Transitivity conditions in infinite graphs

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Abstract

We study transitivity properties of connected graphs with more than one end. We completely classify the distance-transitive such graphs and, for all $k \geq 3$, the k-CS-transitive such graphs.

1 Introduction

A *k-distance-transitive graph* is a graph G such that for every two pairs (x_1, x_2) and (y_1, y_2) of vertices with distances $d(x_1, x_2) = d(y_1, y_2) \leq k$ there is an automorphism α of G with $x_i^\alpha = y_i$ for $i = 1, 2$, where x_i^α is the image of x_i under α. A graph is called *distance-transitive* if it is k-distance-transitive for all $k \in \mathbb{N}$. Macpherson [11] and Ivanov [8] independently classified the connected locally finite distance-transitive graphs. They are exactly the graphs $X_{k,l}$, the infinite graphs of connectivity 1 such that each block is a K^k, a complete graph on k vertices, and every vertex lies in l distinct blocks. Here, k and l are integers, but we shall use the notation of $X_{\kappa,\lambda}$ also when κ or λ are infinite cardinals.

Answering a question of Thomassen and Woess [16], Möller [13] showed that the 2-distance-transitive locally finite connected graphs with more than one end are still only the graphs $X_{k,l}$.

For graphs that are not locally finite, little is known. Our first main result is the following common generalization of the theorems of Macpherson and Möller to arbitrary connected graphs with more than one end:

Theorem 1.1. *Let G be a connected infinite graph with more than one end. The following properties are equivalent:*

(i) G is distance-transitive;

(ii) G is 2-distance-transitive;

(iii) $G \cong X_{\kappa,\lambda}$ for some cardinals κ and λ with $\kappa, \lambda \geq 2$.

A graph is called *n-transitive* or also *n-arc-transitive* if it has no cycle of length at most n and for every two paths $x_0 \dots x_m$ and $y_0 \dots y_m$ with $0 \leq m \leq n$ it admits an automorphism α with $x_i^\alpha = y_i$ for all i.

Thomassen and Woess [16] characterized the locally finite connected graphs with more than one end that are 2-transitive. These are precisely the r-regular trees for some $r \in \mathbb{N}$. As a consequence of Theorem 1.1 we get the following characterization of all such graphs, not necessarily locally finite, which we prove at the end of Section 3.

Corollary 1.2. *If G is a connected 2-transitive graph with more than one end, then G is a λ-regular tree for some cardinal $\lambda \geq 2$.*

In the second part of this paper we investigate graphs with the property that the existence of an isomorphism φ between two finite induced subgraphs implies that there is an automorphism ψ of the entire graph mapping one of the subgraphs to the other. This area divides into two parts: In one part φ has to induce ψ on these subgraphs, while in the other part they may differ. More precisely, a graph G is *k-CS-transitive* if for every two connected isomorphic induced subgraphs of order k some isomorphism between them extends to an automorphism of G. On the other hand, G is called *k-CS-homogeneous* if *every* isomorphism between two induced connected subgraphs of order k of G extends to an automorphism of G. A graph is *CS-transitive* if it is k-CS-transitive for all $k \in \mathbb{N}$, and *CS-homogeneous* if it is k-CS-homogeneous for all $k \in \mathbb{N}$. Furthermore, a graph is *end-transitive* if its automorphism group acts transitively on the set of its ends.

Gray [6] classified the connected locally finite 3-CS-transitive graphs with more than one end and showed that these graphs are end-transitive. He asked whether all locally finite k-CS-transitive graphs for $k \geq 3$ are end-transitive. We give a positive answer to his question, and also show that the ends of k-CS-transitive graphs of arbitrary cardinality have at most two orbits under the action of the automorphism group of the graph.

Since 1-CS-transitive graphs are the transitive graphs and 2-CS-transitive graphs are the edge-transitive graphs, there is not much hope to classify them. Thus we investigate $k \geq 3$. We shall give a complete classification of these k-CS-transitive graphs with more than one end. This is formulated in Theorem 1.3.

In order to state our characterization we have to introduce some classes of graphs. For a graph H let $X_{\kappa,\lambda}(H)$ be the graph which arises from the graph $X_{\kappa,\lambda}$ by replacing each vertex with a copy of H and adding all edges between two copies replacing adjacent vertices of $X_{\kappa,\lambda}$.

For $\kappa \geq 3$, let Y_κ denote a connected graph that has two different kinds of blocks, single edges and blocks that are complete graphs of order κ, and in which every vertex lies in exactly one block of each kind.

Let H_1, H_2 be graphs, and let $\kappa, \lambda \geq 2$ be cardinals. We construct the graph $Z_{\kappa,\lambda}(H_1, H_2)$ as follows. Let T be an infinite tree, viewed as a bipartite graph with bipartition A, B, and assume that the vertices in A have degree κ and the vertices in B have degree λ. We replace every vertex from A by an isomorphic copy of H_1 and every vertex from B by an isomorphic copy of H_2. We add all edges between vertices that belong to graphs that replaced adjacent vertices. The resulting graph is a $Z_{\kappa,\lambda}(H_1, H_2)$.

We also need some finite *homogeneous*[1] graphs. These are graphs such that any isomorphism between two finite induced subgraphs (not necessarily connected) extends to an automorphism of the whole graph. These graphs were determined by Gardiner [5]. Interestingly, Ronse [15] showed that the class of finite homogeneous graphs coincides with its 'transitive' counterpart, the class of graphs such that for any two isomorphic induced subgraphs (not necessarily connected) there *exists* an isomorphism between them that extends to an automorphism of the whole graph.

The classes of finite homogeneous graphs featuring in our characterization will be the classes, denoted as $\mathcal{E}_{k,m,n}$, that occur in Enomoto's article [4] on combinatorially homogeneus graps. Each of these classes consists of all finite homogeneous graphs with the property that every vertex has at most m neighbours, every subgraph of order at least n is connected, and no two non-adjacent vertices have $k-2$ or more common neighbours. Furthermore, we exclude the complete graphs and the complements of complete graphs from $\mathcal{E}_{k,m,n}$ for technical reasons.

Now we are able to state our second main result, the classification of all connected k-CS-transitive graphs with more than one end if k is at least 3.

Theorem 1.3. *Let $k \geq 3$. A connected graph with more than one end is k-CS-transitive if and only if it is isomorphic to one of the following graphs*[2]*:*

(1) $X_{\kappa,\lambda}(K^1)$ *with arbitrary* κ *and* λ*;*

(2) $X_{2,\lambda}(K^n)$ *with arbitrary* λ *and* $n < \frac{k}{2} + 1$*;*

(3) $X_{\kappa,2}(\overline{K^m})$ *with arbitrary* κ *and* $m < \frac{k+2}{3}$*;*

(4) $X_{2,2}(E)$ *with* $E \in \mathcal{E}_{k,m,n}$*,* $m \leq k-2$*,* $n < \frac{k-|E|}{2} + 2$*, and* $2|E| - 2 < k$*;*

(5) Y_κ *with arbitrary* κ *(if k is odd);*

(6) $Z_{2,2}(\overline{K^m}, K^n)$ *with* $2m + n \leq k + 1$ *(if k is even);*

(7) $Z_{\kappa,\lambda}(K^1, K^n)$ *with* $n \leq k-1$*, arbitrary* κ, λ *with* $\kappa = 2$ *or* $\lambda = 2$ *(if k is even);*

(8) $Z_{2,2}(K^1, E)$ *with* $E \in \mathcal{E}_{k,m,n}$*,* $m \leq k-2$*,* $n \leq \frac{k}{2} + 1$ *(if k is even).*

Gray [6] characterized the connected locally finite 3-CS-homogeneous graphs with more than one end. As a corollary of Theorem 1.3 we obtain in Section 7 the following classification of connected k-CS-homogeneous graphs for $k \geq 3$ with more than one end.

[1] *ultrahomogeneous* in [5]

[2] By the definition of these graphs, κ and λ are at least 2 and κ is at least 3 in case (5).

Corollary 1.4. *Let $k \geq 3$. A connected graph with more than one end is k-CS-homogeneous if and only if it is isomorphic to $X_{\kappa,\lambda}(H)$ for one of the following values of κ, λ and graphs H:*

(1) arbitrary κ and λ and $H \cong K^1$;

(2) $\kappa = 2$, arbitrary λ, $n < \frac{k}{2}$ and $H \cong K^n$;

(3) arbitrary κ, $\lambda = 2$, $m < \frac{k}{3}$ and $H \cong \overline{K^m}$;

(4) $\kappa = \lambda = 2$, $H \in \mathcal{E}_{k,m,n}$ for $m \leq k-2$, $n < \frac{k-|E|}{2} + 1$, and $2|E| < k$.

Gray and Macpherson [7] classified the countable CS-homogeneous graphs[3]. Such graphs, connected and with more than one end, are those described in our Theorem 1.1 for countable cardinals κ, λ. As a further corollary of Theorem 1.3 we can extend their classification to arbitrary connected graphs with more than one end.

Corollary 1.5. *For connected graphs with at least two ends the notions of being distance-transitive, CS-transitive, or CS-homogeneous coincide. (These graphs are described in Theorem 1.1.)* □

Let us say a word about the techniques we use for our proofs. The proofs of the corresponding theorems for locally finite graphs are all based on Dunwoody's *structure trees* corresponding to finite edge cuts that are invariant under the action of the automorphism group of the graph. This structure tree theory is described in the book of Dicks and Dunwoody [1]; see Möller [12, 14] and Thomassen and Woess [16] for introductions. Since those edge cuts must be finite, these structure trees can in general only be applied to locally finite graphs.

Recently, Dunwoody and Krön [3] developed a similar structure tree theory based on vertex cuts, providing a similarly powerful tool for the investigation of graphs that are not necessarily locally finite. We use this new theory in our proofs.

2 The structure tree

Throughout this paper we use the terms and notation from [2] if not stated otherwise. In particular, a *ray* is a one-way infinite path and a *double ray* is a two-way infinite path. Two rays in a graph G are *equivalent* if there is no finite vertex set S in G such that the two rays lie eventually in distinct components of $G - S$. (For an induced subgraph H and a subset S of the vertex set of G, we use $H - S$ to denote the induced subgraph $G[V(H) \setminus S]$ and $H + S$ to denote $G[V(H) \cup S]$.) The equivalence of rays is an equivalence relation whose classes are the *ends* of G.

Let G be a connected graph and $A, B \subseteq V(G)$ two vertex sets. The pair (A, B) is called a *separation* of G if

[3]They call these graphs *connected-homogeneous* graphs.

(i) $A \cup B = V(G)$ and

(ii) $E(G[A]) \cup E(G[B]) = E(G)$.

The *order* of a separation (A, B) is the cardinality of its *separator* $A \cap B$ and the *wings* of (A, B) are the induced subgraphs $G[A \setminus B]$ and $G[B \setminus A]$. With $(A, \sim)$ we refer to the separation $(A, (V(G) \setminus A) \cup N(V(G) \setminus A))$. A *cut* is a separation (A, B) of finite order with non-empty wings such that the wing $G[A \setminus B]$ is connected and such that no proper subset of $A \cap B$ separates the wings of (A, B)[4]. A *cut system* $\mathcal{S}$ is a non-empty set of cuts of G satisfying the following properties.

(1) If $(A, B) \in \mathcal{S}$ then there is an $(X, Y) \in \mathcal{S}$ with $X \subseteq B$.

(2) Let $(A, B) \in \mathcal{S}$ and C be a component of $G[B \setminus A]$. If there is a separation $(X, Y) \in \mathcal{S}$ with $X \setminus Y \subseteq C$, then the separation $(V(C) \cup N(C), \sim)$ is also in $\mathcal{S}$.

(3) If $(A, B) \in \mathcal{S}$ with wings X, Y and $(A', B') \in \mathcal{S}$ with wings X', Y' then either there is a component C in $X \cap X'$ and a component D in $Y \cap Y'$, or there is a component C in $Y \cap X'$ and a component D in $X \cap Y'$ such that both $(V(C) \cup N(C), \sim)$ and $(V(D) \cup N(D), \sim)$ are $\mathcal{S}$-separations.

The cuts in $\mathcal{S}$ are also called *$\mathcal{S}$-separations* and an *$\mathcal{S}$-separator* is a vertex set that is the separator of an $\mathcal{S}$-separation.

Two vertices, vertex sets or subgraphs X, Y of G are *separated* by a separation (A, B)—not necessarily in $\mathcal{S}$—if either $X \subseteq A$ and $Y \subseteq B$, or $Y \subseteq A$ and $X \subseteq B$. They are separated *properly* if both, X and Y, meet components C and D of their corresponding wings such that every vertex in $A \cap B$ is adjacent to a vertex in C and a vertex in D. A vertex set S *separates* X and Y (*properly*) if there is a separation (A, B) with separator S that separates X and Y (properly). A vertex set or subgraph is *separated properly* by a separation (or its separator) if it contains two vertices that are separated properly by this separation.

Two separations $(A, B), (A', B')$ are *$\mathcal{S}$-nested* if there is one wing of each of them, W, W' say, such that both separators $A \cap B$ and $A' \cap B'$ are disjoint from $W \cup W'$ and such that there is no component C of $W \cap W'$ with $(C \cup N(C), \sim) \in \mathcal{S}$.[5] If it is clear which cut system we are referring to we may drop its identifier and speak of nested only. The cut system $\mathcal{S}$ is *nested* if each two $\mathcal{S}$-separations are ($\mathcal{S}$-)nested. If $\mathcal{S}$ is nested, then no $\mathcal{S}$-separation (A, B) separates any other $\mathcal{S}$-separator S properly, since S meets at most one wing of (A, B).

A cut in the cut system $\mathcal{S}$ is *minimal* if no other cut in $\mathcal{S}$ has smaller order. A *minimal cut system* is a cut system all of whose cuts are minimal and thus

[4]Dunwoody and Krön [3] call the corresponding vertex set $A \setminus B$ a *cut* and the set $B \setminus A$ the $*$-*complement* of this cut.

[5]This means that there is no '$\mathcal{S}$-important' part of G that lies in $W \cap W'$; Dunwoody and Krön [3] call the vertex set of $W \cap W'$ an *isolated corner*.

have the same order. If $\mathcal{S}$ is a minimal cut system, then the *order* $\mathrm{ord}(\mathcal{S})$ of $\mathcal{S}$ is the order of any of its cuts.

Remark 2.1. Let G be a transitive connected graph and let $\mathcal{S}$ be a nested cut system of G. Then any component of $G - S$ for an $\mathcal{S}$-separator S, is the wing of an $\mathcal{S}$-separation [3, Corollary 3.10]. In particular, for any two (nested) $\mathcal{S}$-separations (A, B) and (A', B') there is a wing of each of them, W, W' say, such that $W \subseteq W'$ or $W' \subseteq W$.

An *($\mathcal{S}$-)block* is a maximal induced subgraph X such that

(i) for every $(A, B) \in \mathcal{S}$ there is $V(X) \subseteq A$ or $V(X) \subseteq B$ but not both, that is X is not separated by any $\mathcal{S}$-separation;

(ii) there is some $(A, B) \in \mathcal{S}$ with $V(X) \subseteq A$ and $A \cap B \subseteq V(X)$.

Let $\mathcal{B}$ be the set of $\mathcal{S}$-blocks and let $\mathcal{W}$ be the set of $\mathcal{S}$-separators. If $\mathcal{S}$ is nested and minimal let $\mathcal{T}(\mathcal{S})$ be the graph with vertex set $\mathcal{W} \cup \mathcal{B}$ and edges WB ($W \in \mathcal{W}$ and $B \in \mathcal{B}$) if and only if $W \subseteq B$. Then $\mathcal{T} = \mathcal{T}(\mathcal{S})$ is called the *structure tree* of G and $\mathcal{S}$.

It is the same structure tree that is used by Dunwoody and Krön [3] but we use a different notation for the underlying cut system. They substantiate the term 'structure tree' in one of their theorems.

Theorem 2.2 ([3, Theorem 6.5]). *Let G be a connected graph, and let $\mathcal{S}$ be a nested minimal cut system of G. Then the structure tree of G and $\mathcal{S}$ is a tree.* □

We remark that this implies for every $\mathcal{S}$-separation (X, Y) that (if $\mathcal{S}$ is minimal and nested) there is an edge WB in $\mathcal{T}$ such that W is the $\mathcal{S}$-separator $X \cap Y$ and $V(B) \subseteq X$. On the other hand, it follows from (2) of the definition of a cut system that for any $\mathcal{S}$-block B and any $\mathcal{S}$-separator $S \subseteq B$, there is an $\mathcal{S}$-separation (X, Y) with separator S such that $V(B) \subseteq X$.

In our proofs we use a certain kind of minimal cut system that was introduced by Dunwoody and Krön [3, Example 2.5].

Example 2.3. Let G be a connected infinite graph with at least two ends. Let $\mathcal{S}$ be the set of all cuts (A, B) such that both $G[A]$ and $G[B]$ contain a ray. Then $\mathcal{S}$ is a cut system.

We need a fundamental property of cut systems that is shown in [3, Theorem 8.6] by Dunwoody and Krön. Since we do not use the whole theorem, we only state the part that is applied in this paper.

Theorem 2.4. *Let G be a connected graph with at least two ends and let $\mathcal{C}$ be the cut system of G from Example 2.3. There is a nested cut system $\mathcal{S} \subseteq \mathcal{C}$ consisting only of minimal $\mathcal{C}$-separations that is invariant under* $\mathrm{Aut}(G)$ *such that if two ends are separated by a minimal cut in $\mathcal{C}$, then they are separated by a cut in $\mathcal{S}$.* □

For a connected graph G, a cut system is called *basic* if it is maximal with the following properties: it is nested, minimal and $\mathrm{Aut}(G)$-invariant, all of its separators lie in the same $\mathrm{Aut}(G)$-orbit, both wings of each cut contain a ray and the order of any cut is minimal with regard to separating two ends of G. We may state a useful corollary of Theorem 2.4 which we shall use in the later proofs without further mentioning.

Corollary 2.5. *Every connected graph with at least two ends has a basic cut system.* □

Let us investigate some properties of basic cut systems.

Lemma 2.6. *For a basic cut system $\mathcal{S}$ of a connected graph G with at least two ends and any $\mathcal{S}$-separator S, every component of $G - S$ that contains a ray is a wing of an $\mathcal{S}$-separation.*

Proof. For this proof we invoke [3, Lemma 3.9] which says that no separator of a nested cut system separates any other separator of that cut system properly. Let C be a component of $G - S$ containing a ray. We show that the separation $(V(C) \cup S, \sim)$ lies in $\mathcal{S}$. If there is an $\mathcal{S}$-separation (X, Y) whose separator S' meets C, then $S' \subseteq V(C) \cup S$ as $\mathcal{S}$ is nested and no two vertices of an $\mathcal{S}$-separator are separated properly by any $\mathcal{S}$-separator. Thus either X or Y is contained in $V(C) \cup S$ and $(V(C) \cup S, \sim)$ and (X, Y) are nested.

If there is an $\mathcal{S}$-separation (X, Y) whose separator S' does not meet C, then one wing of (X, Y) is disjoint from C and from S and thus (X, Y) and $(V(C) \cup S, \sim)$ are nested. Thus $(V(C) \cup S, \sim)$ is nested with all $\mathcal{S}$-separations. Clearly, there is a ray in $G - C$ as S is a separator of a basic cut system. Thus $\mathcal{S} \cup (V(C) \cup S, \sim)$ is nested, minimal and $\mathrm{Aut}(G)$-invariant, all of its separators lie in the same $\mathrm{Aut}(G)$-orbit, both wings of each cut contain a ray and the order of any cut is minimal with regard to separating two ends of G. As $\mathcal{S}$ is basic and thus maximal with these properties, it contains the cut $(V(C) \cup S, \sim)$. □

Together with Lemma 3.9 in [3] this implies the following lemma.

Lemma 2.7. *Let G be a connected graph with at least two ends and let $\mathcal{S}$ be a basic cut system of G. For any $\mathcal{S}$-separator S the components of $G - S$ that do not contain a ray are disjoint from any $\mathcal{S}$-separator.* □

For a basic cut system this lemma yields the following remark.

Remark 2.8. Let S, S' be two distinct $\mathcal{S}$-separators of a basic cut system $\mathcal{S}$ of a connected graph G. Then S' meets precisely one component of $G - S$ and this component contains a ray.

Lemma 2.9. *Let $\mathcal{S}$ be a basic cut system of a connected graph G with more than one end. Then, every finite vertex set separating two $\mathcal{S}$-separators separates two ends.*

In particular, less than $\mathrm{ord}(\mathcal{S})$ *vertices do not separate any two $\mathcal{S}$-separators.*

Proof. Let S be a finite vertex set separating two distinct $\mathcal{S}$-separators S_1 and S_2. As $\mathcal{S}$ is nested and according to Remark 2.8, there is a component C_1 of $G - S_1$ containing an end ω_1 but no vertex of S_2 as well as a component C_2 of $G - S_2$ containing an end ω_2 and no vertex of S_1. Let $C = C_1 \cap C_2$. If C contains a vertex v, then for every $s \in S_1 \setminus S_2$ there is a v-s path with its inner vertices in C_1 as S_1 is minimal end separating. By the choice of C_1, this path contains no vertex from S_2. This implies that C_2 contains s contrary to the choice of C_2. Thus C is empty and $\omega_1 \neq \omega_2$.

Suppose that S does not separate ω_1 and ω_2. Then ω_1 and ω_2 live in the same component of $G - S$ and thus there is a double ray R with one tail in ω_1 and another one in ω_2 avoiding S. Every such double ray meets S_1 and S_2 as shown above. Hence R contains an S_1-S_2 path contradicting that S separates these two $\mathcal{S}$-separators. The last assertion holds, as $\mathcal{S}$ is basic, particularly as no vertex set of cardinality less than $\mathrm{ord}(\mathcal{S})$ separates any two ends. □

The following lemma is proved in [3, Lemma 4.1]. We state it here as it nicely shortens some proofs.

Lemma 2.10. *For any k, every pair of vertices in a connected graph is separated properly by only finitely many distinct separators of order k.* □

2.1 Basic cut systems of special graphs

In Theorem 1.1 and 1.3 several classes of graphs arise. Let us give descriptions of basic cut systems and their structure trees for each of them.

The *building blocks* of $X_{\kappa,\lambda}(H)$ and $Z_{\kappa,\lambda}(H_1, H_2)$ are the isomorphic copies of H, H_1, and H_2 that are used for the construction of these graphs. For a Y^κ the copies of K^κ and the bridges are its *building blocks*.

Let G be isomorphic to $X_{\kappa,\lambda}(H)$ for $\kappa, \lambda \geq 2$ and a finite graph H. In this case there is a unique basic cut system of G. Its separators are the building blocks of the $X_{\kappa,\lambda}(H)$, and its separations are of the form $(V(C) \cup S, \sim)$, where S is any of the separators and C any component of $G - S$. Any block consists of the union of a maximal set of pairwise completely adjacent building blocks. The structure tree is a (semi-regular) tree of degrees κ and λ where the blocks have degree κ and the separators have degree λ.

Let G be isomorphic to Y_κ for $\kappa \geq 3$, then G is vertex transitive and every vertex is a separator of G that separates ends. The unique basic cut system has every single vertex as a separator and separations as in the example above. The blocks are precisely the building blocks. The structure tree is the κ-regular tree with every edge subdivided three times. The vertices of degree κ are the blocks corresponding to the K^κ and the vertices with distance two to them are the blocks corresponding to the K^2. The separators are precisely the vertices of the tree that are adjacent to a vertex of degree κ. The automorphism group has two orbits on the blocks. One orbit contains the building blocks of cardinality 2 and the other orbit those of cardinality κ. This shows that even though the automorphism group acts transitively on the separators it may not act transitively on the blocks.

Let G be isomorphic to $Z_{\kappa,\lambda}(H_1, H_2)$ for $\kappa, \lambda \geq 2$ and non-empty finite graphs H_1, H_2. In this case there may be two distinct basic cut systems, this happens only if $|H_1| = |H_2|$ and either $H_1 \not\cong H_2$ or $\kappa \neq \lambda$. Then one may choose $i, j \in \{1, 2\}$ with $i \neq j$ arbitrarily and there is a basic cut system $\mathcal{S}$ of G with the building blocks corresponding to H_i as the $\mathcal{S}$-separators and the building blocks corresponding to H_j plus all its neighbours in G as the $\mathcal{S}$-blocks. If $H_1 \cong H_2$ and $\kappa = \lambda$, then $G \cong X_{2,\lambda}(H_1)$ and the basic cut system is as discussed above. If $|H_i| < |H_j|$ for $i, j \in \{1, 2\}$, then the building blocks corresponding to H_i are precisely the $\mathcal{S}$-separators and any building block corresponding to H_j plus all its neighbours is an $\mathcal{S}$-block. In both cases all cuts are of the form $(V(C) \cup S, \sim)$ where C is a component of the graph minus a separator S. The structure tree is a semi-regular tree with degrees κ and λ, where if H_1 corresponds to the separators they have degree κ and the blocks have degree λ and if H_2 corresponds to the separators the degrees swap.

3 Distance-transitive graphs

In this section we classify the connected distance-transitive graphs with more than one end (Theorem 1.1). Let us give a short outline of the proof, in particular of the implication that every connected 2-distance-transitive graph with more than one end is an $X_{\kappa,\lambda}$ for some cardinals κ and λ. Considering a basic cut system of such graphs, we show that its blocks are complete graphs and that any two of its separators are disjoint. We finish the proof by showing that all separators of the given cut system have cardinality 1 and have to lie in the same number of blocks and that each block consists of the same number of separators.

Proof of Theorem 1.1. Since the graphs $X_{\kappa,\lambda}$ are indeed distance-transitive and distance-transitive graphs are 2-distance-transitve by definition, it suffices to prove that every connected 2-distance-transitive graph with at least two ends is an $X_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$.

Let G be a connected 2-distance-transitive graph with more than one end. Let $\mathcal{S}$ be a basic cut system of G and let $\mathcal{T}$ be the structure tree of G and $\mathcal{S}$. In particular, for every separation $(A, B) \in \mathcal{S}$ and every automorphism α of G, the cuts $(A, B), (A^\alpha, B^\alpha)$ are nested and (A^α, B^α) lies also in $\mathcal{S}$. Furthermore, both wings of any cut in $\mathcal{S}$ contain a ray. As every 2-distance-transitive graph is vertex transitive by definition and thus every vertex lies in an $\mathcal{S}$-separator, which implies that every vertex lies in an $\mathcal{S}$-block.

Let us show first that all $\mathcal{S}$-blocks are complete graphs. Suppose not and let X be such an $\mathcal{S}$-block that is not complete. Let x, y be two non-adjacent vertices in X and let P be a shortest x-y path in G. To get to a contradiction let us find a block containing three consecutive vertices of P. If P is contained in X, let $Y = X$ and $a, b \in V(X \cap P)$ with $d(a, b) = 2$. If P is not contained in X, then there is an $\mathcal{S}$-separator, separating X and a vertex on P properly. Let S be such an $\mathcal{S}$-separator with maximal distance from X in $\mathcal{T}$. Then there is a component C of $G - S$ that avoids X and contains a vertex v from P for which $(V(C)\cup S,\sim)$ lies in $\mathcal{S}$, according to Lemma 2.6. Let Y be the neighbour of S in $\mathcal{T}$ contained in $C+S$, that is Y is the $\mathcal{S}$-block in $C+S$ containing S. The two neighbours of v on P lie in C or S and all vertices of $P\cap C$ lie in Y by the choice of S. Thus, three consecutive vertices on P, the vertex v and its two neighours a and b, lie in Y, and as P is an induced path a and b are not adjacent and $d(a,b)=2$.

As $\mathcal{S}$ is a cut system, for every $\mathcal{S}$-separation (A,B) every vertex s in $A\cap B$ has a neighbour c in $A\setminus B$ and d in $B\setminus A$ such that these neighbours are separated properly by (A,B). As $cd\notin E(G)$, we have $d(c,d)=2$. Since G is 2-distance-transitive, there is an automorphism α of G with $c^\alpha=a$ and $d^\alpha=b$. This contradicts the fact that Y is an $\mathcal{S}$-block as it is separated properly by (A^α,B^α) as c^α and d^α which are both contained in Y have to lie in distinct wings of (A^α,B^α). Thus all $\mathcal{S}$-blocks are complete.

Let us continue by showing that two distinct $\mathcal{S}$-separators S,S' are disjoint. Let (A,B) be an $\mathcal{S}$-separation and $\alpha\in\mathrm{Aut}(G)$ such that $S=A\cap B$ and $S^\alpha=S'$. These choices are valid since $\mathcal{S}$ is basic. As (A,B) and (A^α,B^α) are nested and G is transitive, we know by Remark 2.1 that there are wings, one of each of these two separations, W,W' say, that are disjoint. Suppose $S\cap S'$ is not empty and let $s\in S\cap S'$, $s'\in S'\setminus S$ and $w\in W$, $w'\in W'$ both adjacent to s. As all blocks are complete, s and s' are adjacent. Furthermore, w and s' are not adjacent, since they are separated by S. Thus, there is an automorphism β of G mapping (w,w') to (w,s'), since $ww',ws'\notin E(G)$. This is a contradiction according to Lemma 2.10 which says that there are only finitely many separators of cardinality $\mathrm{ord}(\mathcal{S})$ separating w and w' properly: The existence of β implies that there is the same finite number of $\mathcal{S}$-separators separating w from w' and w from s' properly. This does not hold since all $\mathcal{S}$-separators separating w and s' properly lie in the component of $G-S'$ that contains w and thus these separators also separate w and w' properly. On the other hand S^α separates w and w' properly while it does not separate w and s' properly. Thus, any two distinct $\mathcal{S}$-separators are disjoint.

In the next step let us show that all $\mathcal{S}$-separators have cardinality 1. Suppose not, then there are at least two vertices in some $\mathcal{S}$-separator S and, as all $\mathcal{S}$-blocks are complete, there is an edge e in $G[S]$. On the other hand, there is an edge e' that has precisely one of its end vertices in S. Since G is 2-distance-transitive it is also 1-distance-transitive and thus there is an automorphism α of G that maps e to e'. This is a contradiction, since S and S^α are neither disjoint nor the same. Thus all $\mathcal{S}$-separators have cardinality 1.

As G is 1-distance-transitive any two $\mathcal{S}$-blocks have the same order and 0-distance-transitivity implies that for every vertex the set of $\mathcal{S}$-blocks it lies in has the same cardinality λ. The order κ of an $\mathcal{S}$-block is at least 2, since there are edges in G and every $\mathcal{S}$-separator lies in at least two different $\mathcal{S}$-blocks. Thus G is isomorphic to $X_{\kappa,\lambda}$ for two cardinals $\kappa,\lambda\geq 2$. □

Next, we briefly deduce Corollary 1.2 from Theorem 1.1.

Proof of Corollary 1.2. A 2-transitive graph is also 2-distance-transitive and, if it has at least two ends, then it is an $X_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$. If $\kappa \geq 3$, then there is a path of length 2 in every block whose (adjacent) endvertices can be mapped onto vertices with distance 2 in distinct blocks. Since no adjacent vertices can be mapped onto vertices with distance 2 by any isomorphism, we know that $\kappa = 2$. The graphs $X_{2,\lambda}$ with $\lambda \geq 2$ are precisely the λ-regular trees. □

4 The local structure for some finite subgraphs

In some k-CS-transitive graphs the previously introduced finite homogeneous graphs play a role as building blocks. Enomoto [4] gave a combinatorially characterization of these homogeneous graphs. We apply a corollary of his result [4, Theorem 1] in our proofs.

For a subgraph X of a graph G let $\Gamma(X) = \bigcap_{x \in V(X)} N(x)$, which is the set of all vertices in G that are adjacent to all the vertices in X. A graph G is *combinatorially homogeneous* if $|\Gamma(X)| = |\Gamma(X')|$ for any two isomorphic induced subgraphs X and X'. Furthermore, a graph G is *l-S-transitive* if for every two isomorphic induced subgraphs of order l there is an automorphism of G mapping one onto the other.

Theorem 4.1. *[4, Theorem 1] Let G be a finite graph. The following properties of G are equivalent.*

(1) G is homogeneous;

(2) G is combinatorially homogeneous;

(3) G is isomorphic to one of the following graphs:

- *(a) a disjoint union of isomorphic complete graphs;*
- *(b) a complete t-partite graph K_r^t with r vertices in each partition class and with $2 \leq t, r$;*
- *(c) C_5;*
- *(d) $L(K_{3,3})$ (the line graph of $K_{3,3}$).* □

Whenever we need finite homogeneous graphs as building blocks for k-CS-transitive graphs we use Corollary 4.2 to handle them.

Corollary 4.2. *Let $k \geq 3$, $m \leq k-2$, and $n \leq \frac{k}{2}$ be positive integers. Let G be a finite graph with maximum degree at most m that is neither complete nor the complement of a complete graph. If G is l-S-transitive for all $l \leq k-1$, if any induced subgraph of G on at least n vertices is connected, and if any two non-adjacent vertices do not have $k-2$ common neighbours, then G is (combinatorially) homogeneous and isomorphic to one of the following graphs:*

(1) t disjoint K^r with $2 \leq t$, $1 \leq r-1 \leq m$, and $tr \leq n-1$;

(2) K_r^t *with* $2 \le t$*,* $2 \le r \le n-1$*, and* $(t-1)r \le \min\{m, k-3\}$*;*

(3) C_5 *with* $2 \le m$ *and* $4 \le n$*;*

(4) $L(K_{3,3})$ *with* $4 \le m$ *and* $6 \le n$*.*

Proof. Let us prove first that G is combinatorially homogeneous. If X and X' are isomorphic induced subgraphs of G both of order at most $k-1$, then l-S-transitivity for $l = |X|$ implies that there is an automorphism φ of G with $X^\varphi = X'$. Thus, we have $\Gamma(X)^\varphi = \Gamma(X')$ and $|\Gamma(X)| = |\Gamma(X')|$. If X and X' are isomorphic induced subgraphs of order at least k, then both $\Gamma(X)$ and $\Gamma(X')$ are empty because the maximum degree of G is at most $k-2$. This implies that G is combinatorially homogeneous and that we can apply Theorem 4.1 which provides that, ignoring the boundaries, there are no other cases as (1) to (4). The specific boundaries for each case can be checked easily. For example, in case (2) the '$k-3$' in the inequality $(t-1)r \le \min\{m, k-3\}$ ensures that K_r^t does not contain two non-adjacent vertices with $k-2$ common neighbours if $m = k-2 = (t-1)r$. □

Let $\mathcal{E}_{k,m,n}$ be the class of all those graphs that satisfy the assumptions of Corollary 4.2 for the values k, m and n.

5 k-CS-transitivity for special graphs

This section is dedicated to showing that any graph on the list in Theorem 1.3 is indeed k-CS-transitive for the specific values of k.

Let G be a graph and $k \ge 3$. A graph H is *good for* G if for any two induced isomorphic copies H' and H'' of H in G there is an automorphism of G mapping H' onto H''. Clearly, a graph is k-CS-transitive if and only if all of its connected induced subgraphs of order k are good for it.

Lemma 5.1. *Let* $k \ge 3$ *and let* G *belong to one of the classes (1) to (8) of Theorem 1.3. The complete graph on* k *vertices is good for* G*.*

Proof. If G contains a complete graph on k vertices, then it is isomorphic to $X_{\kappa,\lambda}(K^1)$, $X_{2,\lambda}(K^n)$, $X_{\kappa,2}(\overline{K^m})$, Y_κ, $Z_{\kappa,2}(K^1, K^n)$, or $Z_{2,\lambda}(K^1, K^n)$ with the corresponding values for m and n.

- In $X_{\kappa,\lambda}(K^1)$ and Y_κ any complete graph on k vertices lies completely in some K^κ.
- In $X_{2,\lambda}(K^n)$, as $2n < k+2$, any complete graph on k vertices consists of precisely two building blocks or precisely two building blocks without one vertex depending on the parity of k.
- In $X_{\kappa,2}(\overline{K^m})$ any complete graph on k vertices has no two vertices in the same building block, and all its vertices in building blocks that are pairwise completely adjacent.

- In $Z_{\kappa,2}(K^1, K^n)$ and $Z_{2,\lambda}(K^1, K^n)$, as $n \leq k-1$, any complete graph on k vertices consists of precisely two adjacent building blocks.

In all these cases K^k is good for G by the construction of G. □

Lemma 5.2. *Let $k \geq 3$ and let G belong to one of the classes (1) to (8) of Theorem 1.3. Every connected graph on k vertices with diameter 2 is good for G.*

Proof. Let X be a connected induced subgraph of G on k vertices with diameter 2. If $G \cong Y_\kappa$ then X is isomorphic to some K^{k-1} with one edge attached. For any two such graphs in G, there is an automorphism of G mapping one to the other. Thus we may assume that $G \ncong Y_\kappa$.

If X is contained in a single building block, then—by cardinality and as it is neither complete nor the complement of a complete graph—$G \cong Z_{2,2}(K^1, E)$. Again by cardinality X lies in a building block corresponding to $E \cong K_r^t$ with $2 \leq t$ and $2 \leq r \leq \frac{k}{2}$ and $(t-1)r \leq k-3$. As $(t-1)r \leq k-3$ holds, there are at least 3 vertices of X in any of its necessarily t partition classes. This implies that for any (complete multipartite) induced subgraph Y of G isomorphic to X there is one building block containing Y, since—because of its diameter—it is contained in at most three building blocks and no building block corresponding to K^1 is contained in any complete multipartite induced subgraph of G that consists of t classes, each of which has cardinality at least 3. By the construction of G there is an automorphism α of G mapping the building block containing X to the building block containing Y and, as E is homogeneous, with $X^\alpha = Y$.

Therefore we may assume that X meets at least two building blocks. If X meets precisely two building blocks, then by cardinality $G \cong X_{2,2}(E)$ for some graph $E \in \mathcal{E}_{k,m,n}$ with $m \leq k-2$, $n < \frac{k-|E|}{2} + 2$ and $2|E| - 2 < k$, or k is even and $G \cong Z_{2,2}(K^1, E)$ for some graph $E \in \mathcal{E}_{k,m,n}$ with $m \leq k-2$ and $n \leq \frac{k}{2}+1$.

In the first case, since $2|E| - 2 < k$, either X covers both building blocks it meets (if k is even) or it misses precisely one vertex in one of these building blocks (if k is odd). As E is homogeneous X is good for G.

In the second case there is one vertex v with $k-1$ neighbours that is the building block corresponding to K^1. As $n \leq \frac{k}{2} + 1$ we know that $X - v$ is connected. On the other hand Y contains a vertex with degree $k-1$ and thus is not contained in a single building block. Let $v' \in V(Y)$ be a vertex in a building block of G corresponding to K^1. As X and Y are isomorphic and any two vertices of degree $k-1$ in Y lie in the same $\mathrm{Aut}(Y)$-orbit it holds that $Y - v'$ is connected, and thus $Y - v'$ lies in a single building block of G. As above there is an automorphism α of G mapping the building blocks containing X to the building blocks containing Y with $(X - v)^\alpha = (Y - v')$.

Thus we may assume that X meets at least three building blocks. Let $B \subseteq G$ be a building block that is adjacent to all vertices of $X \setminus B$, which exists by the small diameter of X. If a separator in X does not contain every vertex of $X \cap B$, then it must contain at least all the vertices in $X \setminus B$ as every vertex of X in B is adjacent to every vertex of X not in B. Furthermore, the existence of a separator that separates $X \cap B$ properly implies that B is not complete. If the number $|X \cap B|$ is smaller than $|X \setminus B|$, then $X \cap B$ is the unique smallest separator and for every isomorphic induced copy Y of X in G precisely the vertices of $X \cap B$ are mapped to *the* smallest separator S in Y. We may assume that Y meets three building blocks, as it, and thus X is good for G otherwise. Since S is a smallest separator, we have $S = Y \cap D$ for the unique building block D of G that is adjacent to all vertices of $Y \setminus D$. Each of the smallest separators of these graphs either contains an edge, contains two non adjacent vertices, or is a single vertex. In all these cases B and D correspond to the same kind of building block by the construction of G. Since the building blocks are homogeneous and B is mapped to D by some automorphism of G, every isomorphism from X to Y extends to an automorphism of G. Thus we may assume that $X \cap B$ is not the unique smallest separator of X and also it is not complete.

Let us finish the remainder of the proof on a case by case analysis. The previous arguments cover (1), (2), (5), and (7) of Theorem 1.3. In (3) as $m < \frac{k+2}{3}$ and $k \geq 3$ it holds that $m < \frac{k}{2}$ and thus if there is a building block B, that separates X, then it is unique and $X \cap B$ is the smallest separator in X. If there is no such separating building block, then all building blocks that meet X are pairwise adjacent and X is a complete multipartite graph with at least three partition classes. As vertices of X lie in the same building block if and only if they are not adjacent, X is good for G.

In (4) there is a unique building block $B \cong E$ adjacent to all vertices in $X \setminus B$ and B separates X. If $X \cap B$ is not the smallest separator in X, then $\frac{k}{2} \leq |X \cap B|$ and as $2|E| - 2 < k$ it holds that $|B| < \frac{k}{2} + 1$ and thus $X \cap B = B$. The building block $B \cong E$ is connected, since $n < \frac{k-|E|}{2} + 2$. All connected graphs in $\mathcal{E}_{k,m,n}$ are 2-connected and thus any separator of X not containing $X \cap B$ contains $X \setminus B$ and at least two vertices from B and hence has at least $\frac{k}{2} + 1$ vertices. Again $X \cap B$ is the unique smallest separator in X, which completes this case.

For the case (6) that $G \cong Z_{2,2}(\overline{K^m}, K^n)$, if $n \neq 1$, then $m < \frac{k}{2}$ and thus $X \cap B$ is the smallest separator in X, as it is either complete or lies in a building block corresponding to $\overline{K^m}$ of order less than $\frac{k}{2}$. If $n = 1$, then B is either complete and the smallest separator or B is not complete and the two building blocks adjacent to B together with B cover X. Thus $|B| + 2 \geq k \geq 2m$ and this implies that $m = 2$ and $k = 4$. Since B is not complete it holds that $B \cong \overline{K^2}$ and $X \cong C_4$. Then it is easy to see that X can be mapped to every other copy of C_4 in $G \cong Z_{2,2}(\overline{K^2}, K^1)$ by some automorphism of G.

In (8) $G \cong Z_{2,2}(K^1, E)$ and we may assume that X meets two building blocks corresponding to K^1 and one other building block $B \cong E$, as otherwise the separating building block is complete, consists of only one vertex and is the unique smallest separator of X. Thus every induced subgraph Y of G isomorphic to X is good for G or meets precisely three building blocks, and—by the same arguments as above—two of these building blocks that Y meets correspond to the K^1. Any pair of non-adjacent vertices in X with $k - 2$ common neighbours in X, can be mapped to any other such pair by an automorphism of X. By the construction of G there is an automorphism α of G mapping the two building blocks corresponding to K^1 in X onto those in Y. As E is homogeneous and $X \cap B$ and $Y \cap B^\alpha$ are isomorphic, there is an automorphism of G mapping X onto Y.

Thus X is good for G in all cases. □

Lemma 5.3. *Let $k \geq 3$ and let G belong to one of the classes (1) to (8) of Theorem 1.3. Every connected graph on k vertices with diameter at least 3 is good for G.*

Proof. Let X and Y be isomorphic connected induced subgraphs of G on k vertices with diameter at least 3 and let α be an isomorphism from X to Y. If X is a path, then there is an automorphism of G mapping X to Y according to the construction of G. Thus we may assume that X is not a path.

If $G \cong Y^\kappa$, then there is a maximal clique $K \subseteq X$ with at least 3 vertices. By the construction of Y^κ there is an automorphism α' of G that maps the building block containing K to the building block of G containing K^α and that is an extension of α.

If G is not isomorphic to a Y_κ, let P be a longest induced path in X whose diameter in X is at least 3. We show that every vertex v on P that lies in a building block corresponding to a finite graph B is mapped onto a vertex $v^\alpha \in V(Y)$ that also lies in a building block corresponding to B. This is easy in all the cases that have only one kind of building block. In particular, we have to proof this property in the cases (6), (7), and (8) of Theorem 1.3.

The path P meets at least four building blocks of G, since there is no building block B in any of the possible graphs with an induced path of length 3, except for the C_5, in which case $k > 5$ and X meets a building block adjacent to B and the diameter of $X \cap B$ in X is 2. As X is connected and not a path, there is a vertex v in $X - P$ that is adjacent to P. The cardinality of $N(v) \cap V(P)$ is $1, 2$, or 3, as P is induced and thus meets every building block in at most one vertex. In particular, these neighbours of v have distance at most 2 on P. Let us show that these cases determine in which kinds of building blocks the neighbours of v on P lie.

If v has only one neighbour p on P, then p is not a leaf of P by the maximiality of P. Furthermore, the vertices v and p do not lie in the same building block, as v would be adjacent to the same vertices on P as p otherwise. If $G \cong Z_{\kappa,\lambda}(K^1, K^n)$, then p lies in a building block corresponding to K^1 if and only if $\kappa > 2$, and in one corresponding to K^n if and only if $\lambda > 2$. If $G \ncong Z_{\kappa,\lambda}(K^1, K^n)$, then G belongs to one of the cases (6) or (8) and the vertex v lies in a building block that contains a leaf of P and thus two non-adjacent vertices. Hence p lies in a complete building block.

If v has two neighbours p_1, p_2 on P, and $d_P(p_1, p_2) = 2$ then the vertex on P adjacent to p_1 and p_2 lies in the same building block as v. This building block corresponds to the complement of a complete graph or a graph from $\mathcal{E}_{k,m,n}$ as it contains two non-adjacent vertices. If $d_P(p_1, p_2) = 1$ then one of p_1 or p_2 is a leaf of P and v lies together with this leaf in a common building block that corresponds to K^n or a graph from $\mathcal{E}_{k,m,n}$.

If v has three neighbours p_1, p_2, p_3 on P, then they induce a path of length 2 in P and v lies in the same building block as the middle vertex of that path of length 2 which is a building block corresponding to K^n or a graph from $\mathcal{E}_{k,m,n}$.

In all these cases, it is determined in which kind of building blocks of G the neighbours of v lie. Thus there is (at least) one vertex w on P such that w and w^α lie in building blocks that are in the same $\mathrm{Aut}(G)$-orbit of G. As in (6), (7), and (8) every second vertex on P lies in building blocks of the same $\mathrm{Aut}(G)$-orbit, it holds that for every w' on P the vertices w' and w'^α lie in the same kind of building block of G.

Using this path P, let us recursively construct an automorphism of G that maps X to Y. The arguments above show as all building blocks are homogeneous that there exists an automorphism α_0 of G with $\alpha_0|_P = \alpha|_P$ and that every such automorphism satisfies that p and p^{α_0} lie in building blocks that correspond to the same graph for every vertex $p \in V(P)$.

To define the automorphism α_l of G for $l \geq 1$ let α_i be defined for $i < l$. First, let W be the set of vertices in G with distance at most $l-1$ to the building blocks that contain P. The graphs X and Y induce graphs $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ with $X_j^\alpha = Y_j$ for all $1 \leq j \leq n$ in the components of $G - W$ and $G - W^{\alpha_{l-1}}$, respectively. Let α_l be an automorphism of G with $w^{\alpha_l} := w^{\alpha_{l-1}}$ for $w \in W$, that maps the component of $G - W$ containing X_j to the component of $G - W^{\alpha_{l-1}}$ containing Y_j for all $j \leq n$ so that the vertices of X adjacent to W are mapped precisely to those vertices of Y adjacent to $W^{\alpha_{l-1}}$. Since the diameter of X is less than k, the automorphism α_k of G maps X onto Y. □

Combining these lemmas we obtain the following corollary.

Corollary 5.4. *Let $k \geq 3$ and let G belong to one of the classes (1) to (8) of Theorem 1.3. Every connected graph on k vertices is good for G.*

In particular, G is k-CS-transitive. □

6 The global structure of k-CS-transitive graphs

This section contains the substantial part of the proof of Theorem 1.3. We show that for $k \geq 3$ every connected k-CS-transitive graph with at least two ends is isomorphic to one of the graphs described in Theorem 1.3. At first, we provide some general properties for basic cut systems of such graphs. Later on we distinguish two fundamentally different cases: in Subsection 6.1 we look at those graphs that are covered by the separators of a basic cut system and in Subsection 6.2 at those that are not.

Lemma 6.1. *Let $k \geq 3$. If G is a connected k-CS-transitive graph with at least two ends, then for G and any of its basic cut systems their structure tree has no leaves.*

Proof. Let $\mathcal{S}$ be a basic cut system of G and let $\mathcal{T}$ be the structure tree of G and $\mathcal{S}$. Suppose that $\mathcal{T}$ has a leaf X. By the construction of a structure tree, X is an $\mathcal{S}$-block. Let $(A, B) \in \mathcal{S}$ be a cut with $V(X) \subseteq A$ and $A \cap B \subseteq V(X)$. By the construction of $\mathcal{T}$, we know that X is adjacent to all $\mathcal{S}$-separators that are contained in X. This implies that $A \cap B$ is the only $\mathcal{S}$-separator in X and $V(X) = A$. In particular, no vertex of $A \setminus B = V(X - B)$ lies in an $\mathcal{S}$-separator as $\mathcal{S}$ is nested. Since there is a ray in $G[A]$, the block X is infinite. There is no vertex in X that has distance $k+1$ to B, as otherwise an induced path in $G[A]$ starting at $v \in A \cap B$ could be mapped into $X - B$ by an automorphism of G. The image of $A \cap B$ under this automorphism is not an $\mathcal{S}$-separator as it contains a vertex from $X - B$. This contradicts the $\mathrm{Aut}(G)$-invariance of the basic cut system $\mathcal{S}$. Thus there are vertices of infinite degree in X. Let $x \in V(X)$ be a vertex with infinite degree and minimal distance to B with this property. Let N be an infinite set of neighbours of x with $d(v, B) > d(x, B)$ for all $v \in N$. By the infinite version of Ramsey's Theorem (see for example [2, Theorem 9.1.2]) there is either a $K^{\aleph_0}$ or an infinite independent set in $G[N]$. Suppose there is an independent set of cardinality $k-1$ in N. As $d(v, B) > d(x, B)$ for all $v \in N$, there is a neighbour u of x with $d(u, B) < d(x, B)$ if $d(x, B) \geq 1$ or with $u \in B \setminus A$ if $x \in A \cap B$ such that u is not adjacent to any vertex in N. Any $k-2$ independent vertices in N together with x and u induce a subgraph that could be mapped onto a subgraph induced by x and $k-1$ independent vertices in N. The former subgraph is either properly separated by an $\mathcal{S}$-separator while the latter is not, or it is closer to any $\mathcal{S}$-separator than the latter one. Thus there is no independent set of cardinality $k-1$ in N and there is a $K^{\aleph_0}$ in $G[N]$. Again, this yields to a contradiction. Indeed, let H be a complete graph on k vertices in $G[N]$, and let $v \in V(H)$. Then there is no automorphism of G that maps $H - v + x$ to H as $H - v + x$ contains only vertices of distance at least $d(x, B) + 1$ to the unique $\mathcal{S}$-separator in X, which is a contradiction to the k-CS-transitivity of G. ☐

Lemma 6.2. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G. Then any ray in the structure tree of G and $\mathcal{S}$ contains infinitely many pairwise disjoint $\mathcal{S}$-separators.*

In particular, G does not have finite diameter.

Proof. Let $\mathcal{T}$ be the structure tree of G and $\mathcal{S}$ and let R be a ray in $\mathcal{T}$. The only neighbours of $\mathcal{S}$-blocks in $\mathcal{T}$ are $\mathcal{S}$-separators. Thus, infinitely many different (finite) $\mathcal{S}$-separators lie on R.

Suppose that there is a vertex x in G that lies in infinitely many of the separators on R. Let S_0 be the first separator on R that contains x, and let X be an $\mathcal{S}$-block adjacent to S_0 in $\mathcal{T}$ that does not lie on the tail S_0R of R with initial vertex S_0 which exists as $\mathcal{T}$ has no leaf and thus S_0 has at least two neighbours in $\mathcal{T}$. Let $(A, B) = (V(C) \cup S_0, \sim)$ be the $\mathcal{S}$-separation with separator S_0 for the component C of $G - S_0$ that meets X. Then all separators on R that contain x lie in B. As (A, B) is a cut, there exists a neighbour $y \in A \setminus B$ of x.

Any $\mathcal{S}$-separator separates two $\mathcal{S}$-separators in G properly if and only if it separates them properly in the structure tree. Hence, if a vertex of G lies in two separators, then it also lies in any separator that appears in the structure tree on the unique path between those two. Every $\mathcal{S}$-separator $S \subseteq B$ on R contains x, as it lies between S_0 and one of the infinitely many other separators containing x on R. There is a neighbour y_S of x such that S separates y and y_S properly. By Lemma 2.10 we know that the number of $\mathcal{S}$-separators separating v and w properly is finite for all vertices $v, w \in V(G)$. This implies that there is an infinite set $\mathcal{F}$ of $\mathcal{S}$-separators on R such that for any two distinct separators $S, S' \in \mathcal{F}$ the vertices $y_S, y_{S'}$ are distinct. Thus, $U := \{y_S \in V(G) \mid S \in \mathcal{F}\}$ is infinite. By the infinite version of Ramsey's Theorem there is either a $K^{\aleph_0}$ or an infinite independent set in $G[U]$. In the first case, let $K \subseteq G[U]$ be such an infinite complete graph. The (finite) $\mathcal{S}$-separators do not separate K properly and hence there are infinitely many $\mathcal{S}$-separators separating y from K properly. As K is infinite and all separators in $\mathcal{F}$ have the same cardinality, there exists a vertex $v \in V(G)$ that lies outside of infinitely many separators in $\mathcal{F}$. Each of the infinitely many separators S in $\mathcal{F}$ for which y_S lies in $V(K)$ and that does not contain v separate y and v properly, as every such separator separates y and y_S properly and $vy_S \in E(G)$. This is a contradiction as y and v are separated properly by infinitely many separators of cardinality $\mathrm{ord}(\mathcal{S})$.

Thus there is an infinite independent set $U' \subseteq U$ completely adjacent to x. Remember that y is not adjacent to any vertex in U'. We choose a subset V_1 of U' of cardinality $k-1$. There is a maximal number n of separators of cardinality $\mathrm{ord}(\mathcal{S})$ that separate any two vertices of V_1 properly as for each of the finitely many pairs of vertices in V_1 there is only a finite number of separators of cardinality $\mathrm{ord}(\mathcal{S})$ that separates it properly. Let V_2 be another subset of U' of cardinality $k-2$ that contains a vertex that is separated by more than n separators of cardinality $\mathrm{ord}(\mathcal{S})$ from y properly: pick a separator S in $\mathcal{F}$ such that on the S_0-S path on R there are more than n other $\mathcal{S}$-separators and let $y_S \in V_2$. By k-CS-transitivity there is an automorphism of G that maps $G[V_2 \cup \{x, y\}]$ onto $G[V_1 \cup \{x\}]$ as both these induced subgraphs are stars with $k-1$ leaves. This automorphism has to fix x and map $V_2 \cup \{y\}$ onto V_1. As y and y_S are separated properly by more than n separators of cardinality $\mathrm{ord}(\mathcal{S})$, their respective images in V_1 are separated properly by just as many such separators. This contradicts the choice of n.

Thus no vertex of G lies in infinitely many $\mathcal{S}$-separators on R and we conclude that there are infinitely many pairwise disjoint $\mathcal{S}$-separators on R. Two $\mathcal{S}$-separators S_1, S_2 that have n disjoint $\mathcal{S}$-separators on their S_1-S_2 path in $\mathcal{T}$ have distance at least n in G. As by Lemma 6.1 every structure tree of a basic cut system of G contains a ray, this implies the second assertion. □

The next lemma provides a fundamental tool in the proof of Theorem 1.3.

Lemma 6.3. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, let $\mathcal{S}$ be a basic cut system of G, and let S be an $\mathcal{S}$-separator. For every ray R in the structure tree $\mathcal{T}$ of G and $\mathcal{S}$ that starts at S, there are* ord($\mathcal{S}$) *disjoint induced rays* $R_1, \dots, R_{\mathrm{ord}(\mathcal{S})}$ *in* G *starting at* S *such that for every* $i \le \mathrm{ord}(\mathcal{S})$ *the ray* R_i *intersects with all* $\mathcal{S}$*-separators on* R.

Proof. On R there are infinitely many disjoint $\mathcal{S}$-separators $S_1, S_2, \dots$ all disjoint from $S_0 := S$ as shown in Lemma 6.2. As by Lemma 2.9 no two of them are separated by less than $\mathrm{ord}(\mathcal{S})$ many vertices, Menger's Theorem implies that there are $\mathrm{ord}(\mathcal{S})$ many pairwise disjoint induced S_0-S_i paths for all $0 < i$. Let $\mathcal{P}_i$ be the subgraph of G consisting of these paths. Since $\mathcal{P}_{i-1}$ covers S_{i-1} on R, we may choose $\mathcal{P}_i$ such that $\mathcal{P}_{i-1} \subseteq \mathcal{P}_i$. The union $\bigcup_{i\in\mathbb{N}} \mathcal{P}_i$ is a subgraph of $\mathrm{ord}(\mathcal{S})$ many pairwise disjoint induced rays each starting at S_0. Clearly, each of those rays intersects with every $\mathcal{S}$-separator on R. □

For a connected k-CS-transitive graph G with $k \ge 3$ and at least two ends and a basic cut system $\mathcal{S}$ of G, there are two profoundly different cases. In the first case the graph is covered with $\mathcal{S}$-separators while in the second case there are vertices in G that do not belong to any $\mathcal{S}$-separator.

For an $\mathcal{S}$-block X we define the *open* ($\mathcal{S}$*-*)*block*

$$\mathring{X} := X - \bigcup \{A \cap B \mid (A,B) \in \mathcal{S}\}.$$

Further down the line it turns out that the two cases above correspond to whether there exist non-empty open blocks or not. In Lemma 6.9 we get rid of any vertices that lie neither in an $\mathcal{S}$-separator nor in an open $\mathcal{S}$-block. In the proof of Theorem 1.1 we got this property for free as the graphs considered there are vertex transitive; here it turns out to require some effort.

Lemma 6.4. *Let* $k \ge 3$*, let* G *be a connected* k*-CS-transitive graph with at least two ends, and let* $\mathcal{S}$ *be a basic cut system of* G*. If the* $\mathcal{S}$*-separators do not cover* G*, then* k *is even.*

Proof. Let k be odd. We show that every vertex lies in some $\mathcal{S}$-separator. By Lemma 6.3 there is an induced ray R meeting infinitely many vertices that lie in $\mathcal{S}$-separators. As k is odd, there is an induced path $P \subseteq R$ of length $k-1$ whose middle vertex v belongs to some $\mathcal{S}$-separator. We may map the path anywhere into the ray and thus know that there are k succeeding vertices on the ray that belong to $\mathcal{S}$-separators. Thus every induced path of length $k-1$ in G has all its vertices in $\mathcal{S}$-separators. As the diameter of G is not finite according to Lemma 6.2, every vertex lies on an induced path of length $k-1$. Therefore every vertex lies in some $\mathcal{S}$-separator. □

Lemma 6.5. *Let* $k \ge 3$*, let* G *be a connected* k*-CS-transitive graph with at least two ends, and let* $\mathcal{S}$ *be a basic cut system of* G*. If* G *has a vertex not in any* $\mathcal{S}$*-separator, then every edge on every induced path of length* $k-1$ *in* G *has precisely one of its incident vertices in an* $\mathcal{S}$*-separator.*

In particular, if there is a vertex not in any $\mathcal{S}$*-separator, then every induced path* $P \subseteq G$ *of length at least* $k-1$ *alternates between vertices in* $\mathcal{S}$*-separators and vertices outside every* $\mathcal{S}$*-separator.*

Proof. As there is a vertex outside every $\mathcal{S}$-separator, k is even by Lemma 6.4. Since the structure tree $\mathcal{T}$ of G and $\mathcal{S}$ has no leaves by Lemma 6.1, every $\mathcal{S}$-separator S lies on a double ray in $\mathcal{T}$. By Lemma 6.3 there are two induced rays R_1, R_2 starting at $s \in S$ with all their other vertices in two distinct components of $G - S$. Let $R = R_1 \cup R_2$, that means R is an induced double ray in G. As G is k-CS-transitive, there are automorphisms of G mapping a path $P \subseteq R$ of length $k-1$ with its middle edge incident with s to any other path of length $k-1$ on R. Thus every edge on R is incident with a vertex in an $\mathcal{S}$-separator.

If some edge on R has both its incident vertices in $\mathcal{S}$-separators, this implies by the same argument that every edge on R has both its incident vertices in $\mathcal{S}$-separators. As there is a vertex $v \in V(G)$ not contained in any $\mathcal{S}$-separator and as the diameter of G is not bounded, there is an induced path P of length $k-1$ starting at v. Thus, as by k-CS-transitivity there is an automorphism of G mapping P to R, there is no edge on R with both its incident vertices in $\mathcal{S}$-separators. Hence every edge on R has precisely one of its incident vertices in an $\mathcal{S}$-separator. By k-CS-transitivity, every edge on an induced path of length $k-1$ in G has precisely one of its incident vertices in an $\mathcal{S}$-separator. Thus any induced path of length at least $k-1$ is such an *alternating* path. □

As a corollary of the proof of the previous lemma, we obtain the following result.

Corollary 6.6. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G. Then, any two vertices on an induced path P of length $k-1$ that have a vertex from an $\mathcal{S}$-separator between them on P are separated by some $\mathcal{S}$-separator in G.*

In particular, if two vertices on P have distance at least 3 on P, then they are separated by an $\mathcal{S}$-separator in G.

Proof. We recall the definitions from the proof of Lemma 6.5: We have a double ray R in G such that for every vertex $s \in V(R)$ that lies in an $\mathcal{S}$-separator, there is one $\mathcal{S}$-separator S with $s \in S$ such that the two components of $R - s$ lie in distinct components of $G - S$. Remark that we obtain such a double ray also in the situation that G is covered by $\mathcal{S}$-separators.

The two components of $R - s$ do not meet any common $\mathcal{S}$-block for any vertex s on R that lies in an $\mathcal{S}$-separator. Thus for any path P of length $k-1$ and any vertex s' in the interior of P, that belongs to some $\mathcal{S}$-separator, the two components of $P - s'$ are separated properly by an $\mathcal{S}$-separator. This implies the first assertion, the second one follows immediately since any two vertices on an induced path of length $k-1$ with distance at least 3 have—by Lemma 6.5 or as every vertex lies in an $\mathcal{S}$-separator—a vertex from some $\mathcal{S}$-separator between them on P. □

Corollary 6.7. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G such that some vertex of G is not contained in any $\mathcal{S}$-separator. If two vertices belong to different sets of $\mathcal{S}$-separators then they are not adjacent.*

Proof. Suppose that there is an $\mathcal{S}$-separator S, and vertices $s \in S$ and $s' \notin S$ but in a different $\mathcal{S}$-separator such that s and s' are adjacent. Then there is an induced path of length $k-1$ that contains this edge and lies otherwise in a component of $G-S$ that does not contain s'. But no such path exists according to Lemma 6.5. $\square$

Lemma 6.8. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, let $\mathcal{S}$ be a basic cut system of G, and let S be an $\mathcal{S}$-separator. Every component C of $G - S$ contains an end and the separation $(V(C) \cup S, \sim)$ lies in $\mathcal{S}$.*

In particular, S separates any two vertices in distinct components of $G - S$ properly.

Proof. The claim is true if the $\mathcal{S}$-separators cover G, because then, for every component C of $G-S$, there is an $\mathcal{S}$-separator S' that meets C. Let $(A', B') \in \mathcal{S}$ with separator S'. According to Remark 2.8 the separator S' lies in $C \cup S$ and hence one wing of (A', B') lies in C. Thus, C contains an end and according to Lemma 2.6 is the wing of an $\mathcal{S}$-separation. Since $\mathcal{S}$ is a cut system and C is a component of $G - S$, we know that $(V(C) \cup S, \sim)$ is an $\mathcal{S}$-separation.

Hence, we may assume that there is a vertex outside every $\mathcal{S}$-separator. By Lemma 6.4 this implies that k is even. Let (A, B) be an $\mathcal{S}$-separation with $A \cap B = S$. As $\mathcal{S}$ is a cut system, there is an $\mathcal{S}$-separation (A', B') such that $A' \subseteq B$ and $S \subseteq A' \cap B'$. Since $\mathcal{S}$ is minimal, it holds that $S = A' \cap B'$.

First, let us assume that $G - S$ consists of precisely two components. Both components contain an end as every $\mathcal{S}$-separator separates two ends. Thus, we have $(A, B) = (B', A')$ and the assertion holds.

Therefore, we may assume that $G - S$ contains at least three components. Then there exists a component C of $G - S$ that lies neither in A nor in A'. Let $c \in V(C)$ be a vertex that is adjacent to $s \in S$, and let $d \in A' \setminus B'$ be adjacent to s. By Lemma 6.3 there is an induced path P in G of length $k-1$ that starts at a vertex $x \in A \setminus B$ and ends in c while meeting S only in s. This implies that P has only one vertex in C which is c. By the choice of P and d, the graph Psd is also an induced path of length $k-1$. As G is k-CS-transitive, there is an automorphism $\alpha \in \mathrm{Aut}(G)$ mapping P to Psd. By Lemma 6.5 both these paths alternate between vertices in $\mathcal{S}$-separators and other vertices. Thus, precisely one endvertex of P and one of Psd is contained in an $\mathcal{S}$-separator as k is even and we have $c^\alpha = d$. The component C^α of $G - S^\alpha$ contains $d = c^\alpha$. Both S and S^α separate x from d. Suppose $S \neq S^\alpha$, then either S separates S^α from x properly or S^α separates S from x properly according to Remark 2.8. If S separates S^α from x let $S_1 = S$, $S_2 = S^\alpha$, and $\beta = \alpha$. Otherwise, if S^α separates S from x let $S_1 = S^\alpha$, $S_2 = S$, and $\beta = \alpha^{-1}$. Let S_0 be the $\mathcal{S}$-separator that contains x and—among all those—is closest to S_1 in the structure tree $\mathcal{T}$ of G and $\mathcal{S}$.

The separator S_0^β contains x as $x^\alpha = x$, and thus lies in A. By the choice of S_0 and as S_1 separates S_0 and S_2, it holds that

$$d_{\mathcal{T}}(S_0, S_1) = d_{\mathcal{T}}(S_0^\beta, S_2) > d_{\mathcal{T}}(S_0^\beta, S_1),$$

since the path between S_2 and S_0^β in $\mathcal{T}$ has to contain S_1. Thus, S_0^β is closer to S_1 in $\mathcal{T}$ than S_0 contradicting the choice of S_0. This implies that $S = S^\alpha$ and $(V(C) \cup S, \sim) = (A', B')^{\alpha^{-1}}$. As every component of $G - S$ is a wing of an end separating cut the component contains an end and thus any two components of $G - S$ are separated properly by S. □

Lemma 6.9. *Let $k \geq 3$, let G be a k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G. Then every vertex of G lies in an $\mathcal{S}$-block.*

Proof. Let v be a vertex of G. If v belongs to some $\mathcal{S}$-separator, it lies in an $\mathcal{S}$-block. So we may assume that v lies outside every $\mathcal{S}$-separator. Let S and S' be two distinct $\mathcal{S}$-separators such that S' separates S and v. By Lemma 6.8 and as $\mathcal{S}$ is nested, S' separates S and v properly. There are only finitely many $\mathcal{S}$-separators separating an $\mathcal{S}$-separator and v properly according to Lemma 2.10 and thus there are only finitely many $\mathcal{S}$-separators separating S and v, as v lies outside every $\mathcal{S}$-separator. As $\mathcal{S}$ is basic and by Remark 2.8 there is one $\mathcal{S}$-separator S_0 that separates S and v such that no other $\mathcal{S}$-separator separates S_0 and v. We show that v and S_0 lie in a common $\mathcal{S}$-block. Let C_0 be the component of $G - S_0$ that contains v. Then $(V(C_0) \cup S_0, \sim)$ lies in $\mathcal{S}$ according to Lemma 6.8. There is an $\mathcal{S}$-block X adjacent to S_0 in the structure tree of G and $\mathcal{S}$ whose vertices lie in $V(C_0) \cup S_0$. This block contains S_0 and, as there is no $\mathcal{S}$-separator separating v from $S_0 \subseteq X$, there is no $\mathcal{S}$-separator separating v from X. Thus v lies in X. □

Corollary 6.10. *Let $k \geq 3$, let G be a k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G. There is a non-empty open $\mathcal{S}$-block if and only if there is a vertex outside every $\mathcal{S}$-separator.* □

This corollary shows that the distinction between '$\mathcal{S}$-separators cover G' and 'there is a vertex outside every $\mathcal{S}$-separator' is in fact a distinction between whether all open $\mathcal{S}$-blocks are empty or not. For this reason we characterize the cases by stating if there is a non-empty open $\mathcal{S}$-block or not from now on. In addition we use the fact that for all cut systems we investigate every vertex lies in a block without refering to Lemma 6.9.

In the construction of $X_{\kappa,\lambda}(H)$ and $Z_{\kappa,\lambda}(H_1, H_2)$ the appropriate copies of H and H_1, H_2, respectively, are completely adjacent. The next lemma provides the corresponding property for k-CS-transitive graphs.

Lemma 6.11. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G. Let X be an $\mathcal{S}$-block, let S be an $\mathcal{S}$-separator with $S \subseteq X$, and let $s \in S$. If $\mathring{X} = \emptyset$, let $x \in V(X - S)$ and if $\mathring{X} \neq \emptyset$, let $x \in V(\mathring{X})$. Then s and x are adjacent.*

Proof. Let $\mathcal{T}$ be the structure tree of G and $\mathcal{S}$. Suppose s and x are not adjacent in G. Let P be a shortest s-x path whose inner vertices lie in the component of $G - S$ that contains x. As P is a shortest path it is induced. Let C be a component of $G - S$ not containing x. As there is an induced ray R starting at s with all its other vertices in C, there is an induced path P' of length at least $k-1$ starting at x and containing P.

Suppose that the distance of s and x on P is 2. The common neighbour y of s and x on P does not lie in any $\mathcal{S}$-separator, because of Corollary 6.6 and as s and x do lie in a common $\mathcal{S}$-block. As every vertex lies in an $\mathcal{S}$-block according to Lemma 6.9, this implies that y lies in an open $\mathcal{S}$-block $\mathring{Y}$. By Lemma 6.4 k is even and hence at least 4. As $y \in V(\mathring{Y})$ its neighbours s and x have to lie in Y. Suppose $Y \neq X$, then let $S' \subseteq V(Y)$ be an $\mathcal{S}$-separator separating X and Y and thus containing s and x. As every component of $G - S'$ contains a ray and does not have finite diameter (according to Lemma 6.1, 6.3, and 6.8) there is an induced ray starting at y and avoiding S'. Furthermore we require R to have precisely one other vertex in Y not adjacent to s, which is possible as Lemma 6.5 implies that the neighbour x' of y on R lies in an $\mathcal{S}$-separator and Corollary 6.7 implies that there is no edge between s and x' while all the other vertices on R are separated properly from S' by an $\mathcal{S}$-separator that contains x'. Let P_1 be the subpath of P' that contains x and has length $k-1$. Let v be the other endvertex of P_1 and let $P_2 = x'yP_1v$. Then there is an automorphism α of G that maps P_1 onto P_2. By Lemma 6.5 the automorphism α has to map x to x' and fix the remainder of P_1. By the same lemma, y does not lie in the same $\mathcal{S}$-block as v. So there is an $\mathcal{S}$-separator separating y and v properly. Every such separator lies together with X in the same component of $\mathcal{T} - Y$. As $\mathcal{S}$ is minimal and nested, every $\mathcal{S}$-separator that separates x and v properly also separates any other vertex in S' and v. Thus, according to Remark 2.8, every $\mathcal{S}$-separator that separates x and v properly separates v and S' properly. Since S' separates v and x' properly, Remark 2.8 also implies that every $\mathcal{S}$-separator that separates v and S' properly also separates v and x' properly. By k-CS-transitivity and according to Lemma 2.10, the same finite number of $\mathcal{S}$-separators separates v from x as from x' properly. This yields a contradiction as S' separates x' and v properly but not x and v. This contradiction shows that $X = Y$ and that $\mathring{X}$ is not empty. Thus, we have $x \in \mathring{X}$ and with Lemma 6.5 this implies that s and x have odd distance as they lie on the alternating path P', in particular $d_P(s, x) \neq 2$.

Therefore, the distance between x and s on P is at least 3. If the length of P is at most $k-1$, then we may choose P' as above of length precisely $k-1$. By Corollary 6.6 the vertices x and s are properly separated by some $\mathcal{S}$-separator.

Thus, we may assume that P has length at least k. As P contains a subpath of length $k-1$ containing x, there has to be an $\mathcal{S}$-separator separating a vertex on P from x properly. Let S' be an $\mathcal{S}$-separator furthest away in $\mathcal{T}$ from X such that there is a vertex on P separated properly by S' from X. Let C be a component of $G - S'$ that meets P and avoids X. Then $(V(C) \cup S', \sim) \in \mathcal{S}$ and there is an $\mathcal{S}$-block $Y \subseteq G[C + S']$ adjacent to S' in $\mathcal{T}$. By the choice of S' all vertices of $P \cap C$ lie in Y and S' separates X and Y. In particular P has a vertex y in $Y - S'$ such that $d_P(y, s)$ is smallest possible. Let y_1 be the neighbour of y on yPx. As no induced subpath of length 3 on P lies in one $\mathcal{S}$-block by Corollary 6.6, the neighbour of y_1 on y_1Px must not lie in Y and thus not in $V(C) \cup S'$. Hence we have $y_1 \in S'$.

As above there is an induced ray starting at y and having no other vertex adjacent to S' than y. Let y_2 be the neighbour of y on that ray. Again, we may elongate—if necessary—y_1Ps in C to obtain an induced path P_1 of length $k-1$ that ends in y_1 and either contains y_1Ps or lies on it. Let v be the other endvertex of P_1 and let $P_2 = y_2yP_1$ be the same path as P_1 with y_1 substituted by y_2. As both subgraphs are induced paths of length $k-1$, there is an automorphism α of G mapping P_1 onto P_2. This automorphism has to map the endvertices of P_1 to the endvertices of P_2. By a similar argument as above, we obtain that the number of $\mathcal{S}$-separators that separate v and y_2 properly and the number of $\mathcal{S}$-separators that separate v and y_1 properly differ which is a contradiction as $P_1^\alpha = P_2$. This contradiction shows that x and s are adjacent. □

Corollary 6.12. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G. Then any two distinct $\mathcal{S}$-separators are disjoint.*

Proof. Suppose that there are two distinct $\mathcal{S}$-separators S, S' that are not disjoint. Every $\mathcal{S}$-separator on the S-S' path in $\mathcal{T}$ contains $S \cap S'$. Thus we may assume that $d_{\mathcal{T}}(S, S') = 2$ and hence S and S' lie in a common $\mathcal{S}$-block X. Let $s \in S \cap S'$ and let x_1 be a neighbour of s in a component of $G - S$ avoiding S'. Let $x_2 \in \mathring{X}$ if $\mathring{X}$ is not empty, and if $\mathring{X} = \emptyset$, then let x_2 be a vertex in $S \setminus S'$. By Lemma 6.11 the vertices x_2 and s are adjacent in both cases. Let P be an induced path of length $k-2$ in G that starts at s and has its other vertices in a component of $G - S'$ avoiding S which exists according to Lemma 6.1 and 6.3. Since G is k-CS-transitive there is an automorphism of G mapping x_1P to x_2P, as both are induced paths in G of length $k-1$. Similar to the proof of the previous lemma and as the $\mathcal{S}$-blocks cover G according to Lemma 6.9, the endvertices of x_1P and those of x_2P are separated by a different finite number of $\mathcal{S}$-separators. By contradiction, this shows that $\mathcal{S}$-separators are either equal or disjoint. □

Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G. A *k-spoon* is an induced subgraph of G that consists of a triangle and a path of length $k-2$, its *handle*, starting at one of its triangle vertices with all in all precisely k vertices. A k-spoon H *pokes* in an $\mathcal{S}$-block X, an $\mathcal{S}$-separator S, or two $\mathcal{S}$-separators S, S' if two of its degree 2 vertices[6] of the triangle are contained in $\mathring{X}$, S, or one in S and one in S', respectively. A *k-fork* is another induced subgraph of G on k vertices that consists of its *prongs*, a pair of two non-adjacent vertices, and of its *handle*, a path such that both prongs are adjacent only to the same endvertex of the handle. A k-fork H *pokes* in an $\mathcal{S}$-block X, an $\mathcal{S}$-separator S, two $\mathcal{S}$-blocks X, Y, or two $\mathcal{S}$-separators S, S' if its prongs are contained in $\mathring{X}$, in S, meet $\mathring{X}$ and $\mathring{Y}$, or meet S and S', respectively.

[6]Remark that for $k > 3$ there are precisely two such vertices, but for $k = 3$ a k-spoon is just the triangle.

6.1 Empty open blocks

In this subsection we investigate k-CS-transitive graphs that have a basic cut system all of whose open blocks are empty. Remember that by Lemma 6.4, this is the only case if k is odd.

Lemma 6.13. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G. If every open $\mathcal{S}$-block is empty, then all $\mathcal{S}$-blocks lie in the same* $\mathrm{Aut}(G)$*-orbit, or k is odd and there is a cardinal $\kappa \geq 3$ such that $G \cong Y_\kappa$.*

Proof. Suppose that there are two $\mathcal{S}$-blocks X and Y that lie in distinct $\mathrm{Aut}(G)$-orbits. As every $\mathcal{S}$-block contains an $\mathcal{S}$-separator and $\mathcal{S}$ is basic, there is an automorphism φ of G with $X \cap Y^\varphi = S$ for an $\mathcal{S}$-separator S. Hence we may assume that $X \cap Y = S$. If S contains two distinct vertices, then by Lemma 6.3 there is either a k-spoon with its triangle—the subgraph isomorphic to a K^3—in X and one k-spoon with its triangle in Y or there is a k-fork with both edges incident with its prongs in X and one such k-fork for Y, such that in each case the handle does not contain any vertex from S. As G is k-CS-transitive, there is, in both cases, an automorphism α of G mapping one edge in X that does not lie in any $\mathcal{S}$-separator to one such edge in Y. Thus $X^\alpha \cap Y$ is not contained in an $\mathcal{S}$-separator and $X^\alpha = Y$.

Hence two distinct $\mathcal{S}$-blocks intersect in at most one vertex and $\mathrm{ord}(\mathcal{S}) = 1$. By Lemma 6.11 and as every open $\mathcal{S}$-block is empty, any two $\mathcal{S}$-separators in a common block are completely adjacent and thus every $\mathcal{S}$-block is complete. For any two $\mathcal{S}$-blocks each of which has more than two vertices, there is a k-spoon with its triangle in each of these $\mathcal{S}$-blocks, respectively. Thus these blocks are $\mathrm{Aut}(G)$-isomorphic as G is k-CS-transitive.

Let P be an induced double ray in G whose edges alternate between two orbits of $\mathcal{S}$-blocks. Such a double ray exists, as one may start at any vertex of G and add appropriate edges greedily, since every vertex lies in blocks of all orbits of blocks. Clearly, every induced path of length $k-1$ shares this property with the ray and thus every vertex lies in at most one block of each orbit. As otherwise, if there is a vertex that lies in more than one block of the same orbit, then one may construct an induced path of length $k-1$ without this property. With the same argument for any two kinds of orbits, there is an induced path of length $k-1$ with edges only in these orbits. Since G is k-CS-transitive, this implies that there are precisely two distinct orbits of $\mathcal{S}$-blocks: in one orbit each $\mathcal{S}$-block is isomorphic to a K^2 and in the other one each $\mathcal{S}$-block is isomorphic to a K^κ for some cardinal $\kappa \geq 2$. If $\kappa = 2$ then G is a double ray and this contradicts that there are two distinct $\mathrm{Aut}(G)$-orbits of $\mathcal{S}$-blocks. Thus $\kappa \geq 3$ and $G \cong Y_\kappa$.

Let us suppose that k is even. Then there is a path of length $k-1$ with both outermost edges in $\mathcal{S}$-blocks isomorphic to a K^2 and there is a path of length $k-1$ with both outermost edges in $\mathcal{S}$-blocks isomorphic to a K^κ with $\kappa \geq 3$. As no automorphism of G maps one of these paths to the other, this is a contradiction and hence k is odd. □

Lemma 6.14. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G such that every open $\mathcal{S}$-block is empty. If any two $\mathcal{S}$-blocks lie in the same* $\mathrm{Aut}(G)$*-orbit, then $G \cong X_{2,2}(H)$ for some finite graph H that is neither complete nor the complement of a complete graph, or there are cardinals $\kappa, \lambda \geq 2$ and integers $2 \leq m < \frac{k+2}{3}$ and $2 \leq n < \frac{k}{2} + 1$ such that $G \cong X_{2,\lambda}(K^n)$ or $G \cong X_{\kappa,2}(\overline{K^m})$ or $G \cong X_{\kappa,\lambda}(K^1)$.*

Proof. Let $H = G[S]$ for some $\mathcal{S}$-separator S. According to Lemma 6.11 and Corollary 6.12 it holds that $G \cong X_{\kappa,\lambda}(H)$ for some cardinals $\kappa \geq 2$ and $\lambda \geq 2$. We may assume that $G \not\cong X_{2,2}(H)$ where H is neither complete nor the complement of a complete graph. If there are edges in H and $\lambda \geq 3$ then there are two kinds of k-spoons: one with its triangle meeting three $\mathcal{S}$-separators and one meeting precisely two $\mathcal{S}$-separators. If there are two non-adjacent vertices in H and $\kappa \geq 3$ then there are two kinds of k-forks: one pokes in a single separator and one pokes in two different separators. As G is k-CS-transitive, all k-spoons as well as all k-forks lie in one $\mathrm{Aut}(G)$-orbit, respectively. Thus it holds that either $G \cong X_{\kappa,2}(\overline{K^m})$ with $m \geq 2$, or $G \cong X_{2,\lambda}(K^n)$ with $n \geq 2$, or $G \cong X_{\kappa,\lambda}(K^1)$. It remains to show that $m < \frac{k+2}{3}$ and $n < \frac{k}{2} + 1$.

Let $G \cong X_{\kappa,2}(\overline{K^m})$ and suppose that $m \geq \frac{k+2}{3}$. Let S_1, S_2 be $\mathcal{S}$-separators in different $\mathcal{S}$-blocks both (completely) adjacent to an $\mathcal{S}$-separator S_0. As $3m \geq k+2$ there are sets $A_i \subseteq S_i$ for $i = 0, 1, 2$ such that $A_1 \cup A_0 \cup A_2$ has cardinality $k+2$, is connected in G—that is $A_0 \neq \emptyset$—and such that each of A_1 and A_2 contains at least two vertices. Let $a, b \in A_1$ and $c \in A_2$. By the construction of G it holds that

$$G[(A_1 \setminus \{a, b\}) \cup A_0 \cup A_2] \cong G[(A_1 \setminus \{a\}) \cup A_0 \cup (A_2 \setminus \{c\})].$$

As there is no automorphism of G mapping the first to the second graph, this is a contradiction and thus $m < \frac{k+2}{3}$.

Let $G \cong X_{2,\lambda}(K^n)$ and suppose $n \geq \frac{k}{2} + 1$. Let S_0, S_1 be two (completely) adjacent $\mathcal{S}$-separators. Let $A_i \subseteq S_i$ with $|A_0| = \lceil \frac{k}{2} \rceil + 1$ and $|A_1| = \lfloor \frac{k}{2} \rfloor - 1$, and let $B_i \subseteq S_i$ with $|B_0| = \lceil \frac{k}{2} \rceil$ and $|B_1| = \lfloor \frac{k}{2} \rfloor$ which exist as $n \geq \frac{k}{2} + 1$ implies that $n \geq \lceil \frac{k}{2} \rceil + 1$ for any integer n. It holds that $|A_0 \cup A_1| = |B_0 \cup B_1| = k$, but there is no automorphism of G that maps the complete graph on k vertices $G[A_0 \cup A_1]$ to the complete graph on k vertices $G[B_0 \cup B_1]$. By contradiction we obtain that $n < \frac{k}{2} + 1$. □

Lemma 6.15. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G such that every open $\mathcal{S}$-block is empty and all $\mathcal{S}$-blocks lie in one orbit of* $\mathrm{Aut}(G)$*. If $G \cong X_{2,2}(E)$ for some finite graph E that is neither complete nor the complement of a complete graph, then $2|E| - 2 < k$ and $E \in \mathcal{E}_{k,m,n}$ for $m \leq k - 2$ and $n < \frac{k-|E|}{2} + 2$.*

Proof. By Corollary 4.2 if it holds that (a) the maximum degree of E is at most $k-2$, (b) E is l-S-transitive for all $l \leq k-1$, (c) any induced subgraph of order at least $\frac{k-|E|}{2} + 1$ in E is connected, and (d) no two non-adjacent vertices of E

have $k-2$ common neighbours, then $E \in \mathcal{E}_{k,m,n}$ for $m \leq k-2$ and $n < \frac{k-|E|}{2}+2$. Considering the distinct boundaries in (b) and for n, we note that a graph on at least $\frac{k-|E|}{2}+1$ vertices has at least n vertices.

(a) Let S ($G[S] \cong E$) be an $\mathcal{S}$-separator. Suppose there is a vertex v of degree at least $k-1$ in $G[S]$. Let $A \subseteq S$ consist of v and $k-1$ of its neighbours. Let w be some vertex from an $\mathcal{S}$-separator that is adjacent to S. Then $G[A] - v + w$ is isomorphic to $G[A]$, but there is no automorphism of G mapping one onto the other. Thus no vertex in S has degree at least $k-1$.

(b) Let $A, B \subseteq S$ induce isomorphic graphs with at most $k-1$ vertices for some $\mathcal{S}$-separator S ($G[S] \cong E$). Then there is a common neighbour v of these vertices in an adjacent $\mathcal{S}$-separator S_0. Let P be an induced path of length $k-1-|A|$ that starts at v and each of its other vertices is separated properly from $A \subseteq S$ by S_0. By construction of $X_{2,2}(E)$, the path P meets each $\mathcal{S}$-separator in at most one vertex. As G is k-CS-transitive, there is an automorphism α of G that maps $G[P+A]$ to $G[P+B]$. If $|A| \neq 1$, then α must map S onto S as it is the only $\mathcal{S}$-separator meeting more than one vertex of $G[P+A]$ and of $G[P+B]$; clearly this implies $A^\alpha = B$ and A and B lie in the same $\mathrm{Aut}(G[S])$-orbit. If $|A| = 1$, let S' be an $\mathcal{S}$-separator such that some induced path of length $k-1$ starting at A ends in S'. Let φ, φ' be the isomorphisms from E to S, S', respectively. Let $A' \subseteq S'$ be $(A^{\varphi^{-1}})^{\varphi'}$. Then we may assume that the path P ends in A'. Thus α maps A to B or A' to B and as A and A' are $\mathrm{Aut}(G)$-isomorphic so are A and B. Again A and B lie in the same $\mathrm{Aut}(G[S])$-orbit.

(c) Suppose there is an induced subgraph $X \subseteq E$ of order at least $\frac{k-|E|}{2}+1$ that is not connected. Let S_0, S_1, S_2 be three distinct $\mathcal{S}$-separators such that S_0 is adjacent to the other two. Let $A_i \subseteq S_i$ for $i \geq 1$ be of cardinality at least $\frac{k-|E|}{2}+1$ such that $G[A_1] \cong G[A_2]$ are not connected. Let H be a connected induced subgraph on $k+2$ vertices in $G[S_0 \cup A_1 \cup A_2]$ such that there is an isomorphism φ from $H[A_1]$ onto $H[A_2]$ and $H[A_1]$ is not connected. Such a graph exists as $|S_0 \cup A_1 \cup A_2| \geq |E| + 2(\frac{k-|E|}{2}+1) = k+2$. Let $a, b \in A_1$ be vertices that lie in distinct components of $H[A_1]$. Then there is no automorphism of G that maps one of its two isomorphic induced and connected subgraphs $H - \{a, b\}$ and $H - \{a^\varphi, b\}$ onto the other. Thus every induced subgraph of E of order at least $\frac{k-|E|}{2}+1$ is connected.

(d) Suppose that there are two non-adjacent vertices x, y in an $\mathcal{S}$-separator S' ($G[S'] \cong E$) with at least $k-2$ common neighbours in S' and let $N \subseteq S'$ be $k-2$ of these neighbours. Let S, S'' be distinct $\mathcal{S}$-separators adjacent to S' and let $s \in S$ and $s'' \in S''$. Then $G[N \cup \{x, y\}]$ and $G[N \cup \{s, s''\}]$ are isomorphic induced connected subgraphs of G of order k but there is no automorphism of G mapping one onto the other.

It remains to show that $2|E| - 2 < k$. As the values of k, m, n imply this inequality whenever E is not a K_r^t, we need to show that if $G \cong X_{2,2}(K_r^t)$, then $2|K_r^t| - 2 = 2tr - 2 < k$. Let X be an $\mathcal{S}$-block with $x, x', y \in V(X)$ and $xx' \in E(G)$, such that x and x' belong to the same $\mathcal{S}$-separator and y belongs to the other $\mathcal{S}$-separator in X. In this setting $G[x, x', y]$ is a K^3, and thus the subgraphs $X - \{x, x'\}$ and $X - \{x, y\}$ are isomorphic. Suppose that $2tr = |X| \geq k + 2$, then there is an induced subgraph X' of X of size precisely $k+2$ containing x, x' and y such that $X' - \{x, x'\}$ and $X' - \{x, y\}$ are isomorphic but there is no automorphism of G mapping one onto the other. This shows that the inequality $2|E| - 2 < k$ holds in all cases. □

By Lemma 6.13, 6.14, and 6.15 we may finish the first case.

Theorem 6.16. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G such that every open block is empty. Then there are cardinals $\kappa, \lambda \geq 2$ and integers m, n such that G is isomorphic to one of the following graphs:*

(1) $X_{\kappa,\lambda}(K^1)$*;*

(2) $X_{2,\lambda}(K^n)$ *with* $n < \frac{k}{2} + 1$*;*

(3) $X_{\kappa,2}(\overline{K^m})$ *with* $m < \frac{k+2}{3}$*;*

(4) $X_{2,2}(E)$ *with* $E \in \mathcal{E}_{k,m,n}$*,* $m \leq k - 2$*,* $n < \frac{k-|E|}{2} + 2$ *and* $2|E| - 2 < k$*;*

(5) Y_κ *(if k is odd).* □

6.2 Non-empty open blocks

Let us discuss the connected k-CS-transitive graphs with at least two ends for $k \geq 3$ such that every basic cut system has non-empty open blocks. As mentioned before this case restricts k to be even by Lemma 6.4. According to Lemma 6.9 every vertex not in any separator of a basic cut system lies in an open block.

Let us show that the k-CS-transitive graphs with non-empty open blocks resemble some $Z_{\kappa,\lambda}(H_1, H_2)$ by proving that the automorphism group acts transitively on its open blocks.

Lemma 6.17. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G such that some open $\mathcal{S}$-block is not empty. Then every open $\mathcal{S}$-block is non-empty and the automorphism group of G acts transitively on the $\mathcal{S}$-blocks.*

Proof. Let X be an $\mathcal{S}$-block. By Lemma 6.1, X contains two distinct $\mathcal{S}$-separators and any two such separators are disjoint according to Corollary 6.12. By Lemma 6.11 it holds that X contains an edge sx where s lies in an $\mathcal{S}$-separator $S \subseteq X$ and x lies in $X - S$. As there is an induced path P of length $k - 2$ starting at s with all its other vertices in a component of $G - S$ that avoids X, the neighbour of x on the induced path xP of length $k - 1$ lies in an $\mathcal{S}$-separator, and thus x is not contained in any $\mathcal{S}$-separator according to Lemma 6.5.

Let Y be a further $\mathcal{S}$-block. By the previous argument a vertex $y \in \mathring{Y}$ exists. Let P' be an induced path of length $k-1$ starting at y—such a path exists as showed above. Since G is k-CS-transitive, there is an automorphism α mapping P' to xP. As k is even by Lemma 6.4 it holds that $y^\alpha = x$ according to Lemma 6.5. Thus $\mathring{Y}^\alpha \cap \mathring{X} \neq \emptyset$ and even $Y^\alpha = X$, as the intersection of any two distinct $\mathcal{S}$-blocks lies in an $\mathcal{S}$-separator. □

Lemma 6.18. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G. If some open $\mathcal{S}$-block is not empty, then there are graphs H_1, H_2 and cardinals κ, λ such that G is isomorphic to $Z_{\kappa,\lambda}(H_1, H_2)$.*

Proof. The structure tree $\mathcal{T}$ of G and $\mathcal{S}$ is an infinite tree where vertices of even distance have the same degree, as $\mathcal{S}$ is basic and as the automorphisms of G act transitively on the $\mathcal{S}$-blocks by Lemma 6.17. Let κ be the degree of any $\mathcal{S}$-separator in $\mathcal{T}$ and let λ be the degree of any $\mathcal{S}$-block in $\mathcal{T}$. Let H_1 be isomorphic to $G[S]$ for some $\mathcal{S}$-separator S, and let H_2 be isomorphic to some open $\mathcal{S}$-block. Then again as $\mathcal{S}$ is basic and by Lemma 6.17 all separators induce an isomorphic copy of H_1 in G and all open blocks are isomorphic to H_2. Since, according to Lemma 6.11, every vertex of an open block $\mathring{X}$ is adjacent to all vertices in $\mathcal{S}$-separators that lie in X, it holds that $G \cong Z_{\kappa,\lambda}(H_1, H_2)$. □

As every connected k-CS-transitive graph for $k \geq 3$ with more than one end and some non-empty open block is isomorphic to $Z_{\kappa,\lambda}(H_1, H_2)$ for some graphs H_1 and H_2, it remains to specify the building blocks and possible values for κ and λ of these graphs. In Section 2.1 we describe what a basic cut system for these graphs looks like if H_1 and H_2 are finite.

Lemma 6.19. *Let $k \geq 3$, let $G \cong Z_{\kappa,\lambda}(H_1, H_2)$ be a k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G such that some open $\mathcal{S}$-block is not empty. Then the following holds:*

(i) At least one of κ or λ is 2;

(ii) if H_i contains two non-adjacent vertices, then H_j $(j \neq i)$ is complete and $\kappa = \lambda = 2$;

(iii) if H_i contains an edge, then H_j $(i \neq j)$ contains no edge.

Proof. Either $H_1 \ncong H_2$ or $\kappa \neq \lambda$ since the copies of H_1 and H_2 are not $\mathrm{Aut}(G)$-isomorphic. Suppose both κ and λ are at least 3, then there are two distinct orbits of k-forks. One whose members poke in two distinct open $\mathcal{S}$-blocks, and one whose members poke in two distinct $\mathcal{S}$-separators. As a k-CS-transitive graph has only one orbit of k-forks this proves (i) by contradiction.

Part (ii) follows using an analogous argument: Suppose κ or λ is greater than 2. Then there is a k-fork that pokes in just one copy of an H_i and one that pokes in two distinct copies of H_1 (if $\kappa > 2$) or in two distinct copies of H_2 (if

$\lambda > 2$). Suppose on the other hand that there are two non-adjacent vertices in H_j, then there are two incompatible k-forks, too. One pokes in an open $\mathcal{S}$-block and the other one in an $\mathcal{S}$-separator.

For (iii), suppose that H_i as well as H_j contain edges. Then there are k-spoons that poke in open $\mathcal{S}$-blocks and others that poke in $\mathcal{S}$-separators. □

From the previous lemma we immediately get the following corollary.

Corollary 6.20. *Let $k \geq 3$, let $G \cong Z_{\kappa,\lambda}(H_1, H_2)$ be a k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G such that some open $\mathcal{S}$-block is not empty. If both H_1 and H_2 have at least two vertices, then one is a complete graph, the other one is the complement of a complete graph, and $\kappa = \lambda = 2$.* □

To finish the proof in the situation that both, H_1 and H_2, have at least two vertices, we will restrict the order of these graphs.

Lemma 6.21. *Let $k \geq 3$, let $G \cong Z_{2,2}(H_1, H_2)$ be a k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G such that some open $\mathcal{S}$-block is not empty. If $H_1 \cong \overline{K^m}$ and $H_2 \cong K^n$, then $2m + n \leq k + 1$.*

Proof. Suppose that $2m + n > k + 1$. Let X be a building block corresponding to the complete graph H_2 and let Y, Y' be the two building blocks corresponding to H_1 adjacent to X. If $m \geq 2$, then as k is even there are subsets Y_1, Y_2 in Y and Y_1', Y_2' in Y' with

$$\begin{aligned} |Y_1| &= \min\{m-1, \tfrac{k}{2}-1\}, \\ |Y_1'| &= \min\{m-1, \tfrac{k}{2}-1\}, \\ |Y_2| &= \min\{m-2, \tfrac{k}{2}-2\}, \text{ and} \\ |Y_2'| &= \min\{m, \tfrac{k}{2}\}. \end{aligned}$$

If $n \geq 2$ let X' be a subset of $V(X)$ of cardinality $k - (|Y_1| + |Y_1'|) \geq 2$ which exists as

$$k - (|Y_1| + |Y_1'|) \leq k - 2(m-1) \leq n.$$

The graphs $G[Y_1 \cup X' \cup Y_1']$ and $G[Y_2 \cup X' \cup Y_2']$ are isomorphic, so by k-CS-transitivity, there is an automorphism of G mapping the first onto the second subgraph. This is a contradiction, as Y_2' is larger than Y_1 as well as Y_1' and every automorphism of G has to map a building block onto a building block corresponding to the same H_i by the construction of $Z_{2,2}(H_1, H_2)$ and our choices for H_1 and H_2.

If $n = 1$, then $2m > k$ and hence $2m \geq k+2$ as k is even. By enlarging each of Y_1' and Y_2' by one vertex we obtain a similar contradiction in this case as for $n \geq 2$.

If $m = 1$, then we have $n \geq k$. Let X be a subset of the vertex set of a building block corresponding to the complete graph H_2 of cardinality k, Let $x \in X$, and let y be a vertex adjacent to x but not in the same building block.

By the construction of G we know that y is adjacent to every vertex of X. Thus, the subgraphs $G[X]$ and $G[X] - x + y$ are both complete graphs on k vertices, so there is an automorphism of G that maps the first onto the second by k-CS-transitivity. But again as every automorphism of G maps a building block onto a building block corresponding to the same H_i, we obtain a contradiction. □

Lemma 6.22. *Let $k \geq 3$, let $G \cong Z_{\kappa,\lambda}(H_1, H_2)$ be a k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G such that some open $\mathcal{S}$-block is not empty. If one of κ and λ is not 2, then both H_1 and H_2 are complete, one of order 1 and the other of order at most $k - 1$.*

Proof. It follows directly from Lemma 6.19 (ii) that both H_1 and H_2 are complete. By Lemma 6.19 (iii) we may assume that $|H_1| = 1$. Suppose that H_2 has more than $k - 1$ vertices. Every open $\mathcal{S}$-block $\mathring{X}$ is a building block that corresponds to H_2 and thus contains an isomorphic copy of a K^k. There is a second isomorphic copy of a K^k in G with $k - 1$ vertices in $\mathring{X}$ and one vertex in some $\mathcal{S}$-separator $S \subseteq X$. Since there is no automorphism of G mapping one onto the other, H_2 has at most $k - 1$ vertices. □

The last part in this case of the proof (that there is some non-empty open block) is to determine the graphs H_2 if the graph H_1 has only one vertex and the open blocks are neither complete nor complements of complete graphs.

Lemma 6.23. *Let $k \geq 3$, let $G \cong Z_{2,2}(H_1, H_2)$ be a k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G such that some open $\mathcal{S}$-block is not empty. If H_2 is neither complete nor the complement of a complete graph, then $H_1 \cong K^1$ and $H_2 \in \mathcal{E}_{k,m,n}$ for $m \leq k - 2$ and $n \leq \frac{k}{2} + 1$.*

Proof. As H_2 is not complete, it contains two non-adjacent vertices. This implies by Lemma 6.19 (ii) that H_1 is complete. Since H_2 also contains an edge, H_1 does not and thus is isomorphic to K^1. By Corollary 4.2 it suffices to show that (a) the maximum degree of H_2 is at most $k - 2$, (b) H_2 is l-S-transitive for all $l \leq k - 1$, (c) any induced subgraph of order at least $\frac{k}{2} + 1$ in H_2 is connected, and (d) no two non-adjacent vertices of H_2 have $k - 2$ common neighbours.

The proofs of (a), (b) and (d) are analogous to those of Lemma 6.15 (a), (b) and (d).

(c) Following the argument of Lemma 6.15 (c), an induced subgraph of order at least $\frac{k-1}{2} + 1$ in H_2 is connected. The '-1' in that term corresponds to the '$-|E|$' in Lemma 6.15. Since k is even, every induced subgraph of order at least $\frac{k}{2} + 1$ is connected if and only if every induced subgraph of order at least $\frac{k-1}{2} + 1$ is connected. □

These lemmas let us finish the case of non-empty open blocks.

Theorem 6.24. *Let $k \geq 3$, let G be a connected k-CS-transitive graph with at least two ends, and let $\mathcal{S}$ be a basic cut system of G such that some open $\mathcal{S}$-block is not empty. Then k is even and G is isomorphic to one of the following graphs:*

(6) $Z_{2,2}(\overline{K^m}, K^n)$ *with* $2m + n \leq k + 1$*;*

(7) $Z_{\kappa,\lambda}(K^1, K^n)$ *with* $n \leq k - 1$ *and cardinals* κ, λ *with* $\kappa = 2$ *or* $\lambda = 2$*;*

(8) $Z_{2,2}(K^1, E)$ *with* $E \in \mathcal{E}_{k,m,n}$*,* $m \leq k - 2$ *and* $n \leq \frac{k}{2} + 1$*.* □

With Corollary 5.4 and Corollary 6.10 the Theorems 6.16 and 6.24 imply our second main result, Theorem 1.3.

7 k-CS-homogeneous graphs

In this section we shall prove Corollary 1.4. The first part of the proof will be to exclude those k-CS-transitive graphs that do not occur in the list of Corollary 1.4 and then to prove that the remaining graphs are k-CS-homogeneous.

Proof of Corollary 1.4. As every k-CS-homogeneous graph is k-CS-transitive, the connected k-CS-homogeneous graphs with $k \geq 3$ and at least two ends belong to classes (1) to (8) of Theorem 1.3. Let us first show that for the appropriate k all graphs that occur in the list Theorem 1.3 but not in that of Corollary 1.4 are not k-CS-homogenous.

For odd $k \geq 3$, the graphs Y_κ for $\kappa \geq 3$ are not k-CS-homogeneous, since we cannot map an induced path in Y_κ of length $k-1$ onto itself by an automorphism of Y_κ without being the identity on that path, as its outermost edges lie in buildings blocks of distinct kinds. As the automorphism group of any non-trivial path consists of two elements, these graphs are not k-CS-homogenous.

For even $k \geq 3$, any graph $G \cong Z_{\kappa,\lambda}(H_1, H_2)$ for any distinct graphs H_1, H_2 or distinct cardinals κ, λ is not k-CS-homogeneous as we cannot map an induced path of odd length $k - 1$ in G onto itself by an automorphism of the whole graph without being the identity on that path as its endvertices lie in distinct kinds of building blocks of G. If on the other hand $H_1 \cong H_2$ and $\kappa = \lambda$, then as it has to be k-CS-transitive Theorem 1.3 implies that $H_1 \cong K^1$ and $Z_{\kappa,\lambda}(H_1, H_2) \cong X_{2,\kappa}(K^1)$ and hence G belongs to the graphs in class (1) of Theorem 1.3.

Let us consider a graph $G \cong X_{2,\lambda}(K^n)$ with arbitrary $k \geq 3$. If $\frac{k}{2} \leq n < \frac{k}{2} + 1$, then there is an induced subgraph isomorphic to K^k in two (completely) adjacent building blocks of G. We cannot extend any automorphism of such a subgraph that does not respect the building blocks to an automorphism of the whole graph. This implies $n < \frac{k}{2}$ in this case.

Let $G \cong X_{\kappa,2}(\overline{K^m})$. If $\frac{k}{3} \leq m < \frac{k+2}{3}$, then take an arbitrary subgraph X on k vertices of three building blocks one of which is completely adjacent to the other two that are not adjacent to each other. Then there is at most one vertex of the three building blocks missing in X. Thus we might build an automorphism of X that maps two vertices of the two non-adjacent building blocks onto each other and fixes all the other vertices of X. As $m \geq 2$, this automorphism of X cannot be extended to an automorphism of G.

Let us now assume that $G \cong X_{2,2}(E)$ for an $E \in \mathcal{E}_{k,m,n}$ with $m \leq k-2$, $2|E|-2 < k$ and $n < \frac{k-|E|}{2} + 2$. Suppose that E contains an induced subgraph on at least $\frac{k-|E|}{2}$ vertices that is not connected. Let E_1, E_2, E_3 be three building blocks of G such that E_2 is (completely) adjacent to the other two but E_1 and E_3 are not adjacent. Then there are two induced subgraphs $X \subseteq E_1$ and $Y \subseteq E_3$ each of order at least $\frac{k-|E|}{2}$, both not connected such that $G[X] \cong G[Y]$. By the cardinality of these vertex sets, there is a non-empty vertex set Z in E_2 such that $|X|+|Y|+|Z| = k$. There is an automorphism of $H := G[X \cup Y \cup Z]$ that exchanges a component of $G[X]$ with one of $G[Y]$ and fixes every other vertex in H. As every automorphism of G maps vertices in the same building block again to a common building block, the just described automorphism of H does not extend to an automorphism of G. By contradiction we get that $E \in \mathcal{E}_{k,m,n'}$ for k and m as above and $n' < \frac{k-|E|}{2} + 1$.

It remains to show that $2|E| < k$ in this situation. If E is isomorphic to t disjoint K^r, then the inequalities imply $tr < \frac{k-|E|}{2}$ and hence $3|E| = 3tr < k$. If $E \cong C_5$, then $4 < \frac{k-5}{2} + 1$ implies $11 < k$ and if $E \cong L(K_{3,3})$, then $6 < \frac{k-9}{2} + 1$ implies $19 < k$. Suppose $2|E| \geq k$, then none of these three previous cases may occur and we conclude $E \cong K_r^t$ with $2 \leq t$. Let H be a subgraph of G induced by two adjacent building blocks B_1, B_2. Then H has less than $k+2$ vertices. Let X be an induced subgraph of H on k vertices. Then either $X = H$ or there is one vertex x in H with $X = H - x$. There is a set Y_1 of r independent vertices in $X \cap B_1$ and a set Y_2 of r independent vertices in $X \cap B_2$. As each of these $2r$ vertices is adjacent to all other vertices of X, there is an automorphism of X that maps Y_1 onto Y_2 and vice versa and that fixes every other vertex in X. Such an automorphism of X cannot extend to an automorphism of G as vertices in the same building block have to be mapped into a common building block by every automorphism of G and this is not satisfied by the above described automorphism of X.

It remains to show that the graphs described in Corollary 1.4 are k-CS-homogeneous. In principle, the proof is similar to those in Section 5. Therefore, we just point out the important bits that have to be changed and give a sketch of the remaining part of the proof. Let G be a graph that occurs in the list of Corollary 1.4. For the corresponding assertion of Lemma 5.1, it suffices to see that the only graphs in the list of Corollary 1.4 that have a complete graph on k vertices as subgraph, are the graphs $X_{\kappa,\lambda}(K^1)$ and $X_{\kappa,2}(\overline{K^m})$ and in each of these cases the construction of the graphs admits the extension of every isomorphism between two complete subgraphs on k vertices.

For the proof of Lemma 5.2, remark that any induced subgraph of an induced connected subgraph X on k vertices with diameter 2 has to meet at least three building blocks by cardinality reasons. It easily follows in each case that either there exists a unique smallest separator in X which is precisely $X \cap B$ where B is a building block adjacent to all other building blocks of G that meet X, or $G \cong X_{\kappa,2}(\overline{K^m})$ and X is a subgraph of a complete multipartite graph with partition classes each of the same cardinality. Where the required extension of any isomorphism between two induced connected subgraphs follows from the homogeneity of complete multipartite graphs in the last case, the extension exists for the first case because the building blocks are homogeneous and by the construction of the graphs $X_{\kappa,\lambda}(H)$.

For the situation that the induced isomorphic subgraphs on k vertices are connected and have diameter at least 3, it suffices to see that any isomorphism between any paths of length at least 3 whose diameter in G is at least 3 in these graphs can be extended to an automorphism of the whole graph. The further construction of the automorphisms α_k in the proof of Lemma 5.3 can also be chosen so that they extend the given isomorphism between the two induced connected subgraphs of order k. This completes the sketch of this direction of the proof and hence the whole proof. □

8 Ends of k-CS-transitive graphs

Gray [6] asked whether every locally finite k-CS-transitive graph is end-transitive for $k \geq 3$. With Theorem 1.3 we may answer his question.

Theorem 8.1. *Let $k \geq 3$ and let G be a connected locally finite graph. If G is k-CS-transitive, then it is end-transitive.* □

This theorem does not extend to graphs with vertices of infinite degree. For example the graphs $X_{\kappa,\lambda}$ with $\kappa \geq \aleph_0, \lambda \geq 2$ contain fundamentally different ends. Let us make this precise: a ray is *local* if it meets a set of finite diameter infinitely often. An end is *local* if all its rays are local, and an end is *global* if none of its rays is local. Theorem 1.3 shows that in k-CS-transitive graphs with $k \geq 3$ and more than one end every end is either local or global and that the automorphism group acts transitively on those of each kind.

Theorem 8.2. *Let $k \geq 3$ and G be a connected k-CS-transitive graph with more than one end. Then every end of G is either local or global. The automorphism group of G acts transitively on the local ends, as well as on the global ends.*

Furthermore, G is end-transitive if and only if it has no local end. □

Krön and Möller [9, 10] introduced metric ends. They call rays *metric* if they are not local, that is, if any infinite subset of its vertices does not have finite diameter in G. Two metric rays R_1 and R_2 are *metrically equivalent* if there is no vertex set S of finite diameter such that R_1 and R_2 lie eventually in different components of $G - S$. This is an equivalence relation on metric rays, whose classes are the *metric ends* of the graph. In locally finite graphs the notions of being an end and being a metric end coincide. Thus for connected locally finite k-CS-transitive graphs with $k \geq 3$ and with more than one end its automorphism group acts transitively on its metric ends. In spite of the local ends this extends by inspection of the examples in Theorem 1.3 to graphs that are not necessarily locally finite.

Theorem 8.3. *If $k \geq 3$, then the automorphism group of any connected k-CS-transitive graph with more than one end acts transitively on the metric ends of the graph.* □

Acknowledgement

We gratefully acknowledge the careful reading and helpful comments of a referee.

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