SELF-EMBEDDINGS OF TREES

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Abstract. We prove a fix point theorem for monoids of self-embeddings of trees. As a corollary, we obtain a result by Laflamme, Pouzet and Sauer that a tree either contains a subdivided binary tree as a subtree or has a vertex, and edge, an end or two ends fixed by all its self-embeddings.

1. Introduction

For a group acting on a graph there is always the following choice: either it fixes a point or the group contains a free subgroup $\mathbb{Z} * \mathbb{Z}$. More precisely the following statements are true for a group Γ acting on a graph G.

- Every automorphism of G is either elliptic, hyperbolic, or parabolic;
- Γ fixes either a bounded subset of G or a unique limit point of Γ in its end space, or G has precisely two limit points of Γ, or Γ contains two hyperbolic elements that freely generate a free subgroup.
- there are either none, one, two or infinitely many limit points of Γ;
- there are either none, two or infinitely many hyperbolic limit points of Γ;
- the hyperbolic limit set of Γ is dense in the limit set of Γ;
- if the limit set of Γ is infinite, then it is a perfect set.

We refer to [4, 5, 6, 9, 10, 12, 13] for all these theorems.

All these results carry over almost verbatim to monoids of self-embeddings of trees and in this paper, we will prove these analogues. As a corollary of our results, we obtain a result by Laflamme et al. [7] that a tree either contains a subdivided binary tree or has a vertex, an edge, or a set of at most two ends fixed by all self-embeddings.

Laflamme et al. used their result to varify the tree alternative conjecture of Bonato and Tardif [1] in various situations. We will discuss the connection of our results with that conjecture in Section 5.2. We will also give an outlook on the possible generalisation of the present results to general graphs in Section 5.1.

2. Main results

Let T be a tree. A *self-embedding* is an injective map of $V(T)$ into itself that preserves the adjacency relation.

A *ray* is a one-way infinite path in T and two rays are *equivalent* if they have for every $v \in V(T)$ subrays in the same component of $T - v$. This is an equivalence relation whose classes are the *ends* of T. By $\Omega(T)$ we denote the set of ends of T.

Note that each self-embedding of T maps rays to rays and equivalent rays to equivalent rays. Thus, it induces a map from $\Omega(T)$ into itself.

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By Emb(T) we denote the monoid of all self-embeddings of T. We call a self-embedding $g \in \mathrm{Emb(T)}$

- *elliptic* if it fixes a non-empty finite subtree of T;
- *hyperbolic* if it is not elliptic and if it fixes precisely two ends;
- *parabolic* if it is not elliptic and if it fixes precisely one end.

Note that Halin [4, Lemma 2] proved that any automorphism of a finite tree fixes either a vertex or an edge. It follows that a self-embedding is elliptic if and only if it fixes either a vertex or an edge. Halin proved the following theorem.

Theorem 2.1. [4, Theorem 5] *Let g be a self-embedding of a tree T. Then either g fixes a vertex or an edge or there is a ray R with $g(R) \subsetneq R$. This ray can be extended to either a g-invariant double ray or a maximal ray R' with $g(R') \subsetneq R'$.* □

Note that any $g \in \mathrm{Emb(T)}$ that is not elliptic fixes at least one end due to Halin's theorem, namely that end that contains the ray R of Theorem 2.1. If R can be extended to a g-invariant double ray R', then both ends defined by R' are g-invariant. Furthermore, no other end ω can be fixed, since the unique ray with precisely one vertex on R' that lies in ω will be mapped to a disjoint ray. Since disjoint rays lie in distinct ends of a tree, g is hyperbolic. A similar argument shows that in the case that R' is a ray, g is parabolic. Thus, we obtain as a corollary of Theorem 2.1 the following.

Corollary 2.2. *Every self-embedding of a tree is either elliptic, hyperbolic, or parabolic.* □

Note that for any non-elliptic self-embedding g all rays that are preserved by g are equivalent, since otherwise the double ray between the two ends these non-equivalent rays lie in had non-equivalent tails R_1 and R_2 with $g(R_1) \subsetneq R_1$ and $g(R_2) \subsetneq R_2$, which is impossible. We call this end the *direction* g^+ of g. If g is hyperbolic, we denote by g^- the unique g-invariant end other than g^+.

Since we also talk about convergence to ends, we need a topology on trees with their ends. For this we consider the tree as a 1-complex. The sets C where C is a component of $T - x$ for some vertex x that contain an end ω form a neighbourhood basis of ω.[1] For $x \in V(T)$ and $\omega \in \Omega(T)$, we denote by $C(T - x, \omega)$ that component of $T - x$ that contains ω, that is, that component of $T - x$ that contains a sequence of vertices converging to ω.

Trees with their ends are *projective*: whenever $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are sequences such that the distances $d(x_i, y_i)$ are bounded, then $(x_i)_{i\in\mathbb{N}}$ converges to an end ω if and only if $(y_i)_{i\in\mathbb{N}}$ converges to ω, see [5, 13]. It follows that for any $v \in V(T)$ and any non-elliptic self-embedding, the sequence $(g^i(v))_{i\in\mathbb{N}}$ converges to g^+.

The following lemma gives a condition under which a self-embedding is non-elliptic.

Lemma 2.3. *Let g be a self-embedding of a tree T. If there is an edge xy such that y and $g(x)$ separate x and $g(y)$, then g is non-elliptic and xy lies on the maximal (double) ray of T that is preserved by g.*

Proof. If g is elliptic, then it fixes a subtree T' on either one or two vertices. Since $d(x, T') = d(g(x), T')$ and $d(y, T') = d(g(y), T')$, we conclude that T' lies

[1] For a general approach to locally finite graphs with their ends seen as topological spaces, we refer to Diestel [3, 2].

in that component of $T - \{x, g(x)\}$ that contains y and also in that component of $T - \{y, g(y)\}$ that contains $g(x)$. But then we have $d(y, T') < d(x, T')$ and $d(g(x), T') < d(g(y), T')$. This is a contradiction as self-embeddings preserve the distance function.

So g is not elliptic and hence preserves a maximal (double) ray R by Theorem 2.1. If xy lies not on R, then if $d(x, R) < d(y, R)$, then also $d(g(x), R) < d(g(y), R)$, which is a contradiction to the assumption that y and $g(x)$ separate x and $g(y)$. Similarly, we obtain a contradiction if $d(x, R) > d(y, R)$. Thus, xy lies on R. $\square$

Let M be a submonoid of Emb(T). The *limit set* $\mathcal{L}(M)$ of M is the set of accumulation points of $\{g(v) \mid g \in M\}$ in $\Omega(T)$ for any $v \in V(T)$. Note that $\mathcal{L}(M)$ is independent from the choice of v by projectivity. By $\mathcal{D}(M)$ we denote the set of all directions of non-elliptic elements in M.

Theorem 2.4. *Let T be a tree and M a submonoid of* Emb(T)*.*

(i) *If $|\mathcal{L}(M)| \geq 2$, then $\mathcal{D}(M)$ is dense in $\mathcal{L}(M)$.*
(ii) *The set $\mathcal{L}(M)$ has either none, one, two, or infinitely many elements.*
(iii) *The set $\mathcal{D}(M)$ has either none, one, two, or infinitely many elements.*

Proof. To prove (i), let $\omega \in \mathcal{L}(M)$ and $v \in V(T)$. Then there is a sequence $(g_i)_{i\in\mathbb{N}}$ in M such that $(g_i(v))_{i\in\mathbb{N}}$ converges to ω. If there are infinitely many non-elliptic among the g_i, then we may assume that all g_i are non-elliptic. For $i \in \mathbb{N}$, let R_i be the unique maximal (double) ray of T that is preserved by g_i, which exists by Theorem 2.1. Then we have $d(v, R_i) \geq d(g_i(v), R_i)$. Let $u \in V(T)$. If infinitely many R_i have a tail in g_i^+ that lies in $C(T - u, \omega)$, we conclude that their directions g_i^+ converge to ω. Let us suppose that only finitely many R_i have a tail in $C(T - u, \omega)$. Since $(g_i(v))_{i\in\mathbb{N}}$ converges to ω, there are infinitely many $i \in \mathbb{N}$ with $d(g_i(v), u) > d(u, v)$. For all of those we conclude

$$\begin{array}{rcl} d(v, R_i) & \leq & d(v, u) + d(u, R_i) \\ & < & d(g_i(v), u) + d(u, R_i) \\ & = & d(g_i(v), R_i). \end{array}$$

This contradiction to $d(v, R_i) \geq d(g_i(v), R_i)$ shows that ω lies in the closure of $\mathcal{D}(M)$ if infinitely many g_i are non-elliptic.

By considering an finite subsequence of $(g_i)_{i\in\mathbb{N}}$, we may assume that all g_i are elliptic. Similarly, let $\omega' \in \mathcal{L}(M)$ with $\omega' \neq \omega$ and let $(h_i)_{i\in\mathbb{N}}$ be a sequence in M such that $(h_i(v))_{i\in\mathbb{N}}$ converges to ω'. Let $x \in V(T)$ lie on the unique double ray between ω and ω' and let x' be the neighbour of x that separates x from ω'.

If there are infinitely many h_i elliptic, we may assume, by looking at a subsequence, that all of them are elliptic. Also by taking subsequences, we may assume that all $g_i(x)$ lie in $C(T-x, \omega)$ and all $h_i(x)$ and $h_i g_i(x)$ lie in $C(T-x, \omega')$. We consider the sequence $(g_i h_i)_{i\in\mathbb{N}}$. Then x, $g_i(x)$, and $g_i h_i(x)$ separate x' from $g_i h_i(x')$. By Lemma 2.3, we obtain that $g_i h_i$ is not elliptic. We claim that $(g_i h_i(v))_{i\in\mathbb{N}}$ converges to ω. To see this, let $y \in V(T)$. Then there is some $n \in \mathbb{N}$ such that for all $i \geq n$ we have that $g_i(x)$ lies in $C(T - y, \omega)$. But as $g_i(x)$ separates x and $g_i h_i(x)$, also $g_i h_i(x)$ lies in $C(T - y, \omega)$. Thus, $(g_i h_i(x))_{i\in\mathbb{N}}$ converges to ω.

So at most finitely many h_i are elliptic. Again, we consider a subsequences such that all h_i are not elliptic and such that all $g_i(x)$ lie in $C(T - x, \omega)$ and all $h_i(x)$ and $h_i g_i(x)$ lie in $C(T - x, \omega')$. Let R_i be the maximal (double) ray R that is preserved by h_i First assume that $g_i(x')$ lies on R_i. As x, x' separate $g_i(x')$ from

$h_ig_i(x')$, we conclude that x and x' lies on R as well and we have that $h_ig_i(x')$ and x' separate $h_ig_i(x)$ from x. We consider $g_ih_ig_i$ is this situation. Since $g_ih_ig_i(x')$ and $g_i(x')$ separate $g_ih_ig_i(x)$ from $g_i(x)$, we conclude that $f_i := g_ih_ig_i$ is not elliptic by Lemma 2.3.

If $g_i(x')$ does not lie on R_i, but x lies on R_i, set $f_i := g_ih_i$. Then x' and $h_ig_i(x)$ separate $h_ig_i(x')$ from x. So $g_i(x')$ and $g_ih_ig_i(x)$ separate $g_i(x)$ from $g_ih_ig_i(x')$. We conclude by Lemma 2.3 that f_i is not elliptic.

If neither $g_i(x')$ nor x lie on R_i, set $f_i := g_ih_i$. Lemma 2.3 implies that there is some double ray in T that contains $h_i(x), h_i(x'), x', x, g_i(x), g_i(x'), g_ih_i(x'), g_ih_i(x)$ in this particular order. This implies that f_i is not elliptic.

So we obtain in all three cases that f_i is not elliptic and that $g_i(x)$ separates x from $f_i(x)$. If we prove that $f_i(x)$ converges to ω, we may replace $(g_i)_{i\in\mathbb{N}}$ by $(f_i)_{i\in\mathbb{N}}$ and are done by our first case, where all g_i were non-elliptic.

To prove that $f_i(x)$ converges to ω, let $x_1, x_2, \ldots$ be the ray that starts at x and lies in ω. It suffices to show that $C := C(T - x_i, \omega)$ contains all but finitely many $f_i(x)$. But this is a direct consequence of the facts that C contains all but finitely many $g_i(x)$ and that $g_i(x)$ separates x from $f_i(x)$ in all cases. Thus, the first case proves (i).

To prove (ii), let us assume $|\mathcal{L}(M)| \geq 3$. Let $f, g \in M$ be non-elliptic and such that f^+ is not fixed by g. These exist as every non-elliptic self-embedding fixes at most two ends. Then all $g^i(f^+)$ are distinct ends and all of them are distinct from g^+ and g^-. Then $g^jf^i(v) \to g^j(f^+)$. So all $g^j(f^+)$ lie in $\mathcal{L}(M)$ and hence $\mathcal{L}(M)$ contains infinitely many ends.

Finally, (iii) is a direct consequence of (i) and (ii). □

In the case of automorphisms[2], the situation $|\mathcal{L}(M)| \geq 2$ implies the existence of hyperbolic automorphisms, cp. [5, 13]. For self-embeddings this is no longer the case, as we shall illustrate with the following example. But if we add the extra assumption that some end of T is fixed by M, we obtain the existence of some hyperbolic element of M in Proposition 2.7.

Example 2.5. Let T_0 be the rooted binary tree, i. e. the tree where one vertex has degree 2 and all others have degree 3. We add a new finite non-trivial path P to T_0 that stars at the vertex of degree 2 and obtain a tree T. We claim that the monoid M of all self-embeddings of T contains no hyperbolic element. Seeking a contradiction, let us suppose that $f \in M$ is hyperbolic. Let R be the f-invariant double ray. Then at all vertices x but one of R, say u, there is a binary tree with root x that is otherwise disjoint from R. But as some vertex of R is mapped onto u, also its binary tree must be mapped onto the tree hanging of R at u. But this is impossible as the tree hanging of at u is a proper subtree of the rooted binary tree, which has a vertex of degree 1. This shows that M has no hyperbolic element.

We note that for a hyperbolic $g \in M$ it can happen that g^- does not lie in $\mathcal{L}(M)$. To see this, we modify Example 2.5 a bit.

Example 2.6. Let T_0 be the rooted binary tree, i. e. the tree where one vertex has degree 2 and all others have degree 3. We add a new ray R to T_0 that stars at the vertex of degree 2 and obtain a tree T. A similar argumentation as in Example 2.5

[2] We refer to [5, 13] for the corresponding definitions in case of automorphisms instead of self-embeddings.

shows that all hyperbolic elements g of M fix the end ω that contains R and that $\omega = g^-$. Note that there are hyperbolic elements in this situation.

Proposition 2.7. *Let T be a tree and M a submonoid of* Emb(T)*.*

(i) If $|\mathcal{L}(M)| \geq 2$ and some end is fixed by M, then M contains a hyperbolic self-embedding.
(ii) If $|\mathcal{L}(M)| = 2$, then all non-elliptic elements of M are hyperbolic.

Proof. Let $|\mathcal{L}(M)| \geq 2$, and let $\eta \in \mathcal{L}(M)$ be fixed by M. Since $\mathcal{D}(M)$ is dense in $\mathcal{L}(M)$ by Theorem 2.4 (i), we find some $\mu \in \mathcal{D}(M)$ with $\mu \neq \eta$. Let $g \in M$ with $g^+ = \mu$. As g fixes η, it must leave the double ray between η and μ invariant. Thus, g is hyperbolic.

Similarly, if $|\mathcal{L}(M)| = 2$, all non-elliptic self-embeddings in M must leave the double ray between the two limit points invariant. Thus, they are hyperbolic self-embeddings. □

Now we are able to prove our fixed point theorem for self-embeddings.

Theorem 2.8. *Let T be a tree and M a submonoid of* Emb(T)*. Then one of the following holds.*

(i) *M fixes a either a vertex or an edge of T;*
(ii) *M fixes a unique element of $\mathcal{L}(M)$;*
(iii) *$\mathcal{L}(M)$ consists of precisely two elements;*
(iv) *M contains two non-elliptic elements that do not fix the direction of the other.*

Proof. Let us assume that neither a vertex nor an edge is fixed by M and that no subset of $\mathcal{L}(M)$ of size at most 2 is fixed by M. In particular, $\mathcal{L}(M)$ and $\mathcal{D}(M)$ are infinite by Theorem 2.4 and we find two non-elliptic self-embeddings in M with distinct directions.

Let us suppose that (iv) does not hold. First, we show that there is a unique $\eta \in \mathcal{L}(M)$ fixed by all non-elliptic self-embeddings in M. If M contains some parabolic self-embedding g, then its direction must be fixed by all non-elliptic self-embeddings since (iv) does not hold. As g fixes no other end, its direction is the unique element of $\mathcal{L}(M)$ fixed by all non-elliptic self-embeddings.

If all non-elliptic elements of M are hyperbolic, let $f, g, h \in M$ such that $g^+, h^+ \notin \{f^+, f^-\}$ and $g^+ \neq h^+$. These exist as $\mathcal{D}(M)$ is infinite. As (iv) does not hold, we know that g and h fix f^+ and hence $h^- = f^+ = g^-$. But then g and h satisfy (iv) as g fixes only g^+ and f^+ and h fixes only h^+ and f^+. This contradiction shows that there is a unique $\eta \in \mathcal{L}(M)$ fixed by all non-elliptic self-embeddings.

Since $\mathcal{D}(M)$ is infinite, there are distinct directions $\mu, \nu \in \mathcal{D}(M) \setminus \{\eta\}$. Let $f, g \in M$ be not elliptic such that the directions of f and g are μ and ν, respectively. Since f fixes μ and η, it fixes not other end, in particular it does not fix ν. Similarly, g does not fix μ. This contradiction to the assumption that (iv) does not hold shows the assertion. □

We shall see in Theorem 3.2 that we may pick the non-elliptic elements in Theorem 2.8 (iv) so that they generate a free submonoid of M freely.

3. Infinitely many directions

In this section, we take a closer look at Theorem 2.8 (iv). But before we do that, we prove that self-embeddings that are not elliptic converge uniformly towards g^+.

Lemma 3.1. *Let T be a tree and $g \in \mathrm{Emb(T)}$ be not elliptic. For every neighbourhoods U of g^+ and V of g^-, if g is hyperbolic, there exists $N \in \mathbb{N}$ such that $g^n(T - V) \subseteq U$ for all $n \geq N$.*

Proof. As U is a neighbourhood of g^+, there is some vertex x on the maximal (double) ray R_g with $g(R_g) \subseteq R_g$ such that the component U' of $T - x$ that contains g^+ lies in U. Similarly, if g is hyperbolic, there is a vertex y on R_g such that the component V' of $T - x$ that contain g^- lies in V. If g is parabolic, let y be the first vertex of the ray R_g. As g is either hyperbolic or parabolic, there is some $N \in \mathbb{N}$ such that $g^N(y) \in U'$. Then $g^n(y) \in U'$ for all $n \geq N$. Let y' be the neighbour of y on R_g such that y separates y' and g^+ and let $v \in V(T)$ be any vertex outside of V'. Then $g^n(v)$ lies in a component of $T - g^n(y)$ that does not contain y'. Hence, we have $g^n(v) \in U' \subseteq U$. □

Now we can investigate the situation of Theorem 2.8 (iv) in more detail.

Theorem 3.2. *Let T be a tree and M a submonoid of* $\mathrm{Emb(T)}$. *If M contains two non-elliptic g, h such that neither g fixes h^+ nor h fixes g^+, then there are $m, n \in \mathbb{N}$ such that g^m and h^n generate a free submonoid of M freely. Furthermore, T contains a subdivided 3-regular tree.*

Proof. Let R_g, R_h be the maximal (double) ray with $g(R_g) \subseteq R_g$ and $h(R_h) \subseteq R_h$, respectively. Let $x \in R_g$ be with $d(x, R_h)$ minimum and, if $x \in R_h$, such that the subray of R_h in h^+ starting at x intersects R_g only in x. Let U_g, U_h be connected neighbourhoods of g^+ and of h^+, respectively, such that $U_g \cap (U_h \cup R_h) = \emptyset$ and $U_h \cap (U_g \cup R_g) = \emptyset$, such that $x \notin U_g \cup U_h$, and such that $T - U_g$ and $T - U_h$ are connected, too. In particular, U_g does not contain h^- and U_h does not contain g^- (if they exist), and they also avoid some neighbourhood around those ends. By Lemma 3.1, there are $m, n \in \mathbb{N}$ such that $g^m(\{x\} \cup U_h) \in U_g$ and $h^n(\{x\} \cup U_g) \in U_h$. As U_g is connected and contains g^+, we have $g^m(U_g) \subseteq U_g$ and, analogously, $h^n(U_h) \subseteq U_h$. We claim that $a := g^m$ and $b := h^n$ freely generate a free monoid.

Suppose they do not generate a free monoid freely. Then there are two distinct words w_1, w_2 over $\{a, b\}$ that represent the same self-embedding of T. We choose them such that the length of w_1 is minimum. Since $a(U_g \cup U_h) \subseteq U_g$ and $b(U_g \cup U_h) \subseteq U_h$ we conclude first that $w_i(U_g \cup U_h) \subseteq U_g \cup U_h)$ for $i = 1, 2$ and second that the first letters of w_1 and w_2 must coincide, that is, $w_1 = cw_1'$ and $w_2 = cw_2'$ for some $c \in \{a, b\}$ and words w_1', w_2' over $\{a, b\}$. The choice of w_1 being minimum implies that $w_1 = w_2$ as words. This contradiction to the assumptions shows that a and b freely generate a free monoid M'.

Let us now construct a subtree of T that is a subdivision of the 3-regular tree. Let $u \in R_g \cap U_g$ be closest to x and let $v \in R_h \cap U_h$ be closest to x. Let P be the u-v path. Let T_a be the minimal subtree of T that contains $u, a(u), a(v)$ and let T_b be the minimal subtree of T that contains $v, b(u), b(v)$. Then T_a and T_b are subdivisions of $K_{1,3}$. Set

$$T' := P \cup \bigcup_{w \in M'} w(T_a) \cup \bigcup_{w \in M'} w(T_b).$$

It is straight forward to check that T' is a subdivision of a 3-regular tree. □

As a corollary of Theorems 2.8 and 3.2, we obtain the following, which is a theorem by Laflamme, Pouzet and Sauer [7].

Corollary 3.3. [7, Theorem 1.1] *Let T be a tree that contains no subdivision of the 3-regular tree. Then* Emb(T) *fixes either a vertex, an edge, or a set of at most two ends of T.* □

Our last result of this section deals with the topology on the set of directions.

Theorem 3.4. *Let T be a tree and M a submonoid of* Emb(T)*. If $\mathcal{L}(M)$ is infinite, then it is perfect.*

Proof. We have to show that $\mathcal{L}(M)$ contains no isolated points. Let us suppose that $\eta \in \mathcal{L}(M)$. As $\mathcal{D}(M)$ is dense in $\mathcal{L}(M)$ by Theorem 2.4 (i), we conclude that η lies in $\mathcal{D}(M)$. So there is some non-elliptic $g \in M$ with $g^+ = \eta$. As g fixes at most two ends but $\mathcal{D}(M)$ is infinite by Theorem 2.4, there is some $\mu \in DF(M)$ that is not fixed by g. Then the sequence $(g^n(\mu))_{n\in\mathbb{N}}$ converges to η. Note that every $g^n(\mu)$ lies in $\mathcal{D}(M)$: if $h_i(x) \to \mu$ for $i \to \infty$, we have $gh_i(x) \to g(\mu)$ for $i \to \infty$. This contradicts our assumption. □

4. Fixing an end

A self-embedding g of T *preserves an end ω forwards* if $g(R) \subseteq R$ for some ray $R \in \omega$ and it *preserves ω backwards* if $R \subseteq g(R)$. Note that any non-elliptic self-embedding f preserves f^+ forwards and if f is hyperbolic it additionally preserves f^- backwards. We say that M *preserves ω forwards* if every $g \in M$ preserves ω forwards and M *preserves ω backwards* if every $g \in M$ preserves ω backwards.

Proposition 4.1. *Let T be a tree and M a submonoid of* Emb(T)*. If M preserves some end of T backwards, then all non-elliptic elements of M are hyperbolic.*

Proof. Let ω be an end that is preserved backwards and let $g \in M$ be not elliptic. Since g preserves ω backwards, we have $g^+ \neq \omega$. Since g fixes g^+ and ω, it is hyperbolic. □

Answering a question of Pouzet [11] we construct a graph that has precisely one fixed end, which is preserved backwards by all self-embeddings. We note that Lehner [8] also constructed an example different from ours. Lehner's example is reproduced in [7, Example 6].

Example 4.2. Let T be the rooted binary tree and let x be its root. To obtain T', we add a new ray R to T and join its first vertex with x. To all vertices that have distance 1 modulo 3 to x we add 4 new neighbours and to all vertices of distance 2 modulo 3 to x we add 8 new neighbours. Let T'' be the resulting tree and let ω be the end of T'' that contains R. It is straight forward to check that degree reasons imply that every self-embedding fixes ω and that no self-embedding preserves ω forwards but not backwards.

5. Outlook

In this section, we discuss open problems related to our main theorems. Whereas the first one is a generalisation of the main theorems to graphs, the second one deals with an application to the tree alternative conjecture.

5.1. **Generalisation to graphs.** Our investigations in this paper were focused on a special class of graphs: on trees. Obviously, the following problem arises.

Problem 1. *Generalise our main theorems to self-embeddings of graphs.*

When one looks at this problem, one naturally brings up the question about the behaviour of the ends: the self-embeddings are injective maps on the vertex sets, but shall that extends to the ends as well? A priori, this does not seem clear. Let us give a short example to show that ends of graphs may collapse if we do not make further restriction on the self-embeddings.

Example 5.1. Let G be a complete graph with countably infinitely many vertices. Let $x \in V(G)$. We attach a new ray at x to obtain a new graph H. Then H has two ends. However, there is a self-embedding g of H that maps G into a proper subgraph of G leaving infinitely many vertices of infinite degree outside of $g(G)$ and mapping the attached ray into $G \setminus g(G)$. Both ends of H are mapped by g to the same end of H, the one originating from G.[3]

Because of this possible collapse of ends under self-embeddings, we may distinguish two cases for arbitrary graphs. We call a self-embedding of a graph *strong* if it extends to an injective map on the ends and *weak* otherwise. So Problem 1 asks for a generalisation to weak self-embeddings of graphs. However, a main step might be to consider the following subproblem.

Problem 2. *Generalise our main theorems to strong self-embeddings of graphs.*

5.2. **Tree alternative conjecture.** Let G and H be non-isomorphic graphs. We call H a *twin* of G if there are embeddings $G \to H$ and $H \to G$, i. e. injective maps $V(G) \to V(H)$ and $V(H) \to V(G)$ that preserve the adjacency relation. Bonato and Tardif [1] made the following conjecture.

Tree Alternative Conjecture. *A tree has either none or infinitely many isomorphism classes of twins.*

As, for twins G, H, the embeddings $G \to H$ and $H \to G$ can be composed to a self-embedding $G \to G$, there is a natural connection to the self-embeddings of G and Bonato and Tardif [1] suggested that the structure of the monoid of self-embeddings may help solving their conjecture. Indeed, Laflamme et al. [7] used this monoid to verify the conjecture for large classes of trees. In order to state their result, we need some definitions.

Let $R = x_0x_1\ldots$ be a ray in a tree T. Let T_i be the maximal subtree of T that is rooted at x_i and edge-disjoint from R. Let $\mathcal{T} := \{T_i \mid i \in I\}$ be a maximal set of these trees such that for no pair T_i, T_j with $i \neq j$ we have embeddings $(T_i, x_i) \to (T_j, x_j)$ and $(T_j, x_j) \to (T_i, x_i)$. If $\mathcal{T}$ is finite we call R *regular*. An end is *regular* if it contains a regular ray. It is easy to see that every ray in a regular end is regular.

A tree T is *stable* if one of the following holds.

(i) There is a vertex or an edge fixed by Emb(T);
(ii) two ends of T are fixed by Emb(T);

[3] If we also require the self-embeddings to preserve non-adjacency we can modify the example by subdividing every edge of the complete graph once. Then the self-embedding of Example 5.1 induces a self-embedding of the new graph that preserves not only adjacency but also non-adjacency.

(iii) T has an end that is preserved forwards and backwards by every self-embedding of T;
(iv) T has a ray R with $g(R) \subseteq R$ for all $g \in \mathrm{Emb(T)}$;
(v) T has a non-regular end preserved forwards by every self-embedding of T.

Theorem 5.2. [7, Theorem 1.9] *The tree alternative conjecture holds for stable trees.* □

With the help of our analysis of the monoid of self-embeddings of trees, we are able to give a precise description of the open cases of the tree alternative conjecture.

Corollary 5.3. *The tree alternative conjecture holds for all trees T whose monoid M of self-embeddings does not satisfy the following properties.*

(1) *There is a regular end ω of T fixed by M and all elements of M preserve ω forwards.*
(2) *There is no free submonoid of M generated freely by two non-elliptic elements.*

Proof. Let T be a tree and M be the monoid of its self-embeddings. By Theorem 2.8, one of the following holds.

(i) M fixes either a vertex or an edge of T;
(ii) M fixes a unique element of $\mathcal{L}(M)$;
(iii) $\mathcal{L}(M)$ consists of precisely two elements;
(iv) M contains two non-elliptic elements that do not fix the direction of the other.

While in cases (i) and (iii) Theorem 5.2 directly implies that the tree alternative conjecture holds, case (iv) together with Theorem 3.2 implies (2). So the only case that remains is if M fixes a unique element ω of $\mathcal{L}(M)$.

We continue by analysing $\mathcal{D}(M)$. If $\mathcal{D}(M)$ is empty, then we only have elliptic elements in M. Let $f \in M$. As f is elliptic and fixes ω, it fixes a vertex and hence the ray in ω starting at this vertex. So all elements of M preserve ω forwards and backwards. By Theorem 5.2, the tree alternative conjecture holds for T.

If ω is the only direction, then all elements of M preserve ω forwards. If ω is regular, then we have (1) and, if ω is not regular, then T is stable and the tree alternative conjecture holds by Theorem 5.2.

If there are at least two directions distinct from ω, then Theorem 3.2 implies (2).

It remains to consider the case that there exists precisely one direction g^+ distinct from ω. Since $|\mathcal{D}(M)| = 2$ and $\mathcal{D}(M)$ is dense in $\mathcal{L}(M)$ by Theorem 2.4 (i), we have $|\mathcal{L}(M)| = 2$. So we are in case (iii) and the tree alternative conjecture holds as we have already seen. □

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