

TREE AMALGAMATIONS AND HYPERBOLIC BOUNDARIES

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ABSTRACT. We look at tree amalgamations of locally finite quasi-transitive hyperbolic graphs and prove that the homeomorphism type of the hyperbolic boundary of such a tree amalgamation only depends on the homeomorphism types of the hyperbolic boundaries of their factors. Additionally, we show that two locally finite quasi-transitive hyperbolic graphs have homeomorphic hyperbolic boundaries if and only if the homeomorphism types of the hyperbolic boundaries of the factors of their terminal factorisations coincide.

1. INTRODUCTION

Tree amalgamations offer a way to construct new graphs out of existing ones similar as new groups can be constructed via free products with amalgamation or HNN-extensions. (We refer to Section 2.2 for the definition of tree amalgamations.) In order to investigate geometric properties of multi-ended quasi-transitive graphs, it is therefore interesting to see how such properties behave with respect to tree amalgamations. In this paper, we are looking at the interaction of tree amalgamations with hyperbolicity. Hyperbolic groups, graphs or spaces play an important role since Gromov's paper [6]. A first observation in [9] is the following.

Theorem 1.1. [9, Theorem 7.10] *Two connected locally finite quasi-transitive graphs are hyperbolic if and only if any of their tree amalgamations of bounded adhesion is hyperbolic.* \square

Hyperbolic graphs are equipped with a natural boundary, the hyperbolic boundary. Our first main result says essentially that in a tree amalgamation of hyperbolic graphs changing the factors without changing the homeomorphism types of their hyperbolic boundary still leads to a tree amalgamation whose hyperbolic boundary is homeomorphic to the original one. For this, a *factorisation* of a quasi-transitive graph G is a tuple (G_1, \dots, G_n) such that G is obtained by iterated tree amalgamations of all the graphs G_i .

Theorem 1.2. *Let $(G_1, \dots, G_n), (H_1, \dots, H_m)$ be factorisations of infinitely-ended quasi-transitive locally finite hyperbolic graphs G, H , respectively, such that the set of homeomorphism types of the hyperbolic boundaries of the factors in (G_1, \dots, G_n) are the same as those for (H_1, \dots, H_m) . Then the hyperbolic boundaries ∂G and ∂H are homeomorphic.*

The obvious question that arises is what can be said about the reverse implication of Theorem 1.2? While it is false in general, we will prove that it holds if we ask it for *terminal factorisations*. These are factorisations (G_1, \dots, G_n) , where each G_i has at most one end. In [7] it was shown that quasi-transitive locally finite

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hyperbolic graphs are accessible in the sense of Thomassen and Woess [13]. Thus, they have a terminal factorisation by [9, Theorem 6.3].

Theorem 1.3. *Let $(G_1, \dots, G_n), (H_1, \dots, H_m)$ be terminal factorisations of infinitely-ended quasi-transitive locally finite hyperbolic graphs G, H , respectively. Then the hyperbolic boundaries ∂G and ∂H are homeomorphic if and only if the set of homeomorphism types of the hyperbolic boundaries of the factors in (G_1, \dots, G_n) are the same as those for (H_1, \dots, H_m) .*

Martin and Świątkowski [11] proved group theoretic versions of Theorems 1.2 and 1.3. However, our results do not follow from theirs since it is not known whether every locally finite hyperbolic quasi-transitive graph is quasi-isometric to some hyperbolic group.

The question whether every locally finite hyperbolic quasi-transitive graph is quasi-isometric to some hyperbolic group is a special case of Woess' problem [14, Problem 1] whether every locally finite transitive graph is quasi-isometric to some locally finite Cayley graph. While his problem was settled in the negative by Eskin et al. [4], their counterexamples, the Diestel-Leader graphs, are not hyperbolic and neither is another counterexample by Dunwoody [3].

In Section 2.1 we define and discuss hyperbolicity and in Section 2.2 tree amalgamations. In both sections, we also state the main preliminary results we need for our main results, which we will prove in Section 3.

2. PRELIMINARIES

In this section, we state the main definitions and preliminary results that we need for the proofs of Theorems 1.2 and 1.3. First, we state some general definitions and then we look at hyperbolic graphs and tree amalgamations more closely in Sections 2.1 and 2.2, respectively.

Let G be a graph. A *ray* is a one-way infinite path and a *double ray* is a two-way infinite path. Two rays are *equivalent* if for every finite vertex set $S \subseteq V(G)$ both rays have all but finitely many vertices in the same component of $G - S$. This is an equivalence relation whose equivalence classes are the *ends* of G .

We call G *transitive* if its automorphism group acts transitively on its vertex set and *quasi-transitive* if its automorphism group acts with only finitely many orbits on $V(G)$.

Let G and H be graphs. A map $\varphi: V(G) \rightarrow V(H)$ is a *quasi-isometry* if there are constants $\gamma \geq 1, c \geq 0$ such that

$$\gamma^{-1}d_G(u, v) - c \leq d_H(\varphi(u), \varphi(v)) \leq \gamma d_G(u, v) + c$$

for all $u, v \in V(G)$ and such that $\sup\{d_H(v, \varphi(V(G))) \mid v \in V(H)\} \leq c$. We then say that G is *quasi-isometric* to H .

A finite or infinite path P is *geodesic* if $d_P(x, y) = d_G(x, y)$ for all vertices x, y on P . It is a (γ, c) -*quasi-geodesic* if it is the (γ, c) -quasi-isometric image of a subpath of a geodesic double ray.

2.1. Hyperbolic graphs. In this section, we will give the definitions and state the lemmas regarding hyperbolic graphs that we need for our results. For a detailed introduction to hyperbolic graphs, we refer to [2, 5, 6].

Let G be a graph and $\delta \geq 0$. If for all $x, y, z \in V(G)$ and all shortest paths P_1, P_2, P_3 , one between every two of those vertices, every vertex of P_1 has distance

at most δ to some vertex on either P_2 or P_3 then G is δ -hyperbolic. We call G hyperbolic if it is δ -hyperbolic for some $\delta \geq 0$.

Two geodesic rays in a hyperbolic graph G are *equivalent* if there is some $M \in \mathbb{N}$ such that on each ray there are infinitely many vertices of distance at most M to the other ray. This is an equivalence relation for hyperbolic graphs whose equivalence classes are the *hyperbolic boundary points* of G . By ∂G we denote the *hyperbolic boundary* of G , i. e. the set of hyperbolic boundary points of G , and we set $\widehat{G} := G \cup \partial G$.

For locally finite hyperbolic graphs, it is possible to equip \widehat{G} with a topology such that \widehat{G} is compact and every geodesic ray converges to the hyperbolic boundary point it is contained in, see [5, Proposition 7.2.9]. For us, it suffices to define convergence of vertex sequences to hyperbolic boundary points. Let $o \in V(G)$. Let $(x_i)_{i \in \mathbb{N}}$ be a sequence in $V(G)$. It *converges* to $\eta \in \partial G$ if for some geodesic ray $y_1 y_2 \dots$ and some sequence $(n_i)_{i \in \mathbb{N}}$ that goes to ∞ any geodesic path from x_i to y_i has distance at least n_i to o .

Since the rays in a tree are always geodesic and two rays that are equivalent regarding the definition of ends eventually coincide, these rays are also equivalent with respect to the definition of the hyperbolic boundary. Thus, there is a canonical one-to-one correspondence between the ends of trees and their hyperbolic boundary. The following lemma follows easily.

Lemma 2.1. *The hyperbolic boundaries of locally finite trees are totally disconnected sets.* □

By its definition, the hyperbolic boundary is a refinement of the end space. But even more can be said about this relation, cf. e. g. [10, Section 7]:

Lemma 2.2. *The connected components of the hyperbolic boundary of every locally finite hyperbolic graph correspond canonically to the ends of that graph.* □

We are interested in homeomorphism types of hyperbolic boundaries. That is why the following result is important for us.

Lemma 2.3. [1, III.H Theorem 3.9] *Quasi-isometries between locally finite hyperbolic graphs induce canonical homeomorphisms between their boundaries.* □

The following lemma is a direct consequence of Woess [15, Corollary 5].

Lemma 2.4. *The hyperbolic boundary of every one-ended quasi-transitive locally finite hyperbolic graph is infinite.* □

2.2. Tree-amalgamations. In this section, we state the definition of tree amalgamations and cite several results about them.

Let $p_1, p_2 \in \mathbb{N} \cup \{\infty\}$. A tree is *semiregular* or (p_1, p_2) -*semiregular* if all vertices in V_1 have the same degree p_1 and all vertices in V_2 have the same degree p_2 , where V_1, V_2 is the canonical bipartition of its vertex set.

Let G_1 and G_2 be two graphs and let T be a (p_1, p_2) -semiregular tree with canonical bipartition V_1, V_2 of its vertex set. Let $\{S_k^i \mid 0 \leq k < p_i\}$ be a set of subsets of $V(G_i)$ such that all S_k^i have the same cardinality and let $\varphi_{k,\ell}: S_k^1 \rightarrow S_\ell^2$ be a bijection. Let

$$c: E(T) \rightarrow \{(k, \ell) \mid 0 \leq k < p_1, 0 \leq \ell < p_2\}$$

such that for all $v \in V_i$ the i -th coordinates of the elements of $\{c(e) \mid v \in e\}$ exhaust the set $\{k \mid 0 \leq k < p_i\}$.

For every $v \in V_i$ with $i = 1, 2$, let G_i^v be a copy of G_i and let S_k^v be the copy of S_k^i in G_i^v . Let $H := G_1 + G_2$ be the graph obtained from the disjoint union of all graphs G_i^v by adding an edge between all $x \in S_k^v$ and $\varphi_{k,\ell}(x) \in S_\ell^u$ for every edge $vu \in E(T)$ with $v \in V_1$ and $c(vu) = (k, \ell)$. Let G be the graph obtained from H by contracting all new edges $x\varphi_{k,\ell}(x)$, i. e. all edges outside of the graphs G_i^v . We call G the *tree amalgamation* of G_1 and G_2 over T (with respect to the sets S_k and the maps $\varphi_{k,\ell}$) and we denote it by $G_1 *_T G_2$. If the *amalgamation tree* T is clear from the context, we simply write $G_1 *_T G_2$. The sets S_k and their copies in G are the *adhesion sets* of the tree amalgamation. If the supremum of the diameter of all adhesion sets of a tree amalgamation is finite, then this tree amalgamation has *bounded adhesion*. Let $\psi: V(H) \rightarrow V(G)$ be such that every $x \in V(H)$ is mapped to the vertex of G it ends up after all contractions. A tree amalgamation $G_1 *_T G_2$ is *trivial* if there is some G_i^v such that the restriction of ψ to G_i^v is a bijection $G_i^v \rightarrow G_1 *_T G_2$. So a tree amalgamation of finite adhesion is trivial if $V(G_i)$ is the only adhesion set of G_i and $p_i = 1$ for some $i \in \{1, 2\}$.

The *identification size* of a vertex $x \in V(G)$ is the smallest size of subtrees T' of T such that x is obtained by contracting only edges between vertices in $\bigcup_{u \in V(T')} V(G_j^u)$. The tree amalgamation has *finite identification* if the supremum of all identification sizes is finite.

In [9, Lemma 7.7], it was shown that for a tree amalgamation $G *_T H$ of bounded adhesion if R is a geodesic ray in G , then its image in some G^u in $G *_T H$ is a quasi-geodesic ray. We state a strengthened version of that result for the case that the tree amalgamation is of adhesion 1. The proof of Lemma 2.5 is contained in the proof of [9, Lemma 7.7].

Lemma 2.5. *Let G, H be connected locally finite graphs and $G *_T H$ a tree amalgamation of adhesion 1. Then every geodesic in G is a geodesic in $G *_T H$. \square*

All following results in this section deal with the interplay of tree amalgamations with quasi-isometries and are proved in [8].

Lemma 2.6. [8, Remark 2.1] *Let G and H be locally finite graphs and $G *_T H$ be a tree amalgamation of finite identification and bounded adhesion. Then $G *_T H$ is quasi-isometric to $G + H$. \square*

The following lemma enables us to change the factors in a tree amalgamation a bit while staying quasi-isometric to the original tree amalgamation but in the result we have more control over the adhesion sets and identification sizes.

Lemma 2.7. [8, Lemma 3.1] *Let G be a locally finite connected quasi-transitive graph and let (G_1, G_2) be a factorisation of G . Then there is a locally finite connected quasi-transitive graph H that has a factorisation (H_1, H_2) such that the following hold.*

- (1) G is quasi-isometric to H ;
- (2) G_i is quasi-isometric to H_i for $i = 1, 2$;
- (3) $H_1 *_T H_2$ has adhesion 1;
- (4) all adhesion sets of $H_1 *_T H_2$ are distinct;
- (5) the adhesion sets of H_i cover H_i for $i = 1, 2$. \square

Lemma 2.8. [8, Lemma 2.9] *A connected locally finite quasi-transitive graph that has a terminal factorisation of only finite graphs is quasi-isometric to a 3-regular tree. \square*

The following theorem plays a central role in our proofs and can be seen as an analogue of Theorem 1.2 for quasi-isometries of graphs instead of homeomorphisms of hyperbolic boundaries.

Theorem 2.9. [8, Theorem 1.4] *Let G and H be locally finite quasi-transitive graphs with infinitely many ends and let $(G_1, \dots, G_n), (H_1, \dots, H_m)$ be factorisations of G, H , respectively. If (G_1, \dots, G_n) and (H_1, \dots, H_m) have the same set of quasi-isometry types of infinite factors, then G and H are quasi-isometric. \square*

3. PROOFS OF THE MAIN THEOREMS

In this section we will prove our main results. But we need some preliminary results first.

The following lemma is a special case of a result of Steiner and Steiner [12, Theorem 4]. It is also possible adapt the proof of Martin and Świątkowski [11, Lemma 4.2] to our situation to obtain that lemma.

Lemma 3.1. *Let G, H be locally finite quasi-transitive hyperbolic graphs and let $f: \partial G \rightarrow \partial H$ be a homeomorphism. Then f extends to a homeomorphism $\widehat{G} \rightarrow \widehat{H}$. \square*

The next lemma describes the hyperbolic boundary of a tree amalgamation in terms of its factors and the involved tree.

Lemma 3.2. *Let G and H be locally finite hyperbolic quasi-transitive graphs and let T be a semiregular tree with canonical bipartition $\{U, V\}$ of its vertex set. Then there exists a canonical bijective map*

$$f: \partial T \cup \bigcup_{u \in U} \partial G^u \cup \bigcup_{v \in V} \partial H^v \rightarrow \partial(G +_T H)$$

such that the following hold.

- (1) *The preimage of each connected component of $\partial(G +_T H)$ is a connected component of an element of*

$$\{\partial T\} \cup \{\partial G^u \mid u \in U\} \cup \{\partial H^v \mid v \in V\};$$

- (2) *every sequence in some G^u or H^v that converges to some boundary point $\eta \in \bigcup_{u \in U} \partial G^u \cup \bigcup_{v \in V} \partial H^v$ converges to $f(\eta)$ in $G +_T H$;*
- (3) *every sequence $(v_i)_{i \in \mathbb{N}}$ with $v_i \in G^{t_i}$ or $v_i \in H^{t_i}$ such that $(t_i)_{i \in \mathbb{N}}$ converges to $\eta \in \partial T$ converges to $f(\eta)$ in $G +_T H$.*

Proof. Since quasi-isometries preserve hyperbolicity, we may apply Lemmas 2.7 and 2.6 to assume that $G *_T H$ is a tree amalgamation of adhesion 1 and distinct adhesion sets are disjoint. Note for this that quasi-isometries map distinct boundary points to distinct boundary points and distinct connected components of the boundary to distinct connected components.

Let us define a map

$$f: \partial T \cup \bigcup_{u \in U} \partial G^u \cup \bigcup_{v \in V} \partial H^v \rightarrow \partial(G +_T H).$$

Let $u \in U$ and $\eta \in \partial G^u$. Any geodesic ray in η is a geodesic ray in $G +_T H$ as well by Lemma 2.5. Thus, two geodesic rays in $G *_T H$ that lie in G^u are equivalent in $G +_T H$ and thus lie in the same boundary point μ . We set $f(\eta) := \mu$. Analogously, we define the image of elements of ∂G^v for $v \in V$.

Now we consider a boundary point η of T . Note that since T is a tree, its boundary points are just its ends. Let R be a ray in η . Since the tree amalgamation has adhesion 1, there is for each edge uv of T with $u \in U$ and $v \in V$ a unique edge of $G +_T H$ that corresponds to uv in that its incident vertices lies in G^u and H^v and get identified when constructing the tree amalgamation. Since distinct adhesion sets are disjoint, it follows that for a subpath $u_0u_1u_2u_3$ of R the edges e_1, e_2, e_3 , where e_i corresponds to the edge $u_{i-1}u_i$, have the properties that they are distinct and e_2 separates e_1 and e_3 . By joining e_1 and e_2 by a shortest path inside G^{u_1} or H^{u_1} , we obtain that R defines a geodesic ray and no matter how we choose the shortest paths to connected the edges e_1, e_2 , the resulting rays are equivalent and thus converge to the same boundary point μ of $G +_T H$. We set $f(\eta) := \mu$.

While defining f , we ensured that it is well-defined. Let us show that f is injective. If we consider hyperbolic boundary points η, μ of distinct elements of

$$X := \{\partial T\} \cup \{\partial G^u \mid u \in U\} \cup \{\partial H^v \mid v \in V\},$$

then each of η, μ belongs to either a hyperbolic boundary point of T or the hyperbolic boundary of a vertex of T and there is an edge of T separating these hyperbolic boundary points or vertices of T . The edge of $G +_T H$ corresponding to that edge of T separates the f -image of those hyperbolic boundary points or of the subgraph G^u or H^v . Thus, $f(\eta)$ and $f(\mu)$ lie in distinct ends of $G +_T H$ and hence are distinct. If η and μ are distinct but in a common element of X , then they lie in either ∂G^u or ∂H^v for some $u \in U$ or $v \in V$. But as geodesic paths and rays in G^u or H^v are geodesic paths and rays in $G *_T H$, inequivalent rays in G^u or H^v stay inequivalent in $G +_T H$. Thus, we have $f(\eta) \neq f(\mu)$ in this case, too.

To show that f is surjective, let $\eta \in \partial(G +_T H)$ and let R be a geodesic ray in η . Since the adhesion of $G *_T H$ is 1, there is either a subgraph G^u or H^v such that R has all but finitely many vertices of that subgraph or not. If R meets every such subgraph in only finitely many vertices, let $W \subseteq V(T)$ consist of those vertices $u \in U$ and $v \in V$ for which R meets G^u and H^v . Note that if R leaves G^u or H^v once, it has to do so through an adhesion set and since it cannot use the same edge again, it cannot enter the subgraph G^u or H^v anymore. Thus, W defines a ray in T and it is straight-forwards to see that the hyperbolic boundary point μ of T that contains this ray is mapped to η by f . If R has infinitely many vertices in a subgraph G^u or H^v , say G^u , then it has a subray in G^u by the above argument that if it leaves G^u once, it never reenters G^u . This subray is geodesic in G^u since the adhesion sets have size 1. It lies in some hyperbolic boundary point μ of G^u and we have $f(\mu) = \eta$ by construction.

So far, we constructed a canonical bijective map f that satisfies (2) and (3). It remains to verify (1). For this, we note that the connected components of the hyperbolic boundary correspond to the ends of the graph by Lemma 2.2. Analogously as in the proof that f is injective, it follows that distinct hyperbolic boundary points in the same end of $G +_T H$ are mapped to hyperbolic boundary points of some G^u or H^v that lie in the same end of that graph. \square

The following proposition is the main step towards the proof of Theorem 1.2.

Proposition 3.3. *Let G_1, G_2, H_1, H_2 be locally finite infinite hyperbolic quasi-transitive graphs such that $\partial G_1, \partial G_2$ is homeomorphic to $\partial H_1, \partial H_2$, respectively. Let $G_1 * G_2$ and $H_1 * H_2$ be tree amalgamations of bounded adhesion and finite identification. Then $\partial(G_1 * G_2)$ and $\partial(H_1 * H_2)$ are homeomorphic.*

Proof. According to Lemma 2.7, we find locally finite connected quasi-transitive graphs G'_1, G'_2, H'_1, H'_2 such that G_i, H_i is quasi-isometric to G'_i , to H'_i for $i = 1, 2$, such that $(G'_1, G'_2), (H'_1, H'_2)$ are factorisations of graphs G, H that are quasi-isometric to tree amalgamations $G_1 * G_2, H_1 * H_2$, respectively, such that $G'_1 * G'_2$ and $H'_1 * H'_2$ have adhesion 1, all their adhesion sets are disjoint and every vertex lies in some adhesion set. As hyperbolicity is preserved by quasi-isometries, G'_1, G'_2, H'_1, H'_2 are hyperbolic. By Theorem 1.1, G and H are hyperbolic, too. Since quasi-isometric hyperbolic graphs have homeomorphic hyperbolic boundaries by Lemma 2.3, the hyperbolic boundaries $\partial(G_1 * G_2), \partial(H_1 * H_2)$ are homeomorphic to $\partial G, \partial H$, respectively. Thus, it suffices to prove the assertion under the assumption that the tree amalgamation is of adhesion 1, all adhesion sets are disjoint and the adhesion sets cover all vertices. By Lemma 2.6 the graphs $G_1 * G_2, H_1 * H_2$ are quasi-isometric to $G_1 + G_2, H_1 + H_2$, respectively. Thus, it suffices to consider those graphs instead of the tree amalgamations.

For $i = 1, 2$, let $g_i: \partial G_i \rightarrow \partial H_i$ be a homeomorphism and let $f_i: \widehat{G}_i \rightarrow \widehat{H}_i$ be a homeomorphism that extends g_i , which exists by Lemma 3.1. Let S_k^G, S_k^H be the adhesion sets in G_1 , in H_1 and T_ℓ^G, T_ℓ^H be the adhesion sets in G_2 , in H_2 , respectively. Note that the amalgamation trees in both cases are the countably infinitely regular tree, i. e. we consider the tree amalgamations $G_1 *_T G_2$ and $H_1 *_T H_2$. Let c_G, c_H be the labelings of the edges of T used for the tree amalgamations $G_1 *_T G_2$, for $H_1 *_T H_2$, respectively. We are going to construct a map $f: G_1 + G_2 \rightarrow H_1 + H_2$ with the following properties, which obviously proves the assertion.

- (1) f is a bijection;
- (2) f induces an automorphism f_t of T ;
- (3) f induces homeomorphisms $\widehat{G}_i^u \rightarrow \widehat{H}_i^{f_t(u)}$;
- (4) f induces a homeomorphism $g: \partial(G_1 + G_2) \rightarrow \partial(H_1 + H_2)$.

The homeomorphisms in (3) will be closely related to the homeomorphisms f_i .

Let v_1, v_2, \dots be an enumeration of $V(T)$ such that for every $i \in \mathbb{N}$ the vertices v_1, \dots, v_i induce a subtree of T . Let $f_{i_1}^{v_1}$ be the map $G_{i_1}^{v_1} \rightarrow H_{i_1}^{v_1}$ that is induced by f_{i_1} . For $j > 1$, let u_j be the vertex in $G_{i_j}^{v_j}$ that separates $G_{i_j}^{v_j}$ from $G_{i_1}^{v_1}$ and let $t \in \{v_1, \dots, v_{j-1}\}$ be the unique neighbour of v_j in that set. Let $g_{i_j}^{v_j}$ be the map $G_{i_j}^{v_j} \rightarrow H_{i_j}^{v_j}$ induced by f_i .

For $j > 1$, let w_j be the vertex in $H_{i_j}^{v_j}$ that separates $H_{i_j}^{v_j}$ from $H_{i_1}^{v_1}$. Let $h_{i_j}^{v_j}$ be $g_{i_j}^{v_j}$ but the images of u_j and $(g_{i_j}^{v_j})^{-1}(w_j)$ exchanged. Note that $h_{i_j}^{v_j}$ still induces a homeomorphism $\widehat{G}_{i_j}^{v_j} \rightarrow \widehat{H}_{i_j}^{v_j}$ but now the vertex in $G_{i_3-j}^t$ that is adjacent to u_j in the graph $G_1 + G_2$ is mapped by $h_{i_3-j}^t$ to the neighbour of w_j in $H_{i_3-j}^t$ in the graph $H_1 + H_2$. Hence, the union of all h_i^v for all $v \in V(T)$ defines a map $f: G_1 + G_2 \rightarrow H_1 + H_2$ that maps vertices of $G_1 + G_2$ that gets identified by constructing $G_1 * G_2$ to those that gets identified by constructing $H_1 * H_2$.

By construction, (1)–(3) holds. Since the compactifications of locally finite hyperbolic graphs are Hausdorff, it suffices to prove that f induces a bijective continuous map $g: \partial(G_1 + G_2) \rightarrow \partial(H_1 + H_2)$. For this, we use Lemma 3.2 to get a map

$$\varphi: \partial T \cup \bigcup_{u \in U} \partial G_1^u \cup \bigcup_{v \in V} \partial G_2^v \rightarrow \partial(G_1 +_T G_2)$$

and a map

$$\psi: \partial T \cup \bigcup_{u \in U} \partial H_1^u \cup \bigcup_{v \in V} \partial H_2^v \rightarrow \partial(H_1 +_T H_2)$$

both having the properties as in Lemma 3.2. Then $g := \psi \circ \varphi^{-1}$ is a bijective map and it follows from the construction of φ and ψ that f induces the restriction of g to those boundary points of $G_1 + G_2$ that are not in the image of ∂T by φ or ψ . It is not hard to see that g is also induced by f on the remaining boundary points. In order to show that g is continuous, we show the slightly stronger assertion that $f \cup g$ is continuous. For this, it suffices to consider a sequence $(x_i)_{i \in \mathbb{N}}$ in $G_1 + G_2$ that converges to some $\eta \in \partial(G_1 + G_2)$ and show that $(f(x_i))_{i \in \mathbb{N}}$ converges to $g(\eta)$.

If $\varphi^{-1}(\eta) \in \partial(G_i^u)$ for some $i \in \{1, 2\}$ and $u \in V(T)$, let $(y_i)_{i \in \mathbb{N}}$ be a sequence in G_i^u such that $x_i = y_i$ if $x_i \in V(G_i^u)$ and such that y_i separates x_i from G_i^u otherwise. Then $(y_i)_{i \in \mathbb{N}}$ converges to η , too, and $(f(x_i))_{i \in \mathbb{N}}$ and $(f(y_i))_{i \in \mathbb{N}}$ converge to the same boundary point of $H_1 + H_2$ by construction. Since $(f(y_i))_{i \in \mathbb{N}}$ lies in H_i^u , it converges to $g(\eta)$ and hence $(f(x_i))_{i \in \mathbb{N}}$ converges to $g(\eta)$.

If $\varphi^{-1}(\eta)$ lies in no $\partial(G_i^u)$, then it lies in ∂T . Let $t_1 t_2 \dots$ be a ray in T that converges to $\varphi^{-1}(\eta)$. Let $(y_i)_{i \in \mathbb{N}}$ be such that y_i separates x_i and η and such that y_i lies in some $G_j^{t_k}$. Then $(y_i)_{i \in \mathbb{N}}$ converges to η , too. As is the previous case, $(f(x_i))_{i \in \mathbb{N}}$ and $(f(y_i))_{i \in \mathbb{N}}$ converge to the same boundary point of $H_1 + H_2$, which is $g(\eta)$ by construction. Thus, $f \cup g$ is continuous. \square

Now we are able to prove our main theorems.

Proof of Theorem 1.2. We will prove the assertion by induction on the number of homeomorphism classes of the hyperbolic boundaries ∂G_i . If there are no homeomorphism classes, then all graphs G_i are finite and so are the graphs H_i . By Lemma 2.8, G and H are quasi-isometric to 3-regular trees and hence G is quasi-isometric to H .

Let us now assume that there is at least one homeomorphism class of hyperbolic boundaries ∂G_i . Let $G_{i_1}, \dots, G_{i_k}, H_{j_1}, \dots, H_{j_\ell}$ be representatives of the infinite quasi-isometry types of G_1, \dots, G_n , of H_1, \dots, H_m , respectively. By Theorem 2.9, G is quasi-isometric to either $G_{i_1} * G_{i_1}$, if $k = 1$, or $G_{i_1} * \dots * G_{i_k}$, if $k > 1$, and similarly H is quasi-isometric to either $H_{j_1} * H_{j_1}$ or $H_{j_1} * \dots * H_{j_\ell}$. Note that the homeomorphism types of the G_{i_p} 's and H_{i_q} 's are the same. We apply Theorem 2.9 again to duplicate factors such that G is quasi-isometric to $G' := G'_1, \dots, G'_{m'}$ and H is quasi-isometric to $H' := H'_1, \dots, H'_{n'}$, where G'_i and H'_i have homeomorphic hyperbolic boundaries. As quasi-isometries do not change the homeomorphism type of the hyperbolic boundary by Lemma 2.3, G and G' as well as H and H' have homeomorphic hyperbolic boundaries and by Proposition 3.3 the assertion follows by induction as all factors have a hyperbolic boundary and thus are infinite. \square

Proof of Theorem 1.3. If the factors have the same homeomorphism types of hyperbolic boundaries, it follows from Theorem 1.2 that the hyperbolic boundaries of G and H are homeomorphic.

Let us now assume that ∂G and ∂H are homeomorphic. By Lemmas 2.1, 2.2, 2.4 and 3.2, the non-singular connected components of G are the hyperbolic boundaries of the G_i 's and the non-singular connected components of ∂H are the hyperbolic boundaries of the H_i 's. Thus, the homeomorphism types of $\{\partial G_1, \dots, \partial G_n\}$ are those of $\{H_1, \dots, \partial H_m\}$. \square

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