inequality we have

$$I_2 + I_3 = -\int_\Omega zww'\,dx \le \frac{\nu}{2}\,\|w\|_V^2 + \frac{1}{2\nu}\,\|z\|_V^2\|w\|_H^2.$$

Furthermore, using a estimation technique analogous to (63) we obtain

$$\begin{aligned}I_4 + I_5 + I_6 &\le \frac{\rho h}{2}\,\|B_h(ww' + wz' + zw')\|_H^2 + \frac{\rho}{2h}\,\|B_h w\|_H^2 \\ &\le \frac{3\rho}{4\nu}\left\{\|w\|_V^2\|w\|_H^2 + \|z\|_V^2\|w\|_H^2 + \|w\|_V^2\|z\|_H^2\right\} + \frac{\rho}{2h}\,\|B_h w\|_H^2\end{aligned}$$

and proceed as in the derivation of (52). Due to Lemma 11 this gives

$$\begin{aligned}(22)\quad \frac{1}{2}\frac{d}{dt}\|w\|_H^2 + \left(\frac{\nu(1-\rho)}{2} - \frac{3\rho}{4\nu}(d_H + \|z\|^2_{L^\infty(0,T;H)})\right)\|w\|_V^2 + \frac{\rho}{2h}\,\|w\|_H^2 \\ \le \frac{3\rho+2}{4\nu}\,\|z\|_V^2\|w\|_H^2.\end{aligned}$$

Now let

$$(23)\qquad 0 < \rho \le \frac{2\nu^2}{2\nu^2 + 3(d_H + \|z\|^2_{L^\infty(0,T;H)})}$$

so that $\frac{\nu(1-\rho)}{2} - \frac{3\rho}{4\nu}(d_H + \|z\|^2_{L^\infty(0,T;H)}) > 0$. Then since the right-hand-side in (22) is integrable a further application of Gronwall's inequality, see [22] for instance, proves

Theorem 6. Let ρ satisfy (23) and let y be the unique solution of (20). Then $w = y - z$ satisfies

$$(24)\qquad \begin{aligned}\|w(t)\|_H^2 &\le C\exp\left(-\tfrac{\rho}{h}t\right) && \text{for all } t \in [0,T],\\ \|w\|^2_{L^\infty(0,T;H)} &\le C, && \\ \|w\|^2_{L^2(0,T;V)} &\le C && \text{and}\\ \|w_t\|^2_{L^2(0,T;V')} &\le C\left(1 + \tfrac{\rho}{h}\right), &&\end{aligned}$$

where $C = C(\nu, \|z\|_{W(0,T;V)})$ is a positive constant independent of ρ and h.

Here recall again that the estimate for w_t is a direct consequence of the second and third estimate in (24).

Remark 4. For the result of the previous theorem only $\phi \in H$ and $z \in W(0,T;V)$ is required.

To prove decay also for the V-norm of w use $-w''$ as test function in (20). This leads to

$$\frac{1}{2}\frac{d}{dt}\,\|w\|_V^2 + \nu\,\|w''\|_H^2 - \int_\Omega (yy' - zz')\,w''\,dx = \frac{\rho}{h}\int_\Omega B_h w B_h w''\,dx + \rho\int_\Omega B_h(yy' - zz')B_h w''\,dx. \tag{25}$$

Estimating analogously to the proof of the previous theorem gives

$$\frac{1}{2}\frac{d}{dt}\,\|w\|_V^2 + \frac{\nu}{2}\,\|w''\|_H^2 + \frac{\rho}{h}\left(1 - 2\rho - \frac{3\rho}{\nu^2}\left(d_H + \|z\|^2_{L^\infty(0,T;H)}\right)\right)\|w\|_V^2 \le C_\nu\left(\left[\|w\|_H\|w\|_V + \|z\|_H\|z\|_V + \|z\|_V^2\right]\|w\|_V^2 + \|z\|_V^2\|w\|_H^2\right). \tag{26}$$

Now require

$$0 < \rho \le \frac{\nu^2}{4\nu^2 + 6\left(d_H + \|z\|^2_{L^\infty(0,T;H)}\right)} \tag{27}$$

so that $1 - 2\rho - \frac{3\rho}{\nu^2}(d_H + \|z\|^2_{L^\infty(0,T;H)}) \ge \frac{1}{2}$. Since the right-hand side in (26) is integrable a further application of Gronwall's inequality, see e.g. [22], proves

Theorem 7. Let ρ satisfy (27) and let y be the unique solution of (20). Then $w = y - z$ satisfies

$$\begin{aligned} \|w(t)\|_V^2 &\le C\,\exp\left(-\tfrac{\rho}{h}t\right) && \text{for all } t \in [0,T], \\ \|w\|^2_{L^\infty(0,T;V)} &\le C, \\ \|w\|^2_{L^2(0,T;H^2(\Omega))} &\le C && \text{and} \\ \|w_t\|^2_{L^2(0,T;H)} &\le C\left(1 + \tfrac{\rho}{h}\right), \end{aligned} \tag{28}$$

where $C = C(\nu, \|z\|_{H^{2,1}(Q)})$ is a positive constant independent of ρ and h.

4.3. **Stability of the discrete controller.** We now investigate the stability properties of the discrete control law

$$(I + hA)y^{j+1} = y^j + h\left(b(y^j) - b(z^j)\right) - \rho B_h B_h(y^j - z^j) - h\rho B_h B_h\left(b(y^j) - b(z^j)\right) + z^{j+1} - z^j + hAz^{j+1}, \quad y^0 = \phi, \tag{29}$$

which according to Theorem 3 is the discrete analogon to the continuous control law in (20).

The following proposition will be frequently used in the proof in Theorem 9 below. Its proof can be found in [18].

Proposition 8. There exists a constant $c > 0$ such that

$$\int_\Omega uv'w\,dx \le c\,\|u\|_H^{1/2}\,\|u\|_V^{1/2}\,\|v\|_H^{1/2}\,\|v\|_V^{1/2}\|w\|_V \text{ for all } u, v, w \in V.$$

In order to investigate the decay properties of the difference $w^j = y^j - z^j$ we rewrite (29) as

$$(I + hA)w^{j+1} = w^j + h\left(b(y^j) - b(z^j)\right) - \rho B_h B_h w^j \\ - \rho h B_h B_h \left(b(y^j) - b(z^j)\right), \quad w^0 = \phi - z(0) \tag{30}$$

and prove

Theorem 9. Let $z \in L^\infty(0,T;V)$ and $\phi \in V$. Then for all $\rho \in (0,2)$ there exist constants $\kappa = \kappa(\rho) \in (0,1)$ and $h_0 = h_0(\rho) > 0$ such that for all $h \le h_0$

$$\|w^n\|_H \le \kappa^n \, \|w^0\|_H \text{ for all } n \in \mathbb{N}. \tag{31}$$

Proof. Since

$$2(w^{j+1}, w^j)_H = \|w^{j+1}\|_H^2 + \|w^j\|_H^2 - \|w^{j+1} - w^j\|_H^2$$

holds, testing (30) with w^{j+1} leads to

$$\frac{1}{2}\,\|w^{j+1}\|_H^2 - \frac{1}{2}\,\|w^j\|_H^2 + \frac{1}{2}\,\|w^{j+1} - w^j\|_H^2 + \nu h\,\|w^{j+1}\|_V^2 = \\ h\int_\Omega \left(w^j(w^j)' + z^j(w^j)' + w^j(z^j)'\right) w^{j+1}\,dx - \rho\int_\Omega B_h w^j B_h w^{j+1}\,dx \\ - \rho h \int_\Omega B_h\left(w^j(w^j)' + z^j(w^j)' + w^j(z^j)'\right) B_h w^{j+1}\,dx. \tag{32}$$

In order to estimate the terms on the right-hand side of (32) we make frequent use of Proposition 8 and Young's inequality. We obtain for every $\alpha \in (0,1)$

$$h\int_\Omega w^j(w^j)' w^{j+1}\,dx \le \\ \frac{\nu h}{4}\,\|w^{j+1}\|_V^2 + h^{1-2\alpha}\,\|w^{j+1}\|_H^2 + \frac{h^{1+\alpha}}{2\sqrt{\nu}}\,\|w^j\|_V^2\|w^j\|_H^2. \tag{33}$$

Applying integration by parts we find that

$$h\int_\Omega \left(z^j(w^j)' + w^j(z^j)'\right) w^{j+1}\,dx = -h\int_\Omega z^j w^j (w^{j+1})'\,dx,$$

so that direct estimation using the regularity properties of the state z yields

$$h\int_\Omega z^j w^j (w^{j+1})'\,dx \le \frac{\nu h}{4}\,\|w^{j+1}\|_V^2 + \frac{h}{\nu}\,\|z^j\|_{L^\infty(\Omega)}^2\|w^j\|_H^2. \tag{34}$$

The properties of the operator B_h imply that

$$-\rho\int_\Omega B_h w^j B_h w^{j+1}\,dx \le -\frac{\rho(2-\rho)}{2}\,\|B_h w^{j+1}\|_H^2 + \frac{1}{2}\,\|w^{j+1} - w^j\|_H^2. \tag{35}$$

Applying the Cauchy-Schwarz inequality together with estimate (49) in the appendix yields

$$-\rho h\int_\Omega B_h(w^j(w^j)')B_h w^{j+1}\,dx \le \\ \frac{3\rho h^{\frac{3}{2}}}{8\sqrt{2}(2-\rho)}\,\|w^j\|_H^2\|w^j\|_V^2 + \frac{\rho(2-\rho)}{12}\,\|B_h w^{j+1}\|_H^2. \tag{36}$$

Similarly,

$$-\rho h\int_\Omega B_h(w^j(z^j)')B_h w^{j+1}\,dx \le \frac{3\rho h^{\frac{3}{2}}}{8\sqrt{2}(2-\rho)}\|w^j\|_H^2\|z^j\|_V^2+\frac{\rho(2-\rho)}{12}\,\|B_h w^{j+1}\|_H^2. \tag{37}$$

Recall that $v_h^j = B_h(z^j(w^j)')$ satisfies

$$\|v_h^j\|^2+\nu h\,\|v_h^j\|_V^2=\int_\Omega z^j(w^j)'v^j\,dx=-\int_\Omega w^j(z^j)'v_h^j+z^j(v_h^j)'w^j\,dx.$$

Proposition 8 gives

$$-\int_\Omega w^j(z^j)'v_h^j\,dx\le\frac{\nu h}{2}\,\|v_h^j\|_V^2+\frac{1}{4}\,\|v_h^j\|_H^2+\frac{1}{\sqrt{2\nu h}}\,\|z^j\|_V^2\|w^j\|_H^2. \tag{38}$$

The regularity properties of z yield

$$-\int_\Omega z^j(v_h^j)'w^j\,dx\le\frac{2}{\nu h}\,\|z^j\|_{L^\infty(\Omega)}^2\|w^j\|_H^2+\frac{\nu h}{2}\,\|v_h^j\|_V^2, \tag{39}$$

so that

$$\|v^j\|^2\le\frac{2}{\nu h}\,\|z^j\|_{L^\infty(\Omega)}^2\|w^j\|_H^2+\sqrt{\frac{8}{\nu h}}\,\|z^j\|_V^2\|w^j\|_H^2.$$

With this estimate we obtain

$$-\rho h\int_\Omega B_h(z^j(w^j)')B_h w^{j+1}\,dx\le \frac{6\rho h}{\nu(2-\rho)}\,\|z^j\|_{L^\infty(\Omega)}^2\|w^j\|_H^2+\frac{3\sqrt{8}\rho h^{\frac{3}{2}}}{\nu(2-\rho)}\,\|z^j\|_V^2\|w^j\|_H^2+\frac{\rho(2-\rho)}{12}\,\|B_h w^{j+1}\|_H^2. \tag{40}$$

Next we note, that by (30) and the regularity properties of ϕ and z, it follows that $(v^{j+1})''\in V$. This in turn implies for $v_h^{j+1}=B_h w^{j+1}$ the estimate

$$\|v^{j+1}\|_H^2\ge\|w^{j+1}\|_H^2-2\nu h\,\|w^{j+1}\|_V^2.$$

Finally, since $\frac{1}{2}-\frac{\rho}{2}(2-\rho)=\frac{1}{2}(1-\rho)^2\ge 0$ for $\rho\in(0,2)$ we conclude from the estimates above and (32)

$$c_1(\rho,h)\,\|w^{j+1}\|_H^2-c_2^j(\rho,h)\,\|w^j\|_H^2\le 0, \tag{41}$$

where the functions c_1 and c_2 are given by

$$c_1(\rho,h)=\frac{1}{2}-h^{1-2\alpha}+\frac{\rho}{4}(2-\rho)+\frac{\nu h}{2}(\rho-1)^2$$

and

$$c_2^j(\rho,h)=\frac{1}{2}+\frac{h^{1+\alpha}}{2\sqrt{\nu}}\,\|w^j\|_V^2+\frac{h}{\nu}\,\|z^j\|_{L^\infty(\Omega)}^2+\frac{3\rho h^{\frac{3}{2}}}{8\sqrt{2}(2-\rho)}\,\|w^j\|_V^2+\frac{3\rho h^{\frac{3}{2}}}{8\sqrt{2}(2-\rho)}\,\|z^j\|_V^2+\frac{6\rho h}{\nu(2-\rho)}\,\|z^j\|_{L^\infty(\Omega)}^2+\frac{3\rho\sqrt{8}h^{\frac{3}{2}}}{\sqrt{\nu}(2-\rho)}\,\|z^j\|_V^2.$$

To complete the proof it remains to show that for every $\rho\in(0,2)$ there exists some positive $h_0(\rho)$ such that the quotient $\frac{c_2^j}{c_1}$ remains positive for all positive h smaller than h_0. The crucial term one has to cope with is $\|w^j\|_V^2$, which appears

in $c_2^j(\rho,h)$. In order to estimate its size fix ρ, go back to Eq. (32) and use again estimates (33)-(40). Then, one may conclude

$$h\ \|w^{j+1}\|_V^2 \le \left(\frac{1}{2} + ch(1+h^\alpha\ \|w^j\|_V^2)\right)\|w^j\|_H^2, \tag{42}$$

with a positive constant c depending uncritical on ρ and the state z. Now let $0<\alpha<\frac{1}{2}$, $\rho\in(0,2)$,

$$\hat{c}_2(\rho,h) = \frac{1}{2} + \frac{h^\alpha}{2\sqrt{\nu}}\ \|w^0\|_H^2 + \frac{h}{\nu}\ \|z\|_{C(0,T;V)}^2 + \frac{3\rho\sqrt{h}}{8\sqrt{2}(2-\rho)}\ \|w^0\|_H^2 + \frac{3\rho}{8\sqrt{2}(2-\rho)}h^{\frac{3}{2}}\ \|z\|_{C(0,T;V)}^2 + \frac{6\rho\nu}{2-\rho}h\ \|z\|_{C(0,T;V)}^2 + \frac{3\rho\sqrt{8\nu}}{2-\rho}h^{\frac{3}{2}}\|z\|_{C(0,T;V)}^2.$$

and argue by induction as follows.

1. Set $j=0$ and choose $h_0=h_0(\rho,w_0)\in(0,1]$ such that
   (a) $c(h+h^\alpha\ \|w^0\|_V^2)\le\frac{1}{2}$,
   (b) $\frac{\hat{c}_2(\rho,h)}{c_1(\rho,h)}=\kappa_1<1$ and
   (c) $\frac{c_2^0(\rho,h)}{c_1(\rho,h)}=\kappa_2<1$
   for all $h\le h_0$. This is possible, since for $\rho\in(0,2)$ the term $\frac{\rho}{4}(2-\rho)$ is positive. Define $\kappa=\max(\kappa_1,\kappa_2)$. Then, (42), (a) and $h\le 1$ give
   $$h\ \|w^1\|_V^2 \le \kappa^0\ \|w^0\|_H^2.$$
2. Now assume that for $j\in\mathbb{N}$ that
   (a) $\|w^j\|_H\le\kappa^j\ \|w^0\|_H$ and
   (b) $h\ \|w^j\|_V^2\le\kappa^{j-1}\ \|w^0\|_H^2$
   holds true.
3. From (42) and 1.(a) we conclude
   $$h\ \|w^{j+1}\|_V^2 \quad\le\quad \left(\frac{1}{2}+c\left(h+h^\alpha\kappa^{j-1}\ \|w^0\|_H^2\right)\right)\kappa^j\ \|w^0\|_H^2 \le \kappa^j\ \|w^0\|_H^2$$
   for all $0<h\le h_0$. Moreover, 2.(b) implies $c_2^j(\rho,h)\le\hat{c}_2(\rho,h)$, i.e.
   $$\frac{c_2^j(\rho,h)}{c_1(\rho,h)}\le\kappa_1 \quad \text{for all } 0<h\le h_0.$$
   Using (41) and 2.(a) it follows that $\|w^{j+1}\|_H\le\kappa^{j+1}\ \|w^0\|_H$. This completes the induction.

□

From the previous proof we immediately infer

Corollary 10. Let $z\in L^\infty(0,T;V)$ and $\phi\in V$. Then, for every $\rho\in(0,2)$ there exist constants $\kappa=\kappa(\rho)\in(0,1)$ and $h_0=h_0(\rho)>0$ such that

$$h\ \|w^n\|_V^2\le\kappa^{n-1}\ \|w^0\|_H^2 \quad \text{for all } n\in\mathbb{N}. \tag{43}$$

For h fixed and small enough this means that even the first space derivatives of the state y converge to that of the desired state, i.e.

$$\|w^n\|_V\to 0 \quad (n\to\infty) \text{ for all } 0<h\le h_0. \tag{44}$$

5. Numerical experiments

In this section we present several numerical examples that demonstrate the stability properties of the continuous controller developed in (19) and of the discrete controller given by Algorithm 1 together with Remark 1. Furthermore, we compare optimal control with instantaneous control and we present a numerical example with boundary control based solely on boundary observation. The computer programs are written in MATLAB version 5.3, as work station environment Digital Alpha 21264 computers are used.

Run 1 (Optimal versus instantaneous control)**.** For the Burgers equation we chose the parameters $T = 1$, $\nu = 0.01$, and

$$\phi = \begin{cases} 1 & \text{in } \left(0, \frac{1}{2}\right], \\ 0 & \text{otherwise.} \end{cases}$$

For $n, m \in \mathbb{N}$ the grid was given by $h = \frac{T}{m}$,

$$x_i = \frac{i}{n} \text{ for } i = 0, \dots, n \text{ and } t_j = jh \text{ for } j = 0, \dots, m-1.$$

To solve the Burgers equation with $u = 0$ we apply Newton's method at each time-step. The algorithm needed 0.25 seconds in case of $n = m = 30$. The numerical solution is shown in Fig. 1.

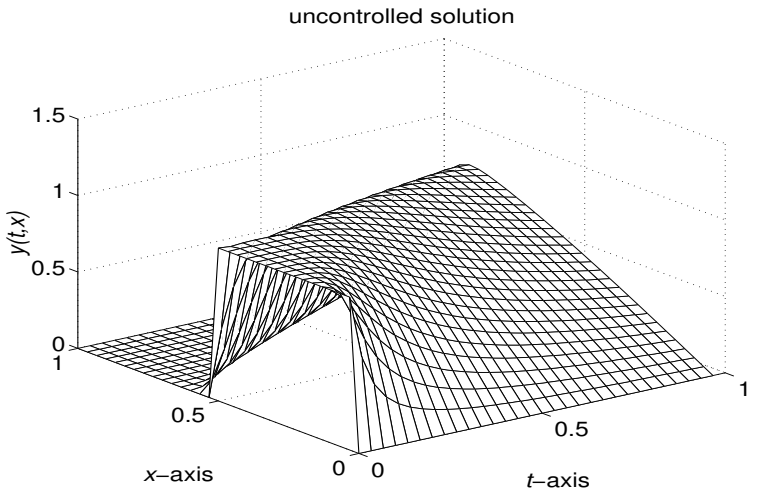

Figure 1.

Now we turn to the optimal control problem. The control acts locally on $Q_c = (0, T) \times (\frac{1}{4}, \frac{3}{4})$. As the desired state we take $z(t) = \phi$ in $(0, T)$. In view of Fig. 1 we can interpret this problem as determining u in such a way that it counteracts the uncontrolled dynamics which smoothes the discontinuity at $x = \frac{1}{2}$ while transporting the resulting disturbance to the left as t increases. The regularization parameter is chosen as $\sigma = 0.001$. We solved the optimal control problem by applying the augmented Lagrangian-SQP method which takes 11 seconds CPU-time. In Fig. 2 the discrete solution is shown.

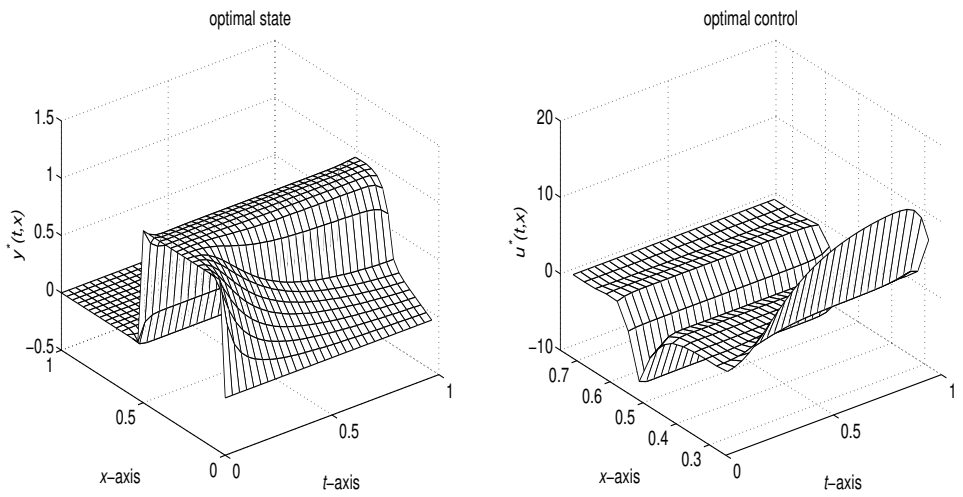

FIGURE 2: Optimal state and optimal control for $n = m = 30$.

The optimal value of the cost functional is 0.056 compared to 0.143 in the uncontrolled case.

Then we applied Algorithm 1 to compute an instantaneous control u_{inst}, which was used as a feedback law for the Burgers equation. Inserting u_{inst} into the Burgers equation as a right hand side we obtain a suboptimal state y_{feed}, which is close to the optimal state. As value of the cost functional we obtain 0.057. In Fig. 3 both y_{feed} and u_{inst} are plotted. As one can see the instantaneous and optimal control are very similar.

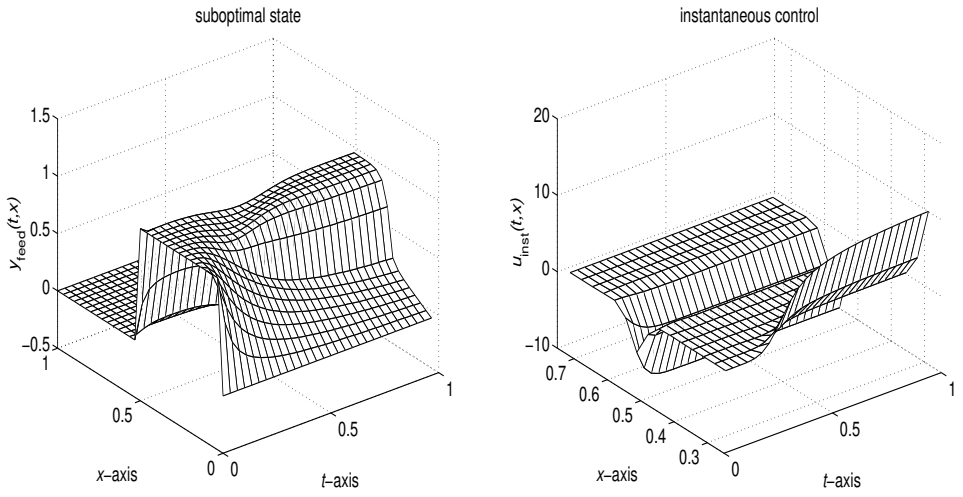

FIGURE 3: Instantaneously controlled state with control for $n = m = 30$.

The behavior of the H-norm of the residuals of the optimal and instantaneous states as functions of time is compared in Fig. 4. It turns out that the instantaneous control approach leads to a solution, which is very close to the optimal one.

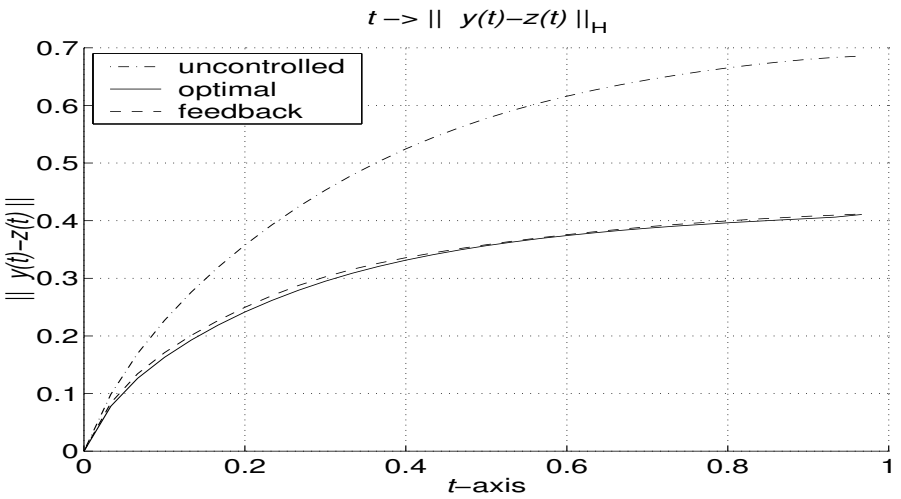

FIGURE 4.

In the following two numerical tests linear finite elements with 98 degrees of freedom for spatial discretization. For the first two runs presented we set $T = 1$, $\nu = 0.1$ and as initial condition for the Burgers equation we use

$$\phi = \begin{cases} 1 & \text{on } (0, \frac{1}{2}], \\ 0 & \text{otherwise.} \end{cases}$$

Run 2 (Continuous controller)**.** As control target we choose the function $z(t, x) = 1$. Thus, z satisfies $z_t = Az - b(z) = 0$, so that control law (14) can be utilized. The corresponding continuous control law (12) is numerically integrated utilizing the backward Euler-Galerkin method. For the time discretization we fix $\delta t = 0.005$. The corresponding uncontrolled solution is presented in Fig. 5.

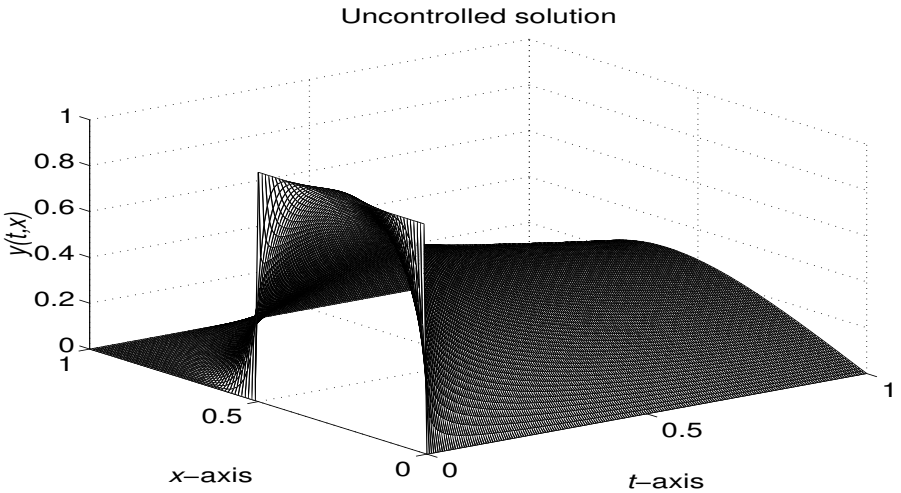

FIGURE 5: $h = 0.1$.

The solution as well as the decay rate for the H-norm for different values for ρ is shown in Fig. 6 and Fig. 7. It turns out that the control law converges exponentially towards the desired state until the accuracy of the time integration scheme is reached. Thereafter the controller steers the difference $y - z$ to a stationary vector

whose norm is of the size of the truncation error of the time integration scheme. As it is expected the stability properties enhance with increasing values for ρ.

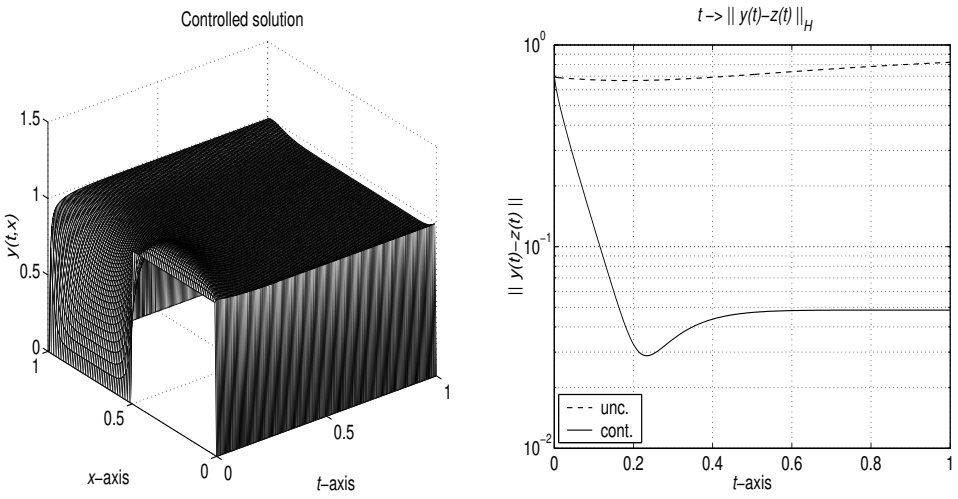

FIGURE 6: $\rho = 1$ and $h = 0.1$.

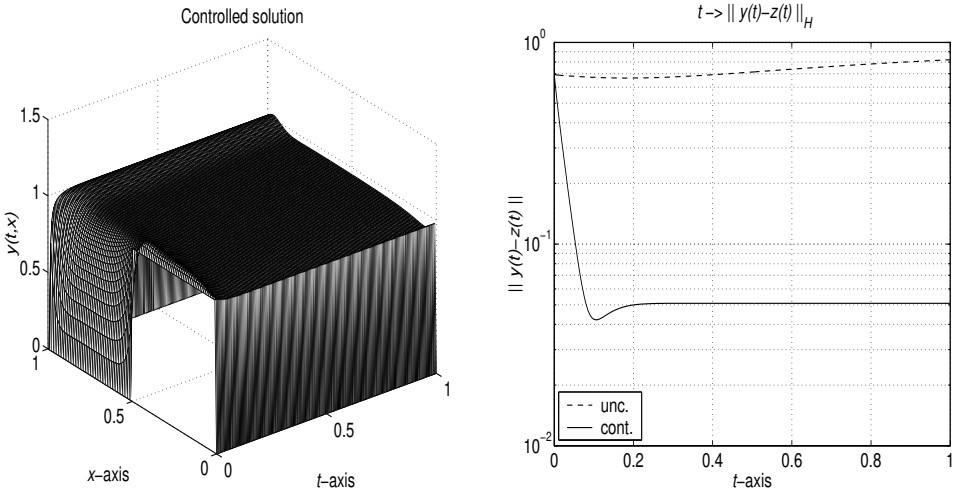

FIGURE 7: $\rho = 3$ and $h = 0.1$.

In Fig. 8 the decay rate for the V-norm is presented. As one can see, the numerical result coincides with them expected from the theoretical investigations.

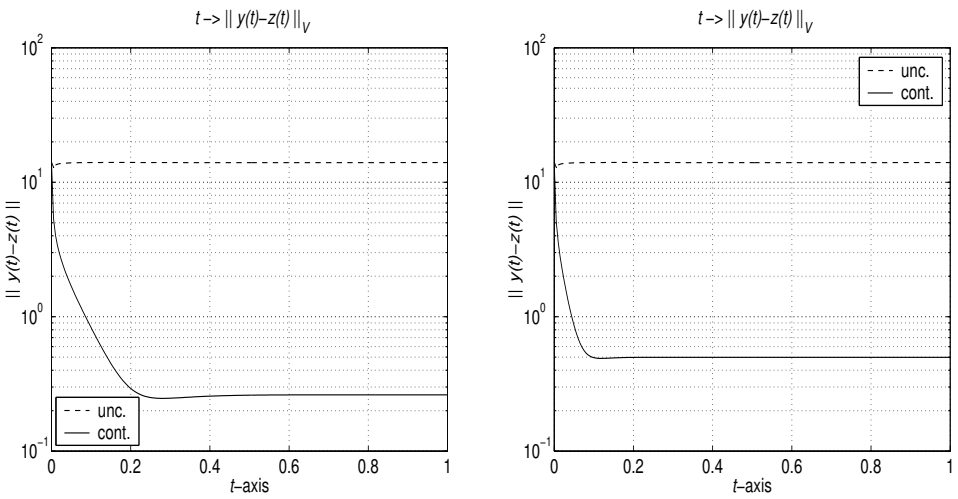

FIGURE 8: $\rho = 1$ (left), $\rho = 3$ (right) and $h = 0.1$.

Run 3 (Discrete controller)**.** We choose spatial discretization and the initial value ϕ as in Run 2. As time step size we take $\delta t = \frac{1}{30}$. Fig. 9 shows the corresponding uncontrolled solution.

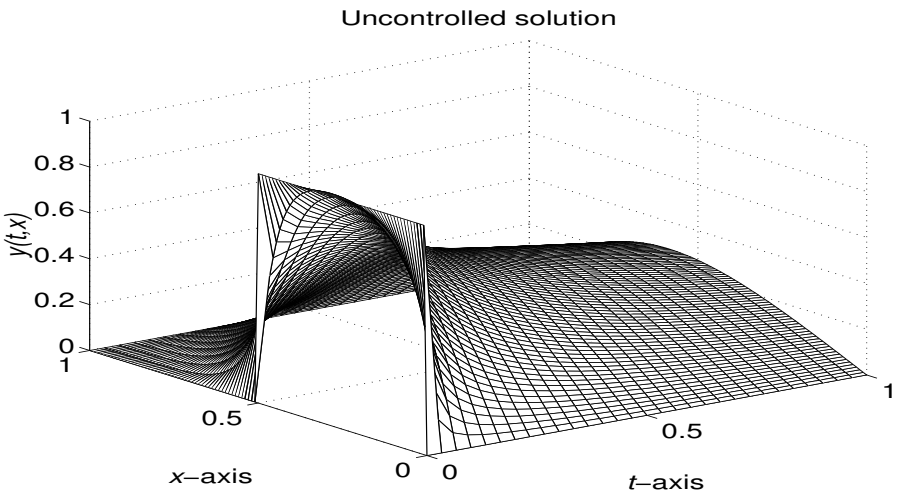

FIGURE 9: $h = \frac{1}{30}$.

The desired state now is given by $z(t) = t$, so that now the control law (20) is used. In Fig. 10 a comparison between the continuous and the discrete controller is shown. Due to implicit treatment of the nonlinear terms the continuous controller has better stability properties in the transient phase ($t \in [0, 0.05]$) than the discrete controller. As Fig. 11 shows the discrete controller converges exponentially. As further numerical experiments showed the order increases for increasing values of ρ. For the continuous controller we observe a behavior similar to that in Run 2.

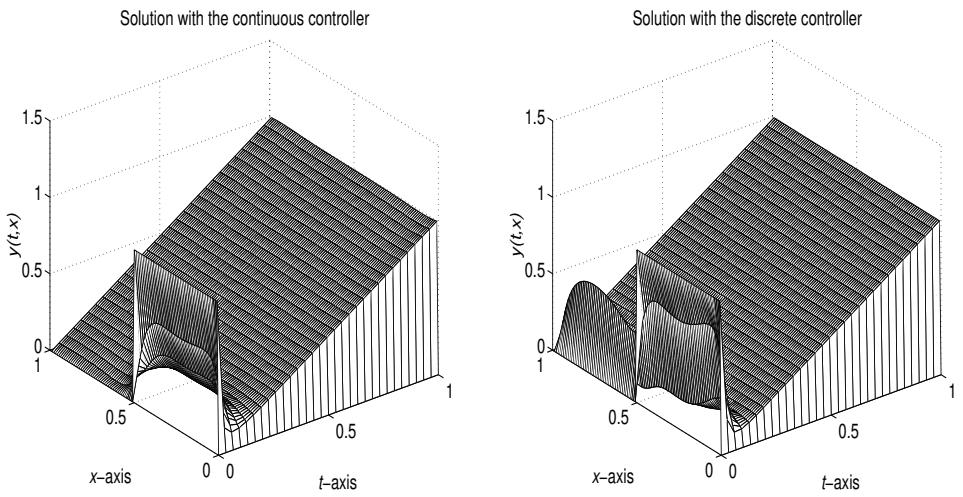

FIGURE 10: Solution for the continuous and the discrete controller, $\rho = 1$, $h = \frac{1}{30}$.

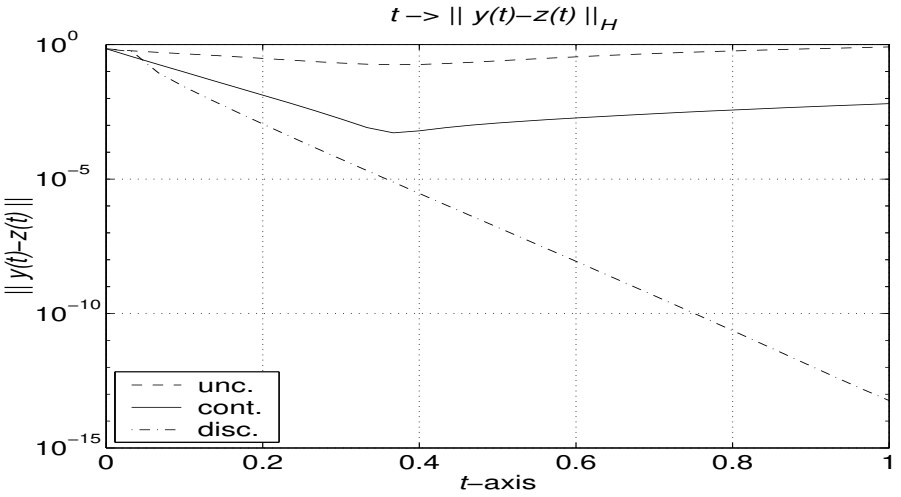

FIGURE 11: $\rho = 1$, $h = \frac{1}{30}$.

For larger ρ we obtain the results presented in Fig. 12 and Fig. 13. The discrete controller now behaves slightly more stable in the transient phase.

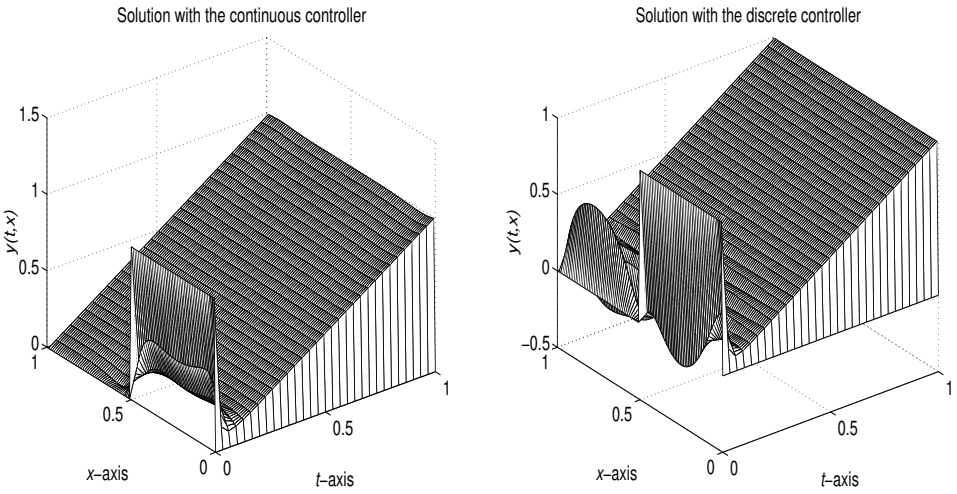

FIGURE 12: Solution for the continuous and the discrete controller, $\rho = 1.9$, $h = \frac{1}{30}$.

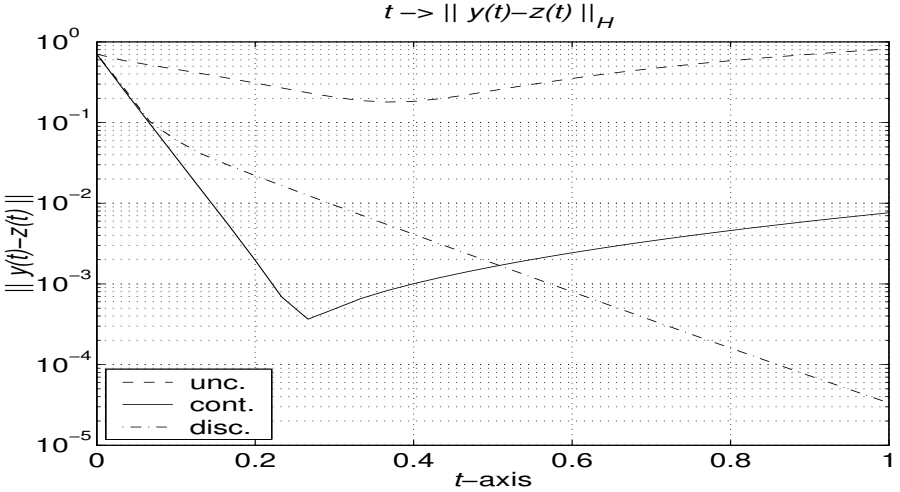

FIGURE 13: $\rho = 1.9$, $h = \frac{1}{30}$.

Run 4 (Boundary control – boundary observation)**.** The control goal in this example is to steer the derivative $y'(t,0)$ of the solution y of the Burgers equation to a prescribed value z controlling the spatial derivative of the state along the spatial boundary utilizing control actions $u(t_k, 1)$. The instantaneous control problem at time slice t_k is given by

$$\textbf{(R)} \quad \begin{cases} \min J(y,u)) \;=\; \frac{1}{2}|y'(0)|^2 \,+\, \frac{\alpha}{2}|u|^2 \\ \text{s.t.} \\ \quad y - \kappa y'' = R_{k-1} \quad \text{in } \Omega, \\ \qquad y(0) = 0, \\ \qquad y'(1) = u, \end{cases}$$

where $R_{k-1} = y^{k-1} + hb(y^{k-1})$ and $\kappa = \nu h$. Let $R_{k-1} \in H$. Then, for every control $u \in \mathbb{R}$ there exists a unique function $y = y(u) \in H^2(\Omega)$ satisfying the linear boundary value problem in **(R)**. Therefore, it makes sense to define the functional

$\hat{J}(u) = J(y(u), u)$. In order to characterize its gradient at u choose $y(u)$ as above and introduce the auxiliary function λ as the unique solution of the boundary value problem

$$\begin{aligned} \lambda - \kappa\lambda'' &= 0 && \text{in } \Omega, \\ \lambda(0) &= -\tfrac{1}{\kappa} y'(0), \\ \lambda'(1) &= 0. \end{aligned} \tag{45}$$

Then we have

$$\nabla \hat{J}(u) \;=\; \alpha u \,-\, \kappa\lambda(1)$$

and we are in the position to apply the instantaneous control method. To relate it to a more practical framework we now assume that the derivative $y'(0)$ can be measured at the boundary of the spatial domain (which is a realistic assumption). Then Algorithm 1 can be formulated in feedback form as follows.

1. Measure the state information $y'(0)$.
2. Compute λ from (45).
3. Update the given control u_0: $u = (1 - \rho\alpha)u_0 + \rho\kappa\lambda(1)$.
4. Apply u to the system and goto 1.

Crucial in this algorithm is the choice of the descent parameter ρ. Since the optimization problem **(R)** has a quadratic objective and linear constraints the optimal value ρ^* can be computed exactly and is given by

$$\rho^* \;=\; \frac{|\nabla \hat{J}(u)|^2}{\alpha |\nabla \hat{J}(u)|^2 - \kappa \xi(1)}. \tag{46}$$

To provide the value $\xi(1)$ for a given control u in addition to the linear problems for y and λ two auxiliary problems have to be solved, namely

$$\begin{aligned} s - \kappa s'' &= 0 && \text{in } \Omega, \\ s(0) &= 0, \\ s'(1) &= \nabla \hat{J}(u) \end{aligned}$$

and

$$\begin{aligned} \xi - \kappa\xi'' &= 0 && \text{in } \Omega, \\ \xi(0) &= -\tfrac{1}{\kappa} s'(0), \\ \xi'(1) &= 0. \end{aligned}$$

In this example we choose $T = 5$, $\nu = 0.1$, $\Delta x = 0.1$, $h = \Delta x/2$, $\alpha = 10^{-6}$ and as initial condition $\phi(x) = \sin(\pi x)$. The desired value for the derivative $y'(0)$ is $z = 0$. As descent parameter in the gradient algorithm we use ρ^* given by (46). In Fig. 14 the instantaneous state as well as the instantaneous control is plotted.

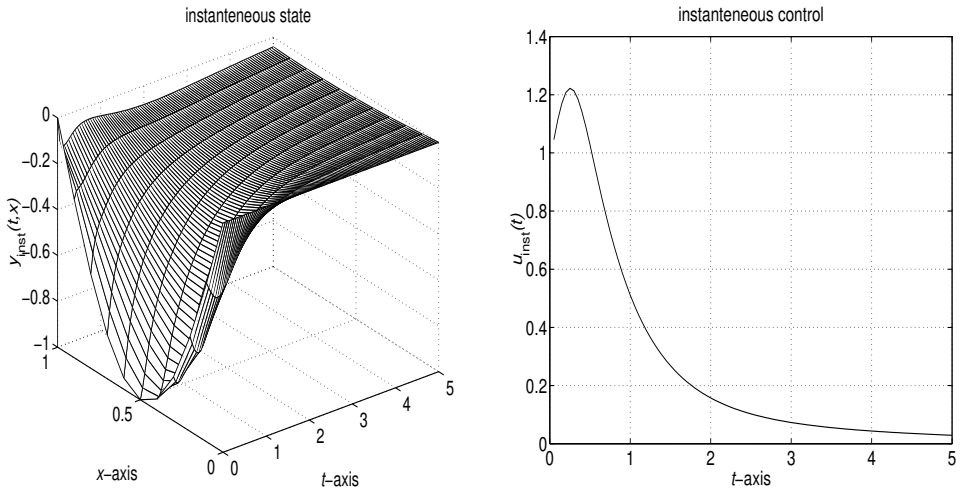

FIGURE 14.

In Fig. 15 the values of the derivative at $x = 0$ of the uncontrolled and the controlled solution is plotted as a function of time. It turns out that the control algorithm behaves as desired.

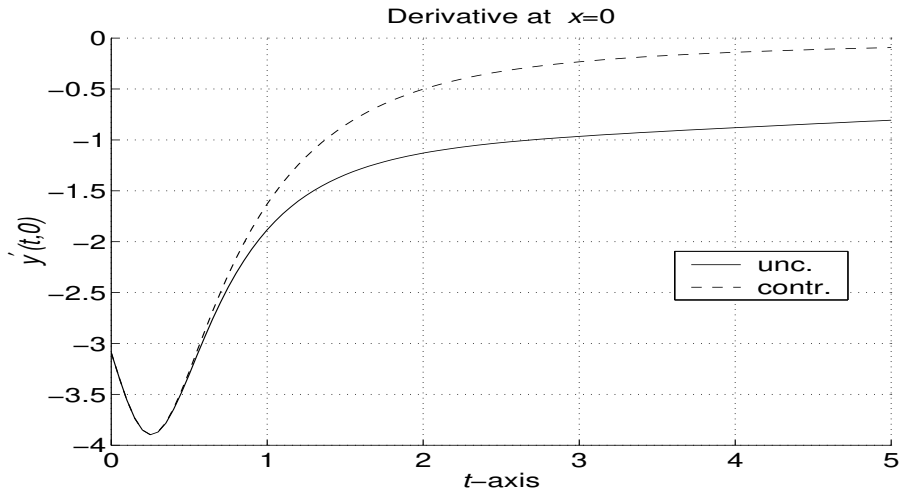

FIGURE 15.

APPENDIX A. PROOF OF THEOREMS 4 AND 5

Proof of Theorem 4. The proof follows the lines of the existence proof for the time-dependent Navier-Stokes equations given in [18, Chap. III]. In addition to the well understood analytical handling of the nonlinearity of the Burgers equation one in essence has to cope with the technical difficulties arising from non-local solution operator $B_h B_h$ applied to the nonlinearity of the Burgers equation. However, as will be demonstrated the smoothing properties of this operator and its sign enhance the stability properties of the Burgers equation.

Recall that ρ_0 is given by

$$\rho_0 = \min\Big(\frac{\sqrt{2}\nu^{3/2}}{4\sqrt{2}\nu^{3/2} + 4C_A\sqrt{h}\ \|\phi\|_H^2}, \frac{6\nu + 3d_H}{2\nu^2}\Big).$$

A.1. **Part 1: A-priori estimates via Galerkin-ansatz.** Let $\{\psi_i\}_{i=1}^\infty$ denote the eigenfunctions of the operator $(-\frac{d^2}{dx^2})^{-1}$. For $m \in \mathbb{N}$ define the discrete ansatz space by $V_m = \text{span}\,\{\psi_1, \ldots, \psi_m\} \subset V$. Set

$$y_m(t,x) = \sum_{i=1}^m y_i^m(t)\psi_i(x),$$

require $y_m(0,\cdot) \mapsto \phi$ in H and note that by the linearity of the operator B_h

$$\begin{aligned}(B_h y_m)(t,x) &= \sum_{i=1}^m y_i^m(t)(B_h\psi_i)(x) \text{ and}\\ (B_h(y_m y_m'))(t,x) &= \sum_{i,l=1}^m y_i^m(t)y_l^m(t)\,(B_h(\psi_i\psi_l'))\,(x)\end{aligned}$$

holds true. Inserting y_m into the state equation and using ψ_j, $j = 1, \ldots, m$, as test functions leads to the system of nonlinear differential equations for the vector $Y(t) = (y_1^m, \ldots, y_m^m)^t$ given next.

$$\begin{aligned}(47)\quad M\dot{Y} + \nu SY + \Big[\sum_{i,l=1}^m y_i^m(t)y_l^m(t)\int_\Omega \psi_i\psi_l'\psi_j\,dx\Big]_{j=1}^m\\ + \frac{\rho}{h}\tilde{M}Y + \rho\Big[\sum_{i,l=1}^m y_i^m(t)y_l^m(t)\int_\Omega B_h(\psi_i\psi_l')B_h(\psi_j)\,dx\Big]_{j=1}^m = 0,\\ Y(0) = \Phi = (\phi_1^0, \ldots, \phi_m^0)^t.\end{aligned}$$

Here, the matrix M denotes the mass matrix, S denotes the stiffness matrix and the matrix $\tilde{M}$ is defined by

$$\tilde{M} = (\tilde{m}_{il})_{i,l=1}^m, \quad \tilde{m}_{il} = \int_\Omega B_h\psi_i B_h\psi_l\,dx.$$

Since the mass matrix is regular, the nonlinear system (47) admits a solution in some maximal time interval $[0, t_m)$. It will follow from the a-priori estimates of the solution that this interval may be chosen as large as desired.

In order to obtain a-priori estimates test with the function y_m (i.e. scalar-multiply with Y in (47)). Applying Young's inequality leads to the differential inequality

$$\begin{aligned}(48)\quad \frac{1}{2}\frac{d}{dt}\|y_m\|_H^2 + \nu\|y_m\|_V^2 + \frac{\rho}{h}\|B_h y_m\|_H^2\\ \le \frac{\rho h}{2}\|B_h(y_m y_m')\|_H^2 + \frac{\rho}{2h}\|B_h y_m\|_H^2,\end{aligned}$$

for the function $t \mapsto \|y_m(t)\|_H^2$. Note that $w_m = B_h(y_m y_m')$ exists in V and is unique, since $y_m y_m'$ is an element of V'. Next, we estimate $\|w_m\|_H^2$. By definition of the operator B_h we have

$$\|w_m\|_H^2 + \nu h\|w_m\|_V^2 = \int_\Omega y_m y_m' w_m\,dx.$$

Utilizing $w_m \in L^\infty(\Omega)$, Agmon's interpolation inequality and again Young's inequality yields

$$\int_\Omega y_m y_m' w_m \, dx \le \frac{C_A^2}{4\sqrt{2\nu h}} \, \|y_m y_m'\|_{L^1}^2 + \nu h \, \|w_m\|_V^2 + \frac{1}{2} \, \|w_m\|_H^2.$$

Estimation of the L^1-norm gives

$$\|w_m\|_H^2 \le \frac{C_A^2}{2\sqrt{2\nu h}} \, \|y_m\|_H^2 \|y_m\|_V^2, \tag{49}$$

and using this estimate together with the differential inequality (48) leads to

$$\frac{1}{2}\frac{d}{dt} \, \|y_m\|_H^2 + \left(\nu - \frac{C_A^2 \rho \sqrt{h}}{2\sqrt{2\nu}} \, \|y_m\|_H^2 \right) \|y_m\|_V^2 + \frac{\rho}{2h} \, \|B_h y_m\|_H^2 \le 0. \tag{50}$$

The outermost addend on the left-hand-side can be estimated with

Lemma 11. For $y \in V$ let $w = B_h y$. Then $w \in V \cap H^3(\Omega)$ and $w'' \in V$. Moreover,

$$\|w\|_H^2 \ge \|y\|_H^2 - 2\nu h \, \|y\|_V^2.$$

Proof. The regularity claim follows from standard arguments. Thus, w'' can be utilized as test function in the equation $w - \nu h w'' = y$. This gives

$$\|w\|_V^2 + \nu h \, \|w\|_V^2 = -\int_\Omega w' y' \, dx$$

or, by using Young's inequality,

$$\|w\|_V^2 + 2\nu h \, \|w\|_V^2 \le \|y\|_V^2. \tag{51}$$

Moreover, the estimate

$$\|y\|_H^2 = \int_\Omega yw + \nu h y' w' \, dx \le \frac{1}{2} \, \|y\|_H^2 + \frac{1}{2} \, \|w\|_H^2 + \nu h \, \|y\|_V \|w\|_V$$

and (51) imply that

$$\|y\|_H^2 \le \|w\|_H^2 + 2\nu h \, \|y\|_V^2,$$

which is the claim. □

From (50) and Lemma 11 it now follows that

$$\frac{1}{2}\frac{d}{dt} \, \|y_m\|_H^2 + \left(\nu(1-\rho) - \frac{C_A^2 \rho \sqrt{h}}{2\sqrt{2\nu}} \, \|y_m\|_H^2 \right) \|y_m\|_V^2 + \frac{\rho}{2h} \, \|y_m\|_H^2 \le 0. \tag{52}$$

Since $y_m(0) \to \phi$ in V one may fix positive thresholds d_H and d_V such that

$$\|y_m(0)\|_H^2 \le 2 \, \|\phi\|_H^2 = d_H \text{ and } \|y_m(0)\|_V^2 \le 2 \, \|\phi\|_V^2 = d_V \text{ for all } m \in \mathbb{N}.$$

Then for all $0 < \rho \le \rho_0$ with ρ_0 from (15) there holds

$$\nu(1-\rho) - \frac{C_A^2 \rho \sqrt{h}}{2\sqrt{2\nu}} \, d_H > \frac{\nu}{2}. \tag{53}$$

We have already proved that (47) possesses a continuous solution on some time interval $[0, t_m)$. It shall be shown by contradiction that $t_m = T$. Suppose that $t_m < T$. Then,

$$\lim_{t \nearrow t_m} \|y_m(t)\|_H = \infty. \tag{54}$$

It follows from (54) and

$$\nu(1-\rho) - \frac{C_A^2\rho\sqrt{h}}{2\sqrt{2\nu}}\,\|y_m(0)\|_H^2 \geq \nu(1-\rho) - \frac{C_A^2\rho\sqrt{h}}{2\sqrt{2\nu}} d_H > \frac{\nu}{2},$$

that there exists a time $t_m^* < t_m$ and $t_m^* > 0$ such that

$$\nu(1-\rho) - \frac{C_A^2\rho\sqrt{h}}{2\sqrt{2\nu}}\,\|y_m(t_m^*)\|_H^2 = \frac{\nu}{2}$$

and

$$\nu(1-\rho) - \frac{C_A^2\rho\sqrt{h}}{2\sqrt{2\nu}}\,\|y_m(t)\|_H^2 > \frac{\nu}{2} \text{ for all } t \in [0, t_m^*).$$

Now (52) together with a continuity argument implies

$$\|y_m(t)\|_H^2 \leq \exp\Big(-\frac{\rho t}{h}\Big)\,\|y_m(0)\|_H^2 \leq \exp\Big(-\frac{\rho t}{h}\Big)\,d_H \text{ for all } t \in [0, t_m^*],$$

and this together with (53) in turn give

$$\begin{aligned}\frac{\nu}{2} &= \nu(1-\rho) - \frac{C_A^2\rho\sqrt{h}}{2\sqrt{2\nu}}\,\|y_m(t_m^*)\|_H^2 \\ &\geq \nu(1-\rho) - \frac{C_A^2\rho\sqrt{h}}{2\sqrt{2\nu}}\,\exp\Big(-\frac{\rho t}{h}\Big) d_H > \frac{\nu}{2},\end{aligned}$$

which is a contradiction. Therefore, $t_m = T$ and

$$\|y_m(t)\|_H^2 \leq \exp\Big(-\frac{\rho t}{h}\Big)\,d_H \text{ for all } t \in [0,T]. \tag{55}$$

Integration of (52) with respect to time then yields

$$\|y_m\|_{L^2(0,T;V)}^2 \leq \frac{2}{\nu}\,d_H \text{ and } \quad \|y_m\|_{L^2(0,T;H)}^2 \leq \frac{2h}{\rho}\,d_H. \tag{56}$$

Utilizing the estimates derived above straightforward estimation yields for the time derivative the estimate

$$\|(y_m)_t\|_{L^2(0,T;V')}^2 \leq\; C(\nu)\Big(1+\frac{\rho}{h}\Big)\,d_H. \tag{57}$$

Collecting the previous results one has

a) For every $T > 0$ the sequence $\{y_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H)$ as well as in $L^2(0,T;V)$ independent of the parameter h and the dimension of the ansatz space m.
b) For every $T > 0$ the sequence $\{(y_m)_t\}_{m\in\mathbb{N}} \subset L^2(0,T;V')$ is bounded by a constant of size $\frac{\rho}{h}$ and independent of the dimension of the Ansatz space m.

A.2. **Part 2: Passage to the limit.** Now a) and b) from above are equivalent to $\{y_m\}_{m\in\mathbb{N}} \subset W(0,T;V)$ bounded and since $W(0,T;V) \hookrightarrow\hookrightarrow L^2(0,T;H)$ and $W(0,T;V) \hookrightarrow C(0,T;H)$ one concludes convergence of a subsequence, again denoted by $\{y_m\}_{m\in\mathbb{N}}$ weak in $W(0,T;V)$, weak-star in $L^\infty(0,T;H)$ and strong in $L^2(0,T;H)$ to a function $y \in W(0,T;V)$. Thus, passing to the limit in the linear terms is straightforward, as is for the non-linear term of the Navier-Stokes equations due to the strong convergence of the subsequence in $L^2(0,T;H)$, see [18]. It finally remains to establish

$$\int_0^T \int_\Omega B_h(y_m - y)\chi(t) B_h\psi_j\,dxdt \to 0 \quad (m \to \infty)$$

and

$$\int_0^T \int_\Omega B_h(y_m y_m' - yy')B_h w_j \chi(t)\,dxdt$$
$$= \int_0^T \int_\Omega (y_m y_m' - yy')B_h B_h w_j \chi(t)\,dxdt \to 0 \quad (m \to \infty),$$

where $\chi \in C_0^\infty(0,T)$. For the first term the boundedness of B_h and the strong convergence of $\{y_m\}_{m\in\mathbb{N}}$ in $L^2(0,T;H)$ gives the result. As $y_m \to y$ and $y_m' \rightharpoonup y'$ in $L^2(0,T;H)$, one concludes $y_m y_m' \rightharpoonup yy'$ in $L^2(0,T;H)$ for $m \to \infty$. Since $B_h B_h w_j \chi(t) \in L^2(0,T;H)$, the convergence of the second integral follows. Uniqueness of the solution is an immediate consequence of inequality (52). This completes the proof of Theorem 4. □

Remark 5. In the proof of Theorem 4 only $\phi \in H$ is required.

Proof of Theorem 5. It is sufficient to establish a-priori estimates for the solution of (16) in the appropriate norms. To begin with denote by $y = y_m$ the solution of (47).

Lemma 12. Let $v = B_h y$ and $w = B_h v$. Then

$$-\int_\Omega B_h y\, B_h y''\,dx = \|y\|_V^2 - \nu h \int_\Omega (v'' + w'')\, y''\,dx$$

and

$$\|v''\|_H^2,\ \|w''\|_H^2 \le \frac{1}{2\nu h}\ \|y\|_V^2. \tag{58}$$

Proof. From the definition of v and w infer $v \in H^3(\Omega) \cap V$ and $w \in H^5(\Omega)^2 \cap V$. Integration by parts gives the first part of the claim. To obtain the second claim test the equation for v with $-v''$. This gives

$$\|v\|_V \le \|y\|_V \text{ and } \|v''\|_H^2 \le \frac{1}{2\nu h}\ \|y\|_V^2.$$

Since the the same estimate holds with v replaced by w and y replaced by v the lemma is proved. □

Now we test (16) with $-y''$ to obtain

$$\frac{1}{2}\frac{d}{dt}\ \|y\|_V^2 + \nu\ \|y''\|_H^2 - \int_\Omega yy'\,y''\,dx = \tag{59}$$
$$\frac{\rho}{h}\int_\Omega B_h y B_h y''\,dx + \rho\int_\Omega B_h(yy')B_h y''\,dx.$$

Applying Hölder's, Agmon's and Young's inequality we obtain

$$\int_\Omega yy'y''\,dx \le \frac{\nu}{6}\ \|y''\|_H^2 + \frac{3C_A^2}{\nu}\ \|y\|_H\|y\|_V^3. \tag{60}$$

From Lemma 12 we conclude

$$\frac{\rho}{h}\int_\Omega B_h y B_h y''\,dx \le \frac{\rho}{h}\ \|y\|_V^2 + \frac{\nu}{6}\ \|y''\|_H^2 + \frac{3\rho^2}{4\nu h}\ \|y\|_V^2. \tag{61}$$

Utilizing Hölder's and Young's inequality it follows that

$$\rho\int_\Omega B_h(yy')B_h y''\,dx \le \frac{3\rho^2}{\nu}\ \|B_h(yy')\|_H^2 + \frac{\nu}{6}\ \|y''\|_H^2. \tag{62}$$

Since

$$\|B_h(yy')\|_H^2 \leq \frac{1}{2\nu h}\,\|yy'\|_{V'}^2 \leq \|y\|_H^2\|y\|_V^2 \tag{63}$$

holds, we infer from the (60), eqrefeinschub2 and (62) that

$$\frac{1}{2}\frac{d}{dt}\,\|y\|_V^2 + \frac{\nu}{2}\,\|y''\|_H^2 + \frac{\rho}{h}\left(1 - \frac{3\rho}{4\nu} - \frac{3\rho}{2\nu^2}\,d_H\right)\|y\|_V^2 \leq \frac{3C_A^2}{\nu}\,\|y\|_H\|y\|_V^3. \tag{64}$$

By (15) we have $1 - 4\rho - \frac{\rho}{2\nu^2}d_H > \frac{1}{2}$ so that Gronwall's lemma applied to the function $v(t) = \exp(\frac{\rho t}{h})\,\|y(t)\|_V^2$ gives

$$\|y\|_V^2 \quad \leq \quad d_V \exp\Big(-\frac{\rho t}{h}\Big)\,\exp\Big(\frac{2C_A^2}{\nu}\,\|y\|_{L^2(0,T;H)}\|y\|_{L^2(0,T;V)}\Big).$$

This is the first inequality in (18) and implies the second one. The third one follows from integrating (64). Finally the fourth inequality in (18) can be obtained by testing (47) with y_t. As a consequence of the a-priori estimates for $\{y_m\}_{m\in\mathbb{N}}$ the unique solution of (16) is an element of $H^{2,1}(Q)$. □

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