

GENERATING THE CYCLE SPACE OF PLANAR GRAPHS

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ABSTRACT. We prove that the cycle space of every planar finitely separable 3-connected graph G is generated by some $\text{Aut}(G)$ -invariant nested set of cycles. We also discuss the situation in the case of smaller connectivity.

1. INTRODUCTION

Dicks and Dunwoody [2] showed that the cut space of every connected graph is generated by some nested set of cuts that is invariant under the automorphisms of the graph, where a set of cuts is *nested* if no cut separates any two edges of any other cut. Recently, Dunwoody [5] strengthened this by showing that there is also a *canonical* such set. This means roughly that no choices were made during the construction of this set or, more precisely, that the construction of the set commutes with graph isomorphisms.

In this note, we present a similar result for the cycle space of a planar graph. Roughly speaking, two cycles in a planar graph are *nested* if their embeddings in the plane do not cross (see Section 2 for a precise definition). Similarly as above, a set of cycles is *canonical* if one of its constructions commutes with graph isomorphisms.¹ We call a graph *finitely separable* if

- (1) *there are only finitely many pairwise internally disjoint paths between any two vertices.*

With these definitions, we are able to state our main theorem:

Theorem 1. *Every planar finitely separable 3-connected graph G has a canonical nested set of cycles that generates its cycle space.*

Note that Theorem 1 is rather trivial for finite graphs: the face boundaries form a nested set of cycles that generates the cycle space and is canonical, as Whitney [12] proved that 3-connected graphs have a unique embedding in the plane (cf. Theorem 4).²

Theorem 1 can be seen as a dual version of Dunwoody's theorem [5], and indeed, our strategy to prove our main theorem is to deduce it from the fact that Dunwoody's theorem holds for its dual graph. Of course, it is necessary to have a (unique) dual graph. Duals of infinite planar graphs are not straightforward analogues of the finite ones, see Thomassen [10]. But Bruhn and Diestel [1] showed that the additional assumption that the graphs are finitely separable implies the existence of unique duals in the 3-connected case.

¹Note that for any graph G every canonical set of cuts and every canonical set of cycles is invariant under the automorphisms of G .

²The same is proof holds for *VAP-free* planar graphs. These are graphs that admit an embedding in the plane without any accumulation point of the vertices.

Theorem 1 fails for graphs that have lower connectivity – regardless which embedding in the plane is considered: take a graph G consisting of two vertices x, y and four distinct paths of length 2 between them. Obviously, this graph is planar. As all cycles in G , each of which has length 4, lie in the same $\text{Aut}(G)$ -orbit, all of them must be chosen in any canonical set that generates the cycle space. But it is easy to find two crossing cycles in G .

Even though Theorem 1 fails for graphs of lower connectivity, it is possible to extend the graph canonically to a graph G' with equal connectivity such that G' has a canonical nested set of cycles generating its cycle space. We will discuss this in Section 4.

It is not clear whether Theorem 1 remains true if we omit the assumption of finite separability:

Problem. Does every planar 3-connected graph G have a canonical nested set of cycles that generates its cycles space?

Our main theorem has applications for infinite graphs and infinite groups: in [7], the author uses it to show that locally finite quasi-transitive planar graphs are accessible, and in [6], Georgakopoulos and the author apply it to obtain planar presentations for planar groups. (*Planar presentations* of planar groups are presentations that directly tell that the group has a planar Cayley graph.)

2. PRELIMINARIES

Throughout the paper, a graph may have loops and multiple edges (what is usually called a multigraph).³ Let G be a graph. A one-way infinite path is a *ray*. Two rays are *equivalent* if they lie eventually in the same component of $G - S$ for every finite $S \subseteq V(G)$. This is an equivalence relation whose equivalence classes are the *ends* of G . By $\Omega(G)$ we denote the set of ends of G .

Now we define a basis of a topology on $G \cup \Omega(G)$. In order to do so, we view every edge as an isometric copy of the unit interval. For every $v \in V(G)$ and every $n \in \mathbb{N}$, the set of all points on edges incident with v that have distance less than $1/n$ to v are open. For every end ω and every finite vertex set S , let $C(S, \omega)$ be the component of $G - S$ that contains all rays in ω eventually and let $\hat{C}(S, \omega)$ be $C(S, \omega)$ together with all ends of G that contains rays of $C(S, \omega)$. The set $\hat{C}(S, \omega)$ together with the interior points of the edges between S and $C(S, \omega)$ is open. We denote by $|G|$ the topological space on $G \cup \Omega(G)$ defined by all these open sets.

A vertex v *dominates* an end ω if for some ray R in ω there are infinitely many v - R paths in G that pairwise meet only in v . We call the end ω *dominated*. We define \tilde{G} to be the quotient space obtained from $|G|$ by identifying every dominated end with its dominating vertices. Note that for finitely separable graphs every end is identified with at most one vertex and that \tilde{G} and $|G|$ coincide if G is locally finite.

A *circle* is a homeomorphic copy of the unit circle in \tilde{G} and its *circuit* is the set of edges that it contains. The *sum* (over \mathbb{F}_2) of finitely many cycles $(C_i)_{i \in I}$ in G is the set of those edges that occur in an odd number of C_i . The set of all sums of

³Note that all cited results that we need for our main result are valid for this general notion of graphs. Most of them are stated in this way, e.g. the result of Dicks and Dunwoody. The only result not stated in this way is Theorem 6, but its proof can easily be adapted to multigraphs. (Note that its forerunner [2] is stated for multigraphs.)

cycles forms a vector space over \mathbb{F}_2 , the *cycle space* $\mathcal{C}(G)$ of G . For every $n \in \mathbb{N}$, let $\mathcal{C}_n(G)$ be the subspace of the cycle space \mathcal{C} that is generated by the cycles of length at most n . So we have $\mathcal{C}(G) = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n(G)$. The automorphisms of G act canonically on $\mathcal{C}(G)$.

A *cut* is a subset B of $E(G)$ that is, for some bipartition $\{X, Y\}$ of $V(G)$, the set of all those edges that have one of its incident vertices in X and the other in Y . We call two cuts B_1 and B_2 *nested* if for bipartitions $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$ that correspond to B_1 and B_2 , respectively, one of the following holds:

$$X_1 \subseteq X_2, \quad X_1 \subseteq Y_1, \quad X_1 \supseteq X_2, \quad \text{or} \quad X_1 \supseteq Y_1.$$

A cut B is *tight* if $G - B$ consists of two components. The *sum* (over \mathbb{F}_2) of finitely many cuts in G is the set of those edges that occur in an odd number of these cuts. The set of all sums of finite cuts form a vector space over \mathbb{F}_2 , the *cut space* $\mathcal{B}(G)$ of G . For $n \in \mathbb{N}$ let $\mathcal{B}_n(G)$ be the subspace of $\mathcal{B}(G)$ induced by the tight cuts of size at most n . So we have $\mathcal{B}(G) = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n(G)$. Note that no element of $\mathcal{B}(G)$ is infinite and that $\text{Aut}(G)$ acts canonically on $\mathcal{B}(G)$.

Let G and G^* be graphs such that G is finitely separable and let

$$*: E(G) \rightarrow E(G^*)$$

be a bijection. We call G^* the *dual* of G if for every (finite or infinite) $F \subseteq E(G)$ the following holds:

(2) F is a circuit in G if and only if $\{f^* \mid f \in F\}$ is a tight non-empty cut.

Theorem 2. [1, Corollary 3.5] *Every planar finitely separable 3-connected graph has a unique dual.* \square

Theorem 3. [10, Theorem 4.5] *Let G and G^* be dual graphs. Then G is 3-connected if and only if G^* is 3-connected.* \square

A *face* of a planar embedding $\varphi: G \rightarrow \mathbb{R}^2$ is a component of $\mathbb{R}^2 \setminus \varphi(G)$. The boundary of a face F is the set of vertices and edges of G that are mapped by φ to the closure of F . A path in G is *facial* if it is contained in the closure of a face. The following theorem is due to Whitney [12, Theorem 11] for finite graphs and due to Imrich [8] for infinite ones.

Theorem 4. [8] *Let G be a 3-connected graph embedded in the sphere. Then every automorphism of G maps each facial path to a facial path.*

In particular, every automorphism of G extends to a homeomorphism of the sphere. \square

So every planar 3-connected graph has, basically, a unique planar embedding into the plane. In the remainder of this paper we always assume this implicitly if we talk about planar embeddings of planar 3-connected graphs.

Duals of finite graphs, can be easily found using the plane, which we will just recall here, as we will use this well-known fact (see e.g. [3, Section 4.6]).

Remark 5. *Let G be a finite planar graph with planar embedding $\varphi: G \rightarrow \mathbb{R}^2$. Let G' be the graph whose vertices are the faces of $\varphi(G)$ and where two vertices are adjacent if their boundaries contain a common edge of G . Then G' is the dual of G .*

For a planar graph G with planar embedding $\varphi: G \rightarrow \mathbb{R}^2$, we call a set \mathcal{D} of cycles *nested* if for no $C, D \in \mathcal{D}$ both faces of $\mathbb{R}^2 \setminus \varphi(C)$ contain vertices of D . So

$\varphi(D)$ lies in one face of $\mathbb{R}^2 \setminus \varphi(C)$ together with its boundary $\varphi(C)$. We call two cycles C_1, C_2 *nested* if $\{C_1, C_2\}$ is nested.

3. PLANAR 3-CONNECTED GRAPHS

Our main tool in the proof of our main theorem will be the following theorem:

Theorem 6. [5, Lemma 3.2] *If G is a connected graph, then there is a sequence $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \dots$ of subsets of $\mathcal{B}(G)$ such that each \mathcal{E}_n is a canonical nested set of tight cuts of order at most n that generates $\mathcal{B}_n(G)$. \square*

Basically, our proof goes as follows. We consider the dual graph G^* of G and apply Theorem 6 to G^* to find canonical nested sets of cuts. These sets then will define the sequence of canonical nested sets of cycles in G . In preparation of that proof, we shall prove a proposition first.

Proposition 7. *Let G be a planar finitely separable 3-connected graph and let G^* be its dual. If C_1 and C_2 are cycles in G such that the cuts $C_i^* := \{e^* \mid e \in C_i\}$ are nested, then C_1 and C_2 are nested.*

Proof. Let $\varphi: G \rightarrow \mathbb{R}^2$ be a planar embedding of G . Let us suppose that C_1 and C_2 are not nested. Let $H \subseteq G^*$ be a finite connected subgraph that contains C_1^* and C_2^* as tight cuts. Then H^* contains the two cycles C_1 and C_2 . For $i = 1, 2$, let A_i be the unbounded face of $\mathbb{R}^2 \setminus \varphi(C_i)$ and let B_i be its bounded face. As C_1 and C_2 are not nested, each of A_1 and B_1 contains edges from C_2 . Due to Remark 5, the tight cut C_1^* separates edges from C_2^* in H . So each of the two components of $G^* - C_1^*$ contains vertices of each of the two components of $G^* - C_2^*$. Thus, C_1^* and C_2^* are not nested in H and hence neither in G^* . \square

Now we are able to prove Theorem 8, a sharpened version of Theorem 1.

Theorem 8. *For every planar finitely separable 3-connected graph G and all $n \in \mathbb{N}$ there exists a canonical nested set \mathcal{D}_n of cycles of length at most n such that \mathcal{D}_n generates $\mathcal{C}_n(G)$.*

In addition, we may choose the sets \mathcal{D}_n such that $\mathcal{D}_{n-1} \subseteq \mathcal{D}_n$.

Note that Theorem 1 is a direct consequence of Theorem 8: just take the set $\bigcup_{n \in \mathbb{N}} \mathcal{D}_n$. Obviously, this is canonical and nested and it generates the whole cycle space of G .

Proof of Theorem 8. Let G be a planar finitely separable 3-connected graph with planar embedding $\varphi: G \rightarrow \mathbb{R}^2$. Due to Theorem 2, it has a unique dual G^* , which is 3-connected due to Thomassen [10, Theorem 4.5]. So we can apply Theorem 6 and obtain a sequence $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \dots$ of subsets of $\mathcal{B}(G^*)$ such that each \mathcal{E}_n is a canonical nested set of tight cuts of order at most n that generates $\mathcal{B}_n(G^*)$.

We shall prove that the sequence of the sets

$$\mathcal{D}_n := \{B^* \mid B \in \mathcal{E}_n\}$$

with $B^* := \{e^* \mid e \in B\}$ satisfies the assertion. Note that \mathcal{D}_n is a set of cycles of length at most n as \mathcal{E}_n is a set of finite tight cuts of size at most n . The set \mathcal{D}_n is nested due to Proposition 7 and it is canonical, as \mathcal{E}_n is canonical and mapping this into the edge set of the dual G^* commutes with graph isomorphisms (for 3-connected graphs). As \mathcal{E}_n generates $\mathcal{B}_n(G^*)$, the set \mathcal{D}_n generates $\mathcal{C}_n(G)$: it suffices to show that every cycle C of length at most n is generated by \mathcal{D}_n , but as the tight

cut C^* is the sum of finitely many elements of \mathcal{E}_n , the dual sets in $E(G)$ of those summands sum to C . This finishes the proof. \square

4. PLANAR GRAPHS OF SMALL CONNECTIVITY

Tutte [11] proved for finite graphs that 2-connected graphs can be decomposed into ‘3-connected parts’ and cycles. Later, his result was extended by Droms et al. [4] to locally finite graphs and by Richter [9] to graphs of arbitrary degree. In order to state this result formally, we need some definitions.

Let G be a graph. A *tree-decomposition* of G is a pair (T, \mathcal{V}) of a tree T and a family $\mathcal{V} = (V_t)_{t \in T}$ of vertex sets $V_t \subseteq V(G)$, one for each vertex of T , such that

- (T1) $V = \bigcup_{t \in T} V_t$;
- (T2) for every $e \in E(G)$ there exists some $t \in V(T)$ such that V_t contains both end vertices of e ;
- (T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_2 lies on the t_1 – t_3 path in T .

The sets V_t are the *parts* of the tree-decomposition (T, \mathcal{V}) and we also call the graph induced by V_t a *part* of (T, \mathcal{V}) . The intersections $V_{t_1} \cap V_{t_2}$ for edges $t_1 t_2$ of T are its *adhesion sets* and the maximum size of the adhesion sets is the *adhesion* of (T, \mathcal{V}) . For a part V_t , its *torso* is the graph whose vertex set is V_t and whose edge set is

$$\{xy \in E(G) \mid x, y \in V_t\} \cup \{xy \mid \{x, y\} \subseteq V_t \text{ lies in an adhesion set}\}.$$

There are several ways of constructing tree-decompositions. If its construction commutes with graph isomorphisms⁴ we call the tree-decomposition *canonical*.

Theorem 9. [9] *Every 2-connected graph G has a canonical tree-decomposition of adhesion 2 each of whose torsos is either 3-connected or a cycle or a complete graph on two vertices.* \square

We call a tree-decomposition as in Theorem 9 a *Tutte decomposition*. Note that there may be more than one Tutte decomposition even though it is a canonical tree-decomposition.

Theorem 10. *Let G be a planar finitely separable 2-connected graph. If G has a Tutte decomposition (T, \mathcal{V}) whose adhesion sets of (T, \mathcal{V}) are complete graphs, then G has a canonical nested set of cycles that generates its cycles space.*

In addition, we may choose the generating cycles so that each of them lies in a unique part of (T, \mathcal{V}) .

Proof. Due to the assumption on the adhesion sets, we know that not only the torsos but also the parts themselves are either cycles or 3-connected. First, we show:

- (3) *Cycles of distinct parts are nested.*

Let φ be the planar embedding of G into \mathbb{R}^2 and let C_1, C_2 be cycles in distinct parts. If C_1 and C_2 are not nested, then both faces of $\mathbb{R}^2 \setminus \varphi(C_1)$ contain vertices of C_2 and thus C_1 and C_2 have two common vertices. As they lie in distinct parts and the intersection of distinct parts lies in some adhesion set, which has size 2,

⁴Note that graph isomorphisms induce isomorphisms between tree-decompositions in the following sense: every graph isomorphism $\varphi: G \rightarrow G'$ induces a canonical bijection between the vertex sets of G and G' and for every tree-decomposition (T, \mathcal{V}) of G , there is a tree-decomposition (T, \mathcal{V}') of G' with $\mathcal{V}' = \{\varphi(V_t) \mid V_t \in \mathcal{V}\}$.

the cycles C_1 and C_2 have precisely two common vertices x, y and these form an adhesion set of (T, \mathcal{V}) . Let V_i be the part that contains C_i for $i = 1, 2$. If one of the parts, say V_1 , is a cycle, then C_1 contains the edge xy . As it contains no other vertex of C_2 , it cannot contain vertices of both faces of $\mathbb{R}^2 \setminus \varphi(C_2)$. Thus, each of the two parts V_i is 3-connected. In particular, we find in V_1 a third path P from every vertex of the bounded face of $\mathbb{R}^2 \setminus \varphi(C_2)$ to every vertex of the unbounded face. But then $\varphi(P)$ crosses $\varphi(C_2)$, so there is a third vertex in $V_1 \cap V_2$. This contradiction shows (3).

Since (T, \mathcal{V}) is canonical, parts that are cycles are mapped to parts that are cycles. So if we take the set \mathcal{D}_1 of cycles that appear as a part in (T, \mathcal{V}) , then this set is canonical and nested due to (3). Let \mathcal{H} be the set of 3-connected parts of (T, \mathcal{V}) . For every $H \in \mathcal{H}$ we find, by Theorem 1, a canonical nested set \mathcal{D}_H of cycles generating the cycles space of H . We may also assume that for $H \in \mathcal{H}$ and $\alpha \in \text{Aut}(G)$, we have $\alpha(\mathcal{D}_H) = \mathcal{D}_{\alpha(H)}$.

As (T, \mathcal{V}) is canonical, the set

$$\mathcal{D}_2 := \bigcup_{H \in \mathcal{H}} \mathcal{D}_H$$

is canonical and it is nested by (3). We claim that $\mathcal{D} := \mathcal{D}_1 \cup \mathcal{D}_2$ satisfies the assertion. The additional part is obvious by definition of \mathcal{D} . As \mathcal{D} is nested and canonical, it remains to prove that it generates the cycle space of G . So let C be any element of the cycles space of G . It suffices to consider the case that C is a cycle, as all cycles generate the cycle space. If C lies in a part of (T, \mathcal{V}) , then it is generated by \mathcal{D} by definition of \mathcal{D} . If C does not lie in any part of (T, \mathcal{V}) , then there is some adhesion set $\{x, y\}$ of (T, \mathcal{V}) that disconnects C . Let P_1, P_2 be the two distinct x - y paths on C . Then C is the sum of $P_1 + yx$ and $P_2 + yx$. By induction on the length of C , the cycles $P_1 + yx$ and $P_2 + yx$ are generated by \mathcal{D} and so is C . \square

Note that we can embed any planar 2-connected graph G in a planar 2-connected graph G' such that G' is obtained from G by making the adhesion sets of a Tutte decomposition (T, \mathcal{V}) of G complete. This is always possible as any adhesion set separates the graph, so lies on the boundary of some face and disconnects this face boundary; thus, it is not possible that two new such edges ‘cross’⁵ since no adhesion set of any tree-decomposition can separate any other of its adhesion sets.

As each cycle of a graph lies in a unique maximal 2-connected subgraph, an immediate consequence of Theorem 10 using the well-known block-cutvertex tree and the fact that cycles in distinct components are disjoint is the following:

Theorem 11. *Let G be a planar finitely separable graph such that for each of its maximal 2-connected subgraphs the adhesion sets of a Tutte decomposition are complete graphs on two vertices. Then G has a canonical nested set of cycles that generates its cycles space.* \square

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⁵This shall mean that adding one of those edges results in a planar graph where the end vertices of the second edge do not lie on any common face boundary.

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