

# TREE AMALGAMATIONS AND HYPERBOLIC BOUNDARIES

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**ABSTRACT.** We look at tree amalgamations of locally finite quasi-transitive hyperbolic graphs and prove that the homeomorphism type of the hyperbolic boundary of such a tree amalgamation only depends on the homeomorphism types of the hyperbolic boundaries of their factors. Additionally, we show that two locally finite quasi-transitive hyperbolic graphs have a homeomorphic hyperbolic boundary if and only if the homeomorphism types of the hyperbolic boundaries of the factors of their terminal factorisations coincide.

## 1. INTRODUCTION

Tree amalgamations offer a way to construct new graphs out of existing ones similar as new groups can be constructed via free products with amalgamation or HNN-extensions. (We refer to Section 2.2 for the definition of tree amalgamations.) In order to investigate geometric properties of multi-ended quasi-transitive graphs, it is therefore interesting to see how such properties behave with respect to tree amalgamations. In this paper, we are looking at the interaction of tree amalgamations with hyperbolicity. Hyperbolic groups, graphs or spaces play an important role since Gromov's paper [6]. A first result is the following.

**Theorem 1.1.** *Let  $G_1$  and  $G_2$  be connected quasi-transitive locally finite graphs and let  $G = G_1 * G_2$  be a tree amalgamation of finite adhesion and finite identification such that, for  $i = 1, 2$ , some group of automorphisms that acts quasi-transitively on  $G_i$  also acts quasi-transitively on the adhesion sets in  $G_i$ . Then  $G$  is hyperbolic if and only if  $G_1$  and  $G_2$  are hyperbolic.*

Hyperbolic graphs are equipped with a natural boundary, the hyperbolic boundary. Our next result says essentially that in a tree amalgamation of hyperbolic graphs changing the factors without changing the homeomorphism types of their hyperbolic boundary still leads to a tree amalgamation whose hyperbolic boundary is homeomorphic to the original one. For this, a *factorisation* of a quasi-transitive graph  $G$  is a tuple  $(G_1, \dots, G_n)$  such that  $G$  is obtained by iterated non-trivial tree amalgamations of all the graphs  $G_i$  that respect the group actions, have finite adhesion and finite identification and distinguish ends.

**Theorem 1.2.** *Let  $(G_1, \dots, G_n)$  and  $(H_1, \dots, H_m)$  be factorisations of infinitely-ended quasi-transitive locally finite hyperbolic graphs  $G$  and  $H$ , respectively, such that the set of homeomorphism types of the infinite hyperbolic boundaries of the factors in  $(G_1, \dots, G_n)$  is the same as that for  $(H_1, \dots, H_m)$ . Then the hyperbolic boundaries  $\partial G$  and  $\partial H$  are homeomorphic.*

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The obvious question that arises is what can be said about the reverse implication of Theorem 1.2? While it is false in general, we will prove that it holds if we ask it for *terminal factorisations*. These are factorisations  $(G_1, \dots, G_n)$ , where each  $G_i$  has at most one end. In [7] it was shown that quasi-transitive locally finite hyperbolic graphs are accessible in the sense of Thomassen and Woess [15]. Thus, they have a terminal factorisation by [9, Theorem 6.3].

**Theorem 1.3.** *Let  $(G_1, \dots, G_n)$  and  $(H_1, \dots, H_m)$  be terminal factorisations of infinitely-ended quasi-transitive locally finite hyperbolic graphs  $G$  and  $H$ , respectively. Then the hyperbolic boundaries  $\partial G$  and  $\partial H$  are homeomorphic if and only if the set of homeomorphism types of the infinite hyperbolic boundaries of the factors in  $(G_1, \dots, G_n)$  is the same as that for  $(H_1, \dots, H_m)$ .*

Martin and Świątkowski [12] proved group theoretic versions of Theorems 1.2 and 1.3. However, our results do not follow from theirs since it is not known whether every locally finite hyperbolic quasi-transitive graph is quasi-isometric to some hyperbolic group.

The question whether every locally finite hyperbolic quasi-transitive graph is quasi-isometric to some hyperbolic group is a special case of Woess' problem [16, Problem 1] whether every locally finite transitive graph is quasi-isometric to some locally finite Cayley graph. While his problem was settled in the negative by Eskin et al. [4], their counterexamples, the Diestel-Leader graphs, are not hyperbolic and neither is another counterexample by Dunwoody [3].

Świątkowski [14] improved the aforementioned result by Martin and himself in order to obtain more structural control over the hyperbolic boundary of group products. As suggested by a referee, in view of the present result here, it seems highly likely that a result similar to the one by Świątkowski holds for the hyperbolic boundary of tree amalgamations of hyperbolic locally finite quasi-transitive graphs as well.

In Section 2.1 we define and discuss hyperbolicity and in Section 2.2 tree amalgamations. In both sections, we also state the main preliminary results we need for our main results, which we will prove in Section 3.

## 2. PRELIMINARIES

In this section, we state the main definitions and preliminary results that we need for the proofs of our theorems. First, we state some general definitions and then we look at hyperbolic graphs and tree amalgamations more closely in Sections 2.1 and 2.2, respectively.

Let  $G$  be a graph. A *ray* is a one-way infinite path and a *double ray* is a two-way infinite path. Two rays are *equivalent* if for every finite vertex set  $S \subseteq V(G)$  both rays have all but finitely many vertices in the same component of  $G - S$ . This is an equivalence relation whose equivalence classes are the *ends* of  $G$ . Two ends  $\omega_1, \omega_2$  are *separated* by a finite edge set  $E$  if there are  $R_i \in \omega_i$  for  $i = 1, 2$  such that the two rays  $R_1, R_2$  lie in different components of  $G - E$ .

We call  $G$  *transitive* if its automorphism group acts transitively on its vertex set and *quasi-transitive* if its automorphism group acts with only finitely many orbits on  $V(G)$ .

A quasi-transitive graph is *accessible in the sense of Thomassen and Woess* if there exists  $n \in \mathbb{N}$  such that every two ends can be separated by at most  $n$  edges.

Let  $G$  and  $H$  be graphs. A map  $\varphi: V(G) \rightarrow V(H)$  is a *quasi-isometry* if there are constants  $\gamma \geq 1$ ,  $c \geq 0$  such that

$$\gamma^{-1}d_G(u, v) - c \leq d_H(\varphi(u), \varphi(v)) \leq \gamma d_G(u, v) + c$$

for all  $u, v \in V(G)$  and such that  $\sup\{d_H(v, \varphi(V(G))) \mid v \in V(H)\} \leq c$ . We then say that  $G$  is *quasi-isometric* to  $H$ . If we want to emphasise the constants  $\gamma$  and  $c$ , then we also use  $(\gamma, c)$ -*quasi-isometric*. We note that being quasi-isometric is an equivalence relation.

A finite or infinite path  $P$  in  $G$  is *geodesic* if  $d_P(x, y) = d_G(x, y)$  for all vertices  $x, y$  on  $P$ . It is  $(\gamma, c)$ -*quasi-geodesic* if it is the  $(\gamma, c)$ -quasi-isometric image of a subpath of a geodesic double ray.

**2.1. Hyperbolic graphs.** In this section, we will give the definitions and state the lemmas regarding hyperbolic graphs that we need for our results. For a detailed introduction to hyperbolic graphs, we refer to [2, 5, 6].

Let  $G$  be a graph and  $\delta \geq 0$ . If for all  $x, y, z \in V(G)$  and all shortest paths  $P_1, P_2, P_3$ , one between every two of those vertices, every vertex of  $P_1$  has distance at most  $\delta$  to some vertex on either  $P_2$  or  $P_3$  then  $G$  is  $\delta$ -*hyperbolic*. We call  $G$  *hyperbolic* if it is  $\delta$ -hyperbolic for some  $\delta \geq 0$ .

Two geodesic rays in a hyperbolic graph  $G$  are *equivalent* if there is some  $M \in \mathbb{N}$  such that on each ray there are infinitely many vertices of distance at most  $M$  to the other ray. This is an equivalence relation for hyperbolic graphs whose equivalence classes are the *hyperbolic boundary points* of  $G$ . By  $\partial G$  we denote the *hyperbolic boundary* of  $G$ , i. e. the set of hyperbolic boundary points of  $G$ , and we set  $\widehat{G} := G \cup \partial G$ .

For locally finite hyperbolic graphs, it is possible to equip  $\widehat{G}$  with a metric such that  $\widehat{G}$  is compact and every geodesic ray converges to the hyperbolic boundary point it is contained in, see [5, Propositions 7.2.9 and 7.3.10]. For us, it suffices to define convergence of vertex sequences to hyperbolic boundary points. Let  $o \in V(G)$ . Let  $(x_i)_{i \in \mathbb{N}}$  be a sequence in  $V(G)$ . It *converges* to  $\eta \in \partial G$  if for some geodesic ray  $y_1 y_2 \dots$  in  $\eta$  and some sequence  $(n_i)_{i \in \mathbb{N}}$  that goes to  $\infty$  any geodesic path from  $x_i$  to  $y_i$  has distance at least  $n_i$  to  $o$ .

Since the rays in a tree are always geodesic and two rays that are equivalent regarding the definition of ends eventually coincide, these rays are also equivalent with respect to the definition of the hyperbolic boundary. Thus, there is a canonical one-to-one correspondence between the ends of trees and their hyperbolic boundary. Since the end space of a tree is totally disconnected, see e. g. Jung [10], we obtain the following lemma.

**Lemma 2.1.** *The hyperbolic boundaries of trees are totally disconnected sets.*  $\square$

By its definition, the hyperbolic boundary is a refinement of the end space: every hyperbolic boundary point lies in a unique end but an end may contain more than one hyperbolic boundary point. But even more can be said about this relation, cf. e. g. [11, Section 7]:

**Lemma 2.2.** *The connected components of the hyperbolic boundary of every locally finite hyperbolic graph correspond canonically to the ends of that graph.*  $\square$

We are interested in homeomorphism types of hyperbolic boundaries. That is why the following result is important for us.

**Lemma 2.3.** [1, III.H Theorem 3.9] *Quasi-isometries between locally finite hyperbolic graphs induce in a canonical way homeomorphisms between their boundaries.*  $\square$

The following lemma is a direct consequence of a result of Woess [17, Corollary 5].

**Lemma 2.4.** *The hyperbolic boundary of every one-ended quasi-transitive locally finite hyperbolic graph is infinite.*  $\square$

**2.2. Tree-amalgamations.** In this section, we state the definition of tree amalgamations and cite several results about them.

Let  $p_1, p_2 \in \mathbb{N} \cup \{\infty\}$ . A tree is *semiregular* or  $(p_1, p_2)$ -*semiregular* if all vertices in  $V_1$  have the same degree  $p_1$  and all vertices in  $V_2$  have the same degree  $p_2$ , where  $V_1, V_2$  is the canonical bipartition of its vertex set.

Let  $G_1$  and  $G_2$  be two graphs and let  $T$  be a  $(p_1, p_2)$ -semiregular tree with canonical bipartition  $V_1, V_2$  of its vertex set. Let  $\{S_k^i \mid 0 \leq k < p_i\}$  be a family of subsets of  $V(G_i)$  such that all  $S_k^i$  have the same cardinality and let  $\varphi_{k,\ell}: S_k^1 \rightarrow S_\ell^2$  be a bijection. Let

$$c: E(T) \rightarrow \{(k, \ell) \mid 0 \leq k < p_1, 0 \leq \ell < p_2\}$$

such that for all  $v \in V_i$  the composition of  $c$  and the projection to the  $i$ -th coordinate form a bijection from  $\{e \in E(T) \mid v \in e\}$  to the set  $\{k \mid 0 \leq k < p_i\}$ .

For every  $v \in V_i$  with  $i = 1, 2$ , let  $G_i^v$  be a copy of  $G_i$  and let  $S_k^v$  be the copy of  $S_k^i$  in  $G_i^v$ . Let  $H := G_1 + G_2$  be the graph obtained from the disjoint union of all graphs  $G_i^v$  by adding an edge between all  $x \in S_k^v$  and  $\varphi_{k,\ell}(x) \in S_\ell^u$  for every edge  $vu \in E(T)$  with  $v \in V_1$  and  $c(vu) = (k, \ell)$ . Let  $G$  be the graph obtained from  $H$  by contracting all new edges  $x\varphi_{k,\ell}(x)$ , i. e. all edges outside of the graphs  $G_i^v$ . We call  $G$  the *tree amalgamation* of  $G_1$  and  $G_2$  over  $T$  (with respect to the sets  $S_k^i$  and the maps  $\varphi_{k,\ell}$ ) and we denote it by  $G_1 *_T G_2$ . If the *amalgamation tree*  $T$  is clear from the context, we simply write  $G_1 *_G G_2$ . The graphs  $G_1$  and  $G_2$  are the *factors* of the tree amalgamation. The sets  $S_k^i$  and their copies in  $G$  are the *adhesion sets* of the tree amalgamation. The tree amalgamation has *finite adhesion* if each adhesion set is finite. It has *adhesion 1* if all adhesion sets have size 1. Let  $\psi: V(H) \rightarrow V(G)$  be such that every  $x \in V(H)$  is mapped to the vertex of  $G$  it ends up after all contractions. A tree amalgamation  $G_1 *_G G_2$  is *trivial* if there is some  $G_i^v$  such that the restriction of  $\psi$  to  $G_i^v$  is a bijection  $G_i^v \rightarrow G_1 *_G G_2$ . So a tree amalgamation of finite adhesion is trivial if all adhesion sets of  $G_i$  are  $V(G_i)$  and either  $p_i = 1$  or also all adhesion sets of  $G_j$  are  $V(G_j)$  for some  $\{i, j\} = \{1, 2\}$ , note that then in particular  $G_i$  completely lies in  $G_j$  and, if  $T$  is not a star, then  $G_i$  and  $G_j$  are equal.

The *identification size* of a vertex  $x \in V(G)$  is the smallest size of subtrees  $T'$  of  $T$  such that  $x$  is obtained by contracting only edges between vertices in  $\bigcup_{u \in V(T')} V(G_j^u)$ . The tree amalgamation has *finite identification* if all vertices have finite identification size and it has *bounded identification* if the supremum of all identification sizes is finite.

The tree amalgamation  $G = G_1 *_G G_2$  *distinguishes* ends if there is some adhesion set  $S_k^v = S_\ell^u$  for adjacent vertices  $u, v$  of  $T$  such that for every component  $C$  of  $T - uv$  the graph induced by  $\bigcup_{w \in C} G_i^w$  contains an end.

The following is a weaker version of [9, Theorem 5.3]. We do not need the full detail of tree amalgamations *respecting group actions* and thus the following result is sufficient for us.

**Theorem 2.5.** [9, Theorem 5.3] *Every connected quasi-transitive locally finite graph with more than one end is a non-trivial tree amalgamation  $G_1 * G_2$  that distinguishes ends and has finite adhesion and finite identification of two connected quasi-transitive locally finite graphs such that the set of adhesion sets in each factor has at most two orbits under some group acting quasi-transitively on that factor.*  $\square$

Note that the properties of the tree amalgamation of Theorem 2.5 imply that it has bounded identification.

Also the notions of (terminal) factorisations need not be as strong as those in the introduction. A *factorisation* of a connected locally finite quasi-transitive graph  $G$  is a tuple  $(G_1, \dots, G_n)$  of connected locally finite quasi-transitive graphs such that  $G$  is obtained from the elements of the tuple by iterated tree amalgamations distinguishing ends of finite adhesion and finite identification such that for each step some group of automorphisms of each factors acts quasi-transitively on this factor and the set of its adhesion sets. A factorisation is *terminal* if every element of the tuple has at most one end. A connected quasi-transitive locally finite graph is *accessible* if it has a terminal factorisation.

All following results in this section deal with the interplay of tree amalgamations with quasi-isometries and are proved in [8].

**Lemma 2.6.** [8, Remark 2.1] *Let  $G$  and  $H$  be locally finite graphs and  $G * H$  be a tree amalgamation of finite adhesion and bounded identification. Then  $G * H$  is quasi-isometric to  $G + H$ .*  $\square$

The following lemma enables us to change the factors in a tree amalgamation a bit while staying quasi-isometric to the original tree amalgamation but in the result we have more control over the adhesion sets and identification sizes.

**Lemma 2.7.** [8, Lemma 3.1] *Let  $G$  be a locally finite connected quasi-transitive graph and let  $(G_1, G_2)$  be a factorisation of  $G$ . Then there is a locally finite connected quasi-transitive graph  $H$  that has a factorisation  $(H_1, H_2)$  such that the following hold.*

- (1)  $G$  is quasi-isometric to  $H$ ;
- (2)  $G_i$  is quasi-isometric to  $H_i$  for  $i = 1, 2$ ;
- (3)  $H_1 * H_2$  has adhesion 1;
- (4) all adhesion sets of  $H_1 * H_2$  are distinct;
- (5) all vertices of  $H_i$  lies in adhesion sets for  $i = 1, 2$ .  $\square$

**Lemma 2.8.** [8, Lemma 2.9] *A connected locally finite quasi-transitive graph that has a terminal factorisation of only finite graphs is quasi-isometric to a 3-regular tree.*  $\square$

The following theorem plays a central role in our proofs and can be seen as an analogue of Theorem 1.2 for quasi-isometries of graphs instead of homeomorphisms of hyperbolic boundaries.

**Theorem 2.9.** [8, Theorem 1.4] *Let  $G$  and  $H$  be locally finite quasi-transitive graphs with infinitely many ends and let  $(G_1, \dots, G_n)$  and  $(H_1, \dots, H_m)$  be factorisations of  $G$  and  $H$ , respectively. If  $(G_1, \dots, G_n)$  and  $(H_1, \dots, H_m)$  have the same set of quasi-isometry types of infinite factors, then  $G$  and  $H$  are quasi-isometric.*  $\square$

### 3. PROOFS OF THE MAIN THEOREMS

In this section we will prove our main results. First, we prove the characterisation of quasi-transitive locally finite hyperbolic graphs in terms of their terminal factorisations.

*Proof of Theorem 1.1.* If the tree amalgamation does not distinguish ends, then one  $G_i$  has a unique adhesion set and the amalgamation tree is a star. Since  $G_i$  is quasi-transitive, it must be finite. Then it is easy to see that  $G$  is quasi-isometric to  $G_{3-i}$ . Since hyperbolicity is preserved by quasi-isometries and finite graphs are always hyperbolic (for a large enough constant), this implies that  $G$  is hyperbolic if and only if  $G_{3-i}$  is hyperbolic. So we may assume that the tree amalgamation distinguishes ends.

Let  $H = H_1 * H_2$  as in Lemma 2.7. In particular,  $H$ ,  $H_1$  and  $H_2$  are quasi-isometric to  $G$ ,  $G_1$  and  $G_2$ , respectively, and the tree amalgamation  $H_1 * H_2$  has adhesion 1. Since hyperbolicity is preserved by quasi-isometries, see e. g. [1, Theorem III.H.1.9], it suffices to prove the assertion in the situation of adhesion 1. So let us assume that  $G_1 * G_2$  already has adhesion 1.

First, let us assume that  $G$  is  $\delta$ -hyperbolic for some  $\delta \geq 0$ . Let  $x, y$  be vertices of some  $G_i^v$ . If an  $x$ - $y$  geodesic leaves  $G_i^v$  through some adhesion set, it must re-enter  $G_i^v$  through the same adhesion set. Since the adhesion is 1, every  $x$ - $y$  geodesic in  $G$  lies completely in  $G_i^v$ . It follows that  $G_i$  is  $\delta$ -hyperbolic.

Now let us assume that  $G_1$  and  $G_2$  are  $\delta$ -hyperbolic for some  $\delta \geq 0$ . Let  $x, y, z \in V(G)$ . For distinct  $u, v \in \{x, y, z\}$ , let  $P_{uv}$  be a  $u$ - $v$  geodesic. Since the adhesion sets have size 1, in any  $G_i^v$  that intersects non-trivially with the paths  $P_{uv}$ , each of the subpaths in  $G_i^v$  induced by  $P_{xy}$ ,  $P_{xz}$  and  $P_{yz}$  lies in the  $\delta$ -neighbourhood of the other two if it is non-trivial. Thus,  $G$  is  $\delta$ -hyperbolic, too.  $\square$

As a corollary of Theorem 1.1, we obtain a characterisation of quasi-transitive locally finite hyperbolic graphs in terms of their terminal factorisations.

**Corollary 3.1.** *A connected quasi-transitive locally finite graph is hyperbolic if and only if it admits a terminal factorisation such that all its factors are connected quasi-transitive locally finite hyperbolic graphs with at most one end.*

*Proof.* Let  $G$  be a connected quasi-transitive locally finite graph. If  $G$  is one-ended, then it is a terminal factorisation of itself and the assertion holds trivially. So let us assume that  $G$  has more than one end.

First, let us assume that  $G$  is hyperbolic. By [7, Theorem 4.3], it is a graph that is accessible in the sense of Thomassen and Woess. Thus it is accessible and has a terminal factorisation by [9, Theorem 6.3]. So there are connected quasi-transitive locally finite graphs  $G_1, \dots, G_n, H_1, \dots, H_{n-1}$  with  $G = H_{n-1}$  such that each  $G_i$  has at most one end and for every  $i \leq n-1$ , the graph  $H_i$  is a tree amalgamation  $H * H'$  of finite adhesion and finite identification, where

$$H, H' \in \{G_j \mid 1 \leq j \leq n\} \cup \{H_j \mid 1 \leq j < i\}.$$

(We may assume that all  $G_i$  are indeed needed at some point during these tree amalgamations.) By repeated application of Lemma 1.1, each  $H_i$ , and thus each  $G_i$  is hyperbolic.

Conversely, if  $G$  has a terminal factorisation into connected finite or connected quasi-transitive locally finite hyperbolic one-ended graphs, then each of the previous

factors we considered for obtaining the terminal factorisation are hyperbolic by Theorem 1.1. In particular,  $G$  is hyperbolic.  $\square$

Now we will turn our attention to the proofs of the results concerning the hyperbolic boundary, Theorems 1.2 and 1.3.

The following lemma is a special case of a result of Steiner and Steiner [13, Theorem 4]. It is also possible to adapt the proof of Martin and Świątkowski [12, Lemma 4.2] to our situation to obtain that lemma.

**Lemma 3.2.** *Let  $G$  and  $H$  be locally finite quasi-transitive hyperbolic graphs and let  $f: \partial G \rightarrow \partial H$  be a homeomorphism. Then  $f$  extends to a homeomorphism  $\widehat{G} \rightarrow \widehat{H}$ .*  $\square$

The next lemma describes the hyperbolic boundary of a tree amalgamation in terms of its factors and the involved tree. The first step of its proof is to reduce the general case to a special case of the so-called block-graph defined by the cutvertices and 2-connected blocks of a graph.

**Lemma 3.3.** *Let  $G$  and  $H$  be locally finite hyperbolic quasi-transitive graphs and let  $T$  be a semiregular tree with canonical bipartition  $\{U, V\}$  of its vertex set. Then there exists a canonical bijective map*

$$f: \partial T \cup \bigcup_{u \in U} \partial G^u \cup \bigcup_{v \in V} \partial H^v \rightarrow \partial(G +_T H)$$

such that the following hold.

- (1) *The preimage of each connected component of  $\partial(G +_T H)$  is a connected component of an element of*

$$\{\partial T\} \cup \{\partial G^u \mid u \in U\} \cup \{\partial H^v \mid v \in V\};$$

- (2) *every sequence of vertices in some  $G^u$  or  $H^v$  that converges to some boundary point  $\eta \in \bigcup_{u \in U} \partial G^u \cup \bigcup_{v \in V} \partial H^v$  converges to  $f(\eta)$  in  $(G +_T H) \cup \partial(G +_T H)$ ;*
- (3) *every sequence  $(v_i)_{i \in \mathbb{N}}$  with  $v_i \in G^{t_i}$  or  $v_i \in H^{t_i}$  such that  $(t_i)_{i \in \mathbb{N}}$  converges to  $\eta \in \partial T$  converges to  $f(\eta)$  in  $(G +_T H) \cup \partial(G +_T H)$ .*

*Proof.* Since quasi-isometries preserve hyperbolicity, we may apply Lemmas 2.7 and 2.6 to assume that  $G *_T H$  is a tree amalgamation of adhesion 1 and distinct adhesion sets are disjoint. Note for this that quasi-isometries map distinct boundary points to distinct boundary points and distinct connected components of the boundary to distinct connected components.

Let us define a map

$$f: \partial T \cup \bigcup_{u \in U} \partial G^u \cup \bigcup_{v \in V} \partial H^v \rightarrow \partial(G +_T H).$$

Let  $u \in U$  and  $\eta \in \partial G^u$ . Since the adhesion is 1, any geodesic ray in  $\eta$  is a geodesic ray in  $G +_T H$  as well. Thus, two geodesic rays in  $G *_T H$  that lie in  $G^u$  are equivalent in  $G +_T H$  and thus lie in the same boundary point  $\mu$ . We set  $f(\eta) := \mu$ . Analogously, we define the image of elements of  $\partial G^v$  for  $v \in V$ .

Now we consider a boundary point  $\eta$  of  $T$ . Note that since  $T$  is a tree, its boundary points are just its ends. Let  $R$  be a ray in  $\eta$ . Since the tree amalgamation has adhesion 1, there is for each edge  $uv$  of  $T$  with  $u \in U$  and  $v \in V$  a unique edge of  $G +_T H$  that corresponds to  $uv$  in that its incident vertices lie in  $G^u$  and  $H^v$  and get identified when constructing the tree amalgamation. Since distinct adhesion

sets are disjoint, it follows that for a subpath  $u_0u_1u_2u_3$  of  $R$  the edges  $e_1, e_2, e_3$ , where  $e_i$  corresponds to the edge  $u_{i-1}u_i$ , have the properties that they are distinct and  $e_2$  separates  $e_1$  and  $e_3$ . By joining  $e_1$  and  $e_2$  by a shortest path inside  $G^{u_1}$  or  $H^{u_1}$ , we obtain that  $R$  defines a geodesic ray and no matter how we choose the shortest paths to connect the edges  $e_1, e_2$ , the resulting rays are equivalent and thus converge to the same boundary point  $\mu$  of  $G +_T H$ . We set  $f(\eta) := \mu$ .

While defining  $f$ , we ensured that it is well-defined. Let us show that  $f$  is injective. If we consider hyperbolic boundary points  $\eta, \mu$  either of distinct elements of

$$X := \{\partial T\} \cup \{\partial G^u \mid u \in U\} \cup \{\partial H^v \mid v \in V\}$$

or both from  $\partial T$ , then each of  $\eta, \mu$  belongs to either a hyperbolic boundary point of  $T$  or the hyperbolic boundary of a vertex of  $T$  and there is an edge of  $T$  separating these hyperbolic boundary points or vertices of  $T$ . The edge of  $G +_T H$  corresponding to that edge of  $T$  separates the  $f$ -image of those hyperbolic boundary points. Thus,  $f(\eta)$  and  $f(\mu)$  lie in distinct ends of  $G +_T H$  and hence are distinct. If  $\eta$  and  $\mu$  are distinct but in a common element of  $X$ , then they lie in either  $\partial G^u$  or  $\partial H^v$  for some  $u \in U$  or  $v \in V$ . But as geodesic paths and rays in  $G^u$  or  $H^v$  are geodesic paths and rays in  $G *_T H$ , inequivalent rays in  $G^u$  or  $H^v$  stay inequivalent in  $G +_T H$ . Thus, we have  $f(\eta) \neq f(\mu)$  in this case, too.

To show that  $f$  is surjective, let  $\eta \in \partial(G +_T H)$  and let  $R$  be a geodesic ray in  $\eta$ . Since the adhesion of  $G *_T H$  is 1, there is either a subgraph  $G^u$  or  $H^v$  such that  $R$  has all but finitely many vertices in that subgraph, or not. If  $R$  meets every such subgraph in only finitely many vertices, let  $W \subseteq V(T)$  consist of those vertices  $u \in U$  and  $v \in V$  for which  $R$  meets  $G^u$  and  $H^v$ . Note that if  $R$  leaves  $G^u$  or  $H^v$  once, it has to do so through an adhesion set and since it cannot use the same edge again, it cannot enter the subgraph  $G^u$  or  $H^v$  anymore. Thus,  $W$  defines a ray in  $T$  and it is straight-forwards to see that the hyperbolic boundary point  $\mu$  of  $T$  that contains this ray is mapped to  $\eta$  by  $f$ . If  $R$  has infinitely many vertices in a subgraph  $G^u$  or  $H^v$ , say  $G^u$ , then it has a subray in  $G^u$  by the above argument that if it leaves  $G^u$  once, it never reenters  $G^u$ . This subray is geodesic in  $G^u$  since the adhesion sets have size 1. It lies in some hyperbolic boundary point  $\mu$  of  $G^u$  and we have  $f(\mu) = \eta$  by construction.

So far, we constructed a canonical bijective map  $f$  that satisfies (2) and (3). It remains to verify (1). For this, we note that the connected components of the hyperbolic boundary correspond to the ends of the graph by Lemma 2.2. Analogously as in the proof that  $f$  is injective, it follows that distinct hyperbolic boundary points in the same end of  $G +_T H$  are mapped to hyperbolic boundary points of some  $G^u$  or  $H^v$  that lie in the same end of that graph.  $\square$

The following proposition is the main step towards the proof of Theorem 1.2.

**Proposition 3.4.** *Let  $G_1, G_2, H_1$  and  $H_2$  be locally finite infinite hyperbolic quasi-transitive graphs such that  $\partial G_1$  and  $\partial G_2$  are homeomorphic to  $\partial H_1$  and  $\partial H_2$ , respectively. Let  $G$  and  $H$  be quasi-transitive locally finite graphs such that  $(G_1, G_2)$  is a factorisation of  $G$  and  $(H_1, H_2)$  is a factorisation of  $H$ . Then  $\partial G$  and  $\partial H$  are homeomorphic.*

*Proof.* According to Lemma 2.7, we find locally finite connected quasi-transitive graphs  $G'_1, G'_2, H'_1, H'_2$  such that the following hold.

- (i)  $G_i$  and  $H_i$  are quasi-isometric to  $G'_i$  and  $H'_i$ , respectively, for  $i = 1, 2$ ,



- (ii)  $(G'_1, G'_2)$  and  $(H'_1, H'_2)$  are factorisations of graphs  $G'$  and  $H'$  that are quasi-isometric to tree amalgamations  $G_1 * G_2$  and  $H_1 * H_2$ , respectively,
- (iii)  $G'_1 * G'_2$  and  $H'_1 * H'_2$  have adhesion 1,
- (iv) all their adhesion sets are disjoint and
- (v) every vertex lies in some adhesion set.

As hyperbolicity is preserved by quasi-isometries,  $G'_1, G'_2, H'_1, H'_2$  are hyperbolic. By Theorem 1.1,  $G'$  and  $H'$  are hyperbolic, too. Since quasi-isometric hyperbolic graphs have homeomorphic hyperbolic boundaries by Lemma 2.3, the hyperbolic boundaries  $\partial(G_1 * G_2)$  and  $\partial(H_1 * H_2)$  are homeomorphic to  $\partial G'$  and  $\partial H'$ , respectively. Thus, it suffices to prove the assertion under the assumption that the tree amalgamation is of adhesion 1, all adhesion sets are disjoint and the adhesion sets cover all vertices. By Lemma 2.6 the graphs  $G_1 * G_2$  and  $H_1 * H_2$  are quasi-isometric to  $G_1 + G_2$  and to  $H_1 + H_2$ , respectively.

For  $i = 1, 2$ , let  $g_i: \partial G_i \rightarrow \partial H_i$  be a homeomorphism and let  $f_i: \widehat{G}_i \rightarrow \widehat{H}_i$  be a homeomorphism that extends  $g_i$ , which exists by Lemma 3.2. Note that the amalgamation trees in both cases are the  $\aleph_0$ -regular tree  $T$ , i. e. we consider the tree amalgamations  $G_1 *_T G_2$  and  $H_1 *_T H_2$ . Let  $c_G$  and  $c_H$  be the labelings of the edges of  $T$  used for the tree amalgamations  $G_1 *_T G_2$  and  $H_1 *_T H_2$ , respectively. We are going to construct a map

$$f: V(G_1 + G_2) \rightarrow V(H_1 + H_2)$$

with the following properties.

- (1)  $f$  is a bijection;
- (2)  $f$  induces an automorphism  $f_t$  of  $T$ ;
- (3)  $f$  induces homeomorphisms  $\widehat{G}_i^u \rightarrow \widehat{H}_i^{f_t(u)}$ ;
- (4)  $f$  induces a homeomorphism  $g: \partial(G_1 + G_2) \rightarrow \partial(H_1 + H_2)$ ;
- (5)  $f$  maps adjacent vertices of  $G_1 + G_2$  from distinct  $G_i^u$  and  $G_j^v$  to adjacent vertices from  $H_i^u$  and  $H_j^v$ .

The homeomorphisms in (3) will be closely related to the homeomorphisms  $f_i$ .

First we note that it suffices to prove the existence of such a map  $f$  in order to prove the assertion. Indeed, by Lemmas 2.6 and 2.3, we obtain that  $\partial(G_1 * G_2)$  and  $\partial(G_1 + G_2)$  are homeomorphic and so are  $\partial(H_1 * H_2)$  and  $\partial(H_1 + H_2)$ . Thus, (4) implies that  $\partial(G_1 * G_2)$  and  $\partial(H_1 * H_2)$  are homeomorphic.

Let us now construct  $f$ . Let  $v_1, v_2, \dots$  be an enumeration of  $V(T)$  such that for every  $i \in \mathbb{N}$  the vertices  $v_1, \dots, v_i$  induce a subtree of  $T$ . Let  $f_{i_1}^{v_1}$  be the map  $G_{i_1}^{v_1} \rightarrow H_{i_1}^{v_1}$  that is induced by  $f_{i_1}$ . For  $j > 1$ , let  $u_j$  be the vertex in  $G_{i_j}^{v_j}$  that separates  $G_{i_j}^{v_j}$  from  $G_{i_1}^{v_1}$  and let  $w_j$  be the vertex in  $H_{i_j}^{v_j}$  that separates  $H_{i_j}^{v_j}$  from  $H_{i_1}^{v_1}$ . Let  $f_{i_j}^{v_j}$  be the map  $G_{i_j}^{v_j} \rightarrow H_{i_j}^{v_j}$  induced by  $f_{i_j}$ . Let  $g_{i_j}^{v_j}$  be  $h_{i_j}^{v_j}$  but with the images of  $u_j$  and  $(f_{i_j}^{v_j})^{-1}(w_j)$  exchanged, i. e.  $u_j$ , the vertex that separates  $G_{i_j}^{v_j}$  from  $G_{i_1}^{v_1}$ , is mapped onto  $w_j$ , the vertex in  $H_{i_j}^{v_j}$  that separates  $H_{i_j}^{v_j}$  from  $H_{i_1}^{v_1}$ .

Exchanging the images of two vertices does not interfere with being a homeomorphism on the hyperbolic compactifications. Thus,  $g_{i_j}^{v_j}$  still induces a homeomorphism

$$h_{i_j}^{v_j}: \widehat{G}_{i_j}^{v_j} \rightarrow \widehat{H}_{i_j}^{v_j}$$

by Lemma 2.3.

Let  $t \in \{v_1, \dots, v_{j-1}\}$  be the unique neighbour of  $v_j$  in that set. Then the vertex in  $G_{3-i_j}^t$  that is adjacent to  $u_j$  in the graph  $G_1 + G_2$  is mapped by  $g_{3-i_j}^t$  to the

neighbour of  $w_j$  in  $H_{i_3-j}^t$  in the graph  $H_1 + H_2$ . Hence, the union of all  $g_i^v$  for all  $v \in V(T)$  defines a map  $f: G_1 + G_2 \rightarrow H_1 + H_2$  that maps vertices of  $G_1 + G_2$  that get identified by constructing  $G_1 * G_2$  to those that get identified by constructing  $H_1 * H_2$ , i.e. the map satisfies (5).

By construction, (1)–(3) hold, too. Since the compactifications of locally finite hyperbolic graphs are Hausdorff, it suffices to prove that  $f$  induces a bijective continuous map  $g: \partial(G_1 + G_2) \rightarrow \partial(H_1 + H_2)$ . For this, we use Lemma 3.3 to get a map

$$\varphi: \partial T \cup \bigcup_{u \in U} \partial G_1^u \cup \bigcup_{v \in V} \partial G_2^v \rightarrow \partial(G_1 +_T G_2)$$

and a map

$$\psi: \partial T \cup \bigcup_{u \in U} \partial H_1^u \cup \bigcup_{v \in V} \partial H_2^v \rightarrow \partial(H_1 +_T H_2)$$

both having the properties as in Lemma 3.3. Let  $g$  be the composition of  $\varphi^{-1}$ , then either the identity on  $\partial T$  or some  $h_{i_j}^{v_j}$ , and then  $\psi$ . Then  $g$  is a bijective map and it follows from the construction of  $\varphi$  and  $\psi$  that  $f$  induces the restriction of  $g$  to those boundary points of  $G_1 + G_2$  that are not in the image of  $\partial T$  by  $\varphi$  or  $\psi$ . It is not hard to see that  $g$  is also induced by  $f$  on the remaining boundary points. In order to show that  $g$  is continuous, we show the slightly stronger assertion that  $f \cup g$  is continuous. For this, it suffices to consider a sequence  $(x_i)_{i \in \mathbb{N}}$  in  $G_1 + G_2$  that converges to some  $\eta \in \partial(G_1 + G_2)$  and show that  $(f(x_i))_{i \in \mathbb{N}}$  converges to  $g(\eta)$ .

If  $\varphi^{-1}(\eta) \in \partial(G_i^u)$  for some  $i \in \{1, 2\}$  and  $u \in V(T)$ , let  $(y_i)_{i \in \mathbb{N}}$  be a sequence in  $G_i^u$  such that  $x_i = y_i$  if  $x_i \in V(G_i^u)$  and such that  $y_i$  separates  $x_i$  from  $G_i^u$  otherwise. Then  $(y_i)_{i \in \mathbb{N}}$  converges to  $\eta$ , too, and  $(f(x_i))_{i \in \mathbb{N}}$  and  $(f(y_i))_{i \in \mathbb{N}}$  converge to the same boundary point of  $H_1 + H_2$  by construction, in particular by (5). Since  $(f(y_i))_{i \in \mathbb{N}}$  lies in  $H_i^u$ , it converges to  $g(\eta)$  and hence  $(f(x_i))_{i \in \mathbb{N}}$  converges to  $g(\eta)$ .

If  $\varphi^{-1}(\eta)$  lies in no  $\partial(G_i^u)$ , then it lies in  $\partial T$ . Let  $t_1 t_2 \dots$  be a ray in  $T$  that converges to  $\varphi^{-1}(\eta)$ . Let  $(y_i)_{i \in \mathbb{N}}$  be such that  $y_i$  separates  $x_i$  and  $\eta$  and such that  $y_i$  lies in some  $G_j^{t_k}$ . Then  $(y_i)_{i \in \mathbb{N}}$  converges to  $\eta$ , too. As in the previous case,  $(f(x_i))_{i \in \mathbb{N}}$  and  $(f(y_i))_{i \in \mathbb{N}}$  converge to the same boundary point of  $H_1 + H_2$ , which is  $g(\eta)$  by construction. Thus,  $f \cup g$  is continuous.  $\square$

Now we are able to prove our main theorems.

*Proof of Theorem 1.2.* If there are no homeomorphism classes of infinite hyperbolic boundaries  $\partial G_i$ , then all graphs  $G_i$  are either finite or two-ended and so are the graphs  $H_i$ . Since two-ended quasi-transitive locally finite graphs are quasi-isometric to the double ray, they have a terminal factorisation with only finite factors, cp. [9, Theorem 7.3]. Thus,  $G$  and  $H$  have factorisations with only finite factors. By Lemma 2.8,  $G$  and  $H$  are quasi-isometric to 3-regular trees and hence  $G$  is quasi-isometric to  $H$ . So Lemma 2.3 implies that  $\partial G$  and  $\partial H$  are homeomorphic.

Let us now assume that there is at least one homeomorphism class of infinite hyperbolic boundaries  $\partial G_i$ . By replacing every two-ended  $G_i$  or  $H_i$  by a factorisation of it into finite graphs, we may assume that all infinite  $G_i$  and  $H_i$  also have an infinite hyperbolic boundary. First, we will throw out the finite graphs  $G_i$  and  $H_i$  since, in the current case, Theorem 2.9 will imply that they can be neglected for Theorem 1.2. For this, let  $G_{i_1}, \dots, G_{i_k}$  and  $H_{j_1}, \dots, H_{j_\ell}$  be representatives of the quasi-isometry types of the infinite  $G_1, \dots, G_n$  and  $H_1, \dots, H_m$ , respectively. By Theorem 2.9,  $G$  is quasi-isometric to either  $G_{i_1} * G_{i_1}$ , if  $k = 1$ , or  $G_{i_1} * \dots * G_{i_k}$ , if

$k > 1$ , and similarly  $H$  is quasi-isometric to either  $H_{j_1} * H_{j_1}$  or  $H_{j_1} * \dots * H_{j_\ell}$ . Since all  $G_{i_p}$ 's and  $H_{i_q}$ 's are infinite, the sets of their homeomorphism types are the same. We apply Theorem 2.9 again to duplicate factors such that  $G$  is quasi-isometric to  $G' := G'_1 * \dots * G'_{m'}$ , and  $H$  is quasi-isometric to  $H' := H'_1 * \dots * H'_{m'}$ , where  $G'_i$  and  $H'_i$  have homeomorphic infinite hyperbolic boundaries. Lemma 2.3 implies that  $G$  and  $G'$  as well as  $H$  and  $H'$  have homeomorphic hyperbolic boundaries. Applying Proposition 3.4 iteratively, we obtain for all  $i \leq n$  that  $G'_1 * \dots * G'_i$  and  $H'_1 * \dots * H'_i$  have homeomorphic hyperbolic boundaries. Thus,  $G'$  and  $H'$  have homeomorphic hyperbolic boundaries, which implies the assertion.  $\square$

*Proof of Theorem 1.3.* If the factors have the same homeomorphism types of infinite hyperbolic boundaries, it follows from Theorem 1.2 that the hyperbolic boundaries of  $G$  and  $H$  are homeomorphic.

Let us now assume that  $\partial G$  and  $\partial H$  are homeomorphic. By Lemmas 2.1, 2.2, 2.4 and 3.3, the non-singular connected components of  $\partial G$  are the hyperbolic boundaries of the  $G_i$ 's and the non-singular connected components of  $\partial H$  are the hyperbolic boundaries of the  $H_i$ 's. Thus, the homeomorphism types of the infinite elements of  $\{\partial G_1, \dots, \partial G_n\}$  are those of infinite elements of  $\{\partial H_1, \dots, \partial H_m\}$ .  $\square$

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#### REFERENCES

1. M.R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, 1999.
2. M. Coornaert, T. Delzant, and A. Papadopoulos, *Notes sur les groupes hyperboliques de Gromov*, Lecture Notes in Mathematics, vol. 1441, Springer-Verlag, 1990.
3. M.J. Dunwoody, *An Inaccessible Graph*, Random walks, boundaries and spectra, Progr. Probab., vol. 64, Birkhäuser/Springer Basel AG, 2011, pp. 1–14.
4. A. Eskin, D. Fisher, and K. Whyte, *Coarse differentiation of quasi-isometries I: Spaces not quasi-isometric to Cayley graphs*, Ann. of Math. (2) **176** (2012), no. 1, 221–260.
5. E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques, d'après M. Gromov*, Progress in Math., vol. 83, Birkhäuser, Boston, 1990.
6. M. Gromov, *Hyperbolic Groups*, Essays in group theory (S.M. Gersten, ed.), MSRI, vol. 8, Springer, New York, 1987, pp. 75–263.
7. M. Hamann, *Accessibility in transitive graphs*, Combinatorica **38** (2018), 847–859.
8. ———, *Tree amalgamations and quasi-isometries*, arXiv:1812.04987, 2018.
9. M. Hamann, F. Lehner, B. Miraftab, and T. Rühmann, *A Stallings type theorem for quasi-transitive graphs*, J. Combin. Theory (Series B) **157** (2022), 40–69.
10. H.A. Jung, *Connectivity in infinite graphs*, Studies in Pure Mathematics (L. Mirsky, ed.), Academic Press, 1971, pp. 137–143.
11. I. Kapovich and N. Benakli, *Boundaries of hyperbolic groups*, Combinatorial and Geometric Group Theory (R. Gilman et al., ed.), Contemporary Mathematics, vol. 296, 2002, pp. 39–94.
12. A. Martin and J. Świątkowski, *Infinitely-ended hyperbolic groups with homeomorphic Gromov boundaries*, J. Group Theory **18** (2015), 273–289.
13. A.K. Steiner and E.F. Steiner, *Graph Closures and Metric Compactifications of  $N$* , Proc. Amer. Math. Soc. **25** (1970), 593–597.
14. J. Świątkowski, *The dense amalgam of metric compacta and topological characterization of boundaries of free products of groups*, Groups Geom. Dyn. **10** (2016), 407–471.
15. C. Thomassen and W. Woess, *Vertex-transitive graphs and accessibility*, J. Combin. Theory (Series B) **58** (1993), no. 2, 248–268.
16. W. Woess, *Topological groups and infinite graphs*, Discrete Math. **95** (1991), no. 1-3, 373–384.

17. ———, *Fixed sets and free subgroups of groups acting on metric spaces*, Math. Z. **214** (1993), no. 3, 425–440.

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