A NOTE ON SIMPLICIAL FUNCTORS AND MOTIVIC HOMOTOPY THEORY

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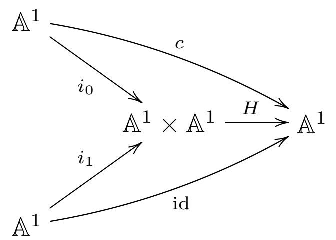
Abstract. We construct models for the motivic homotopy category based on simplicial functors from smooth schemes over a field to simplicial sets. These spaces are homotopy invariant and allow an easier characterization of the fibrant objects as the motivic models are obained by a smaller Bousfield localization from an objectwise structure.

Introduction

In this note, we study certain simplicial functors as an alternative for simplicial presheaves in the construction of the motivic homotopy category. Along the way we give a terse introduction to motivic homology and cohomology and its connection to motivic homotopy theory in order to motivate the technical effort necessary for the setup of the presented results. A short overview of models for the motivic homotopy category based on simplicial presheaves is given and by using a method of Beke we show that there exists an infinite number of them. This serves as a list of examples of model structures lifting to enriched simplicial presheaves. An enriched simplicial presheaf is a simplicial functor from a category of schemes enriched over simplicial sets to the category of simplicial sets enriched over itself. Considering enriched simplicial presheaves instead of simplicial presheaves seems to be quite natural in the spirit of motivic homotopy theory. For example there is a naive homotopy contracting the affine line in the category of schemes. More precisely, for any constant map c there exists a morphism H of smooth schemes over a field, such that the diagram



commutes. The simplicial presheaf represented by $\mathbb{A}^1$ resists to be weakly equivalent to the point until it is finally forced to be weakly contractible by Bousfield localization. In contrast to this the enriched simplicial presheaf represented by $\mathbb{A}^1$ is objectwise contractible (cf. Corollary 3.4). Hence the motivic models can be obtained without the $\mathbb{A}^1$-contracting Bousfield localization.

In the first section we recall the definition of motivic homology and cohomology through an analogy to singular homology and cohomology in algebraic topology. In the second section, the $\mathbb{A}^1$-local injective, projective and intermediate model structures on the category of simplicial presheaves are presented. It is shown that an infinite numbers of non-intermediate model structures with the same weak equivalences exist. In the third section the category of enriched simplicial presheaves is

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introduced and a few properties of enriched simplicial presheaves are verified. Eventually, in the fourth section we construct model structures on enriched simplicial presheaves and discuss some of their properties.

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1. Unstable motivic cohomology

In this section we recall some notations which are used in the sequel. One may regard this section as a brief and topologically flavoured introduction to motivic homology and cohomology.

Throughout this section let G be an abelian group and k be a field. We consider the homological viewpoint first. Motivic homology with coefficients in G consists of a functor

$$H^{\mathrm{mot}}_{p,q}(-,G) : \mathcal{S}\mathrm{m}/k \to \mathrm{Ab}$$

for every pair (p,q) of integers from the category $\mathcal{S}\mathrm{m}/k$ of smooth and separated schemes of finite type over k to the category Ab of abelian groups (one reason for considering only smooth schemes is the Homotopy Purity theorem [MV99, 2.23]).

Motivic homology is strongly related to singular homology of topological spaces. Given an integer p, the p-th singular homology group $H_p(X)$ of a topological space X with coefficients in G is defined as the p-th homology group of the chain complex

$$\ldots \to G \otimes \mathbb{Z}[\hom(\Delta^n, X)] \to \ldots \to G \otimes \mathbb{Z}[\hom(\Delta^0, X)] \to 0$$

with the usual alternating sums as boundary maps. The only topological input to this singular chain complex besides the space X itself is given by the topological cosimplicial object $\Delta^{(\text{-})} : \Delta \to \mathcal{T}\mathrm{op}$ determined by the standard simplices.

An algebraic analogue to Δ^p is given by the object

$$\boldsymbol{\Delta}^p = \operatorname{Spec} k[X_0, \ldots, X_p]/(1 - \textstyle\sum X_i),$$

which is isomorphic to $\mathbb{A}^p$ and $\boldsymbol{\Delta}^{(\text{-})}$ induces a cosimplicial object in $\mathcal{S}\mathrm{m}/k$. Therefore a first attempt to define motivic homology would be to mimic the topological construction with the algebraic cosimplicial object in place of the topological one. However, this naive approach does not work: one notes for example that

$$\hom(\operatorname{Spec}\mathbb{Q}, \operatorname{Spec}\mathbb{Q}(\sqrt{2})) \cong \hom(\mathbb{Q}(\sqrt{2}), \mathbb{Q}) = \emptyset$$

but it is not reasonable that $H^{\mathrm{mot}}_p(\mathbb{Q}(\sqrt{2}), \mathbb{Z})$ is zero and $H^{\mathrm{mot}}_p(\mathbb{Q}, \mathbb{Z}) = \mathbb{Z}$.

The right way to define motivic homology is by using an analogue of the topological Dold-Thom theorem which was discovered by Suslin and Voevodsky [SV96]. For a connected topological space X, there is a weak homotopy equivalence

$$\mathrm{SP}_\infty(X) \to |\mathbb{Z}[\hom(\Delta^{(\text{-})}, X)]|$$

where the right hand side is a realized simplicial set and where $\mathrm{SP}_\infty(X)$ denotes the infinite symmetric product of X, i.e. the colimit of $\mathrm{SP}_n(X) = X_+ \wedge \ldots \wedge X_+/\Sigma_n$ with respect to the structure maps $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, *)$ [DT58]. By interpreting this relation algebraically one defines for a connected scheme $D \in \mathcal{S}\mathrm{m}/k$ and an arbitrary $X \in \mathcal{S}\mathrm{m}/k$ the set $\mathrm{elCor}(D, X)$ of elementary correspondences from D to X as the set of closed and irreducible subsets M of the fiber product $D \times_k X$ such that the canonical map of schemes

$$M_{\mathrm{red}} \subseteq D \times_k X \to D$$

is finite and surjective. The graph of a morphism gives an injective but not surjective map $\hom(D, X) \to \mathrm{elCor}(D, X)$. This makes use of the separability of X.

The above mentioned problem is solved since $\mathrm{elCor}(\mathrm{Spec}\,\mathbb{Q}, \mathrm{Spec}\,\mathbb{Q}(\sqrt{2})) = *$. One writes $\mathrm{Cor}_G(D, X)$ instead of $G \otimes \mathbb{Z}[\mathrm{elCor}(D, X)]$ and omits the reference to the group if integer coefficients are used. This construction is in fact functorial [Voe02].

Set

$$H_p^{\mathrm{mot}}(X, G) = H_p(\mathrm{Cor}_G(\Delta^{(\text{-})}, X))$$

for an integer p and a scheme X of $\mathcal{S}\mathrm{m}/k$. The following theorem of Suslin and Voevodsky underlines the tenability of this definition.

Theorem 1.1 ([SV96])**.** *Let $k = \mathbb{C}$, $X \in \mathcal{S}\mathrm{m}/k$ and l be a prime. Then*

$$H_p^{mot}(X, \mathbb{Z}_l) \cong H_p(X(k), \mathbb{Z}_l)$$

where $X(k)$ denotes the k-rational points of X endowed with the subspace topology.

The Dold-Kan correspondence [Dol58, Theorem 1.9] is an equivalence of categories

$$N : \mathrm{sAb} \rightleftarrows \mathrm{Ch}_{\geq 0}(\mathrm{Ab}) : K$$

with $\pi_*(-, 0)$ corresponding to $H_*(-)$. This expresses the singular homology of a topological space X in the homotopy category of pointed topological spaces by

$$H_p(X) = \Big[\,|\Delta^p/\partial\Delta^p|, |\tilde{\mathbb{Z}}[\hom(\Delta^{(\text{-})}, X_+)]|\,\Big]_{\mathcal{T}\mathrm{op}_*}$$

where $\tilde{\mathbb{Z}}[-] : \mathrm{s}\mathcal{S}\mathrm{et}_* \to \mathrm{sAb}$ is the left adjoint $\tilde{\mathbb{Z}}[* \to A] = \mathbb{Z}[A]/\mathbb{Z}[*]$ of the functor $\tilde{\mathbb{U}}[-]$ which considers a simplicial abelian group as a simplicial set pointed by zero. The usual realization functor $|-| : \mathrm{s}\mathcal{S}\mathrm{et}_* \to \mathcal{T}\mathrm{op}_*$ is part of an adjunction whose unpointed version $|-| : \mathrm{s}\mathcal{S}\mathrm{et} \rightleftarrows \mathcal{T}\mathrm{op} : \mathrm{Sing}$ is obtained by the following standard lemma on left Kan extensions [Bor94a].

Lemma 1.2 (Adjunction Lemma)**.** *Let $\mathcal{D}$ be an essentially small category, $\mathcal{C}$ a cocomplete category and $c : \mathcal{D} \to \mathcal{C}$ a functor. There exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\text{Yoneda}} & \mathrm{Pre}(\mathcal{D}) \\ & {\scriptstyle c}\searrow & \downarrow{\scriptstyle |-|} \\ & & \mathcal{C} \end{array}$$

and an adjunction $|-| : \mathrm{Pre}(\mathcal{D}) \rightleftarrows \mathcal{C} : \mathrm{Sing}$ with $\mathrm{Sing}(X) = \hom(c(-), X)$.

This Lemma fails to be applied directly to the case $c = \Delta^{(\text{-})}$ since the category $\mathcal{S}\mathrm{m}/k$ is not cocomplete. A canonical way to complete and cocomplete a category $\mathcal{C}$ is by considering the category $\mathrm{Pre}(\mathcal{C})$ of presheaves on it. Although one could describe motivic homology analogously to the topological case in some homotopy category of Pre_* [MV99, Proposition 3.14], one enlarges the category of spaces again by embedding it constantly in the simplicial direction into the category sPre of simplicial presheaves since it is more natural to develop a homotopy theory there. There is a diagram of full subcategories

$$\mathcal{S}\mathrm{m}/k \to \mathrm{Pre} \to \mathrm{sPre} \leftarrow \mathrm{s}\mathcal{S}\mathrm{et}$$

where the category of simplicial sets is embedded constantly in the presheaf direction. By applying the Adjunction Lemma to the (embedded) algebraic cosimplicial object $c = \Delta^{(\text{-})}$, one obtains an adjunction

$$|-| : \mathrm{s}\mathcal{S}\mathrm{et}_* \rightleftarrows \mathrm{sPre}_* : \mathrm{Sing}.$$

To describe motivic homology in a suitable homotopy category of pointed simplicial (pre)sheaves, Morel and Voevodsky developed the motivic homotopy category [MV99]. It is the homotopy category, respectively, of several model structures on the category sPre. These motivic model structures are discussed in the next section. For all these models, the above adjunction $(|-|, \mathrm{Sing})$ is a Quillen adjunction. Motivic homology can be expressed as

$$\begin{aligned} H_p^{\mathrm{mot}}(X) &= H_p(\mathrm{Cor}(\Delta^{(\text{-})}, X)) \\ &= \big[\,\Delta^p/\partial\Delta^p\,,\,\mathrm{Cor}(\Delta^{(\text{-})}, X)\,\big]_{\mathrm{s}\mathcal{S}\mathrm{et}_*} \\ &= \Big[\,\Delta^p/\partial\Delta^p\,,\mathrm{Sing}(\tilde{\mathrm{C}}\mathrm{or}(-, X_+))\Big]_{\mathrm{s}\mathcal{S}\mathrm{et}_*} \\ &= \Big[\,S_s^p\,, \tilde{\mathrm{C}}\mathrm{or}(-, X_+)\Big]_{\mathrm{sPre}_*} \end{aligned}$$

where $S_s^p = |\Delta^p/\partial\Delta^p|$ and $\tilde{\mathrm{C}}\mathrm{or}(D, * \to X) = \mathrm{Cor}(D, X)/\mathrm{Cor}(D, *)$. The usual smash product gives rise to a closed symmetric monoidal structure on the category sPre_* and one observes that $S_s^p \simeq S_s^1 \wedge \ldots \wedge S_s^1$. Besides the simplicial sphere S_s^1 with rational points $S_s^1(\mathbb{C}) \simeq S^1 \simeq S_s^1(\mathbb{R})$ there is another reasonable sphere which captures algebraic phenomena. This algebraic sphere S_t^1 is defined as the scheme $\mathbb{A}^1 \setminus \{0\}$ pointed by 1. Note that $S_t^1(\mathbb{C}) \simeq S^1$, $S_t^1(\mathbb{R}) \simeq S^0$ and $S_s^1 \wedge S_t^1 \simeq (\mathbb{P}^1, \infty)$ [Voe98, Lemma 4.1]. This leads to the bi-indexed definition

$$H_{p,q}^{\mathrm{mot}}(X) = \Big[\,S_s^{p-q} \wedge S_t^q\,,\,\tilde{\mathrm{C}}\mathrm{or}(-, X_+)\,\Big]_{\mathrm{sPre}_*}$$

of motivic homology. Smashing with a negative exponent sphere on the left hand side should be interpreted as smashing with the associated positive one on the right.

Now we focus on the cohomological viewpoint. The p-th singular cohomology group of a topological space X consists of the group

$$H^p(X) = \Big[\,X_+, |\,\tilde{\mathbb{Z}}[\hom(\Delta^{(\text{-})}, S^p)]\,|\,\Big]_{\mathcal{T}\mathrm{op}_*}$$

where the latter space is an Eilenberg-MacLane space $K(\mathbb{Z}, p)$. An analogous and appropriate definition of motivic cohomology requires a change in the notation of correspondences since the simplicial sphere is not a scheme.

The category SmCor/k has the same objects as $\mathcal{S}\mathrm{m}/k$ and morphism groups $\mathrm{Cor}(D, X) = \mathbb{Z}[\cup \mathrm{elCor}(D_i, X)]$ where the D_i denote the connected components of D. The cruicial point is to define the composition. A detailed construction can be found in [Voe02]. The graph of a morphism defines a functor $\Gamma : \mathcal{S}\mathrm{m}/k \to \mathrm{SmCor}/k$. Let PST be the abelian category of additive functors from the category $\mathrm{SmCor}/k^{\mathrm{op}}$ to the category of abelian groups. There is a diagram of full additive subcategories

$$\mathrm{SmCor}/k \to \mathrm{PST} \to \mathrm{sPST} \leftarrow \mathrm{sAb}$$

as above. The category PST is symmetric monoidal closed with respect to the canonical tensor product existing by general results on enriched categories [Day70]. The category sPST is also symmetric monoidal closed by taking the tensor product of PST degreewise. Applying the Adjunction Lemma to $c(U, [n]) = \mathrm{Cor}(-, U) \otimes \mathbb{Z}[\Delta^n]$ yields a symmetric monoidal adjunction

$$\tilde{\mathbb{Z}}_{\mathrm{tr}} : \mathrm{sPre}_* \rightleftarrows \mathrm{sPST} : \tilde{\mathbb{U}}_{\mathrm{tr}}$$

where the right adjoint is given by $\tilde{\mathbb{U}}_{\mathrm{tr}}(X) = \tilde{\mathbb{U}}[-] \circ X \circ \Gamma$. The isomorphism $\tilde{\mathrm{C}}\mathrm{or}(-, X_+) \cong \tilde{\mathbb{U}}_{\mathrm{tr}}\tilde{\mathbb{Z}}_{\mathrm{tr}}(X_+)$ gives

$$H_{p,q}^{\mathrm{mot}}(X) = \Big[\,S_s^{p-q} \wedge S_t^q\,,\,\tilde{\mathbb{U}}_{\mathrm{tr}}\tilde{\mathbb{Z}}_{\mathrm{tr}}(X_+)\,\Big]_{\mathrm{sPre}_*}$$

and one defines motivic cohomology (for $p \geq q$ and $q \geq 0$) as

$$H^{p,q}_{\mathrm{mot}}(X) = \Big[\,X_+\,,\,\tilde{\mathbb{U}}_{\mathrm{tr}}\tilde{\mathbb{Z}}_{\mathrm{tr}}(S_s^{p-q} \wedge S_t^q)\,\Big]_{\mathrm{sPre}_*}$$

where the latter space is a motivic Eilenberg-MacLane space $K(\mathbb{Z}, q)[p]$ in sPre_*.

To get rid of the restriction on the indices p, q and to describe the relation to an approach to motivic cohomology using chain complexes [Voe00, MVW06], one defines a model structure on the abelian category sPST by detecting the weak equivalences and the fibrations via the right adjoint $\tilde{\mathbb{U}}_{\mathrm{tr}}$ in the $\mathbb{A}^1$-local projective model structure on sPre_* [Voe07, RØ08]. The adjunction $(\tilde{\mathbb{Z}}_{\mathrm{tr}}, \tilde{\mathbb{U}}_{\mathrm{tr}})$ becomes a Quillen adjunction and one has

$$\begin{aligned} H^{p,q}_{\mathrm{mot}}(X) &= \Big[X_+ , \tilde{\mathbb{U}}_{\mathrm{tr}}\tilde{\mathbb{Z}}_{\mathrm{tr}}(S^{p-q}_s \wedge S^q_t)\Big]_{\mathrm{sPre}_*} \\ &= \Big[\tilde{\mathbb{Z}}_{\mathrm{tr}}(X_+), \tilde{\mathbb{Z}}_{\mathrm{tr}}(S^{p-q}_s \wedge S^q_t)\Big]_{\mathrm{sPST}} \\ &= \Big[\tilde{\mathbb{Z}}_{\mathrm{tr}}(X_+), \tilde{\mathbb{Z}}_{\mathrm{tr}}(S^{p-q}_s) \otimes \tilde{\mathbb{Z}}_{\mathrm{tr}}(S^q_t)\Big]_{\mathrm{sPST}.} \end{aligned}$$

Tensoring with a negative exponent sphere on the right hand side could be interpreted as tensoring with the associated positive one on the left which gives a definition of motivic cohomology without a restriction on the indices.

The category $\mathrm{Ch}_{\geq 0}(\mathrm{PST})$ of non-negative chain complexes of preasheaves with transfers can be equipped with the usual projective model structure where weak equivalences are the quasi-isomorphisms and the fibrations are the epimorphisms in positive degrees [CH02]. This model structure may be localized by the standard technique of Bousfield localization in the same way as it is done to construct the $\mathbb{A}^1$-local projective model structure on sPre_* from the projective one [Voe07]. The outcome is the $\mathbb{A}^1$-local projective model structure on $\mathrm{Ch}_{\geq 0}(\mathrm{PST})$ and each of the adjunctions (N, K) and (K, N) of the Dold-Kan correspondence

$$N : \mathrm{sPST} \rightleftarrows \mathrm{Ch}_{\geq 0}(\mathrm{PST}) : K$$

is a Quillen equivalence. The category $\mathrm{Ch}^-(\mathrm{PST}) = \mathrm{Ch}_+(\mathrm{PST})$ of bounded above cochain complexes of presheaves with transfers admits a $\mathbb{A}^1$-local projective monoidal model structure constructed analogously to the case $\mathrm{Ch}_{\geq 0}(\mathrm{PST})$ [Voe00]. The associated homotopy category is denoted by $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}$. This is Voevodsky's category of effective motives [Voe00]. The inclusion functor

$$\iota : \mathrm{Ch}_{\geq 0}(\mathrm{PST}) \to \mathrm{Ch}^-(\mathrm{PST})$$

is well defined on homotopy classes of maps and one has

$$\begin{aligned} H^{p,q}_{\mathrm{mot}}(X) &= \Big[\tilde{\mathbb{Z}}_{\mathrm{tr}}(X_+), \tilde{\mathbb{Z}}_{\mathrm{tr}}(S^{p-q}_s) \otimes \tilde{\mathbb{Z}}_{\mathrm{tr}}(S^q_t)\Big]_{\mathrm{sPST}} \\ &= \hom_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}} \big(M_{\mathbb{Z}}(X_+), M_{\mathbb{Z}}(S^{p-q}_s) \otimes M_{\mathbb{Z}}(S^q_t)\big) \end{aligned}$$

where $M_{\mathbb{Z}} : \mathrm{sPre}_* \to \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}$ denotes the composed functor $\iota N \tilde{\mathbb{Z}}_{\mathrm{tr}}$ and maps a scheme X of $\mathcal{S}\mathrm{m}/k$ to its associated *motive* $M_{\mathbb{Z}}(X)$. The motive $M_{\mathbb{Z}}(X)$ of a scheme X is represented by the chain complex

$$\underset{}{\ldots} \xrightarrow{0} \underset{2}{\mathrm{Cor}(-,X)} \xrightarrow{\mathrm{id}} \underset{1}{\mathrm{Cor}(-,X)} \xrightarrow{0} \underset{0}{\mathrm{Cor}(-,X)} \to \underset{-1}{0}$$

of presheaves with transfers which is a cofibrant but not generally a fibrant object in $\mathrm{Ch}^-(\mathrm{PST})$. Tensoring a chain complex with the simplicial circle corresponds to a shift of the cochain complex to the left such that its former zeroth entry becomes the first. The chain complex associated to the motivic Eilenberg-MacLane space $\tilde{\mathbb{Z}}_{\mathrm{tr}}(S^{p-q}_s \wedge S^q_t)$ is given by

$$\ldots \to \underset{p-q+k}{\textstyle\bigotimes^q \mathrm{Cor}(-,\mathbb{A}^1 \setminus 0)/\mathbb{Z}} \to \ldots \to \underset{p-q}{\textstyle\bigotimes^q \mathrm{Cor}(-,\mathbb{A}^1 \setminus 0)/\mathbb{Z}} \to \underset{p-q-1}{0}$$

which is also cofibrant but not generally fibrant in the $\mathbb{A}^1$-local projective model structure on $\mathrm{Ch}^-(\mathrm{PST})$. A fibrant replacement for this cochain complex is called the motivic complex $\mathbb{Z}(q)[p]$ [MVW06, Definition 3.1].

2. Model structures for motivic homotopy theory

The main focus of this section is on giving a brief survey of model structures available for localizing $\mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$ to the motivic homotopy category $\mathcal{H}(k)$ in the sense of Morel and Voevodsky [MV99]. This will serve as a collection of examples for model structures lifting to the category of enriched simplicial presheaves in the following section.

Model structures on simplicial presheaves on a general category $\mathcal{C}$ have taken a developement starting at the latest with [BK72] and Jardine's extension [Jar87] of Joyal's model structure [Joy84] from simplicial sheaves to presheaves. In the following paragraphs we will always consider the case $\mathcal{C} = \mathcal{S}\mathrm{m}/k$ and continue to denote the category of simplicial presheaves on $\mathcal{S}\mathrm{m}/k$ by sPre. A morphism in sPre is called a local weak equivalence if it induces isomorphisms on sheaves of homotopy groups. These are defined to be the sheafification of the presheaf given by taking usual homotopy groups of the objectwise realization of a simplicial presheaf. The sheafification process requires a topology on $\mathcal{S}\mathrm{m}/k$ to make sense and the Nisnevich topology has turned out to be most convenient (the Homotopy Purity theorem does not hold when the Zariski topology is used [MV99, 2.23] and Algebraic K-Theory is not representable when the étale topology is used [MV99, 3.9]). We denote by W the class of all local weak equivalences. Let $\mathcal{H}_s(k)$ be the homotopy category of the localizer (sPre, W). A proof of the existence of $\mathcal{H}_s(k)$ through giving a model structure on sPre with W as weak equivalences can be found in [Jar87, Theorem 2.3]. The main reason for defining local weak equivalences this way is to repair the loss of geometric information in the transition $\mathcal{S}\mathrm{m}/k \hookrightarrow \mathrm{sPre}$. The problem can most easily be seen when passing from $\mathcal{S}\mathrm{m}/k$ to presheaves. Consider the usual glueing construction

$$\begin{array}{ccc} \mathbb{A}^1 \setminus 0 & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

of the projective line. The Yoneda embedding takes this diagram to a diagram of presheaves which is a pushout if and only if all presheaves X take it to a pullback diagram

$$\begin{array}{ccc} X(\mathbb{P}^1) & \longrightarrow & X(\mathbb{A}^1) \\ \downarrow & & \downarrow \\ X(\mathbb{A}^1) & \longrightarrow & X(\mathbb{A}^1 \setminus 0) \end{array} \tag{2.1}$$

of sets. Using the presheaf $X = \mathrm{Pic}(-)$ we have $X(\mathbb{P}^1) \cong \mathbb{Z}$ and $X(\mathbb{A}^1) \cong 0$, so the diagram (2.1) is not generally a pullback.

Taking local weak equivalences to be defined by a sufficiently fine topology (e.g. the Nisnevich topology) the induced morphism

$$\mathbb{A}^1 \coprod_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1 \to \mathbb{P}^1$$

of simplicial presheaves is a local weak equivalence as explained in [DHI04]. In fact Dugger, Hollander and Isaksen show in [DHI04, Theorem 6.2] that local weak equivalences emerge from objectwise weak equivalences by left Bousfield localization at the class of all hypercovers. At this point the category of smooth k-schemes still embedds into the homotopy category $\mathcal{H}_s(k)$, but the next step of localization will change this. Inspired by the topological situation and the naive homotopy equivalence $\mathbb{A}^1 \to *$ in $\mathcal{S}\mathrm{m}/k$ one makes the following definition.

Definition 2.1. An object $Z \in \mathrm{sPre}$ is called $\mathbb{A}^1$-local if for any $X \in \mathrm{sPre}$ the projection $X \times \mathbb{A}^1 \to X$ induces a bijection

$$\hom_{\mathcal{H}_s(k)}(X, Z) \to \hom_{\mathcal{H}_s(k)}(X \times \mathbb{A}^1, Z).$$

A morphism $X \to Y$ in sPre is called $\mathbb{A}^1$-local weak equivalence if for any $\mathbb{A}^1$-local object Z the induced map

$$\hom_{\mathcal{H}_s(k)}(Y, Z) \to \hom_{\mathcal{H}_s(k)}(X, Z)$$

is a bijection. Let $W_{\mathbb{A}^1}$ be the class of $\mathbb{A}^1$-local weak equivalences.

The following theorem is a starting point of motivic homotopy theory.

Theorem 2.2 (Morel, Voevodsky [MV99, Theorem 2.3.2])**.** *There is a proper simplicial cofibrantly generated model structure on* sPre *with $\mathbb{A}^1$-local weak equivalences as weak equivalances and monomorphisms as cofibrations.*

This model structure is called the $\mathbb{A}^1$-local injective model structure on simplicial presheaves and the associated *motivic homotopy category* is denoted by $\mathcal{H}(k)$. This model structure is a left Bousfield localization of Jardine's local injective model structure with respect to the class

$$C = \left\{ U \times \mathbb{A}^1 \xrightarrow{\mathrm{pr}} U \mid U \in \mathcal{S}\mathrm{m}/k \right\}$$

of morphisms in sPre. Hence a lot of work for proving the above theorem is done in [Jar87, Theorem 2.3]. As a consequence of the localization the affine line is forced to be contractible and all cylinders on an object X contract to X, but one should be warned that $X \times \mathbb{A}^1$ does not have to be a cylinder object in all of the following model structures.

The $\mathbb{A}^1$-local injective model structure is one end of a family of model structures which follow a more general pattern, outlined by Jardine [Jar06] as a generalization of Larusson's model structure [Lar04]. Before we discuss this general pattern we take a look at the families other end, the so-called $\mathbb{A}^1$-local projective model structure. It arises in the same manner as its injective analogue by Bousfield localization of a local projective structure [Bla01, Theorem 1.6].

Theorem 2.3 (Blander [Bla01, Theorem 2.3])**.** *The category* sPre *admits a proper simplicial cofibrantly generated model structure with $\mathbb{A}^1$-local weak equivalences as weak equivalences and morphmisms with the left lifting property with respect to objectwise acyclic fibrations as cofibrations.*

Generating ($\mathbb{A}^1$-local) projective cofibrations are given by the small class

$$S_0 := \left\{ U \times \partial\Delta^n \xrightarrow{\mathrm{id}_U \times i} U \times \Delta^n \mid U \in \mathcal{S}\mathrm{m}/k, n \geq 0 \right\}$$

of morphisms in sPre, i.e. the cofibrations cof_{proj} are exactly the maps with left lifting property with respect to all maps with the right lifting property with respect to all maps in S_0. Due to a smallness condition fulfilled by all simplicial presheaves the description of cof_{proj} can be made a little more explicit by saying that every cofibration is a retract of a transfinite composition of pushouts along the maps in S_0.

More generally any small class S of morphisms in sPre such that

$$S_0 \subseteq S \subseteq \{\text{monomorphisms}\}$$

gives rise to an *intermediate model structure* $\mathcal{M}_S$ by the following process. Define

$$\bar{S} := \{s \square i_n : B \times \partial\Delta^n \cup_{A \times \partial\Delta^n} A \times \Delta^n \hookrightarrow B \times \Delta^n \mid s \in S, n \geq 0\}$$

to be class of pushout products of the morphisms in S with the boundary inclusions of the standard simplices. The class of *S-cofibrations* cof_S is then given by the saturation $\mathrm{cof}_S = (\bar{S}\text{-inj})\text{-proj}$.

Theorem 2.4 (Jardine, [Jar06]). *The category* sPre *admits a proper simplicial cofibrantly generated model structure* $\mathcal{M}_S$ *with* $\mathbb{A}^1$*-local weak equivalences as weak equivalences and* S*-cofibrations as cofibrations.*

The model structure $\mathcal{M}_S$ is intermediate in the sense that $\mathrm{id}_{\mathrm{sPre}}$ gives a diagram

$$\mathcal{M}_{\text{projective}} \to \mathcal{M}_S \to \mathcal{M}_{\text{injective}}$$

ordered by the perspective of left Quillen equivalences.

An example of an intermediate model structure of particular importance is Isaksen's *flasque model structure* [Isa05]. Let $\mathcal{U}_I := \{U_i \to X\}_{i \in I}$ be a finite (possibly empty) family of monomorphisms in $\mathcal{S}\mathrm{m}/k$ and denote by $\mathcal{U}$ the coequalizer of the diagram

$$\coprod_{i,j} U_i \times_X U_j \rightrightarrows \coprod_i U_i$$

of simplicial presheaves. Define S_{flasque} to be the collection of all monomorphisms $\mathcal{U} \to X$ induced by finite families as above. The S_{flasque}-cofibrations are called flasque cofibrations and we get the following result of Isaksen [Isa05, Section 4.] as a corollary of the above theorem.

Theorem 2.5 (Isaksen). *The category* sPre *admits a proper simplicial cofibrantly generated model structure with* $\mathbb{A}^1$*-local weak equivalences as weak equivalences and flasque cofibrations as cofibrations.*

In [PPR09] Panin, Pimenov and Röndigs use another intermediate model structure which they call closed motivic model structure. It is a slight variation of the flasque model structure, built in the same way but just allowing finite families of closed embeddings as $\mathcal{U}_I$.

Different choices of the class of morphisms S may define the same intermediate model structure. Using a method of Beke [Bek08] based on the Quillen equivalence

$$\mathrm{sd} : \mathrm{s}\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{s}\mathcal{S}\mathrm{et} : \mathrm{Ex}$$

of subdivision we show that a countably infinite number of different model structures with $\mathbb{A}^1$-local weak equivalences as weak equivalences exists.

Lemma 2.6. *Let* $\mathrm{cof}_0 = \mathrm{cof}_{proj}$ *be the class of projective cofibrations. There exists a strictly descending sequence of classes*

$$\mathrm{cof}_0 \supset \mathrm{cof}_1 \supset \mathrm{cof}_2 \supset \dots$$

such that for any $n \geq 0$ *the class* cof_n *fits into a left proper cofibrantly generated model structure* $\mathcal{M}_n$ *with* $\mathbb{A}^1$*-local weak equivalences as weak equivalences and cofibrations* cof_n.

Proof. One starts with the projective model structure on sPre, i.e. with fibrations and weak equivalences being defined objectwise. The functors sd and Ex induce an adjunction

$$\mathrm{sd}_* : \mathrm{sPre} \rightleftarrows \mathrm{sPre} : \mathrm{Ex}_*$$

on simplicial presheaves which can be used to lift the projective model structure along the right adjoint using the Lifting Theorem [Hir03, Theorem 11.3.2.]. Since Ex_* preserves filtered colimits it remains to show that for all pushout squares

$$\begin{array}{ccc} \mathrm{sd}_*(X \times \Lambda^n_k) & \longrightarrow & Y \\ {\scriptstyle \mathrm{sd}_*(j)}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{sd}_*(X \times \Delta^n) & \longrightarrow & Z \end{array}$$

the morphism $\mathrm{Ex}_*(p)$ is a weak equivalence. This can be done by recognising that $\mathrm{sd}_*(j)$ is still an acyclic projective cofibration and that Ex_* preserves objectwise weak equivalences. By induction we obtain a sequence

$$\ldots \to \mathrm{sPre}_2 \xrightarrow{\mathrm{id}} \mathrm{sPre}_1 \xrightarrow{\mathrm{id}} \mathrm{sPre}_0 = \mathrm{sPre}_{\mathrm{proj}}$$

of left Quillen equivalences. The Bousfield localization gives the claimed Quillen equivalence of model structures $\mathcal{M}_n$ with $\mathbb{A}^1$-local weak equivalences using [Hir03, Theorem 3.3.20] by evaluating the derived left adjoint $L\,\mathrm{id}_{\mathrm{sPre}}$ at a class localizing from the projective to the $\mathbb{A}^1$-local projective model structure. By Beke's Theorem [Bek08, Proposition 2.3], for any $n \geq 0$, there exists a simplicial set X, such that $\mathrm{Ex}^n(X)$ is not fibrant, but $\mathrm{Ex}^{n+1}(X)$ is. As a constant simplicial presheaf X is therefore not fibrant in $\mathcal{M}_n$, but it is fibrant in $\mathcal{M}_{n+1}$. Hence this model structures are actually distinct. □

To summarize the results presented in this section the identity on sPre as left Quillen equivalence gives a totally ordered chain

$$\ldots \to \mathcal{M}_1 \to \mathcal{M}_0 \to \mathcal{M}_{\mathrm{proj}} \to \mathcal{M}_{\mathrm{closed}} \to \mathcal{M}_{\mathrm{flasque}} \to \mathcal{M}_{\mathrm{inj}}$$

of the mentioned model structures. Every representable object of $\mathcal{S}\mathrm{m}/k$ is cofibrant in each of these model structures but not fibrant in general. A model structure on the category sPre of simplicial presheaves induces a model structure on the associated pointed category sPre_* by detecting weak equivalences, cofibrations and fibrations with the forgetful functor [Hov99, Proposition 1.1.8].

3. Enriched simplicial presheaves

In this section we introduce the category $\mathcal{S}\mathrm{Pre}$ of *enriched simplicial presheaves* as an alternative for the category sPre of simplicial presheaves. The construction of $\mathcal{S}\mathrm{Pre}$ is based on categories enriched over simplicial sets. In a simplicial category $\mathcal{C}$ there are hom-simplicial sets $\mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A, B)$ instead of just hom-sets associated with any two objects, in a way compatible with an associative and unital composition. The 0-simplices of $\mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A, B)$ can be thought of as morphisms $A \to B$. The relation of being connected by a zig-zag of 1-simplices models a notation of *naive homotopy* depending on the enrichment. In the following we consider $\mathrm{s}\mathcal{S}\mathrm{et}$ as a simplicial category by

$$\mathrm{s}\mathcal{S}\mathrm{et}_{\mathrm{s}\mathcal{S}\mathrm{et}}(A, B)_n = \hom_{\mathrm{s}\mathcal{S}\mathrm{et}}(A \times \Delta^n, B)$$

and the naive homotopy relation turns out to be perfectly sensible in the sense that it coincides with the notation of left homotopy in the usual model structure on simplicial sets. This enrichment is natural in many aspects, for example it is given by the Yoneda embedding and the following straightforward lemma.

Lemma 3.1. *Let $\mathcal{C}$ be a category with finite products. Any cosimplicial object $c : \Delta \to \mathcal{C}$ with c_0 the terminal object of $\mathcal{C}$ gives rise to a simplicial category, which we also denote by $\mathcal{C}$, with underlying category $\mathcal{C}$ and*

$$\mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A, B)_n = \hom_{\mathcal{C}}(A \times c_n, B).$$

Proof. A map $\sigma : [m] \to [n]$ in Δ induces a map $\mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B)_n \to \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B)_m$ by assigning the composite

$$A \times c([m]) \xrightarrow{(\mathrm{pr}_1, c(\sigma)\circ \mathrm{pr}_2)} A \times c([n]) \xrightarrow{f} B$$

to $f \in \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B)_n$. Clearly $\mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B)(\mathrm{id}_{[n]}) = \mathrm{id}_{\mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B)_n}$ and one observes that for composable morphisms σ and τ in Δ the identity

$$\begin{aligned} \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B)(\tau\circ\sigma)(f) &= f \circ (\mathrm{pr}_1, (c(\tau\circ\sigma)\circ \mathrm{pr}_2)) \\ &= \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B)(\sigma)\circ \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B)(\tau)(f) \end{aligned}$$

holds and hence $\mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B)$ is in fact a simplicial set. The composition maps

$$c_{ABC} : \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(B,C) \times \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B) \to \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B), \quad (g,f) \mapsto g \circ (f, \mathrm{pr}_2)$$

are maps of simplicial sets and satisfy the relevant coherence diagrams [Bor94b, 6.9,6.10]. The underlying category $U\mathcal{C}$ has by definition the same objects as $\mathcal{C}$ and the hom-sets are given by

$$\begin{aligned} \hom_{U\mathcal{C}}(A,B) &:= \hom_{\mathrm{s}\mathcal{S}\mathrm{et}}(\Delta[0], \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B)) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{C}}(A,B)_0 \\ &\cong \hom_{\mathcal{C}}(A,B). \end{aligned}$$

The composition in $U\mathcal{C}$ is the same as composition in simplicial dimenstion 0 of the enriched category and therefore $U\mathcal{C} \cong \mathcal{C}$. □

By applying this lemma to the algebraic cosimplicial object Δ one obtains $\mathcal{S}\mathrm{m}/k$ as a simplicial category.

Definition 3.2. The category $\mathcal{S}\mathrm{Pre}$ of *enriched simplicial presheaves* is the category of simplicial functors from $\mathcal{S}\mathrm{m}/k^{\mathrm{op}}$ to $\mathrm{s}\mathcal{S}\mathrm{et}$, i.e. functors X assigning a simplicial set XU to any smooth k-scheme U and a morphism

$$\mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{S}\mathrm{m}/k}(U,V) \to \mathrm{s}\mathcal{S}\mathrm{et}_{\mathrm{s}\mathcal{S}\mathrm{et}}(XV, XU)$$

of simplicial sets to any pair of objects U, V compatible with composition.

The notation of naive homotopy in the simplicial category $\mathcal{S}\mathrm{m}/k$ is not completly convenient, but includes some reasonable aspects as for example

$$\mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{S}\mathrm{m}/k}(S^1_t, S^1_t)_*/\sim_{\mathrm{naive}}$$

equals the integers. A discussion of this naive homotopy relation in $\mathcal{S}\mathrm{m}/k$ can be found in section 2 of [Mor04].

The Adjunction Lemma 1.2 applied to the functor

$$c : \mathcal{S}\mathrm{m}/k \times \Delta \to \mathcal{S}\mathrm{Pre}, \quad (U,[n]) \mapsto \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{S}\mathrm{m}/k}(-,U) \times \Delta^n$$

provides a symmetric monoidal adjunction

$$L : \mathrm{sPre} \rightleftarrows \mathcal{S}\mathrm{Pre} : R \tag{3.1}$$

of cartesian closed categories. The composite functor RL is well known and was already studied in [MV99] as a functor called Sing, defined as

$$\mathrm{Sing}(X)(U)_m = \hom_{\mathrm{Pre}}(U \times \mathbb{A}^m, X_m).$$

Lemma 3.3. *The functors RL and* Sing *coincide.*

Proof. Since the functors R, L and Sing preserve colimits we only need to check their behavior on representable objects.

$$\begin{aligned} RL(U \times \Delta^n)(V,[m]) &= \hom_{\mathcal{S}\mathrm{Pre}}(\mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{S}\mathrm{m}/k}(-,V) \times \Delta^m, \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{S}\mathrm{m}/k}(-,U) \times \Delta^n) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{S}\mathrm{Pre}}(\mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{S}\mathrm{m}/k}(-,V), \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{S}\mathrm{m}/k}(-,U) \times \Delta^n)_m \\ &\cong \hom_{\mathcal{S}\mathrm{m}/k}(V \times \Delta^m, U) \times \Delta^n_m \\ &\cong U(V \times \Delta^m)_m \times \Delta^n_m \\ &\cong \hom_{\mathrm{Pre}}(V \times \Delta^m, U_m) \times \Delta^n_m \\ &\cong \mathrm{Sing}(U \times \Delta^n)(V)_m \end{aligned}$$

□

Corollary 3.4. *The enriched simplicial presheaf represented by the affine line is objectwise contractible.*

Proof. As a corollary of Lemma 3.3 we obtain

$$\begin{aligned} \mathbb{A}^1(U) &= \mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{S}\mathrm{m}/k}(U, \mathbb{A}^1) = L\mathbb{A}^1(U) \\ &= RL\mathbb{A}^1(U) = \mathrm{Sing}(\mathbb{A}^1)(U) \end{aligned}$$

which is contractible by [MV99, Corollary 3.5]. □

The following lemma is a generalization of the bicompleteness of presheaf categories to enriched category theory.

Lemma 3.5. *The category of enriched simplicial presheaves is bicomplete and colimits and limits can be computed objectwise.*

Proof. The category $\mathcal{S}$Pre is the underlying category of a s$\mathcal{S}$et-category in which all weighted s$\mathcal{S}$et-colimits and limits exist [Bor94b, Proposition 6.6.17], so $\mathcal{S}$Pre is bicomplete by [Bor94b, Proposition 6.6.16]. □

We use the conventional terminology and say that a set I of morphisms in a category *permits the small object argument*, if the domains of the elements of I are small relative to transfinite compositions of pushouts of elements in I.

Lemma 3.6. *Let I be a set of morphisms in* sPre. *Then the set LI of morphisms in $\mathcal{S}$Pre permits the small object argument.*

Proof. We make use of the fact that all objects in the locally presentable category sPre are small. So there exists a cardinal κ, such that for all κ-filtered ordinals λ and any λ-sequence $S : \lambda \to \mathcal{S}\mathrm{Pre}$ the following diagram commutes.

$$\begin{array}{ccc} \operatorname{colim}_{\beta<\lambda} \hom_{\mathcal{S}\mathrm{Pre}}(LX, F_\beta) & \xrightarrow{\Phi} & \hom_{\mathcal{S}\mathrm{Pre}}(LX, \operatorname{colim}_{\beta<\lambda} F_\beta) \\ \cong \downarrow & & \downarrow \cong \\ \operatorname{colim}_{\beta<\lambda} \hom_{\mathrm{sPre}}(X, RF_\beta) & \xrightarrow{\cong} & \hom_{\mathrm{sPre}}(X, \operatorname{colim}_{\beta<\lambda} RF_\beta) \end{array}$$

Hence LX is small and LI permits the small object argument. □

Parallel to the construction of enriched simplicial presheaves, there is an additive category of *enriched simplicial presheaves with transfers* denoted by $\mathcal{S}$PST. The category of simplicial abelian groups is canonically enriched by $\mathrm{sAb}(A,B) = \mathrm{Ab}(A \otimes \mathbb{Z}[\Delta^{(\text{-})}], B)$. Also SmCor$/k$ is an sAb-category with $\mathrm{sAb}(X,Y) = \mathrm{Cor}(X \otimes \Delta^{(\text{-})}, Y)$. An additive version of the Adjunction Lemma 1.2 applied to the functor

$$c : \mathrm{SmCor}/k \otimes \mathbb{Z}\Delta \to \mathrm{sPST}, \quad c(U,[n]) = \mathrm{sAb}(-,U) \otimes \mathbb{Z}[\Delta^n]$$

provides an additive adjunction $(L_{\mathrm{tr}}, R_{\mathrm{tr}})$ between sPST and $\mathcal{S}$PST. The composite functor $R_{\mathrm{tr}}L_{\mathrm{tr}}$ coincides with the additive functor $\mathrm{Sing}_{\mathrm{tr}}$ defined by

$$\mathrm{Sing}_{\mathrm{tr}}(X)(U)_m = \mathrm{Ab}(U \otimes \Delta^m, X_m).$$

The set LI permits the small object argument for every set I of morphisms in sPST. One obtains a diagram of symmetric monoidal adjunctions

$$\begin{array}{ccc} \mathrm{sPre}_* & \underset{R_*}{\overset{L_*}{\rightleftarrows}} & \mathcal{S}\mathrm{Pre}_* \\ {\scriptstyle \tilde{\mathbb{Z}}_{\mathrm{tr}}}\downarrow\uparrow{\scriptstyle \tilde{\mathbb{U}}_{\mathrm{tr}}} & & {\scriptstyle \tilde{\mathcal{Z}}_{\mathrm{tr}}}\downarrow\uparrow{\scriptstyle \tilde{\mathcal{U}}_{\mathrm{tr}}} \\ \mathrm{sPST} & \underset{R_{\mathrm{tr}}}{\overset{L_{\mathrm{tr}}}{\rightleftarrows}} & \mathcal{S}\mathrm{PST}, \end{array} \tag{3.2}$$

where the functor $\mathcal{Z}_{\mathrm{tr}}$ comes from an enriched version of the Adjunction Lemma 1.2 and is defined by $\mathcal{Z}_{\mathrm{tr}}(\mathrm{s}\mathcal{S}\mathrm{et}_{\mathcal{S}\mathrm{m}/k}(-,U) \times \Delta^n) = \mathrm{sAb}_{\mathrm{SmCor}/k}(-,U) \otimes \mathbb{Z}[\Delta^n]$. Its right adjoint is given by $\tilde{\mathcal{U}}_{\mathrm{tr}}(X) = \tilde{U}[-] \circ X \circ \Gamma$.

4. Model structures for enriched simplicial presheaves

In this section we construct several model structures on the category $\mathcal{S}$Pre of enriched simplicial presheaves. These model structures correspond to model structures on the category sPre of simplicial presheaves which are presented in the previous sections. Subsequently, Corollary 4.8 gives a characterization of the fibrant objects. Finally motivic homology and motivic cohomology are represented in a homotopy category of enriched simplicial presheaves with transfers.

Definition 4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be a model categories and $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ an adjunction. The model structure on $\mathcal{D}$ is called (L,R)-lifted if a morphism f of $\mathcal{D}$ is a weak equivalence (resp. a fibration) if and only if $R(f)$ is a weak equivalence (resp. a fibration) of $\mathcal{C}$. A cofibrantly generated model category $\mathcal{C}$ is called (I,J)-cofibrantly generated if I is a set of generating cofibrations and J is a set of generating acyclic cofibrations for the model structure on $\mathcal{C}$.

If $\mathcal{C}$ is a model category, $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ an adjunction and $\mathcal{D}$ is equipped with the (L,R)-lifted model structure, then the adjunction (L,R) is necessarily a Quillen adjunction since the right adjoint R preserves fibrations and acyclic fibrations. The lifted model structure on $\mathcal{D}$ is right proper if and only if $\mathcal{C}$ is a right proper model category.

Lemma 4.2 (Lifting Lemma)**.** *Let $\mathcal{C}$ be a (I,J)-cofibrantly generated model category, $\mathcal{D}$ a bicomplete category and $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ an adjunction such that the right adjoint R commutes with colimits and LI and LJ permit the small object argument. Then there exists a (LI,LJ)-cofibrantly generated (L,R)-lifted model structure on $\mathcal{D}$ if and only if for every $j \in J$ and every pushout diagram*

$$\begin{array}{ccc} L(A) & \longrightarrow & X \\ {\scriptstyle L(j)}\downarrow & & \downarrow{\scriptstyle p} \\ L(B) & \longrightarrow & Y \end{array}$$

the morphism $R(p)$ is a weak equivalence of $\mathcal{C}$.

Proof. The existence of a (LI,LJ)-cofibrantly generated (L,R)-lifted model structure on $\mathcal{D}$ is provided by Lifting Theorem [Hir03, Theorem 11.3.2]. Conversely, suppose that there exists a (LI,LJ)-cofibrantly generated (L,R)-lifted model structure on $\mathcal{D}$. Let j be an element of J and let

$$\begin{array}{ccc} L(A) & \longrightarrow & X \\ {\scriptstyle L(j)}\downarrow & & \downarrow{\scriptstyle p} \\ L(B) & \longrightarrow & Y \end{array}$$

be a pushout diagram. Since L is a left Quillen functor and j is an acyclic cofibration, the morphism $L(j)$ is an acyclic cofibration. The class of acyclic cofibrations is closed under pushouts [Hir03, Proposition 7.2.12]. Hence p is a weak equivalence and so $R(p)$ is a weak equivalence of $\mathcal{C}$. □

Theorem 4.3. *Consider the adjunction*

$$L : \mathrm{sPre} \rightleftarrows \mathcal{S}\mathrm{Pre} : R$$

constructed in (3.1). *Let* sPre *be equipped with a cofibrantly generated model structure with* $\mathbb{A}^1$*-local weak equivalences as weak equivalences and with the property that every cofibration is in particular a monomorphism. Then the* (L, R)*-lifted model structure on* $\mathcal{S}$Pre *exists and the adjunction* (L, R) *is a Quillen equivalence.*

Proof. Let I be a set of generating cofibrations and J be a set of generating acyclic cofibrations for the model structure on sPre, j an element of J and

$$\begin{array}{ccc} L(A) & \longrightarrow & X \\ {\scriptstyle L(j)}\downarrow & & \downarrow{\scriptstyle p} \\ L(B) & \longrightarrow & Y \end{array}$$

be a pushout diagram in $\mathcal{S}$Pre. Since R commutes with colimits, the diagram

$$\begin{array}{ccc} RL(A) & \longrightarrow & R(X) \\ {\scriptstyle RL(j)}\downarrow & & \downarrow{\scriptstyle R(p)} \\ RL(B) & \longrightarrow & R(Y) \end{array}$$

is also a pushout. The morphism j is an acyclic cofibration of sPre and therefore in particular an acyclic cofibration in the $\mathbb{A}^1$-local injective model structure on sPre, that is a $\mathbb{A}^1$-local weak equivalence and a monomorphism. Lemma 3.3 identifies the functor RL with the singular functor Sing. The singular functor respects monomorphisms and $\mathbb{A}^1$-local weak equivalences by [MV99, Corollary 3.8]. Therefore $RL(j)$ is an acyclic cofibration in the $\mathbb{A}^1$-injective model structure on sPre. The class of acyclic cofibrations of a model category is closed under pushouts and hence $R(p)$ is a $\mathbb{A}^1$-local weak equivalence. The category $\mathcal{S}$Pre is bicomplete by Lemma 3.5 and Lemma 3.6 provides that LI and LJ permit the small object argument. Hence the category $\mathcal{S}$Pre can be equipped with the (L, R)-lifted model structure by Lemma 4.2. To prove that (L, R) is a Quillen equivalence, let η be the unit of the adjunction (L, R) and let X be a simplicial presheaf. Lemma 3.3 identifies $\eta(X)$ with the canonical morphism $X \to \mathrm{Sing}(X)$ which is a $\mathbb{A}^1$-local weak equivalence by [MV99, Corollary 3.8]. The diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta(X)} & RL(X) \\ & {\scriptstyle f^\sharp}\searrow & \downarrow{\scriptstyle R(f)} \\ & & R(Y) \end{array}$$

shows that a morphism $f : LX \to Y$ is a weak equivalence if and only if its adjoint $f^\sharp$ is a weak equivalence. Therefore (L, R) is a Quillen equivalence. □

As already mentioned, the right properness of a model structure on simplicial presheaves is transfered to the lifted structure on enriched simplicial presheaves. The following lemmas provide that the left properness, the simplicial and the monoidal structure are also preserved.

Lemma 4.4. *Consider the adjunction* $L : \mathrm{sPre} \rightleftarrows \mathcal{S}\mathrm{Pre} : R$ *and let* $(\mathrm{sPre}, \times)$ *be equipped with a monoidal model structure. If the category* $(\mathcal{S}\mathrm{Pre}, \times)$ *is endowed with the* (L, R)*-lifted model structure, then it is a monoidal model category.*

Proof. General results on enriched category theory imply that $\mathcal{S}\mathrm{Pre}$ is cartesian closed [Day70]. Let $i : A \to B$ and $j : C \to D$ be cofibrations. One has to show that the morphism

$$i \square j : (B \times C) \coprod_{(A\times C)} (A \times D) \to B \times D$$

is a cofibration and an acyclic cofibration if i or j is a weak equivalence. This follows from the property of L being a left Quillen functor and from the relation $L(i \square j) \cong L(i) \square L(j)$ holding as the functor L is strong monoidal, which is the case since

$$\begin{aligned} L(X \times Y) &= L(\mathrm{colim}(\mathrm{hom}(-, U) \times \Delta^n) \times \mathrm{colim}(\mathrm{hom}(-, V) \times \Delta^m)) \\ &= L(\mathrm{colim}(\mathrm{hom}(-, U \times V) \times \Delta^n \times \Delta^m)) \\ &= \mathrm{colim}(\mathrm{s}\mathcal{S}\mathrm{et}(-, U \times V) \times \Delta^n \times \Delta^m) \\ &= \mathrm{colim}(\mathrm{s}\mathcal{S}\mathrm{et}(-, U) \times \Delta^n) \times \mathrm{colim}(\mathrm{s}\mathcal{S}\mathrm{et}(-, V) \times \Delta^m) \\ &= L(X) \times L(Y). \end{aligned}$$

□

Lemma 4.5. *Consider the adjunction* $L : \mathrm{sPre} \rightleftarrows \mathcal{S}\mathrm{Pre} : R$ *and let* sPre *be equipped with a simplicial model structure. If the category of enriched simplicial presheaves is endowed with the* (L, R)*-lifted model structure, then it is a simplicial model category.*

Proof. The category $\mathcal{S}\mathrm{Pre}$ is naturally enriched over the category of simplicial sets by $\mathrm{s}\mathcal{S}\mathrm{et}(X, Y) = \mathrm{hom}_{\mathcal{S}\mathrm{Pre}}(X \times \Delta^{(\text{-})}, Y)$. It is tensored with $X \otimes A = X(-) \times A$ and cotensored with $[\![A, X]\!] = \mathrm{hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(A \times \Delta^{(\text{-})}, X(-))$. By Lemma 4.4 a statement equivalent to the (SM7) axiom holds [GJ99, II.3.11]. □

Lemma 4.6. *Every enriched simplicial presheaf* X *is homotopy invariant, that is the map*

$$X(U) \to X(U \times \mathbb{A}^1)$$

induced by the projection is a weak equivalence of $\mathrm{s}\mathcal{S}\mathrm{et}$ *for all objects* U *of* $\mathcal{S}\mathrm{m}/k$.

Proof. An enriched simplicial presheaf X maps a morphism $f : U \to V$ of $\mathcal{S}\mathrm{m}/k$ to a 0-simplex of the simplicial set $\mathrm{s}\mathcal{S}\mathrm{et}(XV, XU)$ and it maps a naive homotopy $H : U \times \Delta^1 \to V$ of $\mathcal{S}\mathrm{m}/k$ to a 1-simplex of $\mathrm{s}\mathcal{S}\mathrm{et}(XV, XU)$, which is a homotopy equivalence of the simplicial sets XV and XU with respect to the cylinder object Δ^1. Therefore X takes naive homotopy equivalences in $\mathcal{S}\mathrm{m}/k$ to weak equivalences in $\mathrm{s}\mathcal{S}\mathrm{et}$. The assertion is obtained from the fact that the affine line $\mathbb{A}^1$ is naive homotopy equivalent to the point $\mathrm{Spec}\,(k)$ in $\mathcal{S}\mathrm{m}/k$ where a homotopy equivalence is given by the map $k[X] \to k[X, Y]$, $X \mapsto XY$ of k-algebras. □

Corollary 4.7. *Let* $\mathcal{S}\mathrm{Pre}$ *be equipped with a simplicial model structure in which every object of* $\mathcal{S}\mathrm{m}/k$ *is cofibrant. Then the class*

$$C = \{U \times \mathbb{A}^1 \xrightarrow{pr} U \mid U \in \mathcal{S}\mathrm{m}/k\}$$

consists of weak equivalences.

Proof. Lemma 4.6 provides that $\mathrm{s}\mathcal{S}\mathrm{et}(U, X) \to \mathrm{s}\mathcal{S}\mathrm{et}(U \times \mathbb{A}^1, X)$ is a weak equivalence of simplicial sets for every enriched simplicial presheaf X by an enriched version of the Yoneda Lemma. Weak equivalences in a simplicial model category are detected by the property of the above morphism being a weak equivalence of simplicial sets for all fibrant objects X [Hir03, Corollary 9.7.5]. □

Corollary 4.8. *Consider the adjunction* $L : \mathrm{sPre} \rightleftarrows \mathcal{S}\mathrm{Pre} : R$ *and the class*

$$C = \{U \times \mathbb{A}^1 \xrightarrow{pr} U \mid U \in \mathcal{S}\mathrm{m}/k\}$$

of morphisms of simplicial presheaves. Let sPre *be equipped with a Bousfield localized model structure* $L_C(\mathrm{sPre})$ *in which every object of* $\mathcal{S}\mathrm{m}/k$ *is cofibrant. Suppose that the* (L, R)*-lifted model structure on* $\mathcal{S}\mathrm{Pre}$ *exists. Then an object* X *of* $\mathcal{S}\mathrm{Pre}$ *is fibrant if and only if the object* $R(X)$ *is fibrant in* sPre *before localizing.*

Lemma 4.9. *Consider the adjunction* $L : \mathrm{sPre} \rightleftarrows \mathcal{S}\mathrm{Pre} : R$ *and let* sPre *be equipped with a left proper cofibrantly generated model structure with* $\mathbb{A}^1$*-local weak equivalences as weak equivalences and with the property that every cofibration is in particular a monomorphism. If the category of enriched simplicial presheaves is endowed with the* (L, R)*-lifted model structure, then it is a left proper model category.*

Proof. It is sufficient to show that the (L, R)-lifted $\mathbb{A}^1$-local injective model structure is left proper. The injective model structure on $\mathcal{S}\mathrm{Pre}$ is left proper and it is the (L, R)-lifted model of the injective structure on sPre [Lur09, Proposition B.1]. Let B be a class of cofibrations in sPre, such that the localization at B is the local injective model structure. Then (L, R) is a Quillen adjunction between the local injective model on sPre and the localization M of the injective model structure on $\mathcal{S}\mathrm{Pre}$ at $L(B)$ [Hir03, Theorem 3.3.20]. We show that M coincides with the (L, R)-lifted $\mathbb{A}^1$-local injective model structure on $\mathcal{S}\mathrm{Pre}$. Let the injective model structure on sPre be (I, J)-cofibrantly generated, then the injective model structure on $\mathcal{S}\mathrm{Pre}$ is (LI, LJ)-cofibrantly generated and so is its left Bousfield localization M. By the same arguments, the (L, R)-lifted $\mathbb{A}^1$-local injective model structure on $\mathcal{S}\mathrm{Pre}$ is (LI, LJ)-cofibrantly generated. Hence both model structures have the same cofibrations. Moreover, their fibrant objects coincide by Corollary 4.8 and the fact that an object X is fibrant in the Bousfield localization M if and only if $\mathrm{s}\mathcal{S}\mathrm{et}(-, X)$ maps B to weak equivalences. Therefore the model structures are the same since a model structure is determined by its cofibrations and its fibrant objects. □

The previous statements might suggest that it is possible to get a model for the motivic homotopy category by lifting a local model structure to the category of enriched simplicial presheaves. In view of Lemma 4.2 one observes that a (I, J)-cofibrantly generated model structure lifts via (L, R) to the category of enriched simplicial presheaves if $\mathrm{Sing}(j)$ is a local weak equivalence for every generating acyclic cofibration j in J, but the singular functor does not preserve local weak equivalences in general.

Now we focus on the $\mathbb{A}^1$-local projective model structure on the category sPre of simplicial presheaves. Theorem 4.3 induces a Quillen equivalence

$$L_* : \mathrm{sPre}_* \rightleftarrows \mathcal{S}\mathrm{Pre}_* : R_*$$

of monoidal model categories [Hov99, Proposition 1.3.17, 4.2.19] since the functor L preserves the terminal object. Parallel to the results of this section, one obtains a Quillen equivalence

$$L_{\mathrm{tr}} : \mathrm{sPST} \rightleftarrows \mathcal{S}\mathrm{PST} : R_{\mathrm{tr}}$$

of monoidal model categories and the Quillen adjunctions of (3.2) are symmetric monoidal. This allows one to describe motivic cohomology as

$$H^{p,q}_{\mathrm{mot}}(X) = \left[L_{\mathrm{tr}}\tilde{\mathbb{Z}}_{\mathrm{tr}}(X_+)\,,\, L_{\mathrm{tr}}\tilde{\mathbb{Z}}_{\mathrm{tr}}(S^{p-q}_s) \otimes L_{\mathrm{tr}}\tilde{\mathbb{Z}}_{\mathrm{tr}}(S^q_t)\right]_{\mathcal{S}\mathrm{PST}}$$

where the first enriched simplicial presheaf $L_{\mathrm{tr}}\tilde{\mathbb{Z}}_{\mathrm{tr}}(X_+)$ is cofibrant as it is the image of a cofibrant object under a left Quillen functor. The enriched simplicial Eilenberg-MacLane presheaf $L_{\mathrm{tr}}\tilde{\mathbb{Z}}_{\mathrm{tr}}(S^{p-q}_s \wedge S^q_t)$ is fibrant as $NR_{\mathrm{tr}} : \mathcal{S}\mathrm{PST} \to \mathrm{Ch}_{\geq 0}(\mathrm{PST})$ maps it to the motivic complex $\mathbb{Z}(q)[p]$ given by

$$\to \underset{\text{p-q+k}}{\bigotimes^q \mathrm{Cor}(- \times \Delta^k, \mathbb{A}^1 \setminus 0)/\mathbb{Z}} \to \ldots \to \underset{\text{p-q}}{\bigotimes^q \mathrm{Cor}(- \times \Delta^0, \mathbb{A}^1 \setminus 0)/\mathbb{Z}} \to \underset{\text{p-q-1}}{0}$$

which is fibrant by [MVW06, Corollary 14.9] whereas the Eilenberg-MacLane simplicial presheaf $K(\mathbb{Z}, q)[p]$ needs to be fibrantly replaced.

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