# TREE AMALGAMATIONS AND QUASI-ISOMETRIES

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ABSTRACT. We investigate the connections between tree amalgamations and quasi-isometries. In particular, we prove that the quasi-isometry type of multi-ended accessible quasi-transitive connected locally finite graphs is determined by the quasi-isometry type of their one-ended factors in any of their terminal factorisations. Our results carry over theorems of Papsoglu and Whyte on quasi-isometries between multi-ended groups to those between multi-ended graphs. In the end, we discuss the impact of our results to a question of Woess.

## 1. INTRODUCTION

Tree amalgamations can be thought of as an analogue of free products with amalgamation or HNN-extensions for graphs. (We refer to Section 2 for the precise definitions.) The following theorem of [5] says that every multi-ended locally finite connected quasi-transitive graph splits over a finite subgraph as a tree amalgamation. It is a graph theoretic version of Stallings' splitting theorem of multi-ended groups [9].

**Theorem 1.1.** [5, Theorem 5.3] *Every connected quasi-transitive locally finite graph with more than one end is a non-trivial tree amalgamation of finite adhesion and finite identification length of two connected quasi-transitive locally finite graphs that respects the group actions and distinguishes ends.* □

Just like Stallings' theorem enables us to prove theorems about multi-ended groups having knowledge of their factors, we are aiming for similar results for graphs. Two such example are already proved in [5] and [4]: tree amalgamations respect hyperbolicity, i. e. two locally finite quasi-transitive graphs are hyperbolic if and only if their tree amalgamation is hyperbolic, see [4, Theorem 1.1], and a connected locally finite quasi-transitive graph has only thin ends if and only if it has a terminal factorisation of finite graphs, see [5, Theorem 7.5]. Here, a terminal factorisation can be seen as analogue of a terminal graph of groups: whenever we split our multi-ended graphs, we may ask if their factors have more than one end and apply to those our splitting theorem, too. If we iterate this splitting and eventually have only factors with at most one end, a *terminal factorisation* of the original graph consists of these finite or one-ended factors. Krön and Möller [6, Theorem 5.5] proved that a connected locally finite quasi-transitive graph has only thin ends if and only if it is quasi-isometric to a tree. Thus, the result of [5, Section 7.2] is the following.

**Theorem 1.2.** *A connected quasi-transitive locally finite graph is quasi-isometric to a tree if and only if it has a terminal factorisation of only finite graphs.* □

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In this paper we are looking more into the connections of tree amalgamations with quasi-isometries. Since tree amalgamations of two graphs differ if we choose different adhesion sets (i. e. if the overlap of the two graphs differ in its size or its position within the graphs), the first natural question is whether the quasi-isometry type of a tree amalgamation depends on these choices. Our first theorem proves that (under some mild assumptions) it does not depend on these choices.

**Theorem 1.3.** *Let $G_1$ and $G_2$ be infinite connected locally finite quasi-transitive graphs and $G$ and $H$ be two non-trivial tree amalgamations of $G_1$ and $G_2$ both having finite adhesion and bounded identification. Assume that for $i = 1, 2$ some group of automorphisms of $G_i$ acts quasi-transitively on $G_i$ and on its adhesion sets for both tree amalgamations. Then $G$ is quasi-isometric to $H$.*

The result stays essentially the same if we take the second tree amalgamation as a tree amalgamation of two graphs, one of which is quasi-isometric to $G_1$ the other to $G_2$. More generally, our next result says that even iterated tree amalgamations only depend on the quasi-isometry type of their infinite factors.

**Theorem 1.4.** *Let $G$ and $H$ be connected locally finite quasi-transitive graphs with infinitely many ends and let $(G_1, \ldots, G_n), (H_1, \ldots, H_m)$ be factorisations of $G, H$, respectively. If $(G_1, \ldots, G_n)$ and $(H_1, \ldots, H_m)$ have the same set of quasi-isometry types of infinite factors, then $G$ and $H$ are quasi-isometric.*

An obvious question is whether we can also obtain a reverse statement of Theorem 1.4, i. e. if $G$ and $H$ are quasi-isometric graphs, do all factorisations of $G$ and $H$ have the same set of quasi-isometry types of infinite parts? In general this is false as an easy example shows: take a tree; this has a factorisation into finite graphs, but also the trivial factorisation into just itself.

In this example, we can still factorise the tree while this is not possible for the finite graphs. So we might just ask for a reverse of Theorem 1.4 for terminal factorisations. We call a connected quasi-transitive locally finite graph *accessible* if it has a terminal factorisation. Note that not all connected locally finite quasi-transitive graphs have a terminal factorisation, i. e. there are inaccessible connected quasi-transitive locally finite graphs. But the class of accessible connected quasi-transitive locally finite graphs is indeed a class where we obtain the reverse of Theorem 1.4 for terminal factorisations, as the following theorem shows.

**Theorem 1.5.** *Let $G$ be a connected accessible locally finite quasi-transitive graph. A connected locally finite quasi-transitive graph $H$ is quasi-isometric to $G$ if and only if it satisfies the following conditions:*

(i) *$H$ has the same number of ends as $G$;*
(ii) *$H$ is accessible; and*
(iii) *any terminal factorisation of $H$ has the same set of quasi-isometry types of one-ended factors as any terminal factorisation of $G$.*

Papasoglu and Whyte [8] proved group theoretic versions of Theorems 1.3, 1.4 and 1.5. Our proofs are inspired by their proof ideas from free products with amalgamations and HNN-extensions of groups. We prove Theorems 1.3 and 1.4 in Section 3 and Theorem 1.5 in Section 4.

In Section 5, we turn our attention towards a question of Woess that seems to be a bit unrelated at a first glance. Woess [10, Problem 1] posed the problem whether there are locally finite transitive graphs that are not quasi-isometric to any locally finite Cayley graph. His problem was settled in the negative by Eskin et al. [2] who proved that the Diestel-Leader graphs are counterexamples. As a corollary of Theorem 1.5 we shall see that in order to find further counterexamples, it suffices to look at either one-ended graphs or inaccessible ones.

## 2. Preliminaries

In this section, we state our major definitions an cite several results that we are going to use in the proofs of our main theorems.

Let $G$ and $H$ be graphs. A map $\varphi\colon V(G)\to V(H)$ is a *quasi-isometry* if there are constants $\gamma\geq 1$, $c\geq 0$ such that

$$\gamma^{-1}d_G(u,v)-c\leq d_H(\varphi(u),\varphi(v))\leq \gamma d_G(u,v)+c$$

for all $u,v\in V(G)$ and such that $\sup\{d_H(v,\varphi(V(G)))\mid v\in V(H)\}\leq c$. We then say that $G$ is *quasi-isometric* to $H$. We call $G$ and $H$ *bilipschitz equivalent* if they are quasi-isometric with $c=0$.

A tree is *semiregular* or $(p_1,p_2)$*-semiregular* if for the canonical bipartition $V_1,V_2$ of its vertex set all vertices in $V_1$ have the same degree $p_1$ and all vertices in $V_2$ have the same degree $p_2$.

Let $G_1$ and $G_2$ be two graphs and let $T$ be a $(p_1,p_2)$-semiregular tree with canonical bipartition $V_1,V_2$ of its vertex set. Let

$$c\colon E(T)\to\{(k,\ell)\mid 0\leq k<p_1, 0\leq \ell<p_2\}$$

such that for all $v\in V_i$ the $i$-th coordinates of the elements of $\{c(e)\mid v\in e\}$ exhaust the set $\{k\mid 0\leq k<p_i\}$. Note that in particular the $i$-th coordinates of the elements of $\{c(e)\mid v\in e\}$ are all distinct. Let $\{S_k^i\mid 0\leq k<p_i\}$ be a set of subsets of $V(G_i)$ such that all $S_k^i$ have the same cardinality. Let $\varphi_{k,\ell}\colon S_k^1\to S_\ell^2$ be a bijection.

For each $v\in V_i$ with $i=1,2$, let $G_i^v$ be a copy of $G_i$. We denote by $S_k^v$ the copy of $S_k^i$ in $G_i^v$. Let $H:=G_1+G_2$ be the graph obtained from the disjoint union of all $G_i^v$ by adding an edge between all $x\in S_k^v$ and $\varphi_{k,\ell}(x)\in S_\ell^u$ for every edge $vu\in E(T)$ with $c(vu)=(k,\ell)$ and $v\in V_1$. Let $G$ be the graph obtained from $H$ by contracting all new edges, i. e. all edges outside of the $G_i^v$. We call $G$ the *tree amalgamation* of $G_1$ and $G_2$ over $T$ (with respect to the sets $S_k$ and the maps $\varphi_{k,\ell}$), denoted by $G_1*_T G_2$. If $T$ is clear from the context, we simply write $G_1*G_2$. The sets $S_k^i$ and their copies in $G$ are the *adhesion sets* of the tree amalgamation. If the adhesion sets of a tree amalgamation are finite, then this tree amalgamation has *finite adhesion*. Let $\psi\colon V(H)\to V(G)$ be the canonical map that maps every $x\in V(H)$ to the vertex of $G$ it ends up after all contractions. A tree amalgamation $G_1*_T G_2$ is *trivial* if there is some $G_i^v$ such that the restriction of $\psi$ to $G_v^i$ is bijective. Note that a tree amalgamation of finite adhesion is trivial if $V(G_i)$ is the only adhesion set of $G_i$ and $p_i=1$ for some $i\in\{1,2\}$.

*Rays* are one-way infinite paths and two rays in a graph $G$ are equivalent if they lie eventually in the same component of $G-S$ for every finite set $S$. The equivalence classes of rays are the *ends* of $G$.

The tree amalgamation $G=G_1*G_2$ *distinguishes* ends if there is some adhesion set $S_k^v=S_\ell^u$ for adjacent vertices $u,v$ of $T$ such that for every component $C$ of $T-uv$ the graph induced by $\bigcup_{w\in C}G_i^w$ contains an end.

The *identification size*[1] of a vertex $x \in V(G)$ is the smallest size of subtrees of $T$ over which the contractions to obtain $x$ take place, i. e. the size $n$ of the smallest subtree $T'$ of $T$ such that $x$ is obtained by contracting only edges between vertices in $\bigcup_{u\in V(T')} V(G_j^u)$. The tree amalgamation has *finite identification* if every vertex has finite identification size. If has *bounded identification* if the supremum of the identification sizes is finite.

**Remark 2.1.** For a tree amalgamation $G_1 * G_2$ of finite identification, the canonical map $\psi\colon V(G_1+G_2) \to V(G_1 * G_2)$ is a quasi-isometry whose constants depend only on the identification sizes of the vertices. Thus, if the identification sizes and the diameters of adhesion sets are bounded, then $G_1$ and $G_2$ embed quasi-isometrically into $G_1 + G_2$ and thus into $G_1 * G_2$.

If $G_1$ is finite we define the *finite extension of $G_2$ by $G_1$ (with respect to $G_1 * G_2$)* to be isomorphic to a subgraph of $G_1 * G_2$ that is induced by a copy $G_2^v$ of $G_2$ and all copies $G_1^u$ of $G_1$ with $uv \in E(T)$. It is straight forward to see that the finite extension of $G_2$ is quasi-isometric to $G_2$ if all adhesion sets in $G_2$ have bounded diameter.

**Remark 2.2.** Let $T$ be a $(p_1, p_2)$-semiregular tree and let $G_1 *_T G_2$ be a tree amalgamation, where $G_1$ is finite. Let $G_2'$ be the finite extension of $G_2$ by $G_1$ with respect to $G_1 *_T G_2$. We will define a tree amalgamation $G_2' *_{T'} G_2$ with $G_2' *_{T'} G_2 = G_1 *_T G_2$, where $T'$ will be a $(p_2(p_1-1), p_2)$-semiregular tree. For that choose $u \in V_2$ and let $U_1$ be the set of vertices $v \in V_2$ with distance 0 mod 4 to $u$ and $U_2 := V_2 \smallsetminus U_1$. To obtain $T'$ we start with $T$ and contract all edges of $T$ that are incident with a vertex of $U_1$. The resulting graph is a $(p_2(p_1-1), p_2)$-semiregular tree. We may assume that the vertex set of $T'$ is $U_1 \cup U_2$. For each vertex in $U_1$ we take a copy of $G_2'$ and for each vertex in $U_2$ we take a copy of $G_2$. There is a canonical way of assigning the labels to the edges of $T'$ so that the tree amalgamation $G_2' *_{T'} G_2$ is the same as the tree amalgamation $G_1 *_T G_2$.

We will use Theorem 1.1 in a slightly different form to avoid the definition of a tree amalgamation with respect to the group actions. We note that the statement about the finite identification varies a bit from [5, Theorem 5.3], but the version we use here is mentioned in its proof in [5].

**Theorem 2.3.** [5, Theorem 5.3] *Every connected quasi-transitive locally finite graph with more than one end is a non-trivial tree amalgamation $G_1 * G_2$ that distinguishes ends and has finite adhesion and finite identification of two connected quasi-transitive locally finite graphs such that the set of adhesion sets in each factor has at most two orbits under some group acting quasi-transitively on that factor.* $\square$

Note that the properties of the tree amalgamation of Theorem 2.3 imply that it has bounded identification.

A *factorisation* of a connected locally finite quasi-transitive graph $G$ is a tuple $(G_1, \ldots, G_n)$ of connected locally finite quasi-transitive graphs such that $G$ is obtained from the elements of the tuple by iterated tree amalgamations of finite adhesion and finite identification such that for each step some group of automorphisms

[1] Equivalently, as in [7], you can define $G$ via identifications of vertices in the disjoint union of the $G_i^v$ instead of contraction of the newly added edges. From this point of view, we get the justification of the term 'identification size'.

of each factors acts quasi-transitively on this factor and the set of its adhesion sets. A factorisation is *terminal* if every element of the tuple has at most one end.[2]

**Remark 2.4.** Let $G$ be an accessible connected quasi-transitive locally finite graph. Then there are connected quasi-transitive locally finite graphs $G_1, \ldots, G_n$, $H_1, \ldots$, $H_{n-1}$ with $G = H_{n-1}$ and trees $T_1, \ldots, T_{n-1}$ such that the following hold:

(1) every $G_i$ has at most one end;
(2) for every $i \leq n-1$, the graph $H_i$ is a tree amalgamation $H *_{T_i} H'$ of finite adhesion and bounded identification, where
$$H, H' \in \{G_j \mid 1 \leq j \leq n\} \cup \{H_j \mid 1 \leq j < i\}.$$

Papasoglu and Whyte used a construction in [8] that we will use for our proofs, too. However, we can express it in terms of tree amalgamations and thus stick to our notations. Let $G_1$ and $G_2$ be graphs and let $v_i \in V(G_i)$ for $i = 1, 2$ be their *base vertices*. Let $(S^i_k)_{k \leq |G_i|}$ be the adhesion sets such that $(S^i_k)_{k \leq |G_i|}$ forms a partition of $V(G_i)$ into sets of size 1. Assume that $S^1_1 = \{v_1\}$ and $S^2_1 = \{v_2\}$. Let $T$ be a $(|G_1|, |G_2|)$-semiregular tree with canonical vertex partition $\{V_1, V_2\}$, and let $u \in V_1$. Let
$$c\colon E(T) \to \{(k, \ell) \mid 0 \leq k < |G_1|, 0 \leq \ell < |G_2|\}$$
be as required for a tree amalgamation with the following additional property for all edges $xy \in E(T)$, where $x$ is closer to $u$ than $y$:

- if $x \in V_1$, let the second coordinate of $c(xy)$ be 0;
- if $x \in V_2$, let the first coordinate of $c(xy)$ be 0.

We denote the graph $G_1 + G_2$ by $G_1 +_{v_1,v_2} G_2$. Note that there is a unique edge in $T$ with label $(0, 0)$. We call the corresponding edge in $G_1 +_{v_1,v_2} G_2$ the *base edge* of $G_1 +_{v_1,v_2} G_2$. It is the unique edge in $G_1 +_{v_1,v_2} G_2$ that connects two base vertices.

Papasoglu and Whyte proved several lemmas about (their version of) this special kind of tree amalgamations. Before we state them, we need a further definition. We call a graph $G$ *unvarying* if there is some $\gamma$ such that for every $u, v \in V(G)$ there is a $\gamma$-bilipschitz map $G \to G$ that maps $u$ to $v$. Note that every quasi-transitive graph is unvarying.

**Lemma 2.5.** [8, Lemma 1.1] *Let $G$ and $H$ be unvarying graphs. Let $F$ be a graph such that there is some $\gamma > 0$ such that the following hold.*

(1) *$F$ contains families $(G_i)_{i \in I}$ and $(H_j)_{j \in J}$ of disjoint subgraphs such $V(F)$ is covered by them.*
(2) *Every vertex of $F$ is incident with a unique edge that lies outside all $G_i, H_j$.*
(3) *Every edge of $F$ outside of all $G_i, H_j$ is incident with a vertex in some $G_i$ and with a vertex in some $H_j$.*
(4) *For every $i \in I$, $j \in J$ there is a $\gamma$-bilipschitz equivalence $G_i \to G$, $H_j \to H$, respectively.*
(5) *Contracting all edges inside the graphs $G_i$ and $H_j$ results in a tree.*

*Then there is a constant $\delta$ (depending only on $\gamma$ and the unvaryingness-constants of $G$ and $H$) such that for any edge $e \in E(F)$ connecting some $G_i$ to some $H_j$ and*

[2] We note that the definitions for (terminal) factorisations are weaker than in [5] as we do not care about the specific group actions for our results here.

*any base points $u$ in $G$ and $v$ in $H$ there is a $\delta$-bilipschitz equivalence $F \to G +_{u,v} H$ that maps $e$ to the base edge of $G +_{u,v} H$.* □

Lemma 2.5 is our main tool to switch from the tree amalgamations we are starting with to tree amalgamations that are defined using base vertices, in order to apply the two following lemmas.

Let $G_1$ and $G_2$ be graphs with base vertices. The *wedge* of $G_1$ and $G_2$ is their disjoint union with an edge joining their base vertices.

**Lemma 2.6.** [8, Lemma 1.3] *Let $G_1$ and $G_2$ be infinite unvarying graphs with base vertices $u, v$, respectively. Then $G := G_1 +_{u,v} G_2$ is bilipschitz equivalent to the wedge of any finite number of copies of $G$.* □

**Lemma 2.7.** [8, Lemma 1.4] *Let $G_1$ and $G_2$ be infinite unvarying graphs and $u, v$ be their base vertices, respectively. Then $G_1 +_{u,v} G_2$ and $G_1 +_{u,v} (G_2 +_{v,v} G_2)$ are bilipschitz equivalent.* □

We end this section by proving two quasi-isometry results regarding infinitely-ended trees.

**Lemma 2.8.** *Every two locally finite trees with infinitely many ends and finitely many orbits of vertices are quasi-isometric.*

*Proof.* It suffices to prove that every locally finite tree $T$ with infinitely many ends and finitely many orbits of vertices is quasi-isometric to the 3-regular tree. To see this, we note that, if $P$ is a path with $k$ edges in a tree such that all vertices of $P$ have degree 3 in that tree, then contracting $P$ to a single vertex leads to a vertex of degree $k+3$. We fix a root $u$ in the 3-regular tree $T_3$ and a root $v$ in $T$. Let $P_v$ be a path in $T_3$ starting at $u$ of length $d(v)-3$, where $d(v)$ denotes the degree of $v$. Now let us assume that for a subtree $T'$ of $T$ that contains $v$ we have the following:

(1) for each vertex $w$ of $T'$ there is a path $P_w$ in $T_3$;
(2) distinct such paths are disjoint; and
(3) $x, y \in V(T')$ are adjacent if and only if $T_3$ has an edge with one end vertex in $P_x$ and and the other in $P_w$.

We pick a vertex $w \in T - T'$ that is adjacent to a vertex $x$ of $T'$. By (3) there is a vertex $a$ in $T_3$ adjacent to $P_x$. Also (3) implies that the vertices in $T_3$ that lie on the paths $P_y$ for $y \in V(T')$ form a subtree. Hence, we find a path $P_w$ in $T_3$ that starts at $a$, is disjoint from any $P_y$ with $y \in V(T')$ and has length $d(w)-3$. We end up with a collection of paths in $T_3$, one for every vertex of $T$, that partition $V(T_3)$. Contracting these paths defines a quasi-isometry, since the paths have bounded length, and results in a tree isomorphic to $T$ due to the requirement on the lengths of the paths $P_w$. □

The last result of this section is a sharpening of the easy direction of Theorem 1.2.

**Lemma 2.9.** *A connected locally finite quasi-transitive graph with infinitely many ends that has a terminal factorisation of only finite graphs is quasi-isometric to a 3-regular tree.*

*Proof.* Let $(G_1, \ldots, G_n)$ be a terminal factorisation of a connected locally finite quasi-transitive graph $G$, where all $G_i$ are finite. First, we proof by induction on $n$ that $G$ is quasi-isometric to a tree with infinitely many ends with at most $n$ orbits on its vertex set. This suffices to prove the assertion since such trees are quasi-isometric to a 3-regular tree by Lemma 2.8. If $n = 2$, then the map $G_1 +_T G_2 \to T$ that maps the vertices in $G_i^u$ to $u$ is a quasi-isometry. So by Remark 2.1 and as the tree amalgamation of a factorisation is of finite identification, $G$ is quasi-isometric to $T$. Now let $n \geq 3$ and let $(H_1, H_2)$ be a factorisation of $G$ such that $H_1$ and $H_2$ have terminal factorisations that are subsequences of $(G_1, \ldots, G_n)$. Note that these are proper subsequences. So by induction $H_i$ is quasi-isometric to some tree $T_i$ with finitely many orbits. Let $\alpha_i \colon H_i \to T_i$ be a quasi-isometry and let $T$ be a tree such that $G = H_1 *_T H_2$. For each $\mathrm{Aut}(G)$-orbit of $E(T)$ we choose an edge $e = uv$ in it and an edge $x_u^e x_v^e \in E(H_1 + H_2)$ with $x_u^e \in H_i^u$ and $x_v^e \in H_j^v$. We extend the definition of $x_u^e$ and $x_v^e$ to all edges of $T$ in such a way that it is compatible with the action of $\mathrm{Aut}(G)$. Now we construct a new tree $T'$. For that, we replace in $T$ each vertex $u$ with a copy $T_i^u$ of $T_i$ and replace the edge $uv \in E(T)$ by an edge $\alpha_i(x_u^{uv})\alpha_j(x_v^{uv})$. By its construction and as $T$, $T_1$ and $T_2$ are trees, $T'$ is a tree, too. There are only finitely many orbits on $V(T')$ as this is true for the other three involved trees and as the construction of $T'$ respects the action of $\mathrm{Aut}(G)$. Finally, we note that the quasi-isometries $\alpha_1$ and $\alpha_2$ extend to a quasi-isometry $\alpha \colon G \to T'$ as the adhesion sets of $H_1 *_T H_2$ have bounded diameter. □

## 3. Tree amalgamations and quasi-isometries

In this section, we shall prove our first main results, Theorems 1.3 and 1.4. In preparations for that, we prove that we may assume – up to quasi-isometry – that the tree amalgamations that we consider have adhesion 1, disjoint adhesion sets and no vertices outside adhesion sets.

**Lemma 3.1.** *Let $G$ be a locally finite connected quasi-transitive graph and let $(G_1, G_2)$ be a factorisation of $G$. Then there is a locally finite connected quasi-transitive graph $H$ that has a factorisation $(H_1, H_2)$ such that the following hold.*

(1) *$G$ is quasi-isometric to $H$;*
(2) *$G_i$ is quasi-isometric to $H_i$ for $i = 1, 2$;*
(3) *$H_1 * H_2$ has adhesion 1;*
(4) *all adhesion sets of $H_1 * H_2$ are distinct;*
(5) *the adhesion sets of $H_i$ cover $H_i$ for $i = 1, 2$.*

*Proof.* We will modify the tree amalgamation $G_1 *_T G_2$ and the involved graphs $G_1, G_2$ so that the resulting graphs and tree amalgamation satisfy our assertions. We will do this step by step. I. e., first we modify them so that (1)–(3) are are true, then modify the resulting so that (1)–(4) are true and finally modify a last time to satisfy all five statements.

First, choose for each edge $uv \in E(T)$ such that $u$ gets replaced by a copy of $G_1$ when moving to $G_1 + G_2$ a vertex $x_{uv}$ in the adhesion set in $G_1^u$ belonging to this edge. We do this so that our choices are invariant under $\mathrm{Aut}(G_1 + G_2)$. Now we delete in $G_1 + G_2$ all edges between copies of $G_1$ and copies of $G_2$ except for those between $x_{uv}$ and $\varphi_{k,\ell}(x_{uv})$ where $(k, \ell) = c(uv)$. Since all adhesion sets have bounded diameter, it follows that the identity map on $V(G_1 + G_2)$ is a quasi-isometry between these two graphs. Contracting all edges between the copies of $G_1$ and $G_2$ leads to a tree amalgamation $F_1$ of $G_1$ and $G_2$ of adhesion 1 that is quasi-isometric to $G$. It follows from the choice of the vertices $x_{uv}$ that $F_1$ is quasi-transitive and $(G_1, G_2)$ is a factorisation of $F_1$. So $F_1, G_1, G_2$ satisfy (1)–(3).

Now we replace each vertex $x$ in $G_1$ that lies in some adhesion set by copies $x_1, \ldots, x_{n_x}$, where $n_x$ denotes the number of adhesion sets in $G_1$ that contain $x$. Note that $n$ is finite as the tree amalgamation has finite identification. We add

- all edges $x_i x_j$ for $1 \leq i, j \leq n_x$ with $i \neq j$ and for every $x$ in adhesion sets,
- all edges $vx_i$ for every $1 \leq i \leq n_x$ and for every edge $vx$ if $v$ lies in no adhesion set but $x$ lies in an adhesion set, and
- all edges $x_i y_j$ for all $1 \leq i \leq n_x$ and $1 \leq j \leq n_y$ and edges $xy$ if $x \neq y$ lie in adhesion sets.

Let $G_1'$ be the new graph. Let $\alpha_1 \colon G_1' \to G_1$ be the map that fixes all vertices of $G_1'$ outside of adhesion sets and maps a vertex $x_i$ to its origin $x$ otherwise. Analogously, we define $G_2'$ for the graph $G_2$ and a map $\alpha_2 \colon G_2' \to G_2$. It is easy to see that $\alpha_1$ and $\alpha_2$ are quasi-isometries. Since $G_1$ and $G_2$ are quasi-transitive, so are $G_1'$ and $G_2'$. Choose the adhesion sets of $G_i'$ such that they are mapped by $\alpha_i$ to the adhesion sets of $G_i$, have size 1, are disjoint and cover all vertices that get mapped into adhesion sets of $G_i$ by $\alpha_i$. Set $F_2 := G_1' * G_2'$. By construction, $F_2$ is quasi-transitive. Obiously, we can extend the maps $\alpha_1$ and $\alpha_2$ to a map $\alpha \colon G_1' + G_2' \to G_1 + G_2$ that is a quasi-isometry. By Remark 2.1, we obtain a quasi-isometry $F_2 \to G_1 * G_2$. Thus, $F_2, G_1', G_2'$ satisfy (1)–(4).

Let $c_i \geq 0$ such that every vertex of $G_i'$ has distance at most $c_i$ to some vertex of the adhesion sets in $G_i'$. We define a new graph $H_i$ whose vertex set consists of the vertices of $G_i'$ that lie in adhesion sets. It has edges between every two vertices that have distance at most $2c_i + 1$ in $G_i'$. Note that this condition ensures that $H_i$ is connected. The identity maps $V(H_1) \to V(G_1')$ and $V(H_2) \to V(G_2')$ are quasi-isometries that extend to a quasi-isometry $H_1 + H_2 \to G_1' + G_2'$. Since $\mathrm{Aut}(G_1'), \mathrm{Aut}(G_2')$ acts quasi-transitively on the adhesion sets in $G_1'$, in $G_2'$, respectively, the analogue is true for $H_1$ and $H_2$ since their construction is compatible with the automorphisms. Also, $H := H_1 * H_2$ is quasi-transitive and quasi-isometric to $H_1 + H_2$ by Remark 2.1. Then $H, H_1, H_2$ satisfy (1)–(5). $\square$

**Lemma 3.2.** *Let $G$ and $H$ be connected graphs. If $G$ is finite and $H$ infinite, then $G * H$ is quasi-isometric to $H * \ldots * H$, where the number of factors equals the number of adhesion sets of $G$.*

*Proof.* Let $m$ be the number of adhesion sets in $G$. Consider the map that fixes in $G + H$ all vertices in copies of $H$ and that maps the vertices in copies of $G$ to a some neighbour of that copy in a copy of $H$. This map is a quasi-isometry $G + H \to H + \ldots + H$ with $m$ summands and its image is quasi-isometric to $H * \ldots * H$ with $m$ factors. $\square$

**Theorem 3.3.** *Let $F$, $G$ and $H$ be connected quasi-transitive locally finite graphs and let $T$ and $T'$ be semi-regular trees with infinitely many ends. If $F$ and $G$ are quasi-isometric, then any locally finite tree amalgamations $F *_T H$ and $G *_{T'} H$ of finite adhesion and bounded identification are quasi-isometric.*

*Proof.* By Lemma 3.1, we may assume that the adhesion sets have size 1, are disjoint and cover $F$, $G$ and $H$.

First, we consider the case that $F$ is finite. Then $G$ is finite, too. If $H$ is finite, then with the notation used in the definition of tree amalgamations we can map $F^v$ to $v$ and $H^w$ to $w$ to obtain a quasi-isometry $F + H \to T$. Similarly, $G + H$ is quasi-isometric to $T'$. Note that finite identification implies that $T$ and $T'$ are locally finite since $F, G$ and $H$ are finite. Since $T$ and $T'$ have infinitely many ends, they are quasi-isometric by Lemma 2.8 and hence $F *_T H$ and $G *_{T'} H$ are quasi-isometric.

Now assume that $H$ is infinite, but $F$ is still finite. By Lemma 3.2, $F *_T H$ is quasi-isometric to $H * \ldots * H$ with $|F|$ factors and $G *_{T'} H$ is quasi-isometric to $H * \ldots * H$ with $|G|$ factors. By Lemma 2.5, these are quasi-isometric to $H +_{v,v} \ldots +_{v,v} H$ and $H +_{v,v} \ldots +_{v,v} H$, respectively, for some $v \in V(H)$. By Lemma 2.7, we know that these are quasi-isometric, and hence $F *_T H$ is quasi-isometric to $G *_{T'} H$.

Let us now consider the case that $F$ and thus also $G$ are infinite. We first assume that $H$ is infinite. We apply Lemma 2.5 and Lemma 2.7 to see that $F * H$ is quasi-isometric to $F * H * H$ and $G * H$ is quasi-isometric to $G * H * H$. By replacing $H$ by $H * H$, we may assume that $H$ is a non-trivial tree amalgamation.

Let $\varphi\colon F \to G$ be a quasi-isometry. Let $B$ be the image of $\varphi$ and let $A \subseteq V(F)$ such that the restriction of $\varphi$ to $A$ is a bijection $A \to B$. Note that the vertices of $F$, of $G$ have bounded distance to $A$, to $B$, respectively, as $\varphi$ is a quasi-isometry.

We consider the graph $F +_{u,v} H$, where $u \in A$ is the base vertex of $F$ and $v \in B$ is the base vertex of $H$. Let $\alpha$ map vertices inside copies $F^t$ of $F$ to copies of $A$ such that $\sup\{d(x, \alpha(x)) \mid x \in \bigcup F^t\}$ is finite and such that every base vertex is the image of itself but of no other vertex. Note that $\alpha$ is finite-to-one as the graphs are locally finite.

Let us modify $F +_{u,v} H$. We replace every edge that is incident with a vertex $x$ in a copy of $F$ and a vertex $y$ in a copy of $H$ by a new edge $\alpha(x)y$. Since $d(x, \alpha(x))$ is bounded, the new graph $X$ is quasi-isometric to $F +_{u,v} H$ and thus it is quasi-isometric to $F * H$ by Lemma 2.5. We equip $A$ with a graph structure by adding edges between any two vertices with distance at most $\sup\{2d(x, \alpha(x)) \mid x \in \bigcup F^t\}$. As noted earlier, this results in a connected graph. Then $A$ with this new metric is bilipschitz equivalent to $A$ with the metric induced by $F$. We change $C$ accordingly, i. e. we replace every copy of $F$ by a copy of $A$ with the new edges, and obtain a graph $X'$. Extending the map $\alpha$ by the identity on the copies of $H$, we obtain a quasi-isometry $F + H \to X'$.

We change $X$ in the following way: for each $a$ in a copy of $A$ we choose a neighbour $u_a$ outside of the copies of $A$ and we replace all edges $au$ with $u$ outside of copies of $A$ by $uu_a$. Let $Y$ be the resulting graph. By the choice of $\alpha$, the base vertex of every copy of $F$ has a unique neighbour outside of its copy of $F$. It follows that every component of $Y$ with all copies of $A$ deleted is a wedge of finitely many copies of $H$. As $H$ is a non-trivial tree amalgamation, Lemma 2.6 shows that each of the components of $Y$ with the copies of $A$ removed is bilipschitz equivalent to $H$. So Lemma 2.5 implies that $F * H$ is quasi-isometric to $A *_{u,v} H$. Analogously, $G * H$ is quasi-isometric to $B *_{w,v} H$, where $w \in B$ is the base vertex of $G$. Since $A$ and $B$ are bilipschitz equivalent, Lemma 2.5 implies that $A *_{u,v} H$ and $B *_{w,v} H$ are quasi-isometric and so are $F * H$ and $G * H$.

If $H$ is finite, then Lemma 3.2 implies that $F * H$ is quasi-isometric to $F * \ldots * F$ with $|H|$ copies and $G * H$ is quasi-isometric to $G * \ldots * G$ with $|H|$ copies. Then Lemmas 2.5 and 2.7 imply that $F * H$ is quasi-isometric to $F * F$ and $G * H$ is quasi-isometric to $G * G$. The previous case with both $F$ and $H$ infinite shows that $F * F$ is quasi-isometric to $F * G$ which in turn is quasi-isometric to $G * G$, which completes the proof. □

Let us prove Theorem 1.3 with the aid of Theorem 3.3.

*Proof of Theorem 1.3.* Let $G = G_1 *_T G_2$ and $H = G_1 *_{T'} G_2$. Since $G_i$ is infinite and some group acts quasi-transitively on its vertices and its adhesion sets, it follows from [5, Proposition 4.6] and the connection between tree amalgamations and tree-decompositions as discussed in [5, Section 5] that all degrees of $T$ are infinite. As the tree amalgamations are non-trivial, both $T$ and $T'$ have infinitely many ends. Thus, the assertion follows from Theorem 3.3. □

Before we turn our attention to Theorem 1.4, we prove a small lemma.

**Lemma 3.4.** *Let $G$ be an infinite locally finite quasi-transitive connected graph and let $(G_1, G_2)$ be a factorisation of $G$ such that $G_1$ is infinite, $G_2$ is finite and the amalgamating tree is not a star. Then $G$ is quasi-isometric to $G_1 * G_1$.*

*Proof.* By Lemma 3.2, $G$ is quasi-isometric to $G_1 * \ldots * G_1$ with as many factors as there are adhesion sets in $G_2$. As the amalgamating tree is not a star, this number is at least 2. Applying Theorem 3.3 repeatedly implies that $G$ is quasi-isometric to $G_1 * G_1$. □

We are going to prove a slightly more technical version of Theorem 1.4 that implies it trivially.

**Theorem 3.5.** *Let $G$ be a locally finite quasi-transitive graph with infinitely many ends and let $(G_1, \ldots, G_m)$ be a factorisation of $G$. Let $H_1, \ldots, H_n$ be infinite factors in $(G_1, \ldots G_m)$ consisting of one representatives per infinite quasi-isometry type. Then one of the following is true.*

(1) *If $H := H_1 *, \ldots * H_n$ has infinitely many ends, then $G$ is quasi-isometric to $H$.*
(2) *If $H$ does not have infinitely many ends, but there exists a finite graph $H_{\text{fin}}$ such that some non-trivial tree amalgamation $H * H_{\text{fin}}$ has infinitely many ends, then $H * H_{\text{fin}}$ is quasi-isometric to $G$.*
(3) *$G$ is quasi-isometric to a 3-regular tree.*

*Proof.* We prove the assertion by induction on the number $n$ of quasi-isometry types of $(G_1, \ldots, G_m)$. If $n = 0$, then $G$ is quasi-isometric to a 3-regular tree by Lemma 2.9. Let $n = 1$. We distinguish the cases whether $(G_1, \ldots, G_m)$ has one or more than one infinite factor. Let us first assume that $(G_1, \ldots, G_m)$ has only one infinite factor, say $G_1$. If it has no finite factor, the assertion follows immediately. So we may assume that $m \geq 2$. We may assume $G = (\ldots(G_1 * G_2) * \ldots * G_m)$. Repeatedly applying Lemma 3.4 shows that $G$ is quasi-isometric to $G_1 * \ldots * G_1$ with at least two factors. Applying Lemma 2.5 and then Lemma 2.7 repeatedly, implies that $G$ is quasi-isometric to $G_1 +_{u,u} G_1$ for the base vertex $u$ of $G_1$. Now if $G_1$ does not have infinitely many ends, we obtain analogously that for any finite graph $H_{\text{fin}}$ and non-trivial tree amalgamation $G_1 * H_{\text{fin}}$ we have that $G_1 * H_{\text{fin}}$ is quasi-isometric to $G_1 * G_1$, which shows the assertion in this case. If $G_1$ has infinitely many ends, then it has a factorisation $(G_1^1, G_1^2)$. If both factors are infinite, we can apply Lemmas 2.5 and 2.7 to see that $G_1 +_{u,u} G_1$ is quasi-isometric to $G_1$, since Lemma 2.7 shows that $(G_1^1 +_{u,v} G_1^2) +_{u,u} (G_1^1 +_{u,v} G_1^2)$ is quasi-isometric to $G_1^1 +_{u,v} G_1^2$. If $G_1^1$ is infinite but $G_1^2$ is finite, $G_1$ is quasi-isometricto $G_1^1 * G_1^1$ by Lemma 3.4. But then Lemmas 2.5 and 2.7 show that $G_1^1 * G_1^1$ is quasi-isometric to $(G_1^1 * G_1^1) +_{u,u} (G_1^1 * G_1^1)$, which is quasi-isometric to $G_1 * G_1$ by Lemma 2.5. If $G_1^1$ and $G_1^2$ are finite, then $G_1$ is quasi-isometric to a 3-regular tree by Lemma 2.9 and so is $G_1 * G_1$. So by Lemma 2.8, $G_1$ is quasi-isometric to $G_1 * G_1$. Thus, we have shown the assertion in the case that $(G_1, \ldots, G_m)$ has only one infinite factor.

So let us assume that $(G_1, \ldots, G_m)$ has more than one infinite factor. Since $n = 1$, all of them are in the same quasi-isometry class. By Theorem 3.3 and Lemma 3.4, we may assume that $G = H_1 * \ldots * H_1$. Lemmas 2.5 and 2.7 implies that $G$ is quasi-isometric to $H_1 * H_1$. If $H_1$ is one-ended, let $H_{\text{fin}}$ be a finite graph and $T$ be a semi-regular tree with infinitely many ends such that $H_{\text{fin}} *_T H_1$ has infinitely many ends. It follows that $H_{\text{fin}} *_T H_1$ is non-trivial. By Lemma 3.4, $H_{\text{fin}} *_T H_1$ is quasi-isometric to $H_1 * H_1$. So $H_{\text{fin}} *_T H_1$ is quasi-isometric to $G$.

If $H_1$ has more than one end, then it splits as a tree amalgamation $H^1 * H^2$ by Theorem 1.1. If $H^1$ and $H^2$ are finite, then $H = H_1$ is quasi-isometric to a 3-regular tree and so is $G$ by Lemma 2.9. Hence, $G$ is quasi-isometric to $H$. If $H^1$ and $H^2$ are infinite, then $H_1 = H^1 * H^2$ is quasi-isometric to $(H^1 * H^1) * (H^2 * H^2) = H_1 * H_1$ by Lemmas 2.5 and 2.7. Theorem 3.3 implies that $G$ is quasi-isometric to $G_{\text{fin}} * H_1$ for any finite $G_{\text{fin}}$ such that $G_{\text{fin}} * H_1$ is non-trivial. If $H^1$ is finite and $H^2$ is infinite, then Lemma 3.4 implies that $H_1$ is quasi-isometric to $H^2 * H^2$. So Lemmas 2.5 and 2.7 imply that $H_1$ is quasi-isometric to $H_1 * H_1$. As $G$ is quasi-isometric to $H_1 * H_1$, it is quasi-isometric to $H_1$.

Let us now assume that $n \geq 2$. For $i = 0, \ldots, n$, let $G_i^1, \ldots, G_i^{k_i}$ be the factors of $(G_1, \ldots, G_m)$ that are quasi-isometric to $H_i$, where $H_0$ is any finite graph. Then

$$G = G_0^1 * \ldots * G_0^{k_0} * \ldots * G_n^1 * \ldots * G_n^{k_n}.$$

If $n \geq 3$ or $k_1 \geq 2$, then $G' := G_1^1 * \ldots * G_{n-1}^{k_{n-1}}$ has infinitely many ends and by induction it is quasi-isometric to either $H_1 * \ldots * H_{n-1}$ or $G_{\text{fin}} * H_1$ for any finite $G_{\text{fin}}$ such that $G_{\text{fin}} * H_1$ is non-trivial. Lemma 3.4 shows that we can replace each finite factor by $H_1$ and thus $G'$ is quasi-isometric to either $H_1 * \ldots * H_{n-1}$ or $H_1 * H_1$. By Theorem 3.3 we may assume that $G_n^j = H_n$ for every $1 \leq j \leq k_n$. By applying Lemmas 2.5 and 2.7 we reduce the multiple factors of $H_1$ and $H_n$ to just one each and obtain that $G$ is quasi-isometric to either $H$ or $H_1 * H_1 * \ldots * H_n$, where another application of Lemmas 2.5 and 2.7 shows that $H_1 * H_1 * \ldots * H_n$ is quasi-isometric to $H$. □

## 4. Accessibility and quasi-isometries

In this section we shall prove Theorem 1.5. The first step is to see that if a one-ended connected quasi-transitive locally finite graph embeds quasi-isometrically into a tree amalgamation, then it embeds already quasi-isometrically into one of the factors.

**Lemma 4.1.** *Let $G$ and $H$ be connected quasi-transitive locally finite graphs and let $(G_1, G_2)$ be a factorisation of $G$. If $H$ has precisely one end and embeds quasi-isometrically into $G$, then it embeds quasi-isometrically into either $G_1$ or $G_2$.*

*Proof.* Let $\varphi\colon H \to G$ be a $(\gamma, c)$-quasi-isometric embedding. Let $S$ be an adhesion set in $G$ and let $S'$ be the set of vertices of $G$ of distance at most $\gamma + c$. If there are vertices of $\varphi(H)$ in different components of $G - S'$, then their preimages are not connected in $H - \varphi^{-1}(S')$ by the choice of $S'$ and as $H$ is one-ended and $S'$ finite, there is only one infinite component $C_S^\infty$ of $H - \varphi^{-1}(S')$ and only finitely many finite components. So we find $\Delta_S$ such that the images of all vertices of finite components of $H - \varphi^{-1}(S')$ have distance at most $\Delta_S$ from $S$. Note that since we have only finitely many orbits of adhesion sets, the numbers $\Delta_S$ are globally bounded by some $\Delta$.

Let $uv$ be an edge of $T$. Then $G_i^u \cap G_j^v$ is an adhesion set. The infinite component of $C^\infty_{G_i^u \cap G_j^v}$ gets mapped into either

$$G_{T_u} := \bigcup_{a \in V(T_u)} G_i^a \qquad \text{or} \qquad G_{T_v} := \bigcup_{a \in V(T_v)} G_i^a$$

but not into both, where $T_w$ is the component of $T - uv$ that contains $w$ for $w \in \{u, v\}$. We orient the edge $uv$ towards $u$ if the infinite component gets mapped into $G_{T_u}$ and we orient it towards $v$ otherwise. It is easy to see that every vertex has at most one out-going edge and that there is at most one vertex without outgoing edges in this orientation of $T$. To see that there is at least one sink, let us suppose that this is not the case. Then there is a directed ray in the orientation of $T$. As $G_1 * G_2$ has finite identification, every $\varphi(a)$ for $a \in V(H)$ lies eventually outside the adhesion sets on that ray and for every adhesion set on that ray, we find a later one that is disjoint from it. Obviously, there is an adhesion set $S$ on that ray separating $\varphi(a)$ from $C^\infty_S$. But this is impossible as $a$ must lie eventually within the $\Delta$-neighbourhood of all later adhesion sets of that ray. So $T$ has a unique sink $G_i^x$ such that every vertex of $H$ gets mapped by $\varphi$ into the $\Delta$-neighbourhood of $G_i^x$. We can easily modify $\varphi$ and obtain a maps $\varphi'$ that maps $H$ quasi-isometrically into $G_i^x$ with respect to the metric of $G$. But since $G_i^x$ has only finitely many orbits of adhesion sets, $\varphi'$ also maps $H$ quasi-isometrically into $G_i^x$ with the metric of $G_i^x$. □

*Proof of Theorem 1.5.* Let us first assume that $H$ satisfies (i) to (iii). If $G$ and $H$ have infinitely many ends, then it follows directly from Theorem 1.4 that $G$ is quasi-isometric to $H$. If they have two ends, then no factor can be one-ended, so all are finite and hence $G$ and $H$ are quasi-isometric to the double ray and thus quasi-isometric to each other. If they have one end, then each of the two terminal factorisations has at most one one-ended factor and all others are finite. Moreover, a tree amalgamation of a one-ended graph and a finite graph is one-ended only if the non-trivial amalgamation tree is a star, i. e. a tree of diameter at most 2. For such tree amalgamations the finite extension of the one-ended factor by the finite factor is just the tree amalgamation itself and thus the tree amalgamation is quasi-isometric to the one-ended factor by Remark 2.2. Thus, $G$ is also quasi-isometric to $H$ in this case.

Now let us assume that $G$ is quasi-isometric to $H$. We refer to [5, Theorem 6.3] to see that accessibility is preserved by quasi-isometries. Since the number of ends is preserved by quasi-isometries, too, it only remains to prove (iii). Let $(G_1, \ldots, G_n)$ be a terminal factorisation of $G$. Then there exist $F_1, \ldots, F_{n-1}$ such that $G = F_{n-1}$ and such that for every $i \le n-1$ the graph $F_i$ is a tree amalgamation of finite adhesion with factors in

$$\{G_j \mid 1 \le j \le n\} \cup \{F_j \mid 1 \le j < i\}.$$

Let $H'$ be a factor in a terminal factorisation $(H_1, \ldots, H_m)$ of $H$. By Remark 2.1, $H'$ maps quasi-isometrically into $H$ and thus into $G$. Applying Lemma 4.1 recursively, we conclude that for some $G_i$ there is a quasi-isometric embedding $\varphi\colon H' \to G_i$. Similarly, $G_i$ embeds quasi-isometrically into some factor $F$ of the terminal factorisation $(H_1, \ldots, H_m)$ of $H$ by a map $\psi$ with $\psi \circ \varphi = id$. Since $\psi \circ \varphi = id$, we know that $F$ must be mapped by $\psi \circ \varphi$ into $H'$. As both are one-ended, we conclude $F = H'$. Thus, $H'$ is quasi-isometric to $G_i$. □

## 5. Quasi-isometries between transitive graphs and Cayley graphs

Woess [10, Problem 1] asked whether there are transitive locally finite graphs that are not quasi-isometric to some locally finite Cayley graph. Eskin et al. [2] showed that the Diestel-Leader graphs are examples of transitive graphs that are not quasi-isometric to some locally finite Cayley graph. Since the Diestel-Leader graphs are one-ended, the question arises what can be said about Woess' question for graphs that need not have one-ended graphs as building blocks in our tree amalgamation sense, i. e. inaccessible graphs. Dunwoody [1] constructed an inaccessible locally finite transitive graphs that is another example for a negative answer to Woess' question. As an application of our previous results, we obtain that one-ended and inaccessible examples are basically the only ones you have to consider when understanding the quasi-isometry differences between locally finite transitive graphs and locally finite Cayley graphs.

**Theorem 5.1.** *Let $G$ be a locally finite transitive accessible graph that is not quasi-isometric to any locally finite Cayley graph. Then there is a one-ended locally finite transitive graph that is quasi-isometric to some factor in a terminal factorisation of $G$ and that is not quasi-isometric to any Cayley graph.*

*Proof.* If $G$ has precisely two ends, then it is quasi-isometric to the double ray by Theorem 1.2, which is a locally finite Cayley graph. So we may assume that $G$ has infinitely many ends. Let $(G_1, \ldots, G_n)$ be a terminal factorisation of $G$. Note that every $G_i$ is quasi-transitive and thus quasi-isometric to some transitive locally finite graph, see e. g. Krön and Möller [6, Theorem 5.1]. Suppose that every $G_i$ is quasi-isometric to some locally finite Cayley graph $H_i$. Then $(G_1, \ldots, G_n)$ and $(H_1, \ldots, H_n)$ have the same quasi-isometry types of infinite factors. Let $\Gamma_i = \langle S_i \mid R_i \rangle$ be a group that has $H_i$ as Cayley graph. Let $\Gamma$ be the free product of all $\Gamma_i$ with presentation $\langle S \mid R \rangle$, where $S = \bigcup_{1 \leq i \leq n} S_i$ and $R = \bigcup_{1 \leq i \leq n} R_i$. Then the Cayley graph $H$ of $\Gamma$ with respect to $\langle S \mid R \rangle$ is the tree amalgamation of all $H_i$, i. e. $H = (\ldots(H_1 * H_2) * \ldots) * H_n$. By Theorem 1.4, $G$ is quasi-isometric to $H$, which is a contradiction. □

We can also restrict Woess' question to certain classes of graphs and groups. If this class is invariant under quasi-isometries, factorisations and tree amalgamations, then the proof of Theorem 5.1 stays true for it.

One such class are the hyperbolic graphs and group: a graph is *hyperbolic* if there is some $\delta \geq 0$ such that for every three vertices $x, y, z$ and every three paths, one between each pair of $\{x, y, z\}$, each of the paths lies in the $\delta$-neighbourhood of the union of the other two paths; and a finitely generated group is *hyperbolic* if it has a locally finite hyperbolic Cayley graph.

It follows from the definition that hyperbolicity is preserved under quasi-isometries. By [5, Theorem 7.10], hyperbolicity is also preserved under factorisations and tree amalgamations. Additionally, hyperbolic quasi-transitive locally finite graphs are accessible by [3, Theorem 4.3]. Thus, if there is a hyperbolic locally finite quasi-transitive graph that shows that Woess' question is false, then there is already a one-ended such graph.

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