TREE AMALGAMATIONS AND HYPERBOLIC BOUNDARIES

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Abstract. We look at tree amalgamations of locally finite quasi-transitive hyperbolic graphs and prove that the homeomorphism type of the hyperbolic boundary of such a tree amalgamation only depends on the homeomorphism types of the hyperbolic boundaries of their factors. Additionally, we show that two locally finite quasi-transitive hyperbolic graphs have a homeomorphic hyperbolic boundary if and only if the homeomorphism types of the hyperbolic boundaries of the factors of their terminal factorisations coincide.

1. Introduction

Tree amalgamations offer a way to construct new graphs out of existing ones similar as new groups can be constructed via free products with amalgamation or HNN-extensions. (We refer to Section 2.2 for the definition of tree amalgamations.) In order to investigate geometric properties of multi-ended quasi-transitive graphs, it is therefore interesting to see how such properties behave with respect to tree amalgamations. In this paper, we are looking at the interaction of tree amalgamations with hyperbolicity. Hyperbolic groups, graphs or spaces play an important role since Gromov's paper [6]. A first observation is the following.

Theorem 1.1. *A connected locally finite quasi-transitive graph is hyperbolic if and only if any of its tree amalgamations respecting the group actions of finite adhesion and finite identification is hyperbolic.*

Hyperbolic graphs are equipped with a natural boundary, the hyperbolic boundary. Our next result says essentially that in a tree amalgamation of hyperbolic graphs changing the factors without changing the homeomorphism types of their hyperbolic boundary still leads to a tree amalgamation whose hyperbolic boundary is homeomorphic to the original one. For this, a *factorisation* of a quasi-transitive graph G is a tuple $(G_1, \ldots, G_n)$ such that G is obtained by iterated non-trivial tree amalgamations of all the graphs G_i that respect the group actions, have finite adhesion and finite identification and distinguish ends.

Theorem 1.2. *Let $(G_1, \ldots, G_n)$ and $(H_1, \ldots, H_m)$ be factorisations of infinitely-ended quasi-transitive locally finite hyperbolic graphs G and H, respectively, such that the set of homeomorphism types of the hyperbolic boundaries of the factors in $(G_1, \ldots, G_n)$ are the same as those for $(H_1, \ldots, H_m)$. Then the hyperbolic boundaries ∂G and ∂H are homeomorphic.*

The obvious question that arises is what can be said about the reverse implication of Theorem 1.2? While it is false in general, we will prove that it holds if we ask it for *terminal factorisations*. These are factorisations $(G_1, \ldots, G_n)$, where each

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G_i has at most one end. In [7] it was shown that quasi-transitive locally finite hyperbolic graphs are accessible in the sense of Thomassen and Woess [13]. Thus, they have a terminal factorisation by [9, Theorem 6.3].

Theorem 1.3. *Let $(G_1, \ldots, G_n)$ and $(H_1, \ldots, H_m)$ be terminal factorisations of infinitely-ended quasi-transitive locally finite hyperbolic graphs G and H, respectively. Then the hyperbolic boundaries ∂G and ∂H are homeomorphic if and only if the set of homeomorphism types of the hyperbolic boundaries of the factors in $(G_1, \ldots, G_n)$ are the same as those for $(H_1, \ldots, H_m)$.*

Martin and Świątkowski [11] proved group theoretic versions of Theorems 1.2 and 1.3. However, our results do not follow from theirs since it is not known whether every locally finite hyperbolic quasi-transitive graph is quasi-isometric to some hyperbolic group.

The question whether every locally finite hyperbolic quasi-transitive graph is quasi-isometric to some hyperbolic group is a special case of Woess' problem [14, Problem 1] whether every locally finite transitive graph is quasi-isometric to some locally finite Cayley graph. While his problem was settled in the negative by Eskin et al. [4], their counterexamples, the Diestel-Leader graphs, are not hyperbolic and neither is another counterexample by Dunwoody [3].

In Section 2.1 we define and discuss hyperbolicity and in Section 2.2 tree amalgamations. In both section, we also state the main preliminary results we need for our main results, which we will prove in Section 3.

2. Preliminaries

In this section, we state the main definitions and preliminary results that we need for the proofs of our theorems. First, we state some general definitions and then we look at hyperbolic graphs and tree amalgamations more closely in Sections 2.1 and 2.2, respectively.

Let G be a graph. A *ray* is a one-way infinite path and a *double ray* is a two-way infinite path. Two rays are *equivalent* if for every finite vertex set $S \subseteq V(G)$ both rays have all but finitely many vertices in the same component of $G - S$. This is an equivalence relation whose equivalence classes are the *ends* of G.

We call G *transitive* if its automorphism group acts transitively on its vertex set and *quasi-transitive* if its automorphism group acts with only finitely many orbits on $V(G)$.

A quasi-transitive graph is *accessible in the sense of Thomassen and Woess* if there exists $n \in \mathbb{N}$ such that every two ends can be separated by at most n edges.

Let G and H be graphs. A map $\varphi\colon V(G) \to V(H)$ is a *quasi-isometry* if there are constants $\gamma \geq 1$, $c \geq 0$ such that

$$\gamma^{-1} d_G(u,v) - c \leq d_H(\varphi(u), \varphi(v)) \leq \gamma d_G(u,v) + c$$

for all $u, v \in V(G)$ and such that $\sup\{d_H(v, \varphi(V(G))) \mid v \in V(H)\} \leq c$. We then say that G is *quasi-isometric* to H.

A finite or infinite path P is *geodesic* if $d_P(x,y) = d_G(x,y)$ for all vertices x, y on P. It is (γ, c)*-quasi-geodesic* if it is the (γ, c)-quasi-isometric image of a subpath of a geodesic double ray.

2.1. **Hyperbolic graphs.** In this section, we will give the definitions and state the lemmas regarding hyperbolic graphs that we need for our results. For a detailed introduction to hyperbolic graphs, we refer to [2, 5, 6].

Let G be a graph and $\delta \geq 0$. If for all $x, y, z \in V(G)$ and all shortest paths P_1, P_2, P_3, one between every two of those vertices, every vertex of P_1 has distance at most δ to some vertex on either P_2 or P_3 then G is *δ-hyperbolic*. We call G *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

Two geodesic rays in a hyperbolic graph G are *equivalent* if there is some $M \in \mathbb{N}$ such that on each ray there are infinitely many vertices of distance at most M to the other ray. This is an equivalence relation for hyperbolic graphs whose equivalence classes are the *hyperbolic boundary points* of G. By ∂G we denote the *hyperbolic boundary* of G, i. e. the set of hyperbolic boundary points of G, and we set $\widehat{G} := G \cup \partial G$.

For locally finite hyperbolic graphs, it is possible to equip $\widehat{G}$ with a topology such that $\widehat{G}$ is compact and every geodesic ray converges to the hyperbolic boundary point it is contained in, see [5, Proposition 7.2.9]. For us, it suffices to define convergence of vertex sequences to hyperbolic boundary points. Let $o \in V(G)$. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in $V(G)$. It *converges* to $\eta \in \partial G$ if for some geodesic ray $y_1y_2\ldots$ in η and some sequence $(n_i)_{i\in\mathbb{N}}$ that goes to ∞ any geodesic path from x_i to y_i has distance at least n_i to o.

Since the rays in a tree are always geodesic and two rays that are equivalent regarding the definition of ends eventually coincide, these rays are also equivalent with respect to the definition of the hyperbolic boundary. Thus, there is a canonical one-to-one correspondence between the ends of trees and their hyperbolic boundary. The following lemma follows easily.

Lemma 2.1. *The hyperbolic boundaries of locally finite trees are totally disconnected sets.* □

By its definition, the hyperbolic boundary is a refinement of the end space. But even more can be said about this relation, cf. e. g. [10, Section 7]:

Lemma 2.2. *The connected components of the hyperbolic boundary of every locally finite hyperbolic graph correspond canonically to the ends of that graph.* □

We are interested in homeomorphism types of hyperbolic boundaries. That is why the following result is important for us.

Lemma 2.3. [1, III.H Theorem 3.9] *Quasi-isometries between locally finite hyperbolic graphs induce canonical homeomorphisms between their boundaries.* □

The following lemma is a direct consequence of a result of Woess [15, Corollary 5].

Lemma 2.4. *The hyperbolic boundary of every one-ended quasi-transitive locally finite hyperbolic graph is infinite.* □

2.2. **Tree-amalgamations.** In this section, we state the definition of tree amalgamations and cite several results about them.

Let $p_1, p_2 \in \mathbb{N} \cup \{\infty\}$. A tree is *semiregular* or *(p_1, p_2)-semiregular* if all vertices in V_1 have the same degree p_1 and all vertices in V_2 have the same degree p_2, where V_1, V_2 is the canonical bipartition of its vertex set.

Let G_1 and G_2 be two graphs and let T be a (p_1, p_2)-semiregular tree with canonical bipartition V_1, V_2 of its vertex set. Let $\{S_k^i \mid 0 \le k < p_i\}$ be a set of subsets of $V(G_i)$ such that all S_k^i have the same cardinality and let $\varphi_{k,\ell} \colon S_k^1 \to S_\ell^2$ be a bijection. Let

$$c\colon E(T) \to \{(k,\ell) \mid 0 \le k < p_1, 0 \le \ell < p_2\}$$

such that for all $v \in V_i$ the i-th coordinates of the elements of $\{c(e) \mid v \in e\}$ exhaust the set $\{k \mid 0 \le k < p_i\}$.

For every $v \in V_i$ with $i = 1, 2$, let G_i^v be a copy of G_i and let S_k^v be the copy of S_k^i in G_i^v. Let $H := G_1 + G_2$ be the graph obtained from the disjoint union of all graphs G_i^v by adding an edge between all $x \in S_k^v$ and $\varphi_{k,\ell}(x) \in S_\ell^u$ for every edge $vu \in E(T)$ with $v \in V_1$ and $c(vu) = (k, \ell)$. Let G be the graph obtained from H by contracting all new edges $x\varphi_{k,\ell}(x)$, i. e. all edges outside of the graphs G_i^v. We call G the *tree amalgamation* of G_1 and G_2 over T (with respect to the sets S_k^i and the maps $\varphi_{k,\ell}$) and we denote it by $G_1 *_T G_2$. If the *amalgamation tree* T is clear from the context, we simply write $G_1 * G_2$. The sets S_k^i and their copies in G are the *adhesion sets* of the tree amalgamation. The tree amalgamation has emphfinite adhesion if each adhesion set is finite. Let $\psi\colon V(H) \to V(G)$ be such that every $x \in V(H)$ is mapped to the vertex of G it ends up after all contractions. A tree amalgamation $G_1 * G_2$ is *trivial* if there is some G_i^v such that the restriction of ψ to G_v^i is a bijection $G_i^v \to G_1 * G_2$. So a tree amalgamation of finite adhesion is trivial if $V(G_i)$ is the only adhesion set of G_i and $p_i = 1$ for some $i \in \{1, 2\}$.

The *identification size* of a vertex $x \in V(G)$ is the smallest size of subtrees T' of T such that x is obtained by contracting only edges between vertices in $\bigcup_{u \in V(T')} V(G_j^u)$. The tree amalgamation has *finite identification* if all vertices have finite identification size and it has *bounded identification* if the supremum of all identification sizes is finite.

The tree amalgamation $G = G_1 * G_2$ *distinguishes* ends if there is some adhesion set $S_k^v = S_\ell^u$ for adjacent vertices u, v of T such that for every component C of $T - uv$ the graph induced by $\bigcup_{w \in C} G_i^w$ contains an end.

The following is a weaker version of [9, Theorem 5.3]. We do not need the full detail of tree amalgamations *respecting group actions* and thus the following result is sufficient for us.

Theorem 2.5. [9, Theorem 5.3] *Every connected quasi-transitive locally finite graph with more than one end is a non-trivial tree amalgamation $G_1 * G_2$ that distinguishes ends and has finite adhesion and finite identification of two connected quasi-transitive locally finite graphs such that the set of adhesion sets in each factor has at most two orbits under some group acting quasi-transitively on that factor.* □

Note that the properties of the tree amalgamation of Theorem 2.5 imply that it has bounded identification.

Also the notions of (terminal) factorisations need not be as strong as those in the introduction. A *factorisation* of a connected locally finite quasi-transitive graph G is a tuple $(G_1, \ldots, G_n)$ of connected locally finite quasi-transitive graphs such that G is obtained from the elements of the tuple by iterated tree amalgamations distinguishing ends of finite adhesion and finite identification such that for each step some group of automorphisms of each factors acts quasi-transitively on this factor and the set of its adhesion sets. A factorisation is *terminal* if every element of the tuple has at most one end. A connected quasi-transitive locally finite graph is *accessible* if it has a terminal factorisation.

All following results in this section deal with the interplay of tree amalgamations with quasi-isometries and are proved in [8].

Lemma 2.6. [8, Remark 2.1] *Let G and H be locally finite graphs and $G * H$ be a tree amalgamation of finite adhesion and bounded identification. Then $G * H$ is quasi-isometric to $G + H$.* □

The following lemma enables us to change the factors in a tree amalgamation a bit while staying quasi-isometric to the original tree amalgamation but in the result we have more control over the adhesion sets and identification sizes.

Lemma 2.7. [8, Lemma 3.1] *Let G be a locally finite connected quasi-transitive graph and let (G_1, G_2) be a factorisation of G. Then there is a locally finite connected quasi-transitive graph H that has a factorisation (H_1, H_2) such that the following hold.*

(1) *G is quasi-isometric to H;*
(2) *G_i is quasi-isometric to H_i for $i = 1, 2$;*
(3) *$H_1 * H_2$ has adhesion 1;*
(4) *all adhesion sets of $H_1 * H_2$ are distinct;*
(5) *the adhesion sets of H_i cover H_i for $i = 1, 2$.* □

Lemma 2.8. [8, Lemma 2.9] *A connected locally finite quasi-transitive graph that has a terminal factorisation of only finite graphs is quasi-isometric to a 3-regular tree.* □

The following theorem plays a central role in our proofs and can be seen as an analogue of Theorem 1.2 for quasi-isometries of graphs instead of homeomorphisms of hyperbolic boundaries.

Theorem 2.9. [8, Theorem 1.4] *Let G and H be locally finite quasi-transitive graphs with infinitely many ends and let $(G_1, \ldots, G_n)$ and $(H_1, \ldots, H_m)$ be factorisations of G and H, respectively. If $(G_1, \ldots, G_n)$ and $(H_1, \ldots, H_m)$ have the same set of quasi-isometry types of infinite factors, then G and H are quasi-isometric.* □

3. Proofs of the main theorems

In this section we will prove our main results. First, we prove the characterisation of quasi-transitive locally finite hyperbolic graphs in terms of their terminal factorisations.

Theorem 3.1. *Let G_1 and G_2 be connected quasi-transitive locally finite graphs and let $G = G_1 * G_2$ be a tree amalgamation of finite adhesion and finite identification such that, for $i = 1, 2$, some group of automorphisms acts quasi-transitively on G_i and on the adhesion sets in G_i. Then G is hyperbolic if and only if G_1 and G_2 are hyperbolic.*

Proof. If the tree amalgamation does not distinguish ends, then one G_i has a unique adhesion set and the amalgamation tree is a star. Since G_i is quasi-transitive, it must be finite. Then it is easy to see that G is quasi-isometric to G_{3-i}. So we may assume that the tree amalgamation distinguishes ends.

Let $H = H_1 * H_2$ as in Lemma 2.7. In particular, H, H_1 and H_2 are quasi-isometric to G, G_1 and G_2, respectively, and the tree amalgamation $H_1 * H_2$ has adhesion 1. Since hyperbolicity is preserved by quasi-isometries, see e. g. [1, Theorem III.H.1.9], it suffices to prove the assertion in the situation of adhesion 1. So let us assume that $G_1 * G_2$ already has adhesion 1.

First, let us assume that G is δ-hyperbolic for some $\delta \geq 0$. Let x, y be vertices of some G_i^v. If a x-y geodesic leaves G_i^v through some adhesion set, it must re-enter G_i^v through the same adhesion set. Since the adhesion is 1, every x-y geodesic in G lies completely in G_i^v. It follows that G_i is δ-hyperbolic.

Now let us assume that G_1 and G_2 are δ-hyperbolic for some $\delta \geq 0$. Let $x, y, z \in V(G)$. For distinct $u, v \in \{x, y, z\}$, let P_{uv} be a u-v geodesic. Since the adhesion sets have size 1, in any G_i^v that intersects non-trivially with the paths P_{uv}, each of the subpaths in G_i^v induced by P_{xy}, P_{xz} and P_{yz} lies in the δ-neighbourhood of the other two if it is non-trivial. Thus, G is δ-hyperbolic, too. □

As a corollary of Theorem 3.1, we obtain a characterisation of quasi-transitive locally finite hyperbolic graphs in terms of their terminal factorisations.

Corollary 3.2. *A connected quasi-transitive locally finite graph is hyperbolic if and only if it admits a terminal factorisation such that all its factors are connected quasi-transitive locally finite hyperbolic graphs with at most one end.*

Proof. Let G be a connected quasi-transitive locally finite graph. If G is one-ended, then it is a terminal factorisation of itself and the assertion holds trivially. So let us assume that G has more than one end.

First, let us assume that G is hyperbolic. By [7, Theorem 4.3], it is a graph that is accessible in the sense of Thomassen and Woess. Thus it is accessible and has a terminal factorisation by [9, Theorem 6.3]. So there are connected quasi-transitive locally finite graphs $G_1, \ldots, G_n$, $H_1, \ldots, H_{n-1}$ with $G = H_{n-1}$ such that each G_i has at most one end and for every $i \leq n-1$, the graph H_i is a tree amalgamation $H * H'$ of finite adhesion, where

$$H, H' \in \{G_j \mid 1 \leq j \leq n\} \cup \{H_j \mid 1 \leq j < i\}.$$

(We may assume that all G_i are indeed needed at some point during these tree amalgamations.) By repeated application of Lemma 3.1, each H_i, and thus each G_i is hyperbolic.

Conversely, if G has a terminal factorisation into connected finite or connected quasi-transitive locally finite hyperbolic one-ended graphs, then each of the previous factors we considered for obtaining the terminal factorisation are hyperbolic by Lemma 3.1. In particular, G is hyperbolic. □

Now we will turn our attention to the proofs of the results concerning the hyperbolic boundary, Theorems 1.2 and 1.3.

The following lemma is a special case of a result of Steiner and Steiner [12, Theorem 4]. It is also possible adapt the proof of Martin and Świątkowski [11, Lemma 4.2] to our situation to obtain that lemma.

Lemma 3.3. *Let G and H be locally finite quasi-transitive hyperbolic graphs and let $f \colon \partial G \to \partial H$ be a homeomorphism. Then f extends to a homeomorphism $\widehat{G} \to \widehat{H}$.* □

The next lemma describes the hyperbolic boundary of a tree amalgamation in terms of its factors and the involved tree.

Lemma 3.4. *Let G and H be locally finite hyperbolic quasi-transitive graphs and let T be a semiregular tree with canonical bipartition $\{U, V\}$ of its vertex set. Then there exists a canonical bijective map*

$$f\colon \partial T \cup \bigcup_{u\in U} \partial G^u \cup \bigcup_{v\in V} \partial H^v \to \partial(G +_T H)$$

such that the following hold.

(1) *The preimage of each connected component of $\partial(G +_T H)$ is a connected component of an element of*

$$\{\partial T\} \cup \{\partial G^u \mid u \in U\} \cup \{\partial H^v \mid v \in V\};$$

(2) *every sequence in some G^u or H^v that converges to some boundary point $\eta \in \bigcup_{u\in U} \partial G^u \cup \bigcup_{v\in V} \partial H^v$ converges to $f(\eta)$ in $G +_T H$;*

(3) *every sequence $(v_i)_{i\in\mathbb{N}}$ with $v_i \in G^{t_i}$ or $v_i \in H^{t_i}$ such that $(t_i)_{i\in\mathbb{N}}$ converges to $\eta \in \partial T$ converges to $f(\eta)$ in $G +_T H$.*

Proof. Since quasi-isometries preserve hyperbolicity, we may apply Lemmas 2.7 and 2.6 to assume that $G *_T H$ is a tree amalgamation of adhesion 1 and distinct adhesion sets are disjoint. Note for this that quasi-isometries map distinct boundary points to distinct boundary points and distinct connected components of the boundary to distinct connected components.

Let us define a map

$$f\colon \partial T \cup \bigcup_{u\in U} \partial G^u \cup \bigcup_{v\in V} \partial H^v \to \partial(G +_T H).$$

Let $u \in U$ and $\eta \in \partial G^u$. Since the adhesion is 1, any geodesic ray in η is a geodesic ray in $G +_T H$ as well. Thus, two geodesic rays in $G *_T H$ that lie in G^u are equivalent in $G +_T H$ and thus lie in the same boundary point μ. We set $f(\eta) := \mu$. Analogously, we define the image of elements of ∂G^v for $v \in V$.

Now we consider a boundary point η of T. Note that since T is a tree, its boundary points are just its ends. Let R be a ray in η. Since the tree amalgamation has adhesion 1, there is for each edge uv of T with $u \in U$ and $v \in V$ a unique edge of $G +_T H$ that corresponds to uv in that its incident vertices lies in G^u and H^v and get identified when constructing the tree amalgamation. Since distinct adhesion sets are disjoint, it follows that for a subpath $u_0u_1u_2u_3$ of R the edges e_1, e_2, e_3, where e_i corresponds to the edge $u_{i-1}u_i$, have the properties that they are distinct and e_2 separates e_1 and e_3. By joining e_1 and e_2 by a shortest path inside G^{u_1} or H^{u_1}, we obtain that R defines a geodesic ray and no matter how we choose the shortest paths to connected the edges e_1, e_2, the resulting rays are equivalent and thus converge to the same boundary point μ of $G +_T H$. We set $f(\eta) := \mu$.

While defining f, we ensured that it is well-defined. Let us show that f is injective. If we consider hyperbolic boundary points η, μ of distinct elements of

$$X := \{\partial T\} \cup \{\partial G^u \mid u \in U\} \cup \{\partial H^v \mid v \in V\},$$

then each of η, μ belongs to either a hyperbolic boundary point of T or the hyperbolic boundary of a vertex of T and there is an edge of T separating these hyperbolic boundary points or vertices of T. The edge of $G +_T H$ corresponding to that edge of T separates the f-image of those hyperbolic boundary points or of the subgraph G^u or H^v. Thus, $f(\eta)$ and $f(\mu)$ lie in distinct ends of $G +_T H$ and hence are distinct. If η and μ are distinct but in a common element of X, then they lie in either ∂G^u or ∂H^v for some $u \in U$ or $v \in V$. But as geodesic paths and rays in G^u or H^v are geodesic paths and rays in $G *_T H$, inequivalent rays in G^u or G^v stay inequivalent in $G +_T H$. Thus, we have $f(\eta) \neq f(\mu)$ in this case, too.

To show that f is surjective, let $\eta \in \partial(G +_T H)$ and let R be a geodesic ray in η. Since the adhesion of $G *_T H$ is 1, there is either a subgraph G^u or H^v such that R has all but finitely many vertices of that subgraph or not. If R meets every such subgraph in only finitely many vertices, let $W \subseteq V(T)$ consist of those vertices $u \in U$ and $v \in V$ for which R meets G^u and H^v. Note that if R leaves G^u or H^v once, it has to do so through an adhesion set and since it cannot use the same edge again, it cannot enter the subgraph G^u or H^v anymore. Thus, W defines a ray in T and it is straight-forwards to see that the hyperbolic boundary point μ of T that contains this ray is mapped to η by f. If R has infinitely many vertices in a subgraph G^u or H^v, say G^u, then it has a subray in G^u by the above argument that if it leaves G^u once, it never reenters G^u. This subray is geodesic in G^u since the adhesion sets have size 1. It lies in some hyperbolic boundary point μ of G^u and we have $f(\mu) = \eta$ by construction.

So far, we constructed a canonical bijective map f that satisfies (2) and (3). It remains to verify (1). For this, we note that the connected components of the hyperbolic boundary correspond to the ends of the graph by Lemma 2.2. Analogously as in the proof that f is injective, it follows that distinct hyperbolic boundary points in the same end of $G +_T H$ are mapped to hyperbolic boundary points of some G^u or G^v that lie in the same end of that graph. □

The following proposition is the main step towards the proof of Theorem 1.2.

Proposition 3.5. *Let G_1, G_2, H_1 and H_2 be locally finite infinite hyperbolic quasi-transitive graphs such that $\partial G_1, \partial G_2$ is homeomorphic to $\partial H_1, \partial H_2$, respectively. Let G and H be quasi-transitive locally finite graphs such that (G_1, G_2) is a factorisation of G and (H_1, H_2) is a factorisation of H. Then ∂G and ∂H are homeomorphic.*

Proof. According to Lemma 2.7, we find locally finite connected quasi-transitive graphs G_1', G_2', H_1', H_2' such that G_i, H_i is quasi-isometric to G_i', to H_i' for $i = 1, 2$, such that (G_1', G_2'), (H_1', H_2') are factorisations of graphs G', H' that are quasi-isometric to tree amalgamations $G_1 * G_2$, $H_1 * H_2$, respectively, such that $G_1' * G_2'$ and $H_1' * H_2'$ have adhesion 1, all their adhesion sets are disjoint and every vertex lies in some adhesion set. As hyperbolicity is preserved by quasi-isometries, G_1', G_2', H_1', H_2' are hyperbolic. By Theorem 1.1, G' and H' are hyperbolic, too. Since quasi-isometric hyperbolic graphs have homeomorphic hyperbolic boundaries by Lemma 2.3, the hyperbolic boundaries $\partial(G_1 * G_2)$, $\partial(H_1 * H_2)$ are homeomorphic to $\partial G'$, $\partial H'$, respectively. Thus, it suffices to prove the assertion under the assumption that the tree amalgamation is of adhesion 1, all adhesion sets are disjoint and the adhesion sets cover all vertices. By Lemma 2.6 the graphs $G_1 * G_2$, $H_1 * H_2$ are quasi-isometric to $G_1 + G_2$, to $H_1 + H_2$, respectively. Thus, it suffices to consider those graphs instead of the tree amalgamations.

For $i = 1, 2$, let $g_i\colon \partial G_i \to \partial H_i$ be a homeomorphism and let $f_i\colon \widehat{G}_i \to \widehat{H}_i$ be a homeomorphism that extends g_i, which exists by Lemma 3.3. Let S_k^G, S_k^H be the adhesion sets in G_1, in H_1 and T_ℓ^G, T_ℓ^H be the adhesion sets in G_2, in H_2, respectively. Note that the amalgamation trees in both cases are the countably infinitely regular tree, i. e. we consider the tree amalgamations $G_1 *_T G_2$ and $H_1 *_T$

H_2. Let c_G, c_H be the labelings of the edges of T used for the tree amalgamations $G_1 *_T G_2$, for $H_1 *_T H_2$, respectively. We are going to construct a map $f\colon G_1+G_2 \to H_1+H_2$ with the following properties, which obviously proves the assertion.

(1) f is a bijection;
(2) f induces an automorphism f_t of T;
(3) f induces homeomorphisms $\widehat{G}_i^u \to \widehat{H}_i^{f_t(u)}$;
(4) f induces a homeomorphism $g\colon \partial(G_1+G_2) \to \partial(H_1+H_2)$.

The homeomorphisms in (3) will be closely related to the homeomorphisms f_i.

Let $v_1, v_2, \ldots$ be an enumeration of $V(T)$ such that for every $i \in \mathbb{N}$ the vertices $v_1, \ldots, v_i$ induce a subtree of T. Let $f_{i_1}^{v_1}$ be the map $G_{i_1}^{v_1} \to H_{i_1}^{v_1}$ that is induced by f_{i_1}. For $j>1$, let u_j be the vertex in $G_{i_j}^{v_j}$ that separates $G_{i_j}^{v_j}$ from $G_{i_1}^{v_1}$ and let $t \in \{v_1, \ldots, v_{j-1}\}$ be the unique neighbour of v_j in that set. Let $g_{i_j}^{v_j}$ be the map $G_{i_j}^{v_j} \to H_{i_j}^{v_j}$ induced by f_i.

For $j>1$, let w_j be the vertex in $H_{i_j}^{v_j}$ that separates $H_{i_j}^{v_j}$ from $H_{i_1}^{v_1}$. Let $h_{i_j}^{v_j}$ be $g_{i_j}^{v_j}$ but the images of u_j and $(g_{i_j}^{v_j})^{-1}(w_j)$ exchanged. Note that $h_{i_j}^{v_j}$ still induces a homeomorphism $\widehat{G}_{i_j}^{v_j} \to \widehat{H}_{i_j}^{v_j}$ but now the vertex in G_{3-j}^t that is adjacent to u_j in the graph G_1+G_2 is mapped by $h_{i_{3-j}}^t$ to the neighbour of w_j in $H_{i_{3-j}}^t$ in the graph H_1+H_2. Hence, the union of all h_i^v for all $v \in V(T)$ defines a map $f\colon G_1+G_2 \to H_1+H_2$ that maps vertices of G_1+G_2 that gets identified by constructing $G_1 * G_2$ to those that gets identified by constructing $H_1 * H_2$.

By construction, (1)–(3) holds. Since the compactifications of locally finite hyperbolic graphs are Hausdorff, it suffices to prove that f induces a bijective continuous map $g\colon \partial(G_1+G_2) \to \partial(H_1+H_2)$. For this, we use Lemma 3.4 to get a map

$$\varphi\colon \partial T \cup \bigcup_{u\in U} \partial G_1^u \cup \bigcup_{v \in V} \partial G_2^v \to \partial(G_1 +_T G_2)$$

and a map

$$\psi\colon \partial T \cup \bigcup_{u\in U} \partial H_1^u \cup \bigcup_{v \in V} \partial H_2^v \to \partial(H_1 +_T H_2)$$

both having the properties as in Lemma 3.4. Then $g := \psi \circ \varphi^{-1}$ is a bijective map and it follows from the construction of φ and ψ that f induces the restriction of g to those boundary points of G_1+G_2 that are are not in the image of ∂T by φ or ψ. It is not hard to see that g is also induced by f on the remaining boundary points. In order to show that g is continuous, we show the slightly stronger assertion that $f \cup g$ is continuous. For this, it suffices to consider a sequence $(x_i)_{i\in\mathbb{N}}$ in G_1+G_2 that converges to some $\eta \in \partial(G_1+G_2)$ and show that $(f(x_i))_{i\in\mathbb{N}}$ converges to $g(\eta)$.

If $\varphi^{-1}(\eta) \in \partial(G_i^u)$ for some $i \in \{1,2\}$ and $u \in V(T)$, let $(y_i)_{i\in\mathbb{N}}$ be a sequence in G_i^u such that $x_i = y_i$ if $x_i \in V(G_i^u)$ and such that y_i separates x_i from G_i^u otherwise. Then $(y_i)_{i\in\mathbb{N}}$ converges to η, too, and $(f(x_i))_{i\in\mathbb{N}}$ and $(f(y_i))_{i\in\mathbb{N}}$ converge to the same boundary point of H_1+H_2 by construction. Since $(f(y_i))_{i\in\mathbb{N}}$ lies in H_i^u, it converges to $g(\eta)$ and hence $(f(x_i))_{i\in\mathbb{N}}$ converges to $g(\eta)$.

If $\varphi^{-1}(\eta)$ lies in no $\partial(G_i^u)$, then it lies in ∂T. Let $t_1t_2\ldots$ be a ray in T that converges to $\varphi^{-1}(\eta)$. Let $(y_i)_{i\in\mathbb{N}}$ be such that y_i separates x_i and η and such that y_i lies in some $G_j^{t_k}$. Then $(y_i)_{i\in\mathbb{N}}$ converges to η, too. As is the previous case, $(f(x_i))_{i\in\mathbb{N}}$ and $(f(y_i))_{i\in\mathbb{N}}$ converge to the same boundary point of H_1+H_2, which is $g(\eta)$ by construction. Thus, $f \cup g$ is continuous. $\square$

Now we are able to prove our main theorems.

Proof of Theorem 1.2. We will prove the assertion by induction on the number of homeomorphism classes of the hyperbolic boundaries ∂G_i. If there are no homeomorphism classes, then all graphs G_i are finite and so are the graphs H_i. By Lemma 2.8, G and H are quasi-isometric to 3-regular trees and hence G is quasi-isometric to H.

Let us now assume that there is at least one homeomorphism class of hyperbolic boundaries ∂G_i. Let $G_{i_1}, \ldots, G_{i_k}$, $H_{j_1}, \ldots, H_{j_\ell}$ be representatives of the infinite quasi-isometry types of $G_1, \ldots G_n$, of $H_1, \ldots, H_m$, respectively. By Theorem 2.9, G is quasi-isometric to either $G_{i_1} * G_{i_1}$, if $k = 1$, or $G_{i_1} * \ldots * G_{i_k}$, if $k > 1$, and similarly H is quasi-isometric to either $H_{j_1} * H_{j_1}$ or $H_{j_1} * \ldots * H_{j_\ell}$. Note that the homeomorphism types of the G_{i_p}'s and H_{i_q}'s are the same. We apply Theorem 2.9 again to duplicate factors such that G is quasi-isometric to $G' := G'_1, \ldots, G'_{m'}$ and H is quasi-isometric to $H' := H'_1, \ldots, H'_{n'}$, where G'_i and H'_i have homeomorphic hyperbolic boundaries. As quasi-isometries do not change the homeomorphism type of the hyperbolic boundary by Lemma 2.3, G and G' as well as H and H' have homeomorphic hyperbolic boundaries and by Proposition 3.5 the assertion follows by induction as all factors have a hyperbolic boundary and thus are infinite. □

Proof of Theorem 1.3. If the factors have the same homeomorphism types of hyperbolic boundaries, it follows from Theorem 1.2 that the hyperbolic boundaries of G and H are homeomorphic.

Let us now assume that ∂G and ∂H are homeomorphic. By Lemmas 2.1, 2.2, 2.4 and 3.4, the non-singular connected components of G are the hyperbolic boundaries of the G_i's and the non-singular connected components of ∂H are the hyperbolic boundaries of the H_i's. Thus, the homeomorphism types of $\{\partial G_1, \ldots, \partial G_n\}$ are those of $\{H_1, \ldots, \partial H_m\}$. □

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