PERIODIC COLORINGS AND ORIENTATIONS IN INFINITE GRAPHS

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Abstract. We study the existence of periodic colorings and orientations in locally finite graphs. A coloring or orientation of a graph G is *periodic* if the resulting colored or oriented graph is *quasi-transitive*, meaning that $V(G)$ has finitely many orbits under the action of the group of automorphisms of G preserving the coloring or the orientation. When such a periodic coloring or orientation of G exists, G itself must be quasi-transitive and it is natural to investigate when quasi-transitive graphs have such periodic colorings or orientations. We provide examples of Cayley graphs with no periodic orientation or non-trivial coloring, and examples of quasi-transitive graphs of treewidth 2 without periodic orientation or proper coloring. On the other hand we show that every quasi-transitive graph G of bounded pathwidth has a periodic proper coloring with $\chi(G)$ colors and a periodic orientation. We relate these problems with techniques and questions from symbolic dynamics and distributed computing and conclude with a number of open problems.

1. Introduction

The purpose of this paper is to investigate when highly-symmetric graphs have highly-symmetric colorings or orientations. Questions of the same flavor have been raised in a number of different settings. In the emerging field of descriptive graph theory (see [Pik21] for a recent survey), the graphs under study are equipped with a topology or a measure, and the goal is to find combinatorial objects, such as proper colorings, that behave well with respect to the underlying topology or measure. This defines notions such as the Borel chromatic number of a Borel graph, and these notions are then compared with the classical chromatic number for various graph classes. Another example is that of the recursive chromatic number of a recursive graph: here the graphs under consideration are recursive sets (adjacency in the graph can be computed by a Turing machine), and the goal is to find a proper coloring which is also recursive [Bea76, Kie81, Sch80].

The early motivation for studying recursive colorings of infinite (recursive) graphs is that a number of classical coloring results for infinite graphs [dBE51, Got51] use compactness arguments and thus do not provide an explicit description of the resulting colorings. If a graph G is highly-symmetric, for instance if G is the Cayley graph of some group Γ, a simple way to give a finite description of a proper coloring of G is to assign colors to a finite subset X of vertices of G, and then transfer the colors of X to all the remaining vertices using the action of a specific subgroup Γ' of Γ on the graph G. This provides an explicit, finite description of a coloring of G, provided for instance that Γ and Γ' are

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automatic. As a simple example of this procedure, consider any Cayley graph G of $\mathbb{Z}^2$, and observe that any precoloring of any large enough square region around the identity, with pairwise distinct colors, can be extended to a proper coloring of G by translation.

We now explain what we mean by highly-symmetric graph or coloring, starting with graphs. A natural choice is the class of *vertex-transitive* graphs, which are graphs G in which for any two vertices u, v of G, there is an automorphism of G that maps u to v (in other words, the automorphism group $\mathrm{Aut}(G)$ of G acts transitively on G). Note that this class contains all Cayley graphs. We will indeed consider the slightly more general class of *quasi-transitive* graphs, which are graphs G such that the vertex set of G has finitely many orbits under the action of $\mathrm{Aut}(G)$. This means that $V(G)$ can be partitioned into $V_1, \ldots, V_k$ so that for any $1 \leqslant i \leqslant k$ and any $u, v \in V_i$, there is an automorphism of G that maps u to v. Every vertex-transitive graph is quasi-transitive. In this paper, all the graphs we consider will be *locally finite*, meaning that every vertex has finite degree. Note that for quasi-transitive graphs, being locally finite is equivalent to having bounded degree.

One motivation for considering quasi-transitive graphs instead of transitive graphs is that these graphs are much less rigid (which allows inductive approaches involving tree-decompositions into simpler quasi-transitive graphs [EGLD24, Ham15, Ham18]) and even though the class is more general, several classical group decomposition results such as Stallings theorem still apply to them [HLMR22]. The definition of quasi-transitive graph can be naturally extended to vertex-colored (resp. edge-colored or oriented) graphs by restricting the group of automorphisms to automorphisms preserving the colors of the vertices (resp. the colors or orientations of the edges).

We say that a vertex- or edge-coloring or an orientation of a graph G is *periodic* if the resulting vertex- or edge-colored or oriented graph is quasi-transitive. In particular, observe that every periodic coloring of a graph G always involves a finite number of colors. Note that this definition corresponds precisely to the procedure described above: in the case of a periodic vertex coloring we only have to fix the colors of a finite number of vertices (one in each vertex-orbit for the action of the subgroup of color-preserving automorphisms), and then this coloring naturally extends to the whole graph by the action of this subgroup. Similarly, in a periodic edge-coloring or orientation, we only have to fix the colors or orientations of a finite set of edges and these extend to the whole graph by a group action.

Note that if a periodic coloring or orientation exists, then G itself must be quasi-transitive. This motivates the following questions, which were raised in [EGLD24].

Problem 1.1 (Problem 6.3 in [EGLD24])**.** *Is it true that any locally finite quasi-transitive graph has a periodic proper vertex-coloring?*

Problem 1.2 (Problem 6.4 in [EGLD24])**.** *Is it true that any locally finite quasi-transitive graph has a periodic orientation?*

These two questions are specific instances of a more general problem, which asks whether every quasi-transitive graph G can be decorated with a non-trivial additional structure such that the graph G is still quasi-transitive if we restrict ourselves to automorphisms that fix the additional structure. It should be noted that the original motivation for raising Problems 1.1 and 1.2 is a bit different from the motivation presented in this introduction (the original goal of the authors was to find periodic structures in specific quasi-transitive graphs in order to simplify some canonical decompositions).

Observe that a positive answer to Problem 1.1 would imply a positive answer to Problem 1.2: the orientation defined from a vertex-coloring by choosing a total ordering on the colors and orienting each edge from the endpoint with the smaller color to the endpoint with the larger color is preserved by every automorphism that preserves the coloring.

The goal of this note is to present examples showing that Problems 1.1 and 1.2 have negative answers in general. The first example showing that Problem 1.1 has a negative answer was constructed by Hamann and Möller. It was then observed by Abrishami, Esperet and Giocanti, and independently by Norin and Przytycki, that a variant of this example could also be used to provide a negative answer to Problem 1.2. Norin and Przytycki furthermore showed that the examples can be chosen to be Cayley graphs (and thus vertex-transitive), rather than merely quasi-transitive. The examples are based on the existence of finitely generated infinite simple groups, whose known constructions are highly non-trivial. As the resulting graphs are 1-ended, a natural question is whether similar negative examples can be obtained for graphs with 2 or infinitely many ends. We indeed present a simple alternative construction of a negative answer to Problems 1.1 and 1.2 which has treewidth 2 (and is in particular planar and ∞-ended) and is perfect. We also construct a graph G which has a periodic proper coloring with $\chi(G)+1$ colors, but admits no periodic proper coloring with $\chi(G)$ colors.

On the positive side, we prove that for graphs of bounded pathwidth, Problems 1.1 and 1.2 have a positive answer (and moreover a periodic proper coloring with $\chi(G)$ colors can always be obtained in this case). The setting of bounded pathwidth is very natural (for infinite Cayley graphs, it corresponds to the case of 2-ended groups) and has interesting connections with symbolic dynamics.

Related work. Problem 1.1 has a natural dual problem: Is it possible to find a vertex colouring so that no non-trivial automorphism of the graph preserves the colouring? Is it possible with just two colours? Assuming that no non-trivial automorphism fixes all but finitely many vertices, Babai proved that two colours suffice [Bab22].

It seems that probabilistic variants of Problems 1.1 and 1.2 have been investigated extensively in probability theory. In this setting, the goal is not to find a specific coloring which is invariant (or almost invariant) under automorphisms, but rather a *random* coloring which is invariant under automorphims (in the sense that the random process that generates the proper coloring is invariant under automorphism). See for instance [Tim24] and the references therein. The results there seem to be more positive than in our setting (in particular it is proved in [Tim24] that a specific type of random proper coloring based on factors of iid with independence at distance more than 4 exists in any graph of bounded degree, using a bounded number of colors – this type of random proper coloring is invariant under automorphisms). Note however that random proper colorings that are invariant under automorphisms do not necessarily produce proper colorings that are invariant under automorphisms.

The connections between our problems and symbolic dynamics are highlighted in Section 5 (we refer the reader to this section for the details).

Finally, we mention an interesting connection between the topic of this paper and techniques from distributed computing in Section 2.

Organization of the paper. We start by giving some definitions and basic results on graph theory and group theory in Section 2. In Section 3, we give our first example providing a negative answer to Problems 1.1 and 1.2. It turns out that the example even provides a negative answer to a much weaker problem, where we only seek a periodic non-trivial coloring (instead of a periodic proper coloring). In Section 4 we show how to construct graphs of bounded treewidth with no periodic orientation, providing a negative answer to Problems 1.1 and 1.2 even for graphs of bounded treewidth. In Section 5 we show that Problems 1.1 and 1.2 have a positive answer if we restrict ourselves to graphs of bounded pathwidth. This is done using connections between our problems and classical results on subshifts of finite type in symbolic dynamics. We conclude in Section 6 with a discussion and a number of open problems.

2. Preliminaries

Graphs. A vertex-coloring of a graph G is *proper* if every two adjacent vertices in G are assigned distinct colors. The *chromatic number* of G, denoted by $\chi(G)$, is the infimum number of colors in a proper vertex-coloring of G. A *path* is a connected acyclic graph in which every vertex has degree at most 2.

A *tree-decomposition* of a graph G is a pair $(T, \mathcal{X})$ such that T is a tree and $\mathcal{X}$ is a collection $(X_t : t \in V(T))$ of subsets of $V(G)$, called the *bags*, such that

- $\bigcup_{t\in V(T)} X_t = V(G)$,
- for every $e \in E(G)$, there exists $t \in V(T)$ such that X_t contains the endpoints of e, and
- for every $v \in V(G)$, the set $\{t \in V(T) : v \in X_t\}$ induces a connected subgraph of T.

For a tree-decomposition $(T, \mathcal{X})$ as above, the sets $X_t \cap X_{t'}$ for $tt' \in E(T)$ are called the *adhesion sets* of $(T, \mathcal{X})$, and the *width* of $(T, \mathcal{X})$ is $\sup_{t\in V(T)}|X_t| - 1$. The *treewidth* of G is the minimum width of a tree-decomposition of G.

If the tree T in a tree-decomposition $(T, \mathcal{X})$ of a graph G is a path, then $(T, \mathcal{X})$ is called a *path-decomposition* of G, and the *pathwidth* of G is the minimum width of a path-decomposition of G.

A *ray* in an infinite graph G is an infinite one-way path in G. Two rays of G are said to be *equivalent* if there are infinitely many disjoint paths between them in G. An *end* of G is an equivalence class of rays in G. It is known that the number of ends of a connected locally finite quasi-transitive graph is either $0, 1, 2$ or ∞ [Bab97, Proposition 2.1]. A graph with k ends (for $k \in \{0, 1, 2, \infty\}$) is said to be *$k$-ended*. For every finite set $S \subseteq V(G)$ and every ray r, note that there exists a unique connected component of $G - S$ containing an infinite number of vertices of r. If X denotes such a component, we say that r *lives* in X. We say that an end ω *lives* in a component X of $G - S$ if some (and thus all) of its rays live in X.

Groups. The *index* of a subgroup Γ' in a group Γ is the cardinality of the family of *left cosets* $\{g\Gamma' : g \in \Gamma\}$, where $g\Gamma' = \{gh : h \in \Gamma'\}$. Equivalently, the index of Γ' in Γ is the cardinality of the family of *right cosets* $\{\Gamma' g : g \in \Gamma\}$, where $\Gamma' g = \{hg : h \in \Gamma'\}$. If a group Γ has a subgroup Γ' of finite index then we say that Γ is *virtually* Γ'.

Given a finitely generated group Γ and a finite set of generators S not containing the identity element 1_Γ, the *(left) Cayley graph* of Γ with respect to the set of generators S is the graph $\mathrm{Cay}(\Gamma, S)$ whose vertex set is the set of elements of Γ and where for every two elements $g, h \in \Gamma$ there is an edge between g and h if and only if there exists $s \in S \cup S^{-1}$

with $h = sg$. Whenever we talk about a Cayley graph $\mathrm{Cay}(\Gamma, S)$, we always assume that S is finite and does not contain 1_Γ (even if this is not stated explicitly). For a graph class $\mathcal{C}$, we say that a finitely generated group is in $\mathcal{C}$ if it has a Cayley graph that is in $\mathcal{C}$.

The number of ends of a Cayley graph of a finitely generated group does not depend on the choice of generators (it follows from the observation that the number of ends of a locally finite graph is a quasi-isometric invariant), so we can also talk about the number of ends of a group. A group is 0-ended if and only if it is finite. A group is 2-ended if and only if it is virtually $\mathbb{Z}$ (i.e., it contains $\mathbb{Z}$ as a subgroup of finite index) [Hop44], if and only if it has finite pathwidth. A group has finite treewidth if and only if it is virtually free (i.e. it contains a free subgroup of finite index) [Ant11].

A subgroup Γ' of a group Γ is *normal* if $gg'g^{-1} \in \Gamma'$ for every $g \in \Gamma$ and $g' \in \Gamma'$ (equivalently, the left and right cosets $g\Gamma'$ and $\Gamma'g$ are equal for every $g \in \Gamma$). A group Γ is *simple* if the only normal subgroups of Γ are Γ itself and the trivial subgroup $\{1_\Gamma\}$ consisting of the identity element 1_Γ of Γ.

We will need the following basic result.

Lemma 2.1. *Let Γ' be a subgroup of finite index in a group Γ. Then Γ has a normal subgroup Γ'' of finite index such that Γ'' is a subgroup of Γ'.*

Proof. The action of Γ on the right cosets $\mathcal{C} = \{\Gamma'g : g \in \Gamma\}$ of Γ' induces a permutation of $\mathcal{C}$ (where each $h \in \Gamma$ maps each right coset $\Gamma'g$ to $\Gamma'gh$), and this action is a group homomorphism f from Γ to $\mathrm{Sym}(\mathcal{C})$, the symmetric group on $\mathcal{C}$. Moreover, since Γ' has finite index, $\mathrm{Sym}(\mathcal{C})$ is finite. The kernel $\ker f$ of f consists of all elements $g \in \Gamma$ which stabilize every right coset of Γ' (in particular, $\ker f$ is contained in Γ'). By the first isomorphism theorem, $\ker f$ is a normal subgroup of Γ and the quotient group $\Gamma/\ker f$ is isomorphic to the image of f (which is a finite subgroup of $\mathrm{Sym}(\mathcal{C})$). Hence, $\ker f$ is a subgroup of finite index in Γ. □

This directly implies the following.

Corollary 2.2. *Let Γ be an infinite simple group. Then Γ has no proper subgroup of finite index.*

Proof. Assume for the sake of contradiction that Γ has proper subgroup Γ' of finite index. By Lemma 2.1, there is a normal subgroup Γ'' of finite index in Γ which is a subgroup of Γ'. Note that since Γ' is a proper subgroup of Γ, Γ'' is also a proper subgroup of Γ. As Γ is a simple group, Γ'' is the singleton consisting of the identity element of Γ. Since Γ is infinite, we obtain that Γ'' has infinite index in Γ, which is a contradiction. □

Periodicity and strong periodicity. We say that a vertex- or edge-coloring or an orientation of a graph G is *periodic* if the subgroup of all automorphisms of G preserving the coloring or the orientation acts *quasi-transitively* on $V(G)$, meaning that $V(G)$ has finitely many orbits under the action of the color-preserving (or orientation-preserving) automorphisms of G. In other words, there is a partition of $V(G)$ into finitely many subsets $V_1, \ldots, V_k$ such that for every $1 \leqslant i \leqslant k$ and every $u, v \in V_i$, there is a color-preserving (or orientation-preserving) automorphism of G that maps u to v. Here, we say that an automorphism of a vertex-colored graph is color-preserving if the image of any vertex is a vertex of the same color. Similarly, a color-preserving automorphism of an edge-colored graph maps any edge to an edge of the same color, and an orientation-preserving automorphism of an oriented graph maps every edge to an edge with the same orientation.

We say that a vertex- or edge-coloring or an orientation of a graph G is *strongly periodic* if the subgroup of all automorphisms of G preserving the coloring (or the orientation) has finite index in $\mathrm{Aut}(G)$, the group of automorphisms of G. Strongly periodic colorings naturally arise in symbolic dynamics (see Section 5) and are related to periodic colorings via the following simple results.

Lemma 2.3. *For every quasi-transitive graph G, every subgroup Γ of finite index of $\mathrm{Aut}(G)$ acts quasi-transitively on $V(G)$.*

Proof. Let $v_1, \ldots, v_k \in V(G)$ be finitely many representatives of the orbits of $V(G)$ under the action of $\mathrm{Aut}(G)$, and let $\Gamma g_1, \ldots, \Gamma g_\ell$ be the finitely many right cosets of Γ, where $g_j \in \mathrm{Aut}(G)$ for any $1 \leqslant j \leqslant \ell$. Note that for every $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \ell$, the group Γ acts transitively on the set $\Gamma g_j(v_i) := \{gg_j(v_i) : g \in \Gamma\}$. As the family of sets $\{\Gamma g_j(v_i) : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \ell\}$ covers the vertex set $V(G)$, it follows that Γ acts quasi-transitively on $V(G)$, as desired. □

The converse of Lemma 2.3 does not hold in general. To see this, consider the free group $F_{a,b}$ on 2 generators a and b, and let $T = \mathrm{Cay}(F_{a,b}, \{a, b\})$ be the natural Cayley graph associated to $F_{a,b}$ and its two generators (T is the infinite 4-regular tree). Now $F_{a,b} \subseteq \mathrm{Aut}(T)$ acts transitively on $V(T)$, but as $\mathrm{Aut}(T)$ is not virtually free, $F_{a,b}$ does not have finite index in $\mathrm{Aut}(T)$.

By taking Γ to be the subgroup of color-preserving or orientation-preserving automorphisms of G, we obtain the following immediate consequence of Lemma 2.3.

Corollary 2.4. *Every strongly periodic coloring or orientation of a quasi-transitive graph G is periodic.*

Graphical rigid representations. Consider a Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ of a finitely generated group Γ. Note that Γ naturally acts by right-multiplication on G (for any $g \in \Gamma$, the map $h \mapsto hg$ induces an automorphism of G). The Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ is said to be a *graphical rigid representation* of Γ if $\Gamma = \mathrm{Aut}(G)$ (in other words, the only automorphisms of G are the right-multiplications by elements of Γ). We now prove that for graphical rigid representations of groups, the converse of Lemma 2.3 holds.

Lemma 2.5. *Let G be a graphical rigid representation of a finitely generated group. Then every subgroup of $\mathrm{Aut}(G)$ acting quasi-transitively on $V(G)$ has finite index in $\mathrm{Aut}(G)$.*

Proof. Let Γ be a finitely generated infinite group and S be a finite set of generators of Γ such that the automorphism group of the Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ is equal to Γ. Let Γ' be a subgroup of $\mathrm{Aut}(G) = \Gamma$ acting quasi-transitively on $V(G) = \Gamma$. The finitely many orbits of $V(G) = \Gamma$ under the action of Γ' are precisely the left cosets of Γ'. It follows that Γ' has finite index in $\Gamma = \mathrm{Aut}(G)$. □

As before, we obtain the following immediate corollary, which shows that the notions of periodicity and strong periodicity are equivalent in the case of graphical rigid representations of groups.

Corollary 2.6. *Let G be a graphical rigid representation of a finitely generated group. Then every periodic coloring or orientation of G is strongly periodic.*

Next, we discuss an observation on the minimum number of colors in a periodic proper vertex-coloring of a graph G (when such a periodic proper vertex-coloring exists). This is connected to classical results and techniques in distributed computing.

Color reduction. The maximum degree of a graph G is denoted by $\Delta(G)$. Every locally finite quasi-transitive graph has bounded maximum degree. Using a classical technique from distributed computing, we show that every periodic proper vertex-coloring of a graph G with more than $\Delta(G)+1$ colors can be turned into a periodic proper vertex-coloring of G with one less color, and thus after several iterations into a periodic proper vertex-coloring of G with $\Delta(G)+1$ colors.

Lemma 2.7. *Let G be a locally finite graph with a periodic proper vertex-coloring. Then G has a periodic proper vertex-coloring with at most $\Delta(G)+1$ colors.*

Proof. As G is quasi-transitive and locally finite, it has finite maximum degree $\Delta = \Delta(G)$. Consider a periodic proper vertex-coloring c of G with $N > \Delta + 1$ colors (call them $1, \ldots, N$). Consider the set S of vertices colored with color N (and note that S is an independent set in G). The set S is divided into finitely many orbits $S_1, \ldots, S_k$ under the action of the group of color-preserving automorphisms of G. For each vertex $v \in S$, assign to v the smallest color from the set $1, \ldots, N$ that does not appear in its neighborhood, and for each vertex $v \notin S$, leave the color of v unchanged. Let c' be the resulting vertex-coloring of G. Since $N > \Delta + 1$, for each vertex of S, the smallest color $c'(v)$ that does not appear in its neighborhood is an element of $1, \ldots, N-1$, so c' uses at most $N-1$ colors. Moreover, for each $1 \leqslant i \leqslant k$, all the vertices of S_i are recolored with the same color (as all the vertices of S_i see the same set of colors in their neighborhood), so every automorphism of G preserving c also preserves c', and thus c' is periodic. Finally, since S is an independent set, the vertex-coloring c' is proper (each vertex has been recolored with a color distinct from that of its neighbors, and no two adjacent vertices have been recolored in the process).

This shows that given a periodic proper vertex-coloring of G with $N > \Delta+1$ colors, we can produce a periodic proper vertex-coloring of G with at most $N-1$ colors. Iterating this procedure at most $N - \Delta - 1$ times, we obtain a periodic proper vertex-coloring of G with at most $\Delta + 1$ colors, as desired. ☐

Remark 2.8. In distributed computing, much faster procedures reducing the number of colors than that of the proof of Lemma 2.7 are usually needed. Together with the color reduction technique described above, a classical algorithm of Cole and Vishkin [CV86] can be used to produce a quasi-transitive proper $(\Delta+1)$-vertex-coloring of G in $\Delta^{O(\Delta)} + \log^* N$ iterations starting from a quasi-transitive proper N-vertex-coloring of G, compared to the $N - \Delta - 1$ iterations needed in the proof of Lemma 2.7. The idea is to view the colors $1, \ldots, N$ as $O(\log N)$-bit words, and to record, for each vertex v and neighbor u of v, a bit in the color of v that differs from the corresponding bit in the color of u. Using this, we can obtain a periodic proper $O(\log N)$-vertex-coloring of G in a single iteration, and a periodic proper $\Delta^{O(\Delta)}$-vertex-coloring in $\log^* N$ iterations.

Basic examples of periodic colorings and orientations. We conclude this section of preliminary results with two simple lemmas describing special cases where Problems 1.1 and 1.2 have positive answers.

Lemma 2.9. *Let G be a connected quasi-transitive bipartite graph (not necessarily locally finite). Then, G has a periodic proper vertex-coloring with 2 colors.*

Proof. Let $\{X, Y\}$ be the bipartition of G ($\{X, Y\}$ is unique since G is connected). Fix $v \in X$, and let Γ consist of every automorphism g of G such that $g(v) \in X$. Let $g \in \Gamma$ and $u \in X$. Since automorphisms are distance-preserving, the distance between u and v has the same parity as the distance between $g(u)$ and $g(v)$. It follows that $g(u) \in X$ if and only if $u \in X$. Therefore, every automorphism of Γ respects the partition (X, Y), seen as a proper vertex-coloring with 2 colors.

Next, we observe that Γ is a group: it contains the identity; for all $g \in \Gamma$, it holds that $g^{-1}(v) \in X$ since $v \in X$, so $g^{-1} \in \Gamma$; and for $g_1, g_2 \in \Gamma$, it holds that $g_1(g_2(v)) \in X$ since $g_2(v) \in X$, so Γ is closed.

Finally, we show that Γ acts quasi-transitively on G. Let $g_1, g_2 \notin \Gamma$. For $i = 1, 2$, since $g_i \notin \Gamma$, it follows that $g_i(v) \in Y$, and so, since automorphisms are distance-preserving, $g_i(u) \in Y$ for every $u \in X$ and $g_i(w) \in X$ for every $w \in Y$. But now $g_1(g_2(v)) \in X$, so $g_1 g_2 \in \Gamma$. Therefore, Γ has index 2 in $\mathrm{Aut}(G)$. By Lemma 2.3, Γ acts quasi-transitively on G.

Since $\Gamma \subseteq \mathrm{Aut}(G)$ preserves (X, Y) and acts quasi-transitively on G, we conclude that (X, Y) is a periodic proper vertex-coloring of G with 2 colors. □

We say that an automorphism g of G *inverts an edge* $uv \in E(G)$ if $g(u) = v$ and $g(v) = u$. It is straightforward to observe that for any subgroup Γ of automorphisms that preserve some proper vertex-coloring or orientation of a graph G, Γ does not contain any automorphism that inverts an edge of G. The next lemma shows that this condition is also sufficient to be orientation-preserving.

Lemma 2.10. *Let G be a locally finite quasi-transitive graph. Then, G has a periodic orientation if and only if there is a subgroup $\Gamma \subseteq \mathrm{Aut}(G)$ acting quasi-transitively such that no automorphism of Γ inverts an edge of G.*

Proof. If G has a periodic orientation, then no element of the automorphism group that preserves the orientation inverts an edge of G (otherwise it would not preserve the orientation of that edge). Conversely, suppose $\Gamma \subseteq \mathrm{Aut}(G)$ acts quasi-transitively and is such that no element of Γ inverts an edge of G. We will construct an orientation preserved by Γ. Let $E_1, \ldots, E_k$ be the orbits of $E(G)$ under Γ (since Γ acts quasi-transitively on $V(G)$ and G is locally finite, Γ also acts quasi-transitively on $E(G)$). Fix an edge $e_i \in E_i$ and an orientation $\overrightarrow{e_i}$ of e_i for all $i = 1, \ldots, k$. For every $e'_i \in E_i$, since no element of Γ inverts an edge of G, there is exactly one orientation $\overrightarrow{e'_i}$ of e'_i so that an automorphism of Γ maps $\overrightarrow{e_i}$ to $\overrightarrow{e'_i}$. Now, the fixed orientations of $e_1, \ldots, e_k$ define an orientation of $E(G)$ that is preserved by Γ. □

A corollary of Lemma 2.10 is that Problem 1.2 has a positive answer when G is the Cayley graph of some finitely generated group Γ with respect to a generating set containing no elements of order 2 (i.e., elements $s \neq 1_\Gamma$ such that $s^{-1} = s$).

3. Periodic non-trivial colorings

A vertex-coloring of a graph G is said to be *trivial* if for each connected component C of G, all the vertices of C are assigned the same color. The vertex-coloring is said to be *non-trivial* otherwise. A vertex-coloring of G is trivial if and only if for every edge uv of G, u and v are assigned the same color. Hence, if G contains at least one edge, for every non-trivial coloring of G there is a pair u, v of adjacent vertices which are assigned different colors. In particular, if G contains at least one edge, then every proper vertex-coloring of G is non-trivial. Also note that non-trivial eigenvectors are a specific case of non-trivial vertex-colorings, so we find a graph G with no periodic non-trivial vertex-coloring, then G has no periodic non-trivial eigenvector either.

Remark 3.1. For non-trivial vertex-colorings, color reduction (as described in the previous section) is much simpler. Given a non-trivial vertex-coloring c with colors $1, \ldots, N$ (and in which each of these colors appears) we can recolor all vertices $\{v : c(v) > 1\}$ with color 2. The resulting coloring is non-trivial, and if c is periodic then the resulting coloring is also periodic. It follows that a graph has a periodic non-trivial vertex-coloring if and only if it has a periodic non-trivial vertex-coloring with 2 colors.

Recall that Problems 1.1 and 1.2 ask whether every locally finite quasi-transitive graph has a periodic proper vertex-coloring and a periodic orientation. We now give a negative answer for graphical rigid representations of finitely generated infinite simple groups using the results from the previous section, relating periodicity and strong periodicity. For vertex-colorings, we show that even if the notion of proper coloring is replaced by the much weaker notion of non-trivial coloring, the answer to Problem 1.1 remains negative.

Lemma 3.2. *Let $G = \mathrm{Cay}(\Gamma, S)$ be a graphical rigid representation of a finitely generated infinite simple group Γ, for some finite set S of generators. Then every periodic vertex-coloring of G is trivial. Moreover, if S contains an element of order 2, then G does not have a periodic orientation.*

Proof. For the sake of contradiction, consider some non-trivial vertex-coloring c of G which is periodic, and denote by Γ' the subgroup of $\mathrm{Aut}(G) = \Gamma$ of automorphisms of G preserving the colors of the vertices of G. By Lemma 2.5, Γ' has finite index in Γ.

Since c is non-trivial and G contains at least one edge (G is infinite and connected) there is an edge uv in G such that $c(u) \neq c(v)$. As G is a Cayley graph, it is vertex-transitive and thus there is an automorphism $g \in \mathrm{Aut}(\Gamma)$ that maps u to v. This automorphism g does not preserve the colors of the vertices, so this shows that Γ' must be a proper subgroup of Γ. By Corollary 2.2, this contradicts the fact that Γ' has finite index in Γ.

It now remains to prove the second part of the statement. So assume that S contains some element s of order 2. Consider any $g \in \Gamma = V(G)$, and its neighbor $h = sg$ in G. As s has order 2, the corresponding automorphism of G maps g to h and h to g, so it maps the pair (g, h) to (h, g). If G has a periodic orientation, then the subgroup of orientation-preserving automorphisms of G does not contain s, and is thus a proper subgroup of Γ. Using Corollary 2.2 again, the same argument as in the previous paragraph then implies that no such periodic orientation of G exists. ∎

Remark 3.3. As any proper vertex-coloring of a graph containing at least one edge is non-trivial, Lemma 3.2 implies in particular that no graph satisfying the assumptions of the lemma has a periodic proper vertex-coloring, and thus any graph satisfying the conditions of Lemma 3.2 provides a negative answer to Problem 1.1.

It remains to prove that a Cayley graph satisfying the conditions of Lemma 3.2 exists. A specific example is R. Thompson's group V [CFP96], which is a finitely presented infinite simple group satisfying the following properties.

Lemma 3.4. *R. Thompson's group V has two finite sets S and T of generators such that S contains an element of order 2, T does not contain any element of order 2, and both $\mathrm{Cay}(V, S)$ and $\mathrm{Cay}(V, T)$ are graphical rigid representations of V.*

Proof. Bleak and Quick [BQ17, Theorem 1.2] gave a presentation of V with three generators a, b, c, two of them having order 2. Using the fact that V has rank 2 and contains elements of arbitrarily large order, [LdlS21, Corollary 11] by Leemann and de la Salle implies that the set $\{a, b, c\}$ of generators of V is a subset of a finite set S of generators of V such that $\mathrm{Cay}(V, S)$ is a graphical rigid representations of V.

Bleak and Quick [BQ17, Theorem 1.3] also gave a presentation of V with two generators u, v, none of them having order 2. Using [LdlS21, Theorem 9] by Leemann and de la Salle, this implies that there is a finite set T of generators of V containing u and v, such that T does not contain any element of order 2 and $\mathrm{Cay}(V, T)$ is a graphical rigid representations of V, as desired. □

Using Lemma 3.2, this immediately implies the following.

Corollary 3.5. *R. Thompson's group V has a finite generating set S such that* $\mathrm{Cay}(V, S)$ *has no periodic orientation and no periodic non-trivial vertex-coloring (and thus no periodic non-trivial eigenvector).*

Remark 3.6. We have chosen R. Thompson's group V for simplicity but it should be mentioned that other families of infinite finitely generated simple groups provide even more striking negative answers to Problem 1.1 (and Problem 1.2, with some additional work). For instance, if we consider again the setting of non-trivial vertex-coloring, we can take a graphical rigid representation G of a *Tarski monster group* of exponent p (see [LdlS21, Theorem 14] for the existence of such a representation for sufficiently large p). This group is infinite, finitely generated, and has the property that any proper subgroup is isomorphic to the cyclic group of order p. As in the proof of Lemma 3.2, observe that for any non-trivial vertex-coloring of G, the subgroup of $\mathrm{Aut}(G)$ of color-preserving automorphisms of G must be a proper subgroup of $\mathrm{Aut}(G)$, and thus it must be finite (which is stronger than being of infinite index).

It was pointed out to us by Emmanuel Jeandel that another infinite finitely generated simple group, constructed by Osin [Osi10], provides a negative answer to Problem 1.1 which is even more remarkable. The group Γ constructed by Osin is an infinite, finitely generated simple group with the property that any two non-identity elements are conjugate. Consider any graphical rigid representation $G = \mathrm{Cay}(\Gamma, S)$ of such a group, for some finite generating set S (the fact that such a representation exists follows directly from [LdlS22, Theorem 1.1], since infinite simple groups are not virtually abelian), and a proper vertex-coloring c of G. Let $g \in \mathrm{Aut}(G) = \Gamma$ be any non-trivial automorphism of G, and let $s \in S$. As g and s are conjugate, there exists $h \in \Gamma$ such that $gh = hs$. So the automorphism g of G maps h to its neighbor hs in G. As $c(h) \neq c(hs)$, g is not color-preserving. It follows that the only color-preserving automorphism of G is the identity. Hence, G is an infinite Cayley graph with the property that for every proper vertex-coloring of G, the subgroup of color-preserving automorphisms of G is trivial! We note that this seems to only apply to proper vertex-colorings (and not to the more general setting of non-trivial vertex-colorings).

Alternative version (without graphical rigidity). In the previous paragraphs we have used specific graphical rigid representations (that is, specific Cayley graphs) of simple groups. We now explain how to turn any Cayley graph of a finitely generated infinite simple group into a quasi-transitive graph with no periodic proper vertex-coloring (and without periodic orientation, with an additional assumption). This does not require any result on graphical rigidity, but the downside is that the resulting examples are only quasi-transitive (instead of being Cayley graphs).

Take *any* finitely generated infinite simple group Γ and *any* finite set S of generators (if we want to give a negative answer to Problem 1.2, we only ask that S contains an element of order 2). Write $S = s_1, \ldots, s_k$ and consider the graph G obtained from $\mathrm{Cay}(\Gamma, S)$ by doing the following: for any vertex $g \in \Gamma$ and element $s_i \in S$ we add a path of length (number of edges) $3i$ between g and gs_i, say $P_{g,s_i} = v_0, v_1, \ldots, v_{3i}$ with $v_0 = g$ and $v_{3i} = gs_i$, and we add a new vertex whose unique neighbor is v_{3i-1}. We also keep in G the (simple) edges of from $\mathrm{Cay}(\Gamma, S)$. Note that G is quasi-transitive, and we claim that $\mathrm{Aut}(G) = \Gamma$. To see this, observe first that for the labelled, directed version $\hat{G}$ of $\mathrm{Cay}(\Gamma, S)$ where arcs are labelled with the corresponding generators (i.e., in $\hat{G}$ we add an arc labelled s from g to gs for any $g \in \Gamma$ and $s \in S$), we have $\mathrm{Aut}(\hat{G}) = \Gamma$. Observe now that the paths P_{g,s_i} in G force every automorphism of G to coincide with an automorphism of $\hat{G}$ on $V(\hat{G})$.

A proof along the lines of that of Lemma 3.2 now shows that G has no periodic proper vertex-coloring, and if S contains an element of order 2, then G has no periodic orientation. On the other hand, G has a periodic non-trivial vertex-coloring (obtained by coloring all the vertices of the original Cayley graph $\mathrm{Cay}(\Gamma, S)$ with color 1, and all the newly added internal vertices of the paths with color 2). More generally, any quasi-transitive graph which is not vertex-transitive has a periodic non-trivial vertex-coloring.

4. Infinitely-ended graphs and finite treewidth

Recall from Section 2 that the number of ends of a quasi-transitive graph is either $0, 1, 2$ or ∞. The graphs presented in Section 3 providing negative answers to Problems 1.1 and 1.2 are all quasi-isometric to finitely generated infinite simple groups and are thus 1-ended[1]. It is thus natural to investigate whether we can also construct examples with 2 or infinitely many ends (graphs with 0 ends are finite, and therefore not of interest for these problems). In this section we will construct ∞-ended graphs providing negative answers to Problems 1.1 and 1.2, and in the next section we will show on the other hand that the answer to both problems is positive for 2-ended graphs. More precisely, the negative examples we will construct in this section have bounded treewidth, while all locally finite quasi-transitive 2-ended graphs have bounded pathwidth, so this shows that graphs of bounded pathwidth and bounded treewidth behave very differently in the context of our problems.

4.1. **Graphs of bounded treewidth.** We now describe a simple construction of a quasi-transitive locally finite graph of bounded treewidth that does not admit any periodic orientation (this provides a negative answer to Problem 1.2, and thus also to Problem 1.1). This example is a particular case of a more general family of examples that we will describe in the next subsection. We start by giving a self-contained proof that the graph does not admit a periodic orientation, and then we show more generally that this holds for a wider family of examples.

[1] See Corollary 1.3 in [AMO07], and the discussion in the introduction of that paper explaining why SQ-universal groups have uncountably many non-isomorphic quotients, which shows that they cannot be simple.

Consider an orientation $\overrightarrow{T}$ of the infinite 3-regular tree T where each vertex has in-degree 2 and out-degree 1. Next, for every arc (u, v) in $\overrightarrow{T}$, replace (u, v) by a (non-oriented) path $u - r_{uv} - s_{uv} - v$, and add a new vertex t_{uv} adjacent to s_{uv} only. Finally, for each original vertex u of $\overrightarrow{T}$, replace u by a triangle $\triangle_u = u_1u_2u_3$, where each u_i is adjacent to a unique neighbor of u and where u_1 is the closest vertex from the unique out-neighbor of u in $\overrightarrow{T}$. The resulting graph is denoted by G (see Figure 1 for an illustration).

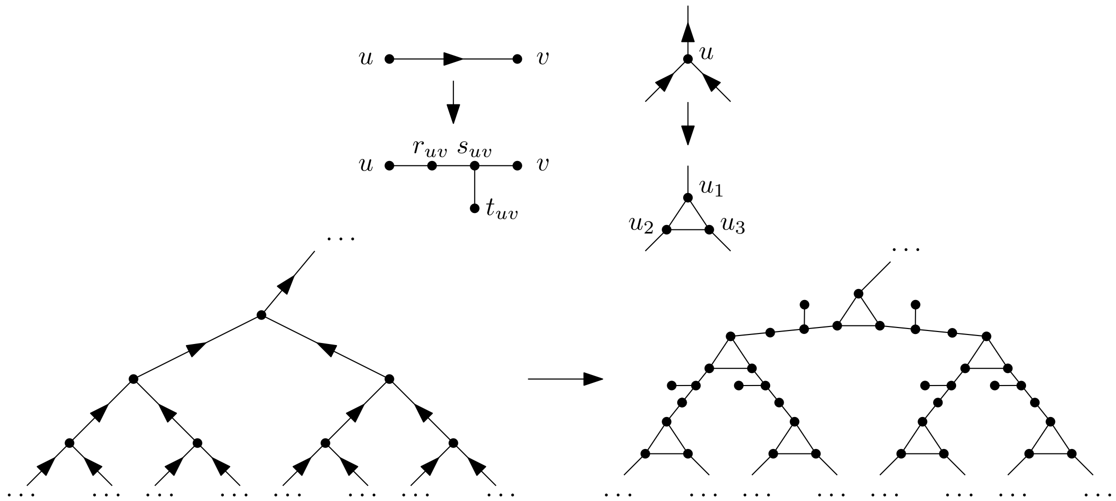

FIGURE 1. The orientation $\overrightarrow{T}$ of the infinite 3-regular tree T (left), and the resulting graph G (right). The local modifications performed to construct G from $\overrightarrow{T}$ are illustrated at the top.

It is easily seen that G has treewidth at most 2 (and is planar). The fact that G is quasi-transitive easily follows from the fact that $\overrightarrow{T}$ is vertex-transitive, and that there is a natural bijection between the automorphisms of $\overrightarrow{T}$ and the automorphisms of G: given an automorphism g of $\overrightarrow{T}$, the corresponding automorphism of G maps u_1 to $g(u)_1$ and $\{u_2, u_3\}$ to $\{g(u)_2, g(u)_3\}$ for any vertex $u \in V(\overrightarrow{T})$, and r_{uv}, s_{uv}, t_{uv} to $r_{g(u)g(v)}, s_{g(u)g(v)}, t_{g(u)g(v)}$ for any arc (u, v) of $\overrightarrow{T}$. In the other direction, observe that any automorphism h of G must map each triangle $\triangle_u$ to some triangle $\triangle_{g(u)}$, and the resulting map g is an automorphism of $\overrightarrow{T}$.

Remark 4.1. Every automorphism of $\overrightarrow{T}$ translates some 2-way infinite directed path or stabilizes some vertex (this easily follows from a classical result of Halin [Hal73] but can also be proved directly). In the latter case, if the automorphism is distinct from the identity, then there is a node v such that the two in-neighbors of v are exchanged by the automorphism.

This remark implies the following.

Claim 4.2. *For every subgroup* Γ *of* $\mathrm{Aut}(G)$ *acting quasi-transitively on* $V(G)$, *there exists some automorphism* $g \in \Gamma$ *and some* $u \in V(\overrightarrow{T})$ *such that* $g(u_2) = u_3$ *and* $g(u_3) = u_2$.

Proof of the Claim: Recall that there is a natural bijection between $\mathrm{Aut}(G)$ and $\mathrm{Aut}(\overrightarrow{T})$. If some automorphism $g \in \Gamma$ distinct from the identity has the property that the corresponding automorphism of $\overrightarrow{T}$ stabilizes a node of $\overrightarrow{T}$, then the result follows from Remark 4.1. So we only need to prove the existence of such an automorphism $g \in \Gamma$.

We say that v is an *ancestor* of u in $\overrightarrow{T}$ if there is a directed path from u to v in $\overrightarrow{T}$. The *lowest common ancestor* of two vertices u and v is the minimum common ancestor of u and v in the ancestor relation (viewed as a partial order on $V(\overrightarrow{T})$). We define a relation $\sim$ on $V(\overrightarrow{T})$ as follows. Given two vertices $u, v \in V(\overrightarrow{T})$, $u \sim v$ if and only if $d_T(u, w) = d_T(v, w)$, where w denotes the lowest common ancestor of u and v and the function d_T denotes the distance in T (or equivalently $\overrightarrow{T}$). Note that $\sim$ is an equivalence relation, and each equivalence class is infinite. In particular, since Γ acts quasi-transitively on $V(G)$, there exist an automorphism $g \in \Gamma$ such that the corresponding automorphism of $\overrightarrow{T}$ maps some vertex u to some different vertex $v \sim u$. But then such a non-trivial automorphism must stabilize all common ancestors of u and v in $\overrightarrow{T}$, as desired. $\Diamond$

Now, assume for the sake of contradiction that G has a periodic orientation. Then, by definition, the subgroup of orientation-preserving automorphisms of G acts quasi-transitively on $V(G)$. By Claim 4.2, some orientation-preserving automorphism g of G must exchange two vertices u_2 and u_3, for some $u \in V(\overrightarrow{T})$. Assume by symmetry that (u_2, u_3) is an arc in the periodic orientation of G. Then g maps the arc (u_2, u_3) to (u_3, u_2), which contradicts the property that g is orientation-preserving (cf. Lemma 2.10). This contradiction shows that G does not admit a periodic orientation (and thus does not admit a periodic proper vertex-coloring either).

Edge-colorings. We now describe a similar construction of a quasi-transitive locally finite tree that does not admit any periodic proper edge-coloring. Consider the tree T' obtained from the orientation $\vec{T}$ of the infinite 3-regular tree we described above, after replacing every arc (u, v) by a (non-oriented) path $u - r_{uv} - s_{uv} - v$ and after adding a vertex t_{uv} adjacent to s_{uv}. Equivalently, T' is obtained from the graph G we constructed above after contracting every triangle Δ_u into a single vertex u (see Figure 2). Then the exact same arguments used in the proof of Claim 4.2 apply to show that for every subgroup Γ of $\mathrm{Aut}(T')$ acting quasi-transitively on T', there exists some non-trivial element $g \in \Gamma \setminus \{1_\Gamma\}$ that fixes a node $w \in V(T)$ while exchanging two of its neighbors. In particular, g maps some edge e to some edge f incident to e, which contradicts the fact that g is color-preserving (since e and f have distinct colors in any proper edge-coloring). This contradiction shows that T' cannot admit a periodic proper edge-coloring.

4.2. **Obstructions to periodic orientations and colorings.** We say that a tree-decomposition $(T, \mathcal{X})$ of a graph G is *canonical*, if $\mathrm{Aut}(G)$ induces a group action on T such that for every $g \in \mathrm{Aut}(G)$ and $t \in V(T)$, $g(X_t) = X_{t \cdot g}$. By definition of a group action on a graph, $t \mapsto t \cdot g$ is an automorphism of T for any $g \in \mathrm{Aut}(G)$. In particular, every automorphism of G sends bags of $(T, \mathcal{X})$ to bags and adhesion sets to adhesion sets.

Recall that the action of a group Γ on a set X is *free* if the only element of Γ stabilizing an element of X is the identity element 1_Γ. We will need the following result, whose proof is based on the same arguments we used to prove Claim 4.2.

Lemma 4.3. *Let Γ be a group acting quasi-transitively on a ∞-ended tree T and stabilizing an end of T. Then the action of Γ on T cannot be free.*

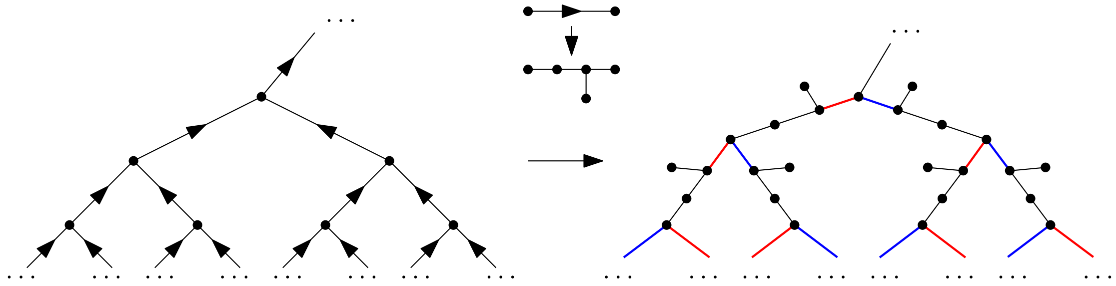

FIGURE 2. The construction of a quasi-transitive tree T' with no periodic proper edge-coloring (right), starting from the orientation $\overrightarrow{T}$ of the infinite 3-regular tree (left). In every periodic edge-coloring of T', some color-preserving automorphism of T' has to exchange a pair of incident edges (depicted in red and blue on the figure).

Proof. Let ω denote an end of T fixed by the action of Γ on T, and consider the orientation $\vec{T}$ of T obtained after orienting every edge $e = uv$ toward ω, i.e., if T_u, T_v denote the two components of $T - e$ containing respectively u and v, then we add the arc (u, v) in $\vec{T}$ if ω lives in T_v. Now every vertex of $\vec{T}$ has out-degree exactly 1, and Γ induces a group action on $\vec{T}$.

The remainder of the proof is identical to the one in the proof of Claim 4.2: we define similarly an equivalence relation $\sim$ on $\vec{T}$ by letting $u \sim v$ if and only if u and v lie at the same distance from their lowest common ancestor. As T is quasi-transitive and has infinitely many ends, the equivalence classes of $\sim$ must be infinite. In particular, as Γ acts quasi-transitively on T, there exist $u, v \in V(T)$ with $u \neq v$, $u \sim v$ and some $g \in \Gamma$ such that $g(u) = v$. Then g is non-trivial, and stabilizes every common ancestor of u and v, so it follows that the action of Γ on T is not free. □

The results of this subsection deal with the situation of locally finite infinitely ended graphs with an automorphism group fixing an end and acting transitively on the graph. It is known [Möl92] that these graphs are quasi-isometric to trees and hence have bounded treewidth. It follows that the same is true if the action fixes an end but only act quasi-transitively on the graph.

The example presented in the previous subsection is a particular case of a more general family of graphs whose structure is described by the following lemma.

Lemma 4.4. *Let G be a locally finite quasi-transitive graph and let $(T, \mathcal{X})$ be a canonical tree-decomposition of G such that:*

- *for every automorphism g of G that stabilizes the bag of some node $t \in V(T)$ but does not stabilize the bag of some neighbor of t in T, g exchanges two adjacent vertices in G, and*
- *there is an end of G which is stabilized by all automorphisms of G.*

Then G does not have a periodic orientation (and thus does not have a periodic proper vertex-coloring either).

Proof. Assume for the sake of contradiction that G has a periodic orientation, and let Γ be the group of orientation-preserving automorphisms of G, acting quasi-transitively on G, and thus also on T. Note that Γ stabilizes the end of T corresponding to the end of G stabilized by $\mathrm{Aut}(G)$. By Lemma 4.3, the action of Γ on T cannot be free, so there is an automorphism $g \in \Gamma$ that stabilizes some bag X_t of $(T, \mathcal{X})$, but does not stabilize some bag $X_{t'}$ where t' is a neighbor of t in T. It follows that g inverts an edge uv in G, which contradicts the fact that Γ is orientation-preserving. □

The following slightly different class of graphs also provides examples of locally finite quasi-transitive graphs of bounded treewidth with no periodic proper vertex-coloring.

Lemma 4.5. *Let G be a locally finite quasi-transitive graph and let $(T, \mathcal{X})$ be a canonical tree-decomposition of G such that:*

- *every bag is a finite clique, and*
- *for every adhesion set X there is a unique edge $tt' \in E(T)$ such that X is the intersection of the bags of t and t', and*
- *there is an end of G which is stabilized by all automorphisms of G.*

Then G does not have a periodic proper vertex-coloring.

Proof. Assume for the sake of contradiction that G has a periodic proper vertex-coloring c, and let Γ be the subgroup of color-preserving automorphisms of G (acting quasi-transitively on G, and thus also on T). For $g \in \Gamma$, we denote by g^T the action of g on T. Suppose for a contradiction that Γ does not act freely on T. Then, there exists an edge $tt' \in E(T)$ and $g_1, g_2 \in \Gamma$ such that $g_1^T(t) = g_2^T(t)$ but $g_1^T(t') \neq g_2^T(t')$. Since the bag X_t of t in $(T, \mathcal{X})$ is a clique, every vertex of X_t is colored with a distinct color, and since Γ respects the coloring, it follows that $g_1(v) = g_2(v)$ for all $v \in X_t$. In particular, g_1 and g_2 agree on $X_t \cap X_{t'}$, the adhesion set of $(T, \mathcal{X})$ corresponding to the edge tt' of T. But since the adhesions sets of X_t are unique, it follows that $g_1^T(t') = g_2^T(t')$, a contradiction. This shows that Γ acts freely on T, a contradiction with Lemma 4.3. □

4.3. **Limits of color reduction.** Lemma 2.7 shows that Problem 1.1 is equivalent to the stronger version where we ask for a periodic proper $(\Delta(G)+1)$-vertex-coloring. This raises the following natural question.

Problem 4.6. *Is it true that every locally finite graph G with a periodic proper vertex-coloring has a periodic proper vertex-coloring with $\chi(G)$ colors?*

We will see in the next section that the answer to this question is positive for 2-ended graphs, but in the remainder of this section we show that the answer is negative for ∞-ended graphs.

Consider the unique orientation $\overrightarrow{T}$ of the infinite 3-regular tree where every vertex has out-degree exactly one (see Figure 3, left), and replace every vertex $v \in V(\overrightarrow{T})$ by a copy H_v of the finite graph H depicted in Figure 3, top right, connecting incident edges as illustrated there. The resulting graph G is depicted in Figure 3, right. This graph is locally finite, ∞-ended, quasi-transitive, and satisfies $\chi(G) = 3$. Figure 3 (right) illustrates the fact that G has a periodic proper vertex-coloring with $4 = \chi(G) + 1$ colors. On the other hand, we claim that G does not have such a periodic coloring with $3 = \chi(G)$ colors. To see this, we first observe that automorphisms of G map copies of H to copies of H, so any automorphism of G induces an automorphism of $\overrightarrow{T}$. Moreover, every automorphism of $\overrightarrow{T}$ must stabilize the end ω of $\overrightarrow{T}$ containing all infinite 1-way directed paths.

Assume for the sake of contradiction that G has a periodic proper coloring c with 3 colors, and let Γ be the subgroup of $\mathrm{Aut}(G)$ of automorphisms preserving the coloring c. By Lemma 4.3, the action of Γ on $\overrightarrow{T}$ cannot be free, so there is an automorphism $g \in \Gamma \setminus \{1_\Gamma\}$ whose induced action on $\overrightarrow{T}$ stabilizes some node $v \in \overrightarrow{T}$, but does not stabilize

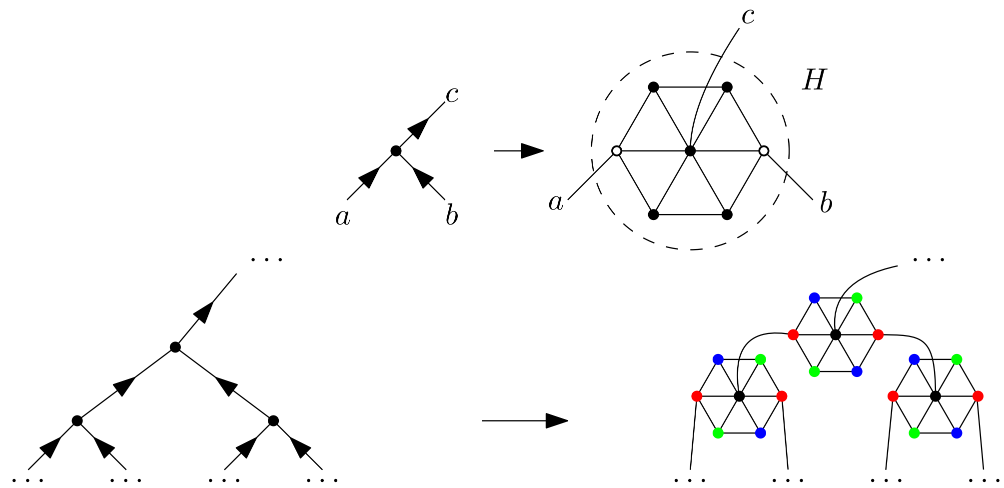

FIGURE 3. The construction of a quasi-transitive graph G with no periodic proper vertex-coloring with $\chi(G) = 3$ colors but with a periodic proper vertex-coloring with $\chi(G)+1 = 4$ colors (right), starting from an orientation $\overrightarrow{T}$ of the infinite 3-regular tree (left).

some in-neighbor u of v in $\overrightarrow{T}$. In particular, g stabilizes H_v, and maps H_u to some disjoint copy $H_{u'}$ of H, for some $u' \neq u$. This automorphism must exchange the two vertices of H_v connected to the copies of H associated to the in-neighbors of v in $\overrightarrow{T}$ (these vertices are depicted in white in Figure 3, top right). However it can be checked easily that in any proper vertex-coloring of H with $\chi(H) = 3$ colors, these 2 vertices have distinct colors, which contradicts the fact that g is color-preserving. This shows that G does not have a periodic proper vertex-coloring with $\chi(G)$ colors, while it has a periodic proper vertex-coloring with $\chi(G) + 1$ colors.

Remark 4.7. Consider the graph G described above, and replace in G every vertex by a clique of size k, and every edge be a complete bipartite graph $K_{k,k}$. Then the resulting graph has chromatic number $3k$, and a proof similar to that above shows that every periodic proper vertex-coloring requires at least $4k$ colors.

Remark 4.8. We have proved in Lemma 2.7 that any graph G with a periodic proper vertex-coloring has a periodic proper vertex-coloring with $\Delta(G) + 1$ colors. The example constructed above has maximum degree $\Delta(G) = 4$, and no periodic proper vertex-coloring with $3 = \Delta(G) - 1$ colors, which shows that Lemma 2.7 is close to best possible.

Our construction in this section only rules out Problem 4.6, and the following weaker problem remains open (even for $f(x) = \frac{4}{3}x + 1$).

Problem 4.9. *Is there a function f such that every locally finite graph G with a periodic proper vertex-coloring has a periodic proper vertex-coloring with $f(\chi(G))$ colors?*

5. 2-ENDED GRAPHS AND BOUNDED PATHWIDTH

In previous sections we have seen 1-ended and ∞-ended examples of quasi-transitive locally finite graphs with no periodic orientation (and thus no periodic proper vertex-coloring). In this section, we show that every 2-ended quasi-transitive locally finite graph has a periodic vertex-coloring (and thus a periodic orientation). We start with the case of Cayley graphs of finitely generated groups, for which the result directly follows from classical results on subshifts of finite type in symbolic dynamics. We then show that the same ideas can be applied more generally to any 2-ended quasi-transitive locally finite graph.

5.1. **2-ended groups and subshifts of finite type.** Consider a finitely generated group Γ. Let A be a finite set of colors and consider any coloring $\sigma : \Gamma \to A$. For every $g \in \Gamma$, we define the coloring $g \cdot \sigma : \Gamma \to A$ by setting for each $x \in \Gamma$, $g \cdot \sigma(x) := \sigma(g^{-1}x)$. Note that this defines a group action of Γ on the set A^Γ of A-colorings of Γ.

Let F be a finite subset of Γ, and let $\alpha : F \to A$ be a coloring of F. A coloring $\sigma : \Gamma \to A$ of the elements of Γ is said to *avoid the pattern* α (we also say that *the pattern* α *is forbidden in* σ) if for any $g \in \Gamma$, the restriction of $g \cdot \sigma$ to F is distinct from α. In other words, this means that for all sets S in the Γ-orbit of F, the corresponding coloring α of S is avoided in σ.

A *subshift of finite type in* Γ is the set of all colorings of Γ avoiding a given finite set of patterns. We say that a coloring $\sigma : \Gamma \to A$ is *strongly periodic* if $\mathrm{Stab}_\Gamma(\sigma)$ has finite index in Γ, or equivalently if the orbit $\Gamma \cdot \sigma$ is finite. We say that Γ is *strongly periodic* if any non-empty subshift of finite type in G contains a strongly periodic coloring.

Remark 5.1. Let $\sigma : \Gamma \to A$, and fix a finite generating set S of Γ. Then σ is strongly periodic if and only if Γ induces a quasi-transitive group action on the colored graph $(\mathrm{Cay}(\Gamma, S), \sigma)$. This follows from the fact that the elements of Γ that induce automorphisms of the colored graph $(\mathrm{Cay}(\Gamma, S), \sigma)$ are exactly the elements from $\mathrm{Stab}_\Gamma(\sigma)$. Thus the Γ-orbits of the vertices of the colored graph $(\mathrm{Cay}(\Gamma, S), \sigma)$ correspond to the different right-cosets $\mathrm{Stab}_\Gamma(\sigma)g$. (This is just a rephrasing of the arguments in the proofs of Lemmas 2.3 and 2.5 in Section 2.)

It was proved in [CP15] that every group which is virtually $\mathbb{Z}$ is strongly periodic (see also [Coh17]). This directly implies the following, which gives a positive answer to Problem 4.6 in the case of Cayley graphs of bounded pathwidth.

Theorem 5.2. *For every finitely generated 2-ended group* Γ *and every finite generating set* S*, the graph* $G = \mathrm{Cay}(\Gamma, S)$ *has a periodic proper vertex-coloring with at most* $\chi(G)$ *colors.*

Proof. Let c be an optimal proper vertex-coloring of G, with colors from a finite set A with $|A| = \chi(G)$, and consider the subshift of finite type $\mathcal{X}$ in Γ where for each edge uv in G, the vertices u and v are required to have different colors. This can clearly be encoded by a finite set of forbidden patterns in Γ (all colorings of pairs $(1_\Gamma, s)$ for $s \in S$ where 1_Γ and s have the same color). The coloring c witnesses that the subshift of finite type $\mathcal{X}$ defined above is non-empty. Since Γ is a finitely generated 2-ended group, it is virtually $\mathbb{Z}$ and thus strongly periodic [CP15]. It follows that G has a proper vertex-coloring c' with colors from A which has a finite orbit under the action of Γ. By Remark 5.1, this implies in particular that c' is periodic, as desired. □

We note here that the approach using subshifts of finite type is inherently limited: it was proved in [Pia08] that non-abelian free groups are not strongly periodic, and it was even conjectured in [CP15] that a group is strongly periodic if and only if it is virtually cyclic (that is, finite or virtually $\mathbb{Z}$).

5.2. **2-ended graphs and bounded pathwidth.** We now extend Theorem 5.2 to all 2-ended locally finite quasi-transitive graphs (or equivalently, to all locally finite quasi-transitive graphs of bounded pathwidth).

A *separation* in a graph G is a triple (Y, S, Z) where Y, S, Z are pairwise disjoint subsets of $V(G)$ with $V(G) = Y \cup S \cup Z$ and no edge of G has an endpoint in Y and the other in Z. The *order* of (Y, S, Z) is $|S|$. A separation (Y, S, Z) is said to *separate* two ends ω_1, ω_2 of G if all but finitely many vertices of each ray of ω_1 lie in Y, and similarly all but finitely many vertices of each ray of ω_2 lie in Z.

Lemma 5.3. *Let G be a locally finite quasi-transitive graph with two ends. Then G has a separation (Y, S, Z) of finite order separating the two ends of G and there is an element $g \in \mathrm{Aut}(G)$ of infinite order such that $g(S \cup Z) \subseteq Z$.*

Proof. Let $k \in \mathbb{N} \setminus \{0\}$ be the minimum order of a separation (Y, S, Z) separating the two ends of G, i.e. such that both $G[Y]$ and $G[Z]$ contain an infinite component of $G - S$, and let (Y, S, Z) be a separation of order k separating the two ends of G such that $G[Z]$ is connected. Let $g_1 \in \mathrm{Aut}(G)$ be such that $g_1(S) \subseteq Z$ (as Z is infinite and quasi-transitive, such an element g_1 exists), and set $(Y_1, S_1, Z_1) := (g_1(Y), g_1(S), g_1(Z))$. If $Z_1 \subseteq Z$, we set $g := g_1$, and claim that g must have infinite order, as an easy induction shows that then $g^{i+1}(S \cup Z) \subseteq g^i(Z) \subsetneq Z$ for all $i \geqslant 0$. If Z_1 is not included in Z, then as $G[Z]$ is connected (and thus $G[Z_1]$ also is) we must have $Y \subseteq Z_1$ and $Y_1 \subseteq Z$. Let $g_2 \in \mathrm{Aut}(G)$ be such that $g_2(S) \subseteq Y_1 \subseteq Z$, and set $(Y_2, S_2, Z_2) := (g_2(Y), g_2(S), g_2(Z))$. Again, if $Z_2 \subseteq Y_1$, then $Z_2 \subseteq Z$ and we conclude as before choosing $g := g_2$; thus we can assume that we do not have $Z_2 \subseteq Y_1$. As $G[Z_2]$ is connected we have $S_1 \subseteq Z_2$ and thus $S_1 \cup Z_1 \subseteq Z_2$. We now conclude by choosing $g := g_1 g_2^{-1}$, which satisfies $g(S_2 \cup Z_2) = S_1 \cup Z_1 \subseteq Z_2$. Again, the fact that g has infinite order immediately follows. □

We now prove the main result of this section, giving a positive answer to Problem 4.6 for connected locally finite quasi-transitive graphs with 2 ends.

Theorem 5.4. *Let G be a connected locally finite quasi-transitive graph with 2 ends. Then G has a periodic proper vertex-coloring with $\chi(G)$ colors.*

Proof. We let (Y, S, Z) and $g \in \mathrm{Aut}(G)$ be given by Lemma 5.3. For each $i \in \mathbb{Z}$, let $(Y_i, S_i, Z_i) := (g^i(Y), g^i(S), g^i(Z))$. Then $S_j \cup Z_j \subseteq Z_i$ for all $i < j$ and (Y_i, S_i, Z_i) also separates the two ends of G. Let $c : V(G) \to [\chi(G)]$ be a proper vertex-coloring of G. By the pigeonhole principle there exist $i < j$ such that $c(g^i(x)) = c(g^j(x))$ for all $x \in S$. Up to replacing g by g^{j-i}, we may assume that $i = 0$ and $j = 1$, i.e., that $c(g(x)) = c(x)$ for all $x \in S$.

For all $i \in \mathbb{Z}$, we let $V_i := V(G) \setminus (Y_i \cup S_{i+1} \cup Z_{i+1})$. Then for each $i \in \mathbb{Z}$, $S_i \subseteq V_i$, $V_{i+1} = g(V_i)$ and as G has two ends and bounded degree, the graph $G_i := G[V_i]$ is finite. Moreover, note that $\{V_i : i \in \mathbb{Z}\}$ is a partition of $V(G)$.

We now define a vertex-coloring $\widetilde{c} : V(G) \to [\chi(G)]$ by setting for each $i \in \mathbb{Z}$ and $v \in V_i$, $\widetilde{c}(v) := c(g^{-i}(v))$. In other words, the vertex-coloring $\widetilde{c}$ is obtained by repeating periodically $c|_{V_0}$ on each G_i. First, note that $\widetilde{c}$ is well-defined on $V(G)$ as for every $v \in V(G)$ there exists a unique $i \in \mathbb{Z}$ such that $v \in V_i$ and then $g^{-i}(v) \in V_0$. Moreover, by definition g is color-preserving, i.e., $\widetilde{c}(g(v)) = \widetilde{c}(v)$ for all $v \in V(G)$.

We show that $\widetilde{c}$ is a proper vertex-coloring. Let $uv \in E(G)$ and $i, j \in \mathbb{Z}$ be such that $u \in V_i$ and $v \in V_j$. For each $i \in \mathbb{Z}$, note that $(Y_i, V_i, S_{i+1} \cup Z_{i+1})$ is a separation such that $V_{i-1} \subseteq Y_i$ and $V_{i+1} \subseteq S_{i+1} \cup Z_{i+1}$ so we must have $|i - j| \leqslant 1$. If $i = j$, then $g^{-i}(u)$ and $g^{-i}(v)$

are adjacent in G_0 and thus $\widetilde{c}(u) = c(g^{-i}(u)) \neq c(g^{-i}(v)) = \widetilde{c}(v)$. Assume that $j = i+1$, the other case being symmetric. Then $g^{-i}(u) \in V_0$ and $g^{-i}(v)$ is adjacent to $g^{-i}(u)$. Moreover $g^{-i}(v) = g(g^{-j}(v)) \in V_1$. Thus we must have $g^{-i}(u) \in Y_1$ and $g^{-i}(v) \notin Y_1$ so $g^{-i}(v) \in S_1$ and $g^{-j}(v) \in S_0$. Then by the choice of g, $c(g^{-j}(v)) = c(g^{-i}(v))$. As c is a proper vertex-coloring and by definition of $\widetilde{c}$ we then have $\widetilde{c}(u) = c(g^{-i}(u)) \neq c(g^{-i}(v)) = c(g^{-j}(v)) = \widetilde{c}(v)$, proving that $\widetilde{c}$ is a proper vertex-coloring of G.

As the sets V_i are finite and cover $V(G)$, and $g(V_i) = V_{i+1}$ for each $i \in \mathbb{Z}$, the subgroup of $\mathrm{Aut}(G)$ generated by g induces a quasi-transitive action on $V(G)$. We conclude that G has a periodic proper $\chi(G)$-coloring. □

We recall that the *chromatic index* of a graph G, denoted by $\chi'(G)$, is the minimum number of colors in a proper edge-coloring of G. This is well defined when the maximum degree of the graph is finite, which is the case for locally finite quasi-transitive graphs. The *line-graph* $L(G)$ of a graph G is the graph with vertex set $E(G)$ in which two vertices are adjacent if and only if the corresponding edges of G share a vertex. Note that $\chi'(G) = \chi(L(G))$ for any graph G.

We obtain the following consequence of Theorem 5.4 for edge-colorings of 2-ended locally finite quasi-transitive graphs.

Corollary 5.5. *If G is a connected locally finite quasi-transitive graph with 2 ends, then there exists a periodic proper edge-coloring of G with $\chi'(G)$ colors.*

Proof. Note that the line-graph $L(G)$ is also locally finite, connected, and quasi-transitive (if there are k orbits of $V(G)$ under the action of $\mathrm{Aut}(G)$, there are at most $\binom{k}{2}$ orbits of $E(G)$ under the action of $\mathrm{Aut}(G)$). Moreover, as rays in $L(G)$ are in correspondence with rays of G and similarly finite subsets separating the ends of G are in correspondence with finite subsets separating the ends of $L(G)$ (in both cases we use the fact that G has bounded degree), $L(G)$ is also 2-ended. We can thus conclude by applying Theorem 5.4 to $L(G)$. □

Remark 5.6. Corollary 5.5 can also be proved directly by a simple modification of the proof of Theorem 5.4, and by similar modifications the proof can be adapted to work with other types of objects in graphs, for instance periodic perfect matchings (assuming G has a perfect matchings), periodic nowhere-zero k-flows (assuming G has a nowhere-zero k-flow), periodic non-trivial eigenvectors (assuming G has a non-trivial eigenvector), etc. In all these cases, if a connected locally finite 2-ended quasi-transitive graph G can be decorated with an additional structure $\mathcal{S}$, then G can be decorated with a periodic version of $\mathcal{S}$.

6. Discussion and open problems

Recall our original question: is it true that every quasi-transitive graph G can be decorated with a non-trivial additional structure such that the graph G is still quasi-transitive if we restrict ourselves to automorphisms preserving the additional structure? We have considered two specific instances of this problem (Problems 1.1 and 1.2) and proved that each of them has a negative answer, even for Cayley graphs and graphs of bounded treewidth. The examples providing a negative answer to Problem 1.1 even rule out the existence of a non-trivial vertex-coloring that is periodic, which immediately rules out the existence of a number of other natural structures in graphs (any non-trivial vertex subset for instance) that would be periodic, in the same sense as before.

The corresponding problem for edge-colorings or edge-subsets appears to be more difficult. We say that an edge-coloring of a graph G is *trivial* if for each connected component C of G, all the edges of C are assigned the same color.

Problem 6.1. *Is it true that every locally finite quasi-transitive graph has a periodic non-trivial edge-coloring?*

Color reduction arguments also work for non-trivial edge-colorings. Consider a non-trivial edge-coloring c of a graph G. If a component C of G contains at least two edges, then c uses at least two different colors from C, say colors $1, 2$. For every $e \in E(G)$, let $c'(e) = 1$ if $c(e) = 1$, and $c'(e) = 2$ otherwise. Note that c' is non-trivial and only uses 2 colors. It follows that Problem 6.1 is equivalent to the following problem.

Problem 6.2. *It it true that every locally finite quasi-transitive graph has a periodic non-trivial edge-coloring with two colors?*

Note that coloring the edges of G with two colors is equivalent to choosing a spanning subgraph H of G (say the spanning subgraph induced by the edges colored 1). So we can ask equivalently:

Problem 6.3. *Is it true that every locally finite quasi-transitive graph has a periodic non-trivial spanning subgraph?*

Here, by *non-trivial* subgraph H of a graph G, we mean that at least one connected component of G contains an edge of H and a non-edge of H (that is an edge of G which is not in H). By *periodic subgraph* we mean a subgraph H of G such that the subgroup of automorphisms of G stabilizing H (that is, mapping edges of H to edges of H and non-edges of H to non-edges of H) acts quasi-transitively on G.

Note that these problems have a trivial (positive) answer for graphs that are not edge-transitive (by taking H to be an orbit of the action of $\mathrm{Aut}(G)$ on the edge-set of G), so any possible counterexample must be edge-transitive.

While we were not able to provide a negative answer to these problems, the examples constructed in the previous sections severely restrict the properties of such a spanning subgraph H in general. Let us say that a class of graphs $\mathcal{C}$ is *periodizing* if any locally finite quasi-transitive graph G has a periodic non-trivial spanning subgraph $H \in \mathcal{C}$. Problem 6.3 asks whether the class of all graphs is periodizing. We now give a number of sufficient conditions on periodizing graph classes.

Lemma 6.4. *Every periodizing class contains a vertex-transitive (and thus regular) infinite graph of minimum degree at least 2.*

Proof. Consider the Cayley graph $G = \mathrm{Cay}(V, T)$ given by Lemma 3.4, where we consider Thompson's group V, which is a finitely generated infinite simple group, and where T is a finite generating set of V containing no element of order 2, and such that G is a graphical rigid representation of V (that is, $\mathrm{Aut}(G) = V$, see Section 3). Assume that G has a non-trivial spanning subgraph H such that the subgroup Γ of automorphisms of G stabilizing H acts quasi-transitively on G. As in the proof of Lemma 3.2, it directly follows from Corollary 2.2 that $\Gamma = V$, that is, all the automorphisms of G stabilize H. Since G is vertex-transitive, H is also vertex-transitive and thus d-regular with $d \geqslant 1$, since H is non-trivial. Assume for the sake of contradiction that $d = 1$ and consider two adjacent vertices u, v in H. Consider an automorphism g of G that maps u to v, and observe that g must map v to u, since otherwise v would have degree 2 in H. It then follows that the element $s \in T$ corresponding to the edge uv in G has order 2 in V, a contradiction. □

A *k-factor* in a graph is a spanning subgraph in which every vertex has degree k. Observe that Lemma 6.4 implies that Problem 6.3 has a negative answer if we require H to be a forest, and in particular a 1-factor (i.e., a perfect matching). The simplest version of Problem 6.3 we can think of is thus the following.

Problem 6.5. *Is it true that every 4-regular vertex-transitive infinite graph has a periodic 2-factor?*

A classical result of Petersen [Pet91] asserts that every finite Eulerian graph G has a spanning subgraph H such that for every vertex $v \in V(G)$, $d_H(v) = d_G(v)/2$, and the result can be extended to infinite locally finite graphs by compactness. Hence, every infinite 4-regular graph contains a 2-factor.

Planar graphs. Planar Cayley graphs are a well-studied topic (see [GH19] and the reference therein), and by extension planar quasi-transitive graphs have been extensively studied (see for instance [Bab97, Ham15, EGLD24, Mac23]). We have seen examples of planar quasi-transitive graphs with no periodic orientation in Section 4 (these examples are all ∞-ended), and in Section 5 we have seen that all 2-ended quasi-transitive graphs have a periodic proper coloring (and thus a periodic orientation). A natural question is whether all planar 1-ended quasi-transitive graphs have a periodic proper coloring. Using a theorem of Babai [Bab97] providing embeddings of planar 1-ended 3-connected quasi-transitive graphs in the Euclidean plane or the hyperbolic plane, it can be proved that in the Euclidean case the graphs have a periodic proper coloring. However, it is unclear whether the same holds in the hyperbolic case.

Problem 6.6. *Are there 1-ended quasi-transitive planar graphs without periodic proper coloring?*

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