A Stallings’ type theorem
for quasi-transitive graphs

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Abstract

We consider locally finite, connected, quasi-transitive graphs and show
that every such graph with more than one end is a tree amalgamation of
two other such graphs. This can be seen as a graph-theoretical version of
Stallings’ splitting theorem for multi-ended finitely generated groups and
indeed it implies this theorem. Our result also leads to a characterisation
of accessible graphs. We obtain applications of our results for planar
graphs (answering a variant of a question by Mohar in the affirmative)
and graphs without thick ends.

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1 Introduction

Stallings [25] proved that every finitely generated group with more than one end is either a free product with amalgamation over a finite subgroup or an HNN-extension over a finite subgroup. While amalgamated free products and HNN-extensions are group theoretical concepts, they can also be interpreted as operations on the Cayley graphs. For instance, if $\Gamma_1$ and $\Gamma_2$ are groups both of which contain a finite subgroup isomorphic to $C$, then Cayley graphs $G_1$ of $\Gamma_1$ and $G_2$ of $\Gamma_2$ can be glued together along copies of cosets of $C$ in a tree-like way in order to obtain a Cayley graph of the amalgamated free product $\Gamma_1 *_C \Gamma_2$.

Mohar [19] proposed a similar operation (called tree amalgamation) for arbitrary graphs. Roughly speaking, a tree amalgamation of two graphs $G_1$ and $G_2$ is obtained by gluing copies of $G_1$ onto $G_2$ and vice versa in a tree-like way along finite subgraphs, called adhesion sets; we refer the reader to Section 5 for a precise definition. The main result of this paper can thus be seen as an analogue of Stallings’ theorem for connected, quasi-transitive graphs.

**Theorem 1.1.** Every connected, quasi-transitive, locally finite graph with more than one end is a non-trivial tree amalgamation of finite adhesion of two connected, quasi-transitive, locally finite graphs.

The connection between Theorem 1.1 and Stallings’ theorem goes even further: we show that we can restrict ourselves to two specific kinds of tree amalgamations which we refer to as ‘Type 1’ and ‘Type 2’, respectively. It turns out that the Cayley graph (with respect to the generators of the factors) of an amalgamated free product can always be seen as a Type 1 tree amalgamation, and similarly the Cayley graph of an HNN-extension can be seen as a Type 2 tree amalgamation, see Examples 5.3 and 5.4. Moreover, if $\Gamma$ is a group acting quasi-transitively on a graph $G$, then the tree amalgamation of $G$ obtained by Theorem 1.1 gives rise to an action of $\Gamma$ on a semiregular tree. In the case that $G$ is a Cayley graph of $\Gamma$, we can apply Bass-Serre theory to recover Stallings’ theorem from Theorem 1.1, see Section 7.1.

We also consider several applications of our result. The first application concerns accessibility of graphs. Recall that a group is called accessible if it can be obtained from finite and one-ended groups by iterated amalgamated free products and HNN-extensions over finite subgroups. Similarly, one can ask what graphs can be obtained from finite and one-ended quasi-transitive graphs by iterated (Type 1 and 2) tree amalgamations over finite sets of vertices. Thomassen and Woess [27] defined accessibility for graphs as follows: a quasi-transitive, locally finite graph is accessible in the sense of Thomassen of Woess\(^1\) if there is some $n \in \mathbb{N}$ such that every two ends can be separated by at most $n$ edges. They showed in [27] that a finitely generated group is accessible if and only if each of its locally finite Cayley graphs is accessible. We show that an analogous result holds for tree amalgamations of quasi-transitive graphs: the class of accessible, connected, quasi-transitive, locally finite graphs is precisely the class of graphs obtained by iterated tree amalgamations of finite adhesion starting with finite or one-ended connected, locally finite, quasi-transitive graphs.

In Section 7.3 we obtain an answer to a question by Mohar from 1988 [19]. He asked, whether tree amalgamations are powerful enough to yield a classi-
fication of infinitely-ended transitive planar graphs in terms of finite and one-ended infinite planar transitive graphs. More precisely, he asked whether every 3-connected, planar, transitive graph can be obtained by iterated tree amalgamations of finite or one-ended, planar, transitive graphs. Georgakopoulos [9] gives examples where this is not possible, but suggests that Mohar intended to allow subdivisions of finite and one-ended, planar, transitive graphs as well. Our theorems provide an affirmative answer for quasi-transitive graphs because Dunwoody [8] proved that they are accessible in the sense of Thomassen and Woess. It is worth noting that even if we start with a transitive graphs, the parts we end up with may still be quasi-transitive.

Additionally, as mentioned above, we obtain Stallings’ theorem as a corollary to Theorem 1.1, see Section 7.1, and a new characterisation of quasi-transitive, locally finite graphs that are quasi-isometric to trees, see Section 7.2. In Section 7.4 we discuss some further applications of our main result regarding hyperbolic graphs, quasi-isometries of graphs and asymptotic dimensions of graphs.

Our main tool to prove Theorem 1.1 are tree-decompositions that are invariant under the automorphisms of the graph. While some proofs of Stallings’ theorem rely on edge separators and their structure trees, see for instance Dunwoody [5], it turns out that tree-decompositions and vertex separators work better in combination with tree amalgamations. However, due to the similar natures of structure trees and tree-decompositions, it is not surprising that some results that we prove here (in particular Propositions 4.7 and 4.8) have also been proved for structure trees, see e.g. Thomassen and Woess [27] and Möller [20, 21].

2 Preliminaries

We follow the general notations of [4] unless stated otherwise. In the following we will state the most important definitions for convenience.

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a subset $X$ of $V(G)$ we denote by $G[X]$ the subgraph of $G$ induced by $X$, that is, $G[X] = (X, \{ uv \in E(G) \mid u, v \in X \})$; $G - X$ denotes the subgraph $G[V(G) \setminus X]$ induced by the complement of $X$. A geodesic is a shortest path between two vertices. A ray is a one-way infinite path, the infinite subpaths of a ray are its tails. Two rays are equivalent if they have tails contained in the same component of $G - S$ for every finite set $S$ of vertices. The equivalence classes of rays in a graph are its ends. The degree of an end is the maximum number of disjoint rays in that end, if this maximum exists. If the maximum does not exist, that is, if an end contains $n$ disjoint rays for every $n \in \mathbb{N}$, then we say that this end has infinite degree and we call it thick. An end with finite degree is called thin. An end $\omega$ is captured by a set $X$ of vertices if every ray of $\omega$ has infinite intersection with $X$ and it lives in $X$ if every ray of $\omega$ has a tail in $X$.

Let $X \subseteq V(G)$. Let $G'$ be the graph with vertex set $(V(G) \setminus X) \cup \{v_X\}$, where $v_X$ is a new vertex, and and the following edge set:

$$E(G') := \{ uv \in E(G) \mid u, v \in V(G) \setminus X \} \cup \{ v_Xu \mid \exists x \in X: xxu \in E(G) \}.$$  

We call $G'$ the contraction of $X$ in $G$ and we say that $G'$ is obtained from $G$ by contracting $X$. Since edges are just vertex sets of size 2, the definition carries
over to edges.

Let $\Gamma$ be a group acting on $G$ and let $X \subseteq V(G)$. The (setwise) stabilizer of $X$ with respect to $\Gamma$ is the set

$$\Gamma_X := \{ g \in \Gamma \mid g(x) \in X \text{ for all } x \in X \}.$$ 

The $\Gamma$-orbit of a vertex $x \in V(G)$ is the set $\{ g(x) \mid g \in \Gamma \}$; if $\Gamma$ is clear from the context we omit it and speak of the orbit of $x$. We say $\Gamma$ acts transitively on $G$ if $V(G)$ is one $\Gamma$-orbit and $\Gamma$ acts quasi-transitively on $G$ if $V(G)$ consists of finitely many $\Gamma$-orbits.

## 3 Tree-decompositions

In this section we introduce the main tool for our proofs: tree-decompositions. A tree-decomposition of a graph $G$ is a pair $(T, \mathcal{V})$ where $T$ is a tree and $\mathcal{V} = (V_t)_{t \in V(T)}$ is a family of vertex sets of $G$ such that the following three conditions are satisfied:

1. $V(G) = \bigcup_{t \in V(T)} V_t$. 
2. For every edge $e \in E(G)$ there is a $t \in V(T)$ such that $V_t$ contains both vertices that are incident with $e$. 
3. $V_{t_1} \cap V_{t_2} \subseteq V_{t_3}$ whenever $t_3$ lies on the $t_1$-$t_2$ path in $T$.

The sets $V_t$ are called the parts of $(T, \mathcal{V})$, the tree $T$ is called the decomposition tree, and the vertices of $T$ are called its nodes (to distinguish them from the vertices of $G$). The sets $V_{t_1} \cap V_{t_2}$ with $t_1, t_2 \in E(T)$ are called adhesion sets. We say that $(T, \mathcal{V})$ has adhesion at most $k$ for $k \in \mathbb{N}$ if all adhesion sets have size at most $k$ and it has finite adhesion if all adhesion sets are finite.

Remark 3.1. Let $(T, \mathcal{V})$ be a tree-decomposition and let $t_1 t_2$ be an edge of $T$. For $i = 1, 2$, let $T_i$ be the component of $T - t_1 t_2$ that contains $t_i$. It follows from (T3) that $V_{t_1} \cap V_{t_2}$ separates the vertices in $\bigcup_{t \in T_1} V_t$ from those in $\bigcup_{t \in T_2} V_t$.

We say that $(T, \mathcal{V})$ distinguishes two ends $\omega_1$ and $\omega_2$ if there is a finite adhesion set $V_{t_1} \cap V_{t_2}$ such that one end lives in $\bigcup_{t \in T_1} V_t$ and the other one lives in $\bigcup_{t \in T_2} V_t$, where $T_i$ is the maximal subtree of $T - t_1 t_2$ containing $t_i$. It distinguishes them efficiently if no vertex set in $G$ of smaller size than $V_{t_1} \cap V_{t_2}$ separates them. For $k \in \mathbb{N}$, two ends of $G$ are $k$-distinguishable if there is a set of $k$ vertices of $G$ that separates them.

Let $\Gamma$ be a group acting on $G$. If every $\gamma \in \Gamma$ maps each part of $(T, \mathcal{V})$ to a part and thereby induces an automorphism of $T$ we say that $(T, \mathcal{V})$ is $\Gamma$-invariant.

The following theorem by Carmesin et al. will be the main result we are building on.

**Theorem 3.2.** [2] Let $G$ be a locally finite graph, let $\Gamma$ be a group acting on $G$ and let $k \in \mathbb{N}$. Then there is a $\Gamma$-invariant tree-decomposition of $G$ of adhesion at most $k$ that efficiently distinguishes all $k$-distinguishable ends.

\[\square\]
4 Basic tree-decompositions

In this section, we first modify the tree-decomposition of Theorem 3.2, mainly to make its parts connected. Then we will prove some properties of the newly obtained tree-decomposition, in particular, where the tree-decomposition captures the ends of the graph. Our first step in modifying the tree-decomposition of Theorem 3.2 will be to make all adhesion sets connected while keeping the action of $\Gamma$ on $(T, \mathcal{V})$.

**Proposition 4.1.** Let $\Gamma$ be a group acting on a locally finite graph $G$ and let $(T, \mathcal{V}) := (T, (V_t)_{t \in V(T)})$ be a $\Gamma$-invariant tree-decomposition of $G$ of finite adhesion. Then there is a $\Gamma$-invariant tree-decomposition $(T, \mathcal{V}') := (T, (V'_t)_{t \in V(T)})$ of $G$ such that every adhesion set of $(T, \mathcal{V})$ is finite and connected, and such that $V_t \subseteq V'_t$ for every $t \in V(T)$.

**Proof.** Let $u$ and $v$ be two vertices of an adhesion set of $(T, \mathcal{V})$. Let $\mathcal{P}_{uv}$ be the set of all geodesics between $u$ and $v$ and let $V_{uv}$ be the set of all vertices of $G$ that lie on the paths of $\mathcal{P}_{uv}$. For a part $V_t$, let $V'_t$ be the union of $V_t$ with all sets $V_{uv}$ where $u$ and $v$ lie in an adhesion set contained in $V_t$. Let $\mathcal{V}' := \{V'_t \mid t \in V(T)\}$.

We claim that $(T, \mathcal{V}')$ is a tree-decomposition. As every element of $\mathcal{V}'$ is a superset of some element of $\mathcal{V}$, we only have to verify (T3). Let $x \in V'_1 \cap V'_2$ for $t_1, t_2 \in V(T)$, and let $t_3$ be on the $t_1$-$t_2$ path $s_1, \ldots, s_n$ in $T$ with $s_1 = t_1$ and $s_n = t_2$. If $x \in V_{t_1} \cap V_{t_2}$, then we have $x \in V_{t_3} \subseteq V'_3$ as $(T, \mathcal{V})$ is a tree-decomposition. If $x \in (V'_1 \setminus V_{t_1}) \cap V_{t_2}$, then it lies on a geodesic $P$ between two vertices $x_1, x_2$ of an adhesion set of $(T, \mathcal{V})$ in $V_{t_1}$. Since every adhesion set $V_{s_i} \cap V_{s_{i+1}}$ separates $V_{s_i}$ from $V_{s_{i+1}}$ and since $x \in V_{t_2}$, the path $P$ must pass through $V_{s_i} \cap V_{s_{i+1}}$. Thus, either $P$ contains two vertices $u, v$ of $V_{s_i} \cap V_{s_{i+1}}$ such that $x$ lies on the $u$-$v$ subpath $P''$ of $P$, or $x$ lies in $V_{s_i} \cap V_{s_{i+1}}$. In the first case, we added $P'$ to the adhesion set $V_{s_i} \cap V_{s_{i+1}}$ because $P''$ is a geodesic with its end vertices in $V_{s_i} \cap V_{s_{i+1}}$. Thus, in both cases $x$ lies in $V_{s_i} \cap V_{s_{i+1}}$ and thus in $V'_{t_3}$. If $x \in (V'_1 \setminus V_{t_1}) \cap (V'_2 \setminus V_{t_2})$, let $t_4 \in V(T)$ with $x \in V_{t_4}$. By the previous case, $x$ lies in $V'_{t_4}$ for every $t$ on the $t_1$-$t_4$ or $t_2$-$t_4$ paths in $T$. Since $T$ is a tree, these cover the path $s_1, \ldots, s_n$ and hence $x \in V'_{t_4}$. This proves that $(T, \mathcal{V}')$ is a tree-decomposition.

It remains to show that $(T, \mathcal{V}')$ has the desired properties. By construction, every adhesion set is connected and $V_t \subseteq V'_t$. Since $G$ is locally finite and the adhesion sets of $(T, \mathcal{V})$ are finite, every adhesion set of $(T, \mathcal{V}')$ is finite. Since we made no choices when adding all possible geodesics to the adhesion sets, $\Gamma$ acts on $(T, \mathcal{V}')$ in the same way as on $(T, \mathcal{V})$.

We call a tree-decomposition of a graph $G$ connected if all parts induce connected subgraphs of $G$.

The step to make the adhesion sets connected is just an intermediate step for us: we aim for connected tree-decompositions. The connection between these two classes is given by our next lemma.

**Lemma 4.2.** If all adhesion sets of a tree-decomposition $(T, \mathcal{V})$ of a connected graph $G$ are connected, then $(T, \mathcal{V})$ is connected.

**Proof.** Let $u$ and $w$ be two vertices of $V_t$ for some $t \in V(T)$. Since $G$ is connected, there is a path $P = p_1, \ldots, p_n$ with $p_1 = u$ and $p_n = w$. We choose $P$...
with as few vertices outside of \( V_t \) as possible. Let us suppose that \( P \) leaves \( V_t \). Let \( p_i \in V_t \) such that \( p_{i+1} \notin V_t \) and let \( p_j \) be the first vertex of \( P \) after \( p_i \) that lies in \( V_t \). As \( p_0 = w \in V_t \) we know that such a vertex always exists. Let \( t' \in V(T) \) be such that \( p_{i+1} \in V_{t'} \). Then the adhesion set \( V_t \cap V_s \), where \( s \) is the neighbour of \( t \) on the \( t-t' \) path in \( T \), separates \( V_t \) from \( p_{i+1} \). Hence, the definition of a tree-decomposition implies that \( p_j \) must lie in \( V_t \cap V_s \), too. But then we can replace the subpath of \( P \) between \( p_i \) and \( p_j \) by a path in \( V_t \cap V_s \). The resulting walk contains a path between \( u \) and \( w \) with fewer vertices outside of \( V_t \) than \( P \). This contradiction shows that all vertices of \( P \) lie in \( V_t \) and hence \( G[V_t] \) is connected.

Most of the time we do not need the full strength of Theorem 3.2 in that it suffices to consider \( \Gamma \)-invariant tree-decompositions with few \( \Gamma \)-orbits that still distinguish some ends.

Let \( \Gamma \) be a group acting on a connected, locally finite graph \( G \) with at least two ends. A \( \Gamma \)-invariant tree-decomposition \((T, \mathcal{V})\) of \( G \) is a basic tree-decomposition (with respect to \( \Gamma \)) if it has the following properties:

(i) \((T, \mathcal{V})\) distinguishes at least two ends.

(ii) Every adhesion set of \((T, \mathcal{V})\) is finite.

(iii) \( \Gamma \) acts on \((T, \mathcal{V})\) with precisely one orbit on \( E(T) \).

If the group \( \Gamma \) is clear from the context, we just say that \((T, \mathcal{V})\) is a basic tree-decomposition of \( G \). It follows from Theorem 3.2 that basic tree-decompositions always exist.

**Corollary 4.3.** Let \( \Gamma \) be a group acting on a locally finite graph \( G \) with at least two ends. Then there is a basic tree-decomposition \((T, \mathcal{V})\) for \( G \).

**Proof.** By Theorem 3.2, we find a \( \Gamma \)-invariant tree-decomposition \((T, \mathcal{V})\) of bounded adhesion that separates some ends. Let \( tt' \) be an edge of \( T \) such that \( V_t \cap V_{t'} \) separates some ends. Let \( E_{tt'} \) be the orbit of \( tt' \), that is, the set \( \{g(tt') \mid g \in \Gamma\} \), and let \( T' \) be obtained from \( T \) by contracting each component \( C \) of \( T - E_{tt'} \) to a single vertex \( t_C \). We set \( V_{t_C} := \bigcup_{C \in \mathcal{C}} V_C \) and set \( \mathcal{V'} \) be the set of those sets \( V_{t_C} \). It is easy to see that \((T', \mathcal{V'})\) is a basic tree-decomposition with respect to \( \Gamma \); the only non-trivial requirement is that \((T', \mathcal{V'})\) distinguishes at least two ends. But this follows from the fact that \( V_t \cap V_{t'} \) separates two ends.

Let us combine our results on connected, basic tree-decompositions.

**Corollary 4.4.** Let \( \Gamma \) be a group acting on a connected, locally finite graph \( G \) with at least two ends. Then the following hold.

(i) There is a connected, basic tree-decomposition of \( G \) with respect to \( \Gamma \).

(ii) If \((T, (V_t)_{t \in V(T)})\) is a basic tree-decomposition of \( G \) with respect to \( \Gamma \), then there is a connected, basic tree-decomposition \((T, (V'_t)_{t \in V(T)})\) of \( G \) with respect to \( \Gamma \) such that \( V_t \subseteq V'_t \) for every \( t \in V(T) \).

**Proof.** By Corollary 4.3, there is a basic tree-decomposition of \( G \). Given a basic tree-decomposition \((T, (V_t)_{t \in V(T)})\), Proposition 4.1 and Lemma 4.2 imply the existence of a connected, basic tree-decomposition \((T, (V'_t)_{t \in V(T)})\) with \( V_t \subseteq V'_t \) for every \( t \in V(T) \).
Now we investigate connections between the graphs and the parts of any connected, basic tree-decomposition. Some of these connections are similar to connections between graphs are their structure trees based on edge separators, see e.g. Thomassen and Woess [27] and Möller [20, 21]. We start by showing that connected, basic tree-decompositions behave well with respect to the class of quasi-transitive graphs.

**Proposition 4.5.** Let $\Gamma$ be a group acting quasi-transitively on a connected, locally finite graph $G$ with at least two ends and let $(T, V)$ be a connected, basic tree-decomposition of $G$. Then for each part $V_i \in V$ its stabilizer $\Gamma V_i$ acts quasi-transitively on $G[V_i]$.

**Proof.** If $u \in V_i$ does not lie in any adhesion set, then none of its images $v \in V_i$ under elements of $\Gamma$ lie in an adhesion set. Hence, if $\gamma \in \Gamma$ maps $u$ to $v$, it must fix $V_i$ setwise, as it acts on $(T, V)$, so it lies in the stabilizer of $V_i$. Thus, the intersection of $V_i$ with the $\Gamma$-orbit of $u$ is the $\Gamma V_i$-orbit of $u$.

Now we consider the vertices in an adhesion set $V_i \cap V_j$. Let $V_i \cap V_j$ be another adhesion set. As $(T, V)$ is basic, there exists $\gamma \in \Gamma$ that maps $V_i \cap V_j$ to $V_i \cap V_k$. If $\gamma$ stabilizes $V_i$, all vertices of $V_i \cap V_j$ lie in $\Gamma V_i$-orbits of the vertices of $V_j \cap V_k$. Let us assume that $\gamma$ does not stabilize $V_i$ and let $V_i \cap V_j$ be another adhesion set such that the element $\gamma' \in \Gamma$ that maps $V_i \cap V_j$ to $V_i \cap V_k$ does not stabilize $V_i$. Then $\gamma'\gamma^{-1}$ maps $V_i \cap V_j$ to $V_i \cap V_k$ and stabilizes $V_i$. We conclude that the number of $\Gamma V_i$-orbits of vertices in adhesion sets of $V_i$ is at most twice the number of $\Gamma$-orbits of vertices in adhesion sets of $V_i$.

Subtrees of connected, basic tree-decompositions that contain a common adhesion set cannot be too large as the following lemma shows.

**Lemma 4.6.** Let $\Gamma$ be a group acting quasi-transitively on a connected, locally finite graph $G$ with at least two ends and let $(T, V)$ be a connected, basic tree-decomposition of $G$ with respect to $\Gamma$. For an adhesion set $X$ let $T_X$ be the maximal subtree of $T$ such that $X \subseteq V_t$ for all $t \in V(T_X)$. Then the diameter of $T_X$ is at most 2.

**Proof.** Suppose the diameter of $T_X$ is at least 3. We have $V_i \cap V_j = X$ for every $tt' \in E(T_X)$ since $X$ is contained in every adhesion set $V_i \cap V_j$ and since they all have the same size. Let $R = \ldots t_0 t_1 \ldots$ be a maximal path in $T_X$. We shall show that $R$ is a double ray.

Let us suppose that $t_{i+3}$ is the last vertex on $R$. As $(T, V)$ is basic, we find $\gamma \in \Gamma$ such that $\gamma(t_it_{i+1}) = t_{i+2}t_{i+3}$. Note that $\gamma$ fixes $X = V_i \cap V_{i+1} = V_{i+2} \cap V_{i+3}$ setwise. If $\gamma(t_i) = t_{i+2}$, then $\gamma(t_{i+2})$ is a neighbour of $t_{i+3}$ distinct from $t_{i+2}$ that contains $X$, a contradiction to the choice of $i$. If $\gamma(t_i) = t_{i+3}$, then $\gamma$ fixes the edge $t_{i+1}t_{i+2}$ but neither of its incident vertices. Let $\gamma' \in \Gamma$ map $t_{i+1}t_{i+2}$ to $t_{i+2}t_{i+3}$. Note that $\gamma'$ fixes $X$ setwise, too. Then either $\gamma'$ or $\gamma'\gamma$ maps $t_i$ to a neighbour of $t_{i+3}$ distinct from $t_{i+2}$. This is again a contradiction, which shows that $R$ has no last vertex. Analogously, $R$ has no first vertex. So it is a double ray.

Note that the part of some node of $T_X$ contains $X$ properly as $G = X$ is finite otherwise. But as $\Gamma$ acts transitively on $E(T)$, we have at most two $\Gamma$-orbits on $V(T)$. Hence infinitely many parts of $R$ contain $X$ properly. Since each $V_i$ is connected, one vertex of $X$ must have infinitely many neighbours. This contradiction to local finiteness shows the assertion. \qed
Our next result is a characterisation of the finite parts of a connected, basic tree-decomposition.

**Proposition 4.7.** Let \( \Gamma \) be a group acting quasi-transitively on a connected, locally finite graph \( G \) with at least two ends and let \( (T, V) \) be a connected, basic tree-decomposition of \( G \). Then the degree of a node \( t \in V(T) \) is finite if and only if \( V_t \) is finite.

**Proof.** Let \( V_t \) be finite. Since \( (T, V) \) is basic, we have only one \( \Gamma \)-orbit on the adhesion sets. Local finiteness of \( G \) thus implies that each vertex of \( V_t \) lies in only finitely many distinct adhesion sets and that each of these adhesion sets separates the graph in only finitely many components. Therefore, the degree of \( t \) is finite.

Now let us assume that the degree of \( t \) is finite. Let \( U \) be a subset of \( V_t \) that consists of one vertex from each \( \Gamma V_t \)-orbit that meets \( V_t \). By Proposition 4.5 the set \( U \) is finite. The vertices in \( U \) have bounded distance to the union \( W \) of all adhesion sets in \( V_t \). As they meet all \( \Gamma V_t \)-orbits and \( \Gamma V_t \) fixes \( W \) setwise, all vertices in \( V_t \) have bounded distance to \( W \). Note that \( W \) is finite as \( t \) has finite degree. Since \( G \) is locally finite, \( V_t \) must be finite. \( \square \)

Let \( (T, V) \) be a tree-decomposition of a graph \( G \). We say that an end \( \eta \) of \( T \) captures an end \( \omega \) of \( G \) if for every ray \( R = t_1, t_2, \ldots \) in \( \eta \) the union \( \bigcup_{i \in \mathbb{N}} V_{t_i} \) captures \( \omega \). A node of \( T \) captures \( \omega \) if its part does so.

Let us now investigate where the ends of \( G \) lie in \( (T, V) \).

**Proposition 4.8.** Let \( G \) be a graph and let \( (T, V) \) be a connected tree-decomposition of \( G \) such that the maximum size of its adhesion sets is at most \( k \in \mathbb{N} \). Then the following hold.

(i) Each end of \( G \) is captured either by an end or by a node of \( T \).

(ii) Every thick end of \( G \) is captured by a node of \( T \).

(iii) Every end of \( T \) captures a unique thin end of \( G \), which has degree at most \( k \).

(iv) Assume that \( \Gamma \) acts quasi-transitively on \( G \) and that \( (T, V) \) is \( \Gamma \)-invariant with finitely many \( \Gamma \)-orbits on \( E(T) \). Every end of \( G \) that is captured by a node \( t \in V(T) \) corresponds to a unique end of \( G[V_t] \), that is, for every end \( \omega \) of \( G \) that is captured by \( t \in V(T) \) there is a unique end \( \omega_t \) of \( G[V_t] \) with \( \omega_t \subseteq \omega \).

**Proof.** Let \( \omega \) be an end of \( G \) and let \( Q, R \) be two rays in \( \omega \). For an edge \( st \in E(T) \), let \( T_s \) and \( T_t \) be the subtrees of \( T - st \) with \( s \in V(T_s) \) and \( t \in V(T_t) \). If the ray \( Q \) has all but finitely many vertices in \( \bigcup_{x \in V(T_s)} V_x \) and \( R \) has all but finitely many vertices in \( \bigcup_{x \in V(T_t)} V_x \), then we have a contradiction as \( Q \) and \( R \) cannot lie in the same end if they have tails that are separated by the finite vertex set \( V_t \cap V_s \). We now orient the edge \( st \) from \( s \) to \( t \) if tails of \( Q \) and \( R \) lie in \( \bigcup_{x \in V(T_s)} V_x \), and we orient it from \( t \) to \( s \) if tails of \( Q \) and \( R \) lie in \( \bigcup_{x \in V(T_t)} V_x \). Obviously, every node of \( T \) has at most one outgoing edge. Let \( t_Q, t_R \) be nodes of \( T \) such that the first vertex of \( Q \) lies in \( V_{t_Q} \), and the first vertex of \( R \) lies in \( V_{t_R} \), and let \( P_Q \) and \( P_R \) be the maximal (perhaps infinite) directed paths in our orientation of \( T \) that start at \( t_Q \) and \( t_R \), respectively. Note
that if \( PQ \) and \( PR \) meet at a vertex, they continue in the same way. Thus, if they meet, they either end at a common vertex or have a common infinite subpath.

We shall show that \( PQ \) and \( PR \) meet. Let \( P \) be the \( t_Q \)-\( t_R \) path in \( T \). Then there is a unique sink \( x \) on it as every node of \( T \) has at most one outgoing edge. This sink is a common node of \( PQ \) and \( PR \). If \( PQ \) and \( PR \) end at a node, this node captures \( \omega \) and if they share a common infinite subpath, this is a ray whose end captures \( \omega \). We proved (i).

Now let us assume that \( \omega \) has degree at least \( k + 1 \). Then there are \( k + 1 \) pairwise disjoint rays \( R_1, \ldots, R_{k+1} \) in \( \omega \). Let \( t_i, P_i \) be a node and a path of \( T \) defined for \( R_i \) as we defined \( t_R \) and \( P_R \) for the ray \( R \). By an easy induction, we can extend the above argument that \( PQ \) and \( PR \) meet to obtain that all \( P_1, \ldots, P_{k+1} \) have a common node \( x \). Let us suppose that \( \omega \) is captured by an end \( \eta \) of \( T \). Let \( y \) be the node of \( T \) that is adjacent to \( x \) and that separates \( x \) and \( \eta \). Then all rays \( R_i \) must contain a vertex of \( V_x \cap V_y \). This is not possible as \( V_x \cap V_y \) contains at most \( k \) vertices and the rays \( R_i \) are disjoint. This contradiction shows (ii) and the second part of (iii).

Let \( R, Q \) be two rays that lie in ends of \( G \) that are captured by the same end \( \eta \) of \( T \). With the notations \( PQ, PR \) as above, the intersection \( PQ \cap PR \) is a ray in \( \omega \). As \( G \) is locally finite and \( (T, V) \) is a connected tree-decomposition, there are infinitely many disjoint paths between \( Q \) and \( R \) and thus, they are equivalent and lie in the same end of \( G \). This proves (iii).

To prove (iv), let us assume that \( \Gamma \) acts quasi-transitively on \( G \) and has finitely many orbits on the edges of the decomposition tree \( T \). Let \( \omega \) be an end of \( G \) that is captured by a node \( t \in V(T) \) and let \( R \) be a ray in \( \omega \) that starts at a vertex in \( V_t \). Since \( V_t \) captures \( \omega \), there are infinitely many vertices of \( V_t \) on \( R \). Whenever \( R \) leaves \( V_t \) through an adhesion set, it must reenter it through the same adhesion set by Remark 3.1. We replace every such subpath \( P \), where the end vertices of \( P \) lie in a common adhesion set and the inner vertices of \( P \) lie outside of \( V_t \), by a geodesic in \( G[V_t] \) between the end vertices of \( P \). We end up with a walk \( W \) with the same starting vertex as \( R \). We shall see that \( W \) contains a one-way infinite path. First, we recursively delete closed subwalks of \( W \) to end up with a path \( R' \). Since \( G \) is locally finite and \( R \) meets \( V_t \) infinitely often, \( R \) contains vertices of \( V_t \) that are arbitrarily far away from the starting vertex of \( R \). As we only took geodesics to replace the subpaths of \( R \) that were outside of \( V_t \) and as \( \Gamma \) acts on \( (T, V) \) with only finitely many orbits on the edges of \( T \), these replacement paths have a bounded length. Hence, \( W \) eventually leaves every ball of finite diameter around its starting vertex. This implies that \( R' \) is a ray. Obviously, \( R \) and \( R' \) are equivalent. Thus \( G[V_t] \) contains a ray in \( \omega \). Let \( \omega_t \) be the end of \( G[V_t] \) that contains \( R' \) and let \( Q \) be a ray in \( \omega_t \). Since no finite separator can separate \( Q \) and \( R' \) in \( G[V_t] \), the rays are also equivalent in \( G \). Thus, we have shown \( \omega_t \subseteq \omega \).

Let \( \omega'_t \) be an end in \( G[V_t] \) different from \( \omega_t \), let \( S \) be a finite subset of \( V_t \) that separates \( \omega_t \) from \( \omega'_t \), and let \( P \) be a path in \( G \) connecting vertices in different components of \( G[V_t] - S \). As before, whenever \( P \) leaves \( V_t \) through an adhesion set, it must reenter it through the same adhesion set by Remark 3.1. We again replace every such subpath, where the end vertices lie in a common adhesion set and the inner vertices lie outside of \( V_t \), by a geodesic in \( G[V_t] \) to obtain a walk \( P' \) in \( G[V_t] \). Since \( P \) and \( P' \) have the same endpoints and \( P' \) must meet \( S \), we know that \( P \) either contains a vertex in \( S \), or it contains a vertex in an adhesion set which meets \( S \). Let \( S' \) be the set containing all vertices of \( S \) and all vertices
contained in adhesion sets that meet $S$. There are only finitely many orbits of vertices in adhesion sets, hence there is an upper bound on the diameter of the adhesion sets. Since $S$ is finite and $G$ is locally finite, this implies that $S'$ is finite. By definition, there is no path in $G - S'$ connecting vertices in different components of $G[V_t] - S$. In particular, $S'$ separates every ray in $\omega_t$ from every ray in $\omega_t'$, and hence (iv) holds.

\[ \square \]

5 Tree amalgamations

In this section, we prove our main result, Theorem 1.1. Before we move on to that proof, we need to state some definitions, in particular, the main definition: tree amalgamations, a notion introduced by Mohar [19].

For the definition of tree amalgamations, let $G_1$ and $G_2$ be graphs. Let $(S^i_k)_{k \in I_i}$ be a family of subsets of $V(G_i)$. Assume that all sets $S^i_k$ have the same cardinality and that the index sets $I_1$ and $I_2$ are disjoint. For all $k \in I_1$ and $\ell \in I_2$, let $\phi_{k\ell} : S^i_k \to S^i_\ell$ be a bijection and let $\phi_{k\ell} = \phi_{\ell k}^{-1}$. We call the maps $\phi_{k\ell}$ and $\phi_{k\ell}$ bonding maps.

Let $T$ be a $(|I_1|, |I_2|)$-semiregular tree, that is, a tree in which for the canonical bipartition $\{V_1, V_2\}$ of $V(T)$ the vertices in $V_i$ all have degree $|I_i|$. Denote by $D(T)$ the set obtained from the edge set of $T$ by replacing every edge $xy$ by two directed edges $\overline{xy}$ and $yx$. For a directed edge $\overline{e} = \overline{xy} \in D(T)$, we denote by $e = \overline{yx}$ the edge with the reversed orientation. Let $f : D(T) \to I_1 \cup I_2$ be a labelling, such that for every $t \in V_i$, the labels of edges starting at $t$ are in bijection to $I_i$.

For every $i \in \{1, 2\}$ and for every $t \in V_i$, take a copy $G_t$ of the graph $G_i$. Denote by $S^i_k$ the corresponding copies of $S^i_k$ in $V(G_t)$. Let us take the disjoint union of the graphs $G_t$ for all $t \in V(T)$. For every edge $\overline{t} = \overline{st}$ with $f(\overline{t}) = k$ and $f(\overline{\theta}) = \ell$ we identify each vertex $x$ in the copy of $S^i_k$ with the vertex $\phi_{k\ell}(x)$ in $S^i_\ell$. Note that this does not depend on the orientation we pick for $\overline{t}$, since $\phi_{k\ell} = \phi_{\ell k}^{-1}$. The resulting graph is called the tree amalgamation of the graphs $G_1$ and $G_2$ over the connecting tree $T$ and is denoted by $G_1 * T G_2$ or by $G_1 * T G_2$ if we want to specify the tree.

In the context of tree amalgamations the sets $S^i_k$ and $S^i_\ell$ are called the adhesion sets of the tree amalgamation. More specifically, the sets $S^i_k$ are the adhesion sets of $G_1$ and the sets $S^i_\ell$ are the adhesion sets of $G_2$. If the adhesion sets of a tree amalgamation are finite, then this tree amalgamation has finite adhesion. For every node $t \in V(T)$, there is a canonical map mapping each vertex $x \in V(G_t)$ to the vertex of $G_1 * T G_2$ obtained from $x$ by all identifications.

We call a tree amalgamation $G_1 * T G_2$ trivial if for some $t \in V(T)$ this canonical map is a bijection. Note that if the tree amalgamation has finite adhesion, it is trivial if $V(G_i)$ is the only adhesion set of $G_i$ and $|I_i| = 1$ for some $i \in \{1, 2\}$.

The tree amalgamation $G = G_1 * G_2$ distinguishes ends if there is some adhesion set $S^i_k = S^i_\ell$ for adjacent $s, t \in V(T)$ such that for every component $C$ of $T - st$ the subgraph of $G$ induced by $\bigcup_{t \in C} G_t^i$ contains an end.

We remark that the map described in the definition of a trivial tree amalgamation does not necessarily induce a graph isomorphism $G_t \to G_1 * T G_2$: it is a bijection $V(G_t) \to V(G_1 * T G_2)$ but need not induce a bijection $E(G_t) \to E(G_1 * T G_2)$. 

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The **identification size** of a vertex \( x \in V(G_1 *_T G_2) \) is the size of the subtree \( T' \) of \( T \) induced by all nodes \( t \) for which a vertex of \( G_t \) is identified with \( x \). The tree amalgamation has **finite identification** if all identification sizes are finite. The **identification length** of a vertex \( x \in V(G_1 *_T G_2) \) is the diameter of the subtree \( T' \) of \( T \) induced by all nodes \( t \) for which a vertex of \( G_t \) is identified with \( x \). The **identification length** of the tree amalgamation is the supremum of the identification lengths of its vertices. We note that if a quasi-transitive tree amalgamation has finite identification, then its identification length is finite.

We remark that in Mohar’s definition of a tree amalgamation [19] the identification length is always at most 2. But apart from this, our definition is equivalent to his.

It is worth noting that every tree amalgamation gives rise to a tree decomposition in the following sense.

**Remark 5.1.** Let \( G \) be a graph. If \( G \) is a tree amalgamation \( G_1 *_T G_2 \) of finite adhesion, then there is a naturally defined tree-decomposition of \( G \). For \( t \in V(T) \) let \( V_t \) be the set obtained from \( V(G_t) \) after all identifications in \( G_1 * G_2 \). Set \( V := \{ V_t \mid t \in V(T) \} \). Obviously, all vertices of \( G \) lie in \( \bigcup_{t \in V(T)} V_t \) and for each edge there is some \( V_t \in V \) containing it. Property (T3) of a tree-decomposition is satisfied as the copies \( G_v^v \) are arranged in a treelike way and identifications to obtain a vertex take place in subtrees of \( T \). So \( (T, V) \) is a tree-decomposition. If \( G_1 *_T G_2 \) has finite adhesion, so does \( (T, V) \). If the tree amalgamation distinguishes ends, then so does the tree-decomposition.

So far, the tree amalgamations do not interact with any group actions on \( G_1 \) and \( G_2 \). In particular, it is easy to construct a tree amalgamation of two quasi-transitive graphs which is not quasi-transitive: for instance, let \( G_1 \) be a double ray and let \( G_2 \) be any finite non-trivial graph. Pick precisely two adhesion sets in \( G_1 \) sets and at least two adhesion sets in \( G_2 \), all of finite size. Then it is easy to see that the tree amalgamation \( G_1 * G_2 \) is not quasi-transitive by noting that vertices of \( G_1 \) at different distances from the adhesion sets cannot be mapped to one another.

In the following, we describe some conditions on tree amalgamations which will ensure that tree amalgamations of quasi-transitive graphs are again quasi-transitive; this will be proved in Lemma 5.6.

Let \( \Gamma_i \) be a group acting on \( G_i \) for \( i = 1, 2 \), let \( t \in V_i \), let \( \gamma \in \Gamma_i \) and let \( j \in \{1, 2\} \setminus \{i\} \). We say that the tree amalgamation **respects** \( \gamma \) if there is a permutation \( \pi \) of \( I_i \) such that for every \( k \in I_i \) there is \( \ell \in I_j \) and \( \tau \) in the setwise stabiliser of \( S_{\ell} \) in \( \Gamma_j \) such that

\[
\phi_{kl} = \tau \circ \phi_{\pi(k)\ell} \circ \gamma \big|_{S_k}.
\]

Note that this in particular implies that \( \gamma(S^k_{\ell}) = S_{\pi(k)}^{\ell} \). The tree amalgamation **respects** \( \Gamma_i \) if it respects every \( \gamma \in \Gamma_i \).

Let \( k \in I_i \) and let \( \ell, \ell' \in I_j \). We call the bonding maps from \( k \) to \( \ell \) and \( \ell' \) **consistent** if there is \( \gamma \in \Gamma_j \) such that

\[
\phi_{k\ell} = \gamma \circ \phi_{k\ell'}.
\]

We say that the bonding maps between two sets \( J_1 \subseteq I_1 \) and \( J_2 \subseteq I_2 \) are **consistent** if they are consistent for any \( i \in \{1, 2\} \), \( k \in J_i \), and \( \ell, \ell' \in J_j \).
We say that the tree amalgamation \( G_1 \ast G_2 \) is of Type 1 respecting the actions of \( \Gamma_1 \) and \( \Gamma_2 \) or \( (G_1, \Gamma_1) \ast (G_2, \Gamma_2) \) is a tree amalgamation of Type 1 for short if the following holds:

(i) The tree amalgamation respects \( \Gamma_1 \) and \( \Gamma_2 \).

(ii) The bonding maps between \( I_1 \) and \( I_2 \) are consistent.

We say that the tree amalgamation \( G_1 \ast G_2 \) is of Type 2 respecting the actions of \( \Gamma_1 \) and \( \Gamma_2 \) or \( (G_1, \Gamma_1) \ast (G_2, \Gamma_2) \) is a tree amalgamation of Type 2 for short if the following holds:

(o) \( G_1 = G_2 =: G, \Gamma_1 = \Gamma_2 =: \Gamma \), and \( I_1 = I_2 =: I \), and there is \( J \subseteq I \) such that \( f(\tilde{e}) \in J \), if and only if \( f(\tilde{e}) \notin J \).

(i) The tree amalgamation respects \( \Gamma \).

(ii) The bonding maps between \( J \) and \( I \setminus J \) are consistent.

In this second case we also say that \( G_1 \ast G_2 = G \ast G \) is a tree amalgamation of \( G \) with itself.

We say that \( G_1 \ast G_2 \) is a tree amalgamation respecting the actions of \( \Gamma_1 \) and \( \Gamma_2 \) if it is of either Type 1 or Type 2 respecting the actions \( \Gamma_1 \) and \( \Gamma_2 \) and we speak about the tree amalgamation \((G_1, \Gamma_1) \ast (G_2, \Gamma_2)\). It is worth noting that the tree amalgamation also depends on the choices of adhesion sets and bonding maps, but we usually suppress this dependency.

We note that in the case of a tree amalgamation of Type 1, all adhesion sets \( S_k^1 \) lie in the same \( \Gamma_i \)-orbit, while in the case of a tree amalgamation of Type 2 we potentially have two \( \Gamma_i \)-orbits of adhesion sets \( S_k^1 \).

We now give some examples illustrating the behaviour of tree amalgamations which respect group actions. The first set of examples shows that the groups acting on the factors have a substantial impact on the outcome of the tree amalgamation if we insist that it must respect the actions. They also show that sometimes (if the setwise stabilisers of the adhesion sets do not agree), we have to take multiple copies of the same adhesion set in order to be able to amalgamate consistently.

**Example 5.2.** Let \( G_1 \) consist of a single edge with end vertices \( u, v \) and let \( \Gamma_1 = \mathbb{Z}_2 \) acting in the obvious way. Let \( G_2 \) be a 4-cycle on vertices \( x, y, z, w \), and let \( \Gamma_2 = \mathbb{Z}_2 \) where the non-trivial element swaps \( x \) with \( y \) and \( z \) with \( w \).

Let \( S_1^1 = S_2^1 = \{u, v\} \), let \( S_1^2 = \{x, w\} \), and let \( S_2^2 = \{z, y\} \) and define bonding maps by

\[
\begin{align*}
\phi_{1a}(u) &= x, \quad \phi_{1a}(v) = w, & \phi_{1b}(u) &= y, \quad \phi_{1b}(v) = z, \\
\phi_{2a}(u) &= w, \quad \phi_{2a}(v) = x, & \phi_{2b}(u) &= z, \quad \phi_{2b}(v) = y.
\end{align*}
\]

and \( \phi_{ji} = \phi_{ij}^{-1} \). The resulting tree amalgamation is an infinite double ladder with the infinite dihedral group \( D_\infty \) acting on it, and a straightforward case check shows that this tree amalgamation respects the action of the groups.\[\footnote{Technically this is not allowed, in particular since for the definition of \( \phi_{k\ell} \) we needed \( I_1 \) and \( I_2 \) to be disjoint. These technicalities can be easily dealt with by an appropriate notion of isomorphism the details of which we leave to the reader.}
It is worth pointing out that once we have picked $S^1_1$ and $S^2_1$, this is the smallest tree amalgamation which respects the groups. We must have the adhesion set $S^2_2$, otherwise the tree amalgamation cannot respect the action of $\Gamma_2$. Furthermore, the adhesion set $\{u, v\}$ must appear at least twice in order to respect the nontrivial element of $\Gamma_1$ (since the stabilisers of $S^2_1$ and $S^2_2$ are trivial). It is also worth noting that any other choice of bonding maps would either violate consistency or not respect one of the group actions. We would also like to point out that the action on the tree amalgamation is transitive although the action of $\Gamma_2$ on $G_2$ was not transitive.

For a different example of a tree amalgamation, choose $(G_1, \Gamma_1)$ and $(G_2, \Gamma_2)$ as above, pick $S^1_1 = \{u, v\}$, but choose $S^2_2 = \{x, y\}$. Then the bonding map $\phi_{i\ell}(u) = x, \phi_{i\ell}(v) = y$ results in a tree amalgamating respecting the actions. In fact, this is an example of a trivial tree amalgamation. Moreover, the tree amalgamation is isomorphic to $(G_2, \Gamma_2)$: the vertices of $\Gamma_1$ are simply identified with a subset of $G_2$.

The next two examples illustrate the relation of Type 1 and 2 tree amalgamations to amalgamated free products and HNN-extensions of groups.

We first recall the definition of the free product with amalgamation. Let $\Gamma_i = (S_i \mid R_i)$ be a finitely generated group with a subgroup $\Delta_i$ for $i = 1, 2$ such that an isomorphism $\phi: \Delta_1 \rightarrow \Delta_2$ exists. Then the free product of $\Gamma_1$ and $\Gamma_2$ with amalgamation over $\Delta_1$ and $\Delta_2$ is the group defined by

$$
\langle S_1 \cup S_2 \mid R_1, R_2, \{\delta^{-1}\phi(\delta) \mid \delta \in \Delta_1\}\rangle,
$$

and we denote it by $\Gamma_1 *_{\Delta_1, \Delta_2} \Gamma_2$.

**Example 5.3.** Let $\Gamma_i = (S_i)$ be a finitely generated group with a finite subgroup $\Delta_i$ for $i = 1, 2$ such that an isomorphism $\phi: \Delta_1 \rightarrow \Delta_2$ exists. In addition let $G_i$ be the Cayley graph of $\Gamma_i$ with respect to $S_i$ for $i = 1, 2$. We show that the Cayley graph of $\Gamma = \Gamma_1 *_{\Delta_1, \Delta_2} \Gamma_2$ with respect to $S_1 \cup S_2$ is a tree amalgamation of $G_1$ and $G_2$.

In order to make sense of the above statement, we need to define adhesion sets and bonding maps for the tree amalgamation. The adhesion sets are the left cosets of $\Delta_1$ in $\Gamma_i$, formally we can pick a system $\{t^1_k\}_{k \in \ell_1}$ of coset representatives and let $S^1_k = t^1_k \Delta_1$. Note that $(S^1_k)_{k \in \ell_1}$ is a family of pairwise disjoint sets. Next we define the bonding maps. For $k \in I_1$ and $\ell \in I_2$, define $\phi_{k\ell}: S^1_k \rightarrow S^2_\ell$ by $\phi_{k\ell}(t^1_k g) = t^2_\ell \phi(g)$ and $\phi_{k\ell} = \phi_{k\ell}^{-1}$. Let $T$ be a $(|I_1|, |I_2|)$-semiregular with the canonical partition $(V_1, V_2)$ of $V(T)$. We can assume that the vertices of $T$ are the left cosets of $\Gamma_1$ and $\Gamma_2$ and the edges correspond to the left cosets of $\Delta_1$. Set the label under $f: D(T) \rightarrow I_1 \cup I_2$ of the edge corresponding to $g \Delta_1$ directed towards $h \Gamma_1$, as $k \in I_1$ such that $h^{-1} g \Delta_1 = S^1_k$. It is easy to see that the edges directed towards $h \Gamma_1$ have all of $I_1$ as their labels. We have to show that $(G_1, \Gamma_1) * (G_2, \Gamma_2)$ is a tree amalgamation of Type 1. Let $\gamma \in \Gamma_i$ and let $j \in \{1, 2\} \setminus \{i\}$. For every $k$ in $I_1$, there exists $k' \in I_j$ such that $\gamma t^1_k \Delta_1 = t^1_{k'} \Delta_1$ and such that, for every $m \in I_j$ with $m \neq k$, we have $k' \neq m'$. This mapping of $k$ to $k'$ for every $k \in I_1$ defines a permutation $\pi$. Then for $\ell \in I_j$ the following holds

$$
\phi_{k\ell} = \text{id} \circ \phi_{x(\ell)k} \circ \gamma \mid_{S\ell_1}.
$$

In order to see that the bonding maps between $I_1$ and $I_2$ are consistent, let $k \in I_1$ and let $\ell, \ell' \in I_j$. Let $\gamma \in \Gamma_j$ such that $\gamma t^1_{k} \Delta_1 = t^1_{k'} \Delta_1$. Then it is straighfor-
ward to check that
\[ \phi_{k\ell'} = \gamma \circ \phi_{k\ell}. \]

Therefore we proved that \((G_1, \Gamma_1) * (G_2, \Gamma_2)\) is a tree amalgamation of Type 1.
By the choice of \(f\), we obtain an isomorphism \(\Phi\) from \(G\) to \((G_1, \Gamma_1) * (G_2, \Gamma_2)\).

Let \(\Gamma = \langle S | R \rangle\) be a finitely generated group. Let \(\Delta_1\) and \(\Delta_2\) be finite subgroups such that there is an isomorphism \(\phi: \Delta_1 \to \Delta_2\). Moreover, let \(t\) be a symbol which is not an element of \(\Gamma\). Then the HNN-extension of \(\Gamma\) over \(\Delta_t\) with respect to \(\phi\) is given by
\[ \Gamma_{*\phi} = \langle S, t | R, \{\delta t = t\phi(\delta) | \delta \in \Delta\} \rangle. \]

**Example 5.4.** Let \(\Gamma = \langle S \rangle\) be a finitely generated group with isomorphic finite subgroups \(\Delta_1\) and \(\Delta_2\) and let \(\phi: \Delta_1 \to \Delta_2\) be an isomorphism. In addition let \(G\) be the Cayley graph of \(\Gamma\) with respect to \(S\). Let \(G^*\) be the graph obtained from \(G\) by adding for each vertex \(v\) of \(G\) a new vertex \(vt\) and joining these by an edge. We note that the action of \(\Gamma\) on \(G\) extends to an action of \(\Gamma\) on \(G^*\).

Let \(\{g_k^i \Delta_i\}_{k \in J_i}\) be the set of all left cosets of \(\Delta_i\) in \(\Gamma\) for \(i = 1, 2\). The adhesion sets are the elements of \(\{S_k\}_{k \in J_1} = \{g_k^1 \Delta_1 t\}_{k \in J_1}\) and of \(\{T_i\}_{i \in J_2} = \{g_i^2 \Delta_2\}_{i \in J_2}\).

Without loss of generality we can assume \(S_{k_0} = \Delta_1 t\) and \(T_{i_0} = \Delta_2\). The bonding maps are \(\phi_{k_0}: S_{k_0} \to T_{i_0}\) and \(\phi_{k_0}: T_{i_0} \to S_{k_0}\) with \(\phi_{k_0}(g_0, \delta t) = \phi(\delta)\) and \(\phi_{k_0}(g_0, \delta) = \phi^{-1}(\delta)\). Let \(T\) be the tree whose vertices are the left cosets of \(\Gamma\) and whose edges correspond to the left cosets of \(\Delta_1\). Set the label under \(f: D(T) \to J_1 \cup J_2\) of the edges corresponding to \(g_1 \Delta_1\) directed towards \(h\Gamma\) with \(g_1 \Delta_1 \leq h\Gamma\) as \(k \in J_2\) such that \(h^{-1}g_1 \Delta_1 = T_k\) and set the label of the same edge but with the reverse direction as \(t \in J_1\) such that \(S_t = T_k\). To show that \(G^* * G^*\) respects \(\Gamma\), let \(\gamma \in \Gamma\). For every \(k \in J_1\), there exists \(k' \in J_1\) such that \(\gamma g_k^1 \Delta_1 t = g_k' \Delta_1 t\) and such that, for every \(m \in J_1\) with \(m \neq k\), we have \(k' \neq m'\). This mapping of \(k\) to \(k'\) for every \(k \in J_1\) defines a permutation \(\pi\). Then for \(t \in J_2\) the following holds
\[ \phi_{k\ell} = \text{id} \circ \phi_{\pi(k)\ell} \circ \gamma \mid_{S_k}. \]

Analogously, we obtain a permutation \(\pi'\) such that for every \(k \in J_2\) there is \(t \in J_1\) with
\[ \phi_{k\ell} = \text{id} \circ \phi_{\pi'(k)t} \circ \gamma_{T_k}. \]

Thus, the tree amalgamation respects \(\gamma\). In order to prove that the bonding maps between \(J_1\) and \(J_2\) for \(i \neq j\) are consistent, let \(k \in J_i\) and \(t, t' \in J_j\). Let \(\gamma \in \Gamma\) such that \(\gamma g_k^1 \Delta_1 t = g_k' \Delta_1 t\) if \(j = 1\) and such that \(\gamma g_k^2 \Delta_2 = g_k' \Delta_2\) if \(j = 2\). Then it is easy to see that
\[ \phi_{k\ell'} = \gamma \circ \phi_{k\ell}. \]
Thus, \(G^* * G^*\) is a tree amalgamation of Type 2. By the choice of \(f\), we obtain an isomorphism \(\Phi\) from the Cayley graph of \(\Gamma_{*\phi}\) with respect to \(S \cup \{t\}\) to \(G^* * G^*\).

Note that conditions (i) and (ii) in both cases of the definition of tree amalgamations respecting the actions do not depend on the specific labelling of the tree. This is no coincidence. In fact we will show that any two legal labellings of \(D(T)\) give isomorphic tree amalgamations, see Lemma 5.6. Furthermore, any \(\gamma \in \Gamma_1\) (interpreted as an isomorphism between parts of two such tree amalgamations) can be extended to an isomorphism of the tree amalgamations,
which also implies that the tree amalgamations obtained this way are always quasi-transitive.

Before we turn to the proof of these facts, we need some notation. A \textit{legally labelled star} centred at \( V_i \) is a function \( \Lambda \) from \( I_i \) to \( I_j \). If the tree amalgamation is of Type 2, we further require that \( \Lambda(k) \in J \) if and only if \( k \notin J \). Informally, think of this as a star whose labels on directed edges could appear on a subtree of \( T \) induced by a vertex \( t \in V_i \) and its neighbours: for \( e \) with label \( k \), the value \( \Lambda(k) \) tells us the label of \( e \).

An isomorphism of two legally labelled stars \( \Lambda, \Lambda' \) is a triple \( (\gamma, \pi, \gamma_k)_{k \in I_i} \) consisting of some \( \gamma \in \Gamma_1 \), a permutation \( \pi \) of \( I_i \), and a family \( \gamma_k \) of elements of \( \Gamma_j \) such that for every \( k \in I_i \)

\[
\phi_{k, \Lambda(k)} = \gamma_k \circ \phi_{\pi(k)\Lambda'(\pi(k))} \circ \gamma \mid I_k .
\]

In our interpretation of legally labelled stars as subtrees of \( T \), this corresponds to an isomorphism of the corresponding subgraphs of the tree amalgamation.

**Proposition 5.5.** Let \( \Lambda, \Lambda' \) be two legally labelled stars with respect to a tree amalgamation \( (G_1, \Gamma_1) \ast_T (G_2, \Gamma_2) \) centred at \( V_i \) and let \( \gamma \in \Gamma_1 \). Then \( \gamma \) extends to an isomorphism \( (\gamma, \pi, (\gamma_k)_{k \in I_i}) \) of \( \Lambda \) and \( \Lambda' \). Furthermore, if we are given \( \tilde{k}, \tilde{k}' \in I_i \) and \( \tilde{\gamma}_k \in \Gamma_j \) such that

\[
\phi_{k, \Lambda}(\tilde{k}) = \tilde{\gamma}_k \circ \phi_{k', \Lambda'}(\tilde{k}') \circ \gamma \mid I_k,
\]

then we can choose \( \pi(\tilde{k}) = \tilde{k}' \) and \( \tilde{\gamma}_k = \tilde{\gamma}_k \).

**Proof.** Since the tree amalgamation respects \( \gamma \), there are \( \pi \) and \( \tilde{\Lambda} : I_i \to I_j \) and \( \tilde{\gamma}_k \) in the stabiliser of \( S_{\Lambda(k)} \) in \( \Gamma_j \) such that

\[
\phi_{k, \Lambda(k)} = \tau_k \circ \phi_{\pi(k)\tilde{\Lambda}(\pi(k))} \circ \gamma \mid I_k .
\]

Let \( \gamma_k' \in \Gamma_j \) be such that \( \phi_{k, \Lambda}(k) = \gamma_k' \circ \phi_{k, \tilde{\Lambda}(k)} \), and let \( \gamma_k'' \in \Gamma_j \) be such that \( \phi_{\pi(k)\tilde{\Lambda}(k)} = \gamma_k'' \circ \phi_{\pi(k)\tilde{\Lambda}'(k)} \). These exist by (ii); for Type 2 recall that by the definition of legally labelled stars \( k \in J \) if and only if \( \Lambda(k) \notin J \). Now clearly

\[
\phi_{k, \Lambda(k)} = \gamma_k' \circ \tau_k \circ \gamma_k'' \circ \phi_{\pi(k)\tilde{\Lambda}'(\pi(k))} \circ \gamma \mid I_k ,
\]

thus showing that the two stars are isomorphic by means of the isomorphism \( (\gamma, \pi, (\gamma_k)_{k \in I_i}) \), where \( \gamma_k = \gamma_k' \circ \tau_k \circ \gamma_k'' \).

For the second part, let \( (\gamma, \pi, (\gamma_k)_{k \in I_i}) \) be an isomorphism between \( \Lambda \) and \( \Lambda' \). Let \( \tilde{k}'' = \pi^{-1}(\tilde{k}') \). Define \( \rho(k) = k'' \) and \( \rho(\tilde{k}'') = \tilde{k}' \). Let \( \delta_k = \tilde{\gamma}_k \) and let

\[
\delta_{k''} = \gamma_k'' \circ \tilde{\gamma}_k^{-1} \circ \tilde{\gamma}_k.
\]

For the remaining \( k \in I_i \), let \( \rho(k) = \pi(k) \) and \( \delta_k = \gamma_k \). It is straightforward to check that \( \gamma, \rho, \) and \( (\delta_k)_{k \in I_i} \) define an isomorphism between \( \Lambda \) and \( \Lambda' \) with the desired properties. \( \square \)

**Lemma 5.6.** Let \( G_1 \) and \( G_2 \) be connected, locally finite graphs and let \( \Gamma_i \) be a group acting quasi-transitively on \( G_i \) for \( i = 1, 2 \). Then any tree amalgamation \( (G_1, \Gamma_1) \ast_T (G_2, \Gamma_2) \) is quasi-transitive and independent (up to isomorphism) of the particular labelling of \( T \).
Proof. Let $T$ and $T'$ be two labelled trees giving rise to tree amalgamations $G = (G_1, \Gamma_1) *_{T'} (G_2, \Gamma_2)$ and $G' = (G_1, \Gamma_1) *_{T} (G_2, \Gamma_2)$, respectively, such that the adhesion sets as well as the bonding maps for both tree amalgamations are the same. Let $t \in V(T)$ and let $t' \in V(T')$ be such that $G_t$ and $G_{t'}$ are both isomorphic to $G_i$. Let $\gamma_t \in \Gamma_i$. We claim that there is an isomorphism $\bar{\gamma} : G \to G'$ such that

$$\bar{\gamma} |_{G_i} = \text{id}_{G'} \circ \gamma_t \circ \text{id}_{G_i}^{-1},$$

where $\text{id}_i$ and $\text{id}_{G'}$ denote the canonical isomorphisms from $G_i$ to $G_t$ and $G_{t'}$ respectively. Clearly, the lemma follows from this claim.

For the proof of the claim define the star around $s \in V(T)$ by the map $\Lambda_s$ mapping $k$ to the label of $e_k$, where $e_k$ is the unique edge with label $k$ starting at $s$. By Proposition 5.5, there are a bijection $\pi : N(t) \to N(t')$ and a family $(\gamma_s \in \Gamma_{j'}),$ which extend $\gamma_t$ to an isomorphism of the stars around $t$ and $t'$. Iteratively apply Proposition 5.5 to vertices at distance $n = \{1, 2, 3, \ldots \}$ from $t$. We obtain an isomorphism $\pi : T \to T'$ and maps $\gamma_s \in \Gamma_i$ for each $s \in V_i$ such that the restriction of $\pi$ to $s$ and its neighbours and the corresponding maps $\gamma_s$ form an isomorphism between the stars at $s$ and $\pi(s)$.

For $v \in V(G_s)$, define $\bar{\gamma}(v) = \text{id}_{\pi(s)} \circ \gamma_s \circ \text{id}_{s}^{-1}(v)$. If $ss'$ is an edge, $v \in V(G_s)$ and $u \in V(G_{s'})$ such that $u$ and $v$ get identified in the construction of $G$, then $\bar{\gamma}(u) = \gamma(v)$. Hence $\bar{\gamma}$ is well defined, and since it obviously maps edges to edges and non-edges to non-edges, it is the desired isomorphism.

A closer inspection of the proof of Lemma 5.6 together with Remark 5.1 shows that tree amalgamations respecting the actions of quasi-transitive groups give rise to basic tree-decompositions of $(G_1, \Gamma_1) *_{(G_2, \Gamma_2)}$. The following lemma shows that the converse also holds, that is, basic tree-decompositions of quasi-transitive graphs give rise to tree amalgamations respecting the actions of some quasi-transitive group on the parts.

\textbf{Lemma 5.7.} Let $\Gamma$ be a group acting quasi-transitively on a connected, locally finite graph $G$ and let $(T, V)$ be a connected, basic tree-decomposition of $G$ with respect to $\Gamma$. Then one of the following holds.

1. There are $V_i, V_{t'} \in V$ such that $G$ is a non-trivial tree amalgamation

$$G[V_i] *_{T} G[V_{t'}]$$

of Type 1 respecting the actions of the stabilisers of $G[V_i]$ and $G[V_{t'}]$ in $\Gamma$.

2. There is $V_i \in V$ such that $G$ is a non-trivial tree amalgamation

$$G[V_i] *_{T} G[V_i]$$

of Type 2 respecting the actions of the stabiliser of $G[V_i]$ in $\Gamma$.

\textbf{Proof.} Choose an oriented edge $e_0 \in D(T)$. We say that $e \in D(T)$ is positively oriented, if there is $\gamma \in \Gamma$ mapping $e_0$ to $\bar{e}$. Otherwise we say that $\bar{e}$ is negatively oriented. If $\Gamma$ contains an element that reverses an edge of $T$, then let $\Gamma'$ be the subgroup preserving the bipartition of $T$. This subgroup has index 2, and still acts quasi-transitively on $G$ and transitively on the edges of $T$. Hence we can without loss of generality assume that no element of $\Gamma'$ swaps the endpoints of
an edge, and thus every edge is either positively or negatively oriented, but not both.

Let $s$ and $t$ be the start and end point of $e_0$ respectively. Let $(e_{0k})_{k \in K}$ be the positively oriented edges starting at $s$ and let $(e_{0\ell})_{\ell \in L}$ be the negatively oriented edges starting at $t$. Without loss of generality, assume that $K$ and $L$ are disjoint, and that $e_0 = e_{0k} = e_{0\ell}$. For every $k \in K$ pick a $\gamma_k \in \Gamma$ which maps $e_0$ to $\hat{e}_k$ (with $\gamma_{0k} = \text{id}$). For every $\ell \in L$ pick $\gamma_\ell \in \Gamma$ which maps $e_0$ to $\hat{e}_\ell$ (with $\gamma_{0\ell} = \text{id}$). If there is an element $\gamma_{st}$ of $\Gamma$ that maps $s$ to $t$, then for every $k \in K, \ell \in L$ let $\gamma_k = \gamma_k \circ \gamma_{st}$ and $\gamma_\ell = \gamma_\ell \circ \gamma_{st}^{-1}$.

Note that $e_0$ can be mapped to any edge incident to $e_0$ by a unique element of the form $\gamma_k$ or $\gamma_\ell'$ for some $k \in K \cup L$. For an arbitrary edge $e \neq e_0$, let $e'$ be the first edge of the path connecting $e$ to $e_0$. If $\gamma_{e'} \in \Gamma$ maps $e_0$ to $e'$, then by the above remark there is a unique element $\delta_e$ of the form $\gamma_k$ or $\gamma_\ell'$ such that $\gamma_{e'} \circ \delta_e$ maps $e_0$ to $e$. Use this to inductively construct (starting from $\delta_{e_0} = \text{id}$) for each $e \in E(T)$ an automorphism $\gamma_e \in \Gamma$ such that $\gamma_e(e_0) = e$. Let $\hat{e}$ be the orientation of $e$ pointing away from $e_0$ if $e \neq e_0$ and $\hat{e} = e_0$ otherwise. Define the label $f(E)$ to be the unique $k \in K \cup L$ such that the $\delta_e$ from above equals $\gamma_k$ or $\gamma_\ell'$. Note that $k \in K$ if and only if $\hat{e}$ is positively oriented. In this case define $f(\hat{e}) = \ell_0$, otherwise define $f(\hat{e}) = k_0$.

The following observation will be useful later. Let $v$ be a vertex of $T$, and let $\hat{e}$ be the first edge of the path from $v$ to $e_0$ (in case $v$ is $s$ or $t$ this is an orientation of $e_0$). Let $\Delta_v = \{\delta_f \mid v \in f, f \neq e\}$.

- If all edges starting at $v$ are positively (resp. negatively) oriented, then $\Delta_v = \{\gamma_k \mid k_0 \neq k \in K\}$ (resp. $\Delta_v = \{\gamma_\ell \mid \ell_0 \neq \ell \in L\}$).
- Otherwise, if $\hat{e}$ is positively (resp. negatively) oriented, then $\Delta_v = \{\gamma_k, \gamma_\ell' \mid k_0 \neq k \in K, \ell \in L\}$ (resp. $\Delta_v = \{\gamma_k', \gamma_\ell \mid k \in K, \ell_0 \neq \ell \in L\}$).

In particular, taking into account the label of $\hat{e}$, in the first case the edges starting at $v$ are labelled bijectively by $K$ (resp. $L$), while in the second case they are labelled bijectively by $K \cup L$.

Next we show how this labelling defines a tree amalgamation. First assume there is no automorphism $\gamma \in \Gamma$ mapping $s$ to $t$. Then all positively oriented edges must point from $V_1$ to $V_2$, where $V_1 \cup V_2$ is the bipartition of $T$ with $s \in V_1$—this corresponds to the first case in the above observation. Let $G_1$ be isomorphic to $G/V_1$, and let $G_2$ be isomorphic to $G/V_2$. Let $\text{id}_1$ and $\text{id}_2$ be the respective isomorphisms.

For the definition of the adhesion sets let $I_1 = K$ and $I_2 = L$. For $k \in K$ let $t_k$ be the endpoint of $\hat{e}_k$ and define $S_k = \text{id}_{t_k}^{-1}(V_s \cap V_{t_k})$. Similarly, for $\ell \in L$, let $s_{\ell}$ be the endpoint of $\hat{e}_\ell$ and define $S_\ell = \text{id}_{s_{\ell}}^{-1}(V_t \cap V_{s_{\ell}})$. Finally, define the adhesion maps by $\phi_{k\ell} = \text{id}_{t_k}^{-1} \circ \gamma_k \circ \gamma_k^{-1} \circ \text{id}_{s_{\ell}} |_{S_k}$.

The labels of directed edges starting at each vertex are in bijection to $K$ or $L$ depending on whether the vertex is in $V_1$ or $V_2$. Hence the above information together with the labelling defines a tree amalgamation. If $\Gamma_1$ is a group acting on $G_1$ in the same way as the setwise stabiliser (in $\Gamma$) of $G_s$, of $G_1$ acts on $G_s$, on $G_t$ respectively, then it is straightforward to verify that this tree amalgamation is of Type 1 respecting the actions. Note that the possible replacement of $\Gamma$ by $\Gamma'$ changes neither $\Gamma_1$ nor $\Gamma_2$.

It only remains to show that the tree amalgamation is isomorphic to $G$. Let $e_v$ be the first edge on the path from $v \in V(T)$ to $e_0$. If $v \in V_1$, then set
id_v = \gamma_{G_v} \circ \text{id}_v. Otherwise set id_v = \gamma_{G_v} \circ \text{id}_v. It is easy to verify that for an edge \( e = uv \) with labels \( f(e) = k, f'(e) = \ell \) we have that \( \text{id}_v^{-1} \circ \text{id}_u = \phi_{k\ell} \), and this clearly shows that the tree amalgamation is isomorphic to \( G \).

The proof in the case where there is \( \gamma_{G_{st}} \) mapping \( s \) to \( t \) is very similar to the first case. Define \( G_1 \) and \( G_2 \) as before, but make sure that \( \gamma_{G_{st}} \circ \text{id}_s = \text{id}_t \). This ensures that the actions \( \Gamma_1 \) on \( G_1 \) and \( \Gamma_2 \) on \( G_2 \) are the same, hence we can without loss of generality assume that \( G_1 = G_2 \) and \( \Gamma_1 = \Gamma_2 \).

Set \( I_1 = I_2 = K \cup L \) and let \( J = K \). Recall that \( f(e) \in K \) if and only if \( f'(e) \in L \). Since we can map \( s \) to \( t \), there are positively and negatively oriented edges starting at each vertex, hence the labels of edges starting at any vertex are in bijection with \( K \cup L \). Hence (a) for tree amalgamations of Type 2 holds.

Define the adhesion sets and adhesion maps exactly as above (but note that all adhesion sets end up in the same graph since \( G_1 = G_2 \)). This gives a tree amalgamation of Type 2 by construction which is isomorphic to \( G \) by the same argument as above.

Now we are ready to prove the main result of this section, the graph-theoretical analogue of Stallings’ theorem, Theorem 1.1. We are proving a slightly stronger version than the one we stated in the introduction.

**Theorem 5.8.** Let \( \Gamma \) be a group acting quasi-transitively on a connected, locally finite graph \( G \) with more than one end. Then there are connected subgraphs \( G_1, G_2 \) of \( G \) and groups \( \Gamma_1, \Gamma_2 \) acting quasi-transitively on \( G_1, G_2 \), respectively, such that \( G \) is a non-trivial tree amalgamation \( (G_1, \Gamma_1) \ast_T (G_2, \Gamma_2) \) of finite adhesion and finite identification distinguishing ends.

Furthermore, \( \Gamma_i \) can be chosen to be the setwise stabiliser of \( G_i \) in \( \Gamma \).

**Proof.** By Corollary 4.4, \( G \) has a connected, basic tree-decomposition \( (T, V) \). Using Lemma 5.7, \( G \) is a non-trivial tree amalgamation \( (G_1, \Gamma_1) \ast_T (G_2, \Gamma_2) \), where \( \Gamma_i \) is the setwise stabiliser of \( G_i \) in \( \Gamma \) for \( i = 1, 2 \). Proposition 4.5 implies that \( \Gamma_i \) acts quasi-transitively on \( G_i \) for \( i = 1, 2 \). It remains to show that the tree amalgamation has finite identification. Since \( \Gamma \) acts transitively on the adhesion sets and \( G \) is locally finite, there are only finitely many distinct adhesion sets containing a fixed vertex \( x \). In particular for a node \( t \in V(T) \) only finitely many edges incident with \( t \) correspond to adhesion sets that contain \( x \). If the tree amalgamation does not have finite identification, then there must be a ray \( t_1 t_2 \ldots \) in \( T \) such that all edges \( t_i t_{i+1} \) correspond to adhesion sets that contain \( x \). Since there are only finitely many distinct adhesion sets that contain \( x \), we may assume by (T3) that all \( t_i t_{i+1} \) correspond to the same adhesion set \( S \). As \( (T, V) \) is connected, in every \( V_t \neq S \) there is a neighbour of some vertex of \( S \). Applying (T3) shows that these are all distinct neighbours of \( S \). Thus, we have \( V_t = S \) for all but finitely many \( i \in \mathbb{N} \). In particular, there is an edge \( t_i t_{i+1} \) with \( V_{t_i} = V_{t_{i+1}} \). Since \( \Gamma \) acts transitively on \( E(T) \), this implies \( V(G) = V_t \). This is not possible by (i) from the definition of a basic tree-decomposition. Thus, the tree amalgamation has finite identification. It distinguishes ends, since \( (T, V) \) does that.

### 6 Accessible graphs

Let \( G \) be a connected, quasi-transitive, locally finite graph with more than one end and let \( \Gamma \) act quasi-transitively on \( G \). We say that \( G \) splits (non-trivially)
into connected, quasi-transitive, locally finite graphs $G_1, G_2$ if it is a non-trivial tree amalgamation $G = G_1 \ast G_2$ of finite adhesion respecting the actions of groups $\Gamma$, acting quasi-transitively on $G_i$ and if the tree-decomposition defined by $G_1 \ast G_2$ (as in Remark 5.1) is basic with respect to $\Gamma$. Note that the stabilizer in $\Gamma$ of $G_i$ acts quasi-transitively on $G_i$ by Proposition 4.5. Now if one of the factors $G_1$ or $G_2$ also has more than one end, we can split it with respect to its stabilizer, too. We can continue this for every factor and call this a process of splittings. Note that it is important in a process of splittings to use the group action of the stabiliser of the factor in order to split the factor. If we eventually end up with factors that are either finite or have at most one end, that is, if the process of splittings terminates, we call the (multi-)set of these factors a terminal factorisation of $G$. (Also, if $G$ is one-ended, we say it is a terminal factorisation of itself.) We call $G$ accessible if it has a terminal factorisation.

Remark 6.1. Let $G$ be an accessible connected, quasi-transitive, locally finite graph. Then there are connected, quasi-transitive, locally finite graphs $G_1, \ldots, G_n, H_1, \ldots, H_{n-1}$ with $G = H_{n-1}$ and trees $T_1, \ldots, T_{n-1}$ such that the following hold:

(i) every $G_i$ has at most one end;

(ii) for every $i \leq n - 1$, the graph $H_i$ is a tree amalgamation $H \ast T_i H'$ with respect to group actions of finite adhesion, where

$$H, H' \in \{G_j \mid 1 \leq j \leq n\} \cup \{H_j \mid 1 \leq j < i\}.$$ 

Remark 6.2. Let $G_0$ be the class of all connected, quasi-transitive, locally finite graphs with at most one end. For $i > 0$, let $G_i$ be the class obtained by tree amalgamations of finite adhesion of elements in $\bigcup_{j<i} G_j$ respecting group actions. Set $G := \bigcup_{i \in \mathbb{N}} G_i$. Then $G$ is the class of all accessible connected, quasi-transitive, locally finite graphs.

The following result generalizes a graph theoretical characterisation of accessibility of finitely generated groups: Thomassen and Woess [27, Theorem 1.1] proved that a finitely generated group is accessible if and only if it has a locally finite Cayley graph that is accessible in their sense.

Theorem 6.3. Let $G$ be a connected, locally finite, quasi-transitive graph. Then the following statements are equivalent.

(1) $G$ is accessible.

(2) $G$ is accessible in the sense of Thomassen and Woess.

Before we prove Theorem 6.3, we need another result. Recall that a tree-decomposition efficiently distinguishes two ends if there is an adhesion set $V_{t_1} \cap V_{t_2}$ separating them such that no set of smaller size than $V_{t_1} \cap V_{t_2}$ separates them.

Theorem 6.4. Let $G$ be a connected, locally finite graph such that there is an $n \in \mathbb{N}$ such that every two ends of $G$ can be separated by at most $n$ vertices. Let $\Gamma$ be a group acting quasi-transitively on $G$. Then there exists a $\Gamma$-invariant tree-decomposition $(T, V)$ of $G$ of finite adhesion such that $(T, V)$ distinguishes all ends of $G$ efficiently and such that there are only finitely many $\Gamma$-orbits on $E(T)$. 

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Proof. By Theorem 3.2 we find a $\Gamma$-invariant tree-decomposition $(T, V)$ of $G$ of adhesion at most $k$ that distinguishes all ends efficiently.

For every adhesion set $V_i \cap V_j$ that does not separate any two ends efficiently, we contract the edge $tt'$ in $T$ and assign the vertex set $V_i \cup V_j$ to the new node. It is easy to check that the resulting pair $(T', V')$ is again a tree-decomposition. It only has adhesion sets that distinguish ends efficiently. Note that $\Gamma$ still acts on $(T', V')$ as the set of adhesion sets that do not separate ends efficiently is $\Gamma$-invariant. A result of Thomassen and Woess [27, Proposition 4.2] says that there are only finitely many vertex sets $S$ of size at most $k$ containing a fixed vertex such that for two components $C_1, C_2$ of $G - S$ every vertex of $S$ has a neighbour in $C_1$ and in $C_2$. Since $G$ is locally finite and quasi-transitive, it follows that there are only finitely many orbits of adhesion sets that separate ends efficiently. This proves the assertion. \[\square\]

Proof of Theorem 6.3. To prove that (2) implies (1), let $G$ be accessible in the sense of Thomassen and Woess and let $\Gamma$ be a group acting on $G$ with only finitely many orbits. As $G$ is quasi-transitive, there is an $n \in \mathbb{N}$ such that any two ends of $G$ can be separated by at most $n$ vertices. By Theorem 6.4 we find a $\Gamma$-invariant tree-decomposition $(T, V)$ of $G$ of finite adhesion such that $(T, V)$ distinguishes all ends efficiently and such that there are only finitely many $\Gamma$-orbits on $E(T)$. We apply Proposition 4.1, and obtain a tree-decomposition $(T, U)$ with connected adhesion sets and $V_i \subseteq U_i$ for all $i \in V(T)$. We prove the assertion by induction on the number of $\Gamma$-orbits of adhesion sets of $(T, U)$. Let $tt' \in E(T)$. For every edge $t_1t_2 \in E(T)$ that does not lie in the same $\Gamma$-orbit as $tt'$, we contract the edge $t_1t_2$ in $T$ and assign the vertex set $U_{t_1} \cup U_{t_2}$ to the new node. Let $T'$ be the resulting tree and $U' = \{U_s \mid s \in V(T')\}$. It is easy to verify that $(T', U')$ is a tree-decomposition. The only edges of $T'$ are those that have their origin in the $\Gamma$-orbit of the edge $tt' \in E(T)$ and $\Gamma$ still acts on $(T', U')$ so that $(T', U')$ is a connected, basic tree-decomposition of $G$ with only connected adhesion sets. Lemma 5.7 implies that $G$ is a non-trivial tree amalgamation $G_1 *_{T'} G_2$ with respect to group actions, where the graphs $G_1$ and $G_2$ are induced by the parts of $(T', U')$. The tree-decomposition $(T, U)$ induces a tree-decomposition $(T_W, W)$ on the parts $W$ of $(T', U')$ and there are fewer $\Gamma_W$-orbits on the adhesion sets of $(T_W, W)$ than $\Gamma$-orbits on the adhesion sets of $(T, U)$. Thus, we can apply induction on the number of orbits of adhesion sets. This shows (1).

To prove that (1) implies (2), we will use the graph classes $G_i$ and $G$ as defined in Remark 6.2 and show inductively that every $G_i$ contains only connected, quasi-transitive, locally finite graphs for which there exists an $n \in \mathbb{N}$ such that any two of its ends can be separated by at most $n$ vertices. This clearly implies that every $G_i$ only contains graphs that are accessible in the sense of Thomassen and Woess. This is obviously true for $G_0$. Let $G \in G_i$ for $i > 0$. Then there are $G_1, G_2 \in \bigcup_{j < i} G_j$ such that $G$ is a tree amalgamation $G_1 *_{T'} G_2$ of finite adhesion respecting group actions. By induction, we may assume that $G_1$ and $G_2$ are accessible in the sense of Thomassen and Woess and quasi-transitive. Note that quasi-transitivity of $G$ follows from Lemma 5.6 since $G_1$ and $G_2$ are quasi-transitive. For $i = 1, 2$, let $k_i$ be a positive number such that any two ends of $G_i$ can be separated by at most $k_i$ many vertices. Let $(T, V)$ be the tree-decomposition we obtain from the tree amalgamation $G_1 *_{T'} G_2$ according to Remark 5.1. Let $k$ be the maximum of $k_1, k_2$ and the size of adhesion sets
of $G_1 + T G_2$.

Let $Q, R$ be two rays in different ends $\omega_Q, \omega_R$ of $G$, respectively. If there is some adhesion set $V_t \cap V_{t'}$ such that $Q$ and $R$ have tails that are separated by $V_t \cap V_{t'}$, then the ends they lie in must be separated by that adhesion set as well. Hence, they are separable by a separator of order at most $k$. So we may assume that, eventually, they lie on the same side of each adhesion set. By Proposition 4.8 (i) every end of $G$ is captured either by an end or by a node of $T$. Since no separator separates any tails of $Q$ and $R$, their ends are captured by the same node or end of $T$. By Proposition 4.8 (iii) an end of $T$ captures a unique end of $G$. Thus, $\omega_Q$ and $\omega_R$ are captured by the same node of $T$. By Proposition 4.8 (iv) every end of $G$ that is captured by a node $t \in V(T)$ corresponds to a uniquely determined end of $G[V_t]$. These ends can be separated by a separator $S$ in $G[V_t]$ of order at most $k$ by assumption. However, $S$ need not be a separator of $G$ that separates those ends. Still, it is possible to enlarge $S$ to a separator of $G$ that separates $\omega_Q$ and $\omega_R$ and still has bounded size: if $K$ is the maximum diameter of the adhesion sets measured in $G_1$ and in $G_2$, then every vertex of $S$ has distance at most $K$ to only finitely many adhesion sets that are contained in $V_t$ as $G$ is locally finite; so we can add all these adhesion sets to $S$ and obtain a set $S'$. As $G$ is quasi-transitive, the size of $S'$ only depends on $k$, the number of orbits of vertices of $G$, the maximum number of adhesion sets in $V_t$ that have distance at most $K$ to a common vertex and the size of any adhesion set of $(T, V)$, in particular, it is bounded by some $\ell \in \mathbb{N}$ and it is independent of the chosen ends. If we show that $S'$ separates $\omega_Q$ and $\omega_R$, then it follows immediately that there is an $n \in \mathbb{N}$ such that every two ends of $G$ can be separated by at most $n$ vertices.

Let $P = \ldots, x_{-1}, x_0, x_1, \ldots$ be a double ray with its tail $x_0, x_1, \ldots \in \omega_Q$ and its tail $x_0, x_{-1}, \ldots \in \omega_R$. Since both ends $\omega_Q$ and $\omega_R$ are captured by $V_t$, there are infinitely many $x_i$ with $i > 0$ that lie in $V_t$ and infinitely many $x_i$ with $i < 0$ that lies in $V_t$. Let us assume $x_0 \in V_t$. Whenever the ray $P^+ := x_0 x_1 \ldots$ leaves $V_t$ through an adhesion set $V_t \cap V_{t'}$, it must reenter $V_t$ and this must happen through the same adhesion set. Since $S$ is finite and separates $\omega_Q$ and $\omega_R$, there are $i_1, i_2 \in \mathbb{Z}$ such that no $x_i \in V_t$ with $i \geq i_1$ is separated in $G[V_t]$ by $S$ from $\omega_Q$ and no $x_i \in V_t$ with $i \leq i_2$ is separated in $G[V_t]$ by $S$ from $\omega_R$. Then there must be some path $x_{i_1}, \ldots, x_{i_2}$ with $j \geq i + 1$ and whose inner vertices lie outside of $V_t$ such that $x_j$ is not separated by $S$ from $\omega_Q$ and $x_i$ is not separated by $S$ from $\omega_R$. Thus, the shortest $x_i-x_j$ path in $G[V_t]$ meets $S$. As $x_i$ and $x_j$ lie in a common adhesion set, we conclude that this lies in $S'$. Thus, $S'$ separates $\omega_Q$ from $\omega_R$ in $G$. This shows (2).

In the proof of the implication (2) to (1) of Theorem 6.3 we chose a specific way to split the factors. (It was based on a $\Gamma$-invariant tree-decomposition of $G$.) In an earlier version of this paper, we did not know if we can split arbitrarily in each step and still have to end in a terminal factorisation. But we conjectured that this is true.

**Conjecture 6.5.** Let $G$ be an accessible, connected, quasi-transitive, locally finite graph. Every process of splittings must end after finitely many steps.

This conjecture has been verified in [15].

Accessibility of finitely generated groups received a lot of attention after Wall [28] conjectured that all finitely generated groups are accessible and among
the main results in this area are Dunwoody’s results that Wall’s conjecture is false in general [7] but true for (almost) finitely presented groups [6]. In the case of quasi-transitive, locally finite graphs, the investigation focused on graphs that are accessible in the sense of Thomassen and Woess, see [13, 22, 27]. However, Theorem 6.3 enables us to carry over these results to graphs that are accessible in our sense.

7 Applications

7.1 Stallings’ theorem

There are several proofs of Stallings’ theorem in the literature, see [5, 16, 23, 25]. In this section we will discuss how to obtain Stallings’ theorem from our results.

Let $\Gamma$ be a finitely generated group with infinitely many ends and let $G$ be a locally finite Cayley graph of $\Gamma$. Then $G$ has infinitely many ends, too. By Theorem 5.8, $G$ is a non-trivial tree amalgamation $G_1 \ast_T G_2$ of finite adhesion respecting group actions. Since it has finite adhesion and $\Gamma$ acts regularly on $G$ of an edge of $T$, which is a subgroup of the stabiliser in $\Gamma$ of the corresponding adhesion set, is finite. If the induced action of $\Gamma$ on $T$ is with inversion of edges, then we subdivide each edge of $T$ once. On the edges of the resulting tree, the group $\Gamma$ acts transitively but without inversion. Now we apply Bass-Serre theory via the following theorem.

**Theorem 7.1.** [24] Let $T$ be a tree without leaves and let $\Gamma$ act on $T$ without inversion of edges but transitively on $E(T)$. If $\Gamma$ acts transitively on $T$ then $\Gamma$ is an HNN-extension of the stabilizer of a vertex over the pointwise stabilizer of an edge. If there are two $\Gamma$-orbits on $V(T)$, then $\Gamma$ is the free product of the stabilizers of two adjacent vertices with amalgamation over the pointwise stabilizer of the incident edge.

We thus obtain Stallings’ theorem.

**Theorem 7.2.** [25] A finitely generated group has more than one end, if and only if it is either a free product with amalgamation over a finite subgroup or an HNN-extension over a finite subgroup.

Note that tree amalgamations of Type 1 respecting the actions of groups acting on the factors lead to free products with amalgamation via Theorem 7.1 and tree amalgamations of Type 2 respecting the action of groups lead to HNN-extension via Theorem 7.1.

7.2 Graphs without thick ends

In this section, we consider connected, quasi-transitive, locally finite graphs that have only thin ends. We prove that such graphs have are the connected, quasi-transitive, locally finite graphs that have terminal factorisations with only finite factors. But before we go into the proof, we need some definitions.

Let $G$ and $H$ be graphs. A map $\varphi: V(G) \to V(H)$ is a $(\gamma, c)$-quasi-isometry if there are constants $\gamma \geq 1, c \geq 0$ such that

$$\gamma^{-1}d_G(x, y) - c \leq d_H(\varphi(x), \varphi(y)) \leq \gamma d_G(x, y) + c$$

3that is, for every two $u, v \in V(G)$ there is a unique element of $\Gamma$ mapping $u$ to $v$
for all \( x, y \in V(G) \) and such that \( \sup \{ d_H(x, \varphi(V(G))) \mid x \in V(H) \} \leq c \). We then say that \( G \) is quasi-isometric to \( H \).

Krön and Möller [17, Theorem 5.5] showed that a connected, quasi-transitive, locally finite graph has only thin ends if and only if it is quasi-isometric to a tree. Trees are obviously accessible in the sense of Thomassen and Woess and it follows from their definition of accessibility that the class of quasi-transitive, locally finite such graphs is invariant under quasi-isometries. Thus, we have verified the following.

**Proposition 7.3.** Every connected, locally finite, quasi-transitive graph that has only thin ends is accessible in the sense of Thomassen and Woess.  

We mention that Thomassen and Woess [27, Theorem 5.3] showed Proposition 7.3 for transitive graphs directly with a nice graph theoretical argument. It is not too hard to modify their argument in such a way that the proof works for quasi-transitive graphs as well.

Another result we need for our investigation here is due to Thomassen.

**Proposition 7.4.** [26, Proposition 5.6.] If \( G \) is an infinite, connected, quasi-transitive, locally finite graph with only one end, then the end is thick.

Recently, Carmesin et al. [3, Theorem 5.1] extended Proposition 7.4 to graphs that need not be locally finite.

Now we are able to give a new characterisation of connected, quasi-transitive, locally finite graphs with only thin ends.

**Theorem 7.5.** A connected, quasi-transitive, locally finite graph has only thin ends if and only if it has a terminal factorisation of only finite graphs.

**Proof.** Let \( G \) be a connected, quasi-transitive, locally finite graph. First, let us assume that every end of \( G \) is thin. By Proposition 7.3, \( G \) is accessible in the sense of Thomassen and Woess. So Theorem 6.3 implies that \( G \) is accessible and hence has a terminal factorisation. All the factors of that terminal factorisation have at most one end. Since they are quasi-transitive by Proposition 4.5, they cannot have one end due to Proposition 7.4. So they are locally finite graphs without ends, which implies that they are finite graphs.

For the other direction, we follow the steps to factorise \( G \), factorise each of its factors and so on until we end up with a terminal factorisation. Note that by Proposition 4.8 (ii) every thick end of \( G \) is captured by nodes of the involved basic tree-decompositions. So if \( G \) had a thick end, then one of the factors of the terminal factorisation must have a thick end, which is impossible as these factors are finite by assumption. Thus, all ends of \( G \) are thin.

We note that there are several characterisations of (quasi-transitive or Cayley) graphs that are quasi-isometric to trees, see e.g. Antolín [1], Krön and Möller [17], Manning [18] and Woess [29]. We enlarged their list of characterisations by our theorem.

A natural class of quasi-transitive graphs are Cayley graphs. So our theorems apply in particular for such graphs and we obtain as a corollary of Theorem 7.5 a result for virtually free groups. A group \( \Gamma \) is virtually free if it contains a free subgroup of finite index.

Woess [29] showed that a finitely generated group is virtually free if and only if every end of any of its locally finite Cayley graphs is thin. Thus we directly obtain the following corollary.
Corollary 7.6. A finitely generated group is virtually free if and only if any of its locally finite Cayley graphs has a terminal factorisation of only finite graphs.

In [12] the interplay between tree amalgamations and quasi-isometries is investigated further and the results of this section are extended to graphs other than trees in two ways. First, it is shown that the quasi-isometry type of (iterated) tree amalgamations only depend on the quasi-isometry types of the infinite factors. Then, in the case of accessible infinitely-ended graphs, it is shown that the quasi-isometry types of the graphs determine the quasi-isometry types of the infinite factors in any of its terminal factorisations.

7.3 Planar graphs

Mohar, see [19], raised the question whether tree amalgamations are powerful enough to characterise (3-connected) planar, transitive, locally finite graphs in terms of finite or one-ended, locally finite, planar, transitive graphs. We are able to give an affirmative answer in case of planar, quasi-transitive graphs.

Dunwoody [8] proved that planar, quasi-transitive, locally finite graphs are accessible in the sense of Thomassen and Woess, see also [14]. This allows us to apply Theorem 6.3 to these graphs. We directly obtain the following result.

Theorem 7.7. For every planar, connected, quasi-transitive, locally finite graph $G$ there are finitely many planar, connected, quasi-transitive, locally finite graphs $G_1, \ldots, G_n$ with at most one end such that $G$ can be obtained by finitely many (iterated) tree amalgamations of $G_1, \ldots, G_n$.

We point out that examples given in [9] show that we cannot replace the term ‘quasi-transitive’ by ‘transitive’ or ‘Cayley’ in the above theorem; Georgakopoulos [9] suggests that Mohar’s question is to be interpreted in terms of subdivisions, that is, we can replace ‘quasi-transitive’ by ‘transitive’ if in addition to tree amalgamations we allow subdivisions.

7.4 Further applications

In this section we briefly mention further applications of our main results. In [11] we prove that tree amalgamations and hyperbolic graphs fit well together in that we prove that a locally finite, quasi-transitive graph with more than one end is hyperbolic if and only if it is the non-trivial tree amalgamation of two locally finite, quasi-transitive, hyperbolic graphs. Additionally, the homeomorphism type of the hyperbolic boundary is uniquely determined by the homeomorphism types of the hyperbolic boundaries of their factors [11, Theorem 1.2]. Since hyperbolic, locally finite, quasi-transitive graphs are accessible in the sense of Thomassen and Woess [13], this implies by Theorem 6.3 that the homeomorphism type of the hyperbolic boundary is uniquely determined by the homeomorphism types of the hyperbolic boundaries in any terminal factorisation.

Similarly, we consider in [12] quasi-isometry types of tree amalgamations and prove that they only depends on the quasi-isometry types of its factors. In the case of accessible graphs with the same number of ends, we obtain that
two graphs are quasi-isometric if and only if all terminal factorisations have the same quasi-isometry types of infinite factors.

As a third application, we obtain in [10] a sharp upper bound of the asymptotic dimension of tree amalgamations depending on the asymptotic dimensions of their factors.

References


