HYPERBOLICITY IN SEMIMETRIC SPACES, DIGRAPHS AND SEMIGROUPS

MATTHIAS HAMANN

Abstract. Gray and Kambites introduced a notion of hyperbolicity in the setting of semimetric spaces like digraphs or semigroups. We carry over some of the fundamental results of hyperbolic spaces to this new setting. In particular, we prove under some small additional geometric assumption that their notion of hyperbolicity is preserved by quasi-isometries. We also construct a boundary based on quasi-geodesic rays and anti-rays that is preserved by quasi-isometries and, in the case of locally finite digraphs, refines their ends. We show that it is possible to equip the space, if it is finitely based, with its boundary with a pseudo-semimetric and show some further results for the boundary. We also apply our results to semigroups and give a partial solution to a problem of Gray and Kambites.

1. Introduction

Since Gromov's paper [17] on hyperbolic groups, hyperbolicity is a topic that received a lot of interest, mostly for geodesic metric spaces, graphs or groups, see e. g. [1, 2, 10, 14, 21]. Occasionally, some of these assumptions are dropped. One such example is the paper of Väisälä [24], where the metric spaces under consideration need not be geodesic. In the present paper, we try to extend some of the basic results to a generalisation of hyperbolicity in spaces, where the distance function need not be symmetric. The particular notion we are interested in was defined by Gray and Kambites [16] in order to have a geometric notion of hyperbolicity for monoids.

Gray and Kambites were not the first to look for a notion of hyperbolicity for monoids but their notion is the first geometric one that takes directions into account. Previous definitions either were based on context-free rewriting systems, see e. g. [6, 7, 12, 13, 19], or looked at the underlying undirected graph of Cayley digraphs of semigroups, see e. g. [6, 8, 13]. Portilla et al. [22] considered a notion of hyperbolicity in (a certain class of) digraphs that is also defined via hyperbolicity of their underlying undirected graphs. The advantage of a geometric notion for monoids that takes directions into account is that it also applies to other objects such as directed graphs, digraphs for short, or geodesic semimetric spaces. The latter are spaces where the distance function has $[0,\infty]$ as its range and we do ask for all condition needed for a metric except for $d(x,y) = d(y,x)$, see Section 2.1 for the formal definition. In the literature, these spaces are also known as quasi-metric or asymmetric metric spaces, but we use the term "semimetric" here. We refer to Gray and Kambites [15, Section 2] for a discussion about these terms.

In Section 4, we will see that the notion of Gray and Kambites indeed generalizes the one for geodesic metric spaces. In some special situations, we will also consider

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a different notion that is stronger than Gray and Kambite's notion since there are also some interesting results related to that stronger notion. Both notions are based on a thinness condition for geodesic triangles, see Section 3 and 5 for the precise definitions.

While Gray and Kambites were mostly interested in properties of hyperbolic monoids, such as the word problem or other semigroup-theoretic decision problems and the question of finite presentability of those monoids, we focus more on the geometric aspects of the spaces themselves. We look at two particular topics and prove results related to those: first, we consider the question whether this notion of hyperbolicity is preserved by quasi-isometries and then we will define a boundary for hyperbolic semimetric spaces that is defined by an equivalence relation on (quasi-)geodesic rays and prove several results for them. See below for more details on these two topics. But before we turn our attention to some details of those topics, let us mention that most of the results that we obtain need additional geometric assumptions on the spaces.

In geodesic metric spaces X, we find for every point $x \in X$ and points y, z in the ball of radius r around x a y-z geodesic of length at most $2r$. For semimetric spaces Y, this is no longer true: if $x \in Y$ and $y, z \in Y$ with $d(x, y) < r$ and $d(x, z) < r$, then $d(y, z)$ need not be bounded or even be finite; it may also happen that the out-ball of radius r around x contains an infinite geodesic. The same also holds for the in-balls. We will ask our semimetric spaces to satisfy a condition that bounds the lengths of geodesics in out-balls or in-balls of finite radius in terms of the radius. So our results still cover a large proper subclass of the geodesic semimetric spaces that contain all geodesic metric spaces. As a consequence, when applying our results to semigroups, we only obtain results for right cancellative semigroups. All results that we will discuss within the introduction will only be proved for geodesic semimetric spaces that satisfy this condition but for sake of simplicity, we will omit it in the following.

Before moving on to the detailled discussion, let us mention that even though we use some well-known notions such as quasi-isometries or geodesics, they behave slightly different in semimetric spaces compared to metric spaces and we need definitions that take this into account. So for the introduction and as intuition, it is fine to just think of these notions as being similar to the ones in metric spaces and that they carry over the important properties that we will need. We omit these definitions in the introduction to keep the introduction simple and not overburdened by definitions. But later on we will give the precise definitions.

1.1. **Quasi-isometries.**

In the situation of metric spaces, in order to prove that quasi-isometries preserve hyperbolicity, first, it is shown that hyperbolicity is equivalent with geodesic stability and the latter is easily seen to be preserved by quasi-isometries. In order to prove that equivalence, one uses that both properties are also equivalent to exponential divergence of geodesics.

In the situation of semimetric spaces, we do no longer have equivalence between these three properties. More precisely, we will prove that hyperbolicity implies both other properties but neither of the other two implies hyperbolicity nor do they imply each other, see Sections 6 and 7. Still, this is enough to prove that hyperbolicity (and geodesic stability) is preserved by quasi-isometries. It stays an open problem whether (exponential) divergence of geodesics is preserved by quasi-isometries.

1.2. **Geodesic boundary.** In hyperbolic metric spaces, the hyperbolic boundary can be defined by an equivalence relation on geodesic rays, where two such rays are equivalent if they are eventually close to each other. For hyperbolic semimetric spaces, this is no longer true, but the above relation is still a quasiorder on the geodesic rays and anti-rays, see Section 9. Quasiorders canonically gives rise to an equivalence relation. In our situation, the corresponding equivalence classes will be our geodesic boundary points. By ∂X we denote the geodesic boundary of a hyperbolic semimetric space X.

We can define the same relation on quasi-geodesic rays and anti-rays and the boundary points defined by that relation are trivially preserved by quasi-isometries. For the geodesic boundary, however, it is still unknown whether they are preserved by quasi-isometries. While in the case of proper hyperbolic geodesic metric spaces, we can apply the Arzelà-Ascoli theorem to prove that both boundaries are essentially the same, it is not possible to apply an analogue theorem in the case of semimetric spaces, since that is false, in general, see [9]. But for locally finite digraphs, we can use a compactness argument to show that the quasi-geodesic boundary and the geodesic boundary coincide and thus conclude that the geodesic boundary is preserved by quasi-isometries, see Section 10.

For general digraphs, there is another notion for a boundary: ends as defined by Zuther [25]. It turns out that the geodesic boundary is a refinements of the end space for locally finite digraphs. Furthermore, we prove that the ends of locally finite digraphs are also preserved by quasi-isometries, see Section 11.

Semimetric spaces X have two natural topologies associated to them: one has the open out-balls of finite radius as base and the other the open in-balls of finite radius. These two topologies extend to topologies of $X \cup \partial X$ and quasi-isometries between hyperbolic semimetric spaces extend to homeomorphisms on the spaces with their geodesic boundaries with respect to these two topologies (see Section 12). Furthermore, if X has a finite base, we can equip $X \cup \partial X$ with a pseudo-semimetric, whose induced topologies coincide with the just mentioned ones under small additional assumptions, see Section 13.

Further results regarding the geodesic boundary include that, for locally finite digraphs D, the space $D \cup \partial D$ is f-complete and b-complete (see Section 14), two notions that mimic completeness in the setting of semimetrics with respect to the two topologies that we discussed. Additionally, we prove a result that can be seen as a partial analogue of the fact that in the case of metric spaces the ends correspond to the connected components of the hyperbolic boundary. We use that result to obtain some results on the size of the geodesic boundary, see Section 15.

1.3. **Hyperbolic semigroups.** In the final section, Section 16, we apply our results to semigroups. As mentioned above, our additional geometric assumptions imply that the results only hold for right cancellative semigroups.

Gray and Kambites [16] defined hyperbolicity for finitely generated semigroups by asking one of the Cayley digraphs be hyperbolic for a finite generating set. They ask whether for a finitely generated hyperbolic semigroup each of its Cayley digraphs with respect to finite generating sets is hyperbolic. We answer this positively for right cancellative semigroups.

As the geodesic boundary is preserved by quasi-isometries and thus by changing the finite generating set of finitely generated right cancellative semigroups, the geodesic boundary gives rise to a boundary of semigroups. The results on the number of geodesic boundary points implies for finitely generated cancellative hyperbolic semigroups that they have either 0, 1, 2 or infinitely many geodesic boundary points. Moreover, if the semigroup has exactly one end, then is has either 1 or infinitely many geodesic boundary points.

We end by some discussions about finitely generated right cancellative hyperbolic semigroups with at most two geodesic boundary points.

2. Preliminaries

In this section, we define all basic notion related to pseudo-semimetric spaces and to digraphs.

2.1. **Semimetric spaces.** Let X be a set. A map $d\colon X \times X \to [0,\infty]$ is a *pseudo-semimetric* if

(i) $d(x,x) = 0$ for all $x \in X$ and
(ii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

We then call (X,d) a *pseudo-semimetric space*.

A pseudo-semimetric is a *semimetric* if $d(x,y) = 0$ if and only if $x = y$ for all $x,y \in X$. Then we call (X,d) a *semimetric space*.

For x,y in a (pseudo-)semimetric space X, we set

$$d^{\leftrightarrow}(x,y) := \min\{d(x,y), d(y,x)\}$$

(Pseudo-)Semimetric spaces X come along with two natural topologies that are defined via the out-balls and the in-balls. For $r \geq 0$ and $x \in X$, we set the *out-ball* and the *open out-ball* of radius r around x as

$$\mathcal{B}_r^+(x) := \{y \in X \mid d(x,y) \leq r\}, \qquad \mathring{\mathcal{B}}_r^+(x) := \{y \in X \mid d(x,y) < r\}$$

and the *in-ball* and the *open in-ball* of radius r around x as

$$\mathcal{B}_r^-(x) := \{y \in X \mid d(y,x) \leq r\}, \qquad \mathring{\mathcal{B}}_r^-(x) := \{y \in X \mid d(y,x) < r\},$$

respectively.

The open out-balls $\mathring{\mathcal{B}}_r^+(x)$ for all $r \geq 0$ and $x \in X$ generate the *forward* topology $\mathcal{O}_f$ and the open in-balls $\mathring{\mathcal{B}}_r^-(x)$ for all $r \geq 0$ and $x \in X$ generate the *backward* topology $\mathcal{O}_b$.

For $a,b \in \mathbb{R}$ with $a < b$, we consider the interval $[a,b]$ with the above two topologies. A function $P\colon [a,b] \to X$ is a *directed path* if it is continuous with respect to the forward topologies of $[a,b]$ and X and if it is continuous with respect to the backward topologies of $[a,b]$ and X. Then $P(a)$ is the *starting point* and $P(b)$ is the *end point* of P. The *length* $\ell(P)$ of P is defined as

$$\ell(P) := \lim_{N\to\infty} \sum_{i=1}^{N} d(P(t_{i-1}), P(t_i))$$

with $t_i := a + i(b-a)/N$ for all $0 \leq i \leq N$. For $x,y \in X$, a *directed x-y path* is a directed path with starting point x and end point y. For $U,V \subseteq X$, a *directed U-V path* is a directed path with starting point in U and end point in V.

Let $P\colon [a,b] \to X$ and $Q\colon [a',b'] \to X$ be directed paths. They are *parallel* if $P(a) = Q(a')$ and $P(b) = Q(b')$ and they are *composable* if $P(b) = Q(a')$. If they are composable, then we denote by PQ the resulting directed path, their *composition*.

We say that $x \in X$ *lies on* P if it is in the image of P. For x,y on P with $x = P(r)$ and $y = P(R)$ such that $r < R$, we denote by xPy the subpath of P

between x and y. With a slight abuse of notation, we denote for $r \geq 0$ by $\mathcal{B}_r^+(P)$, by $\mathcal{B}_r^-(P)$, the out-ball, the in-ball, of radius r around the image of P, respectively.

Let (X, d_X) and (Y, d_Y) be semimetric spaces. A map $f\colon X \to Y$ is a *quasi-isometric embedding* if there are constants $\gamma \geq 1$ and $c \geq 0$ such that

$$\gamma^{-1} d_X(x, x') - c \leq d_Y(f(x), f(x')) \leq \gamma d_X(x, x') + c$$

for all $x, x' \in X$. It is a *quasi-isometry* if additionally for every $x \in X$ there is $y \in Y$ such that $d(f(x), y) \leq c$ and $d(y, f(x)) \leq c$. Then we say that X is *quasi-isometric* to Y. If we want to emphasize on the particular constants γ and c, we talk about (γ, c)-*quasi-isometries*. By Gray and Kambites [15, Proposition 1], being quasi-isometric is an equivalence relation. An *isometry* is a $(1, 0)$-quasi-isometry.

A *geodesic* from $x \in X$ to $y \in X$, also called an *x-y geodesic*, is a directed path P from x to y with $d(u, v) = \ell(uPv)$ for all u, v on P with v on uPy. For subsets $U, V \subseteq X$, a *U-V geodesic* is a directed U-V path that is a geodesic. For $\gamma \geq 1$ and $c \geq 0$, a (γ, c)-*quasi-geodesic* from x to y is a directed path from x to y with

$$\ell(uPv) \leq \gamma d(u, v) + c$$

for all u, v on P with v on uPy. We call X *geodesic* if there exists an x-y geodesic for all $x, y \in X$ with $d(x, y) < \infty$.

Note that x-y geodesics are not isometric images of $[0, d(x, y)]$ since we make no restrictions on $d(P(b), P(a))$ for $0 \leq a \leq b \leq 1$. Similarly, quasi-geodesics are not quasi-isometric images of $[0, d(x, y)]$.

Let $(x_i)_{i \in \mathbb{N}}$ be a sequence in X. It *f-converges* to $x \in X$ if it converges to x with respect to $\mathcal{O}_f$ and it *b-converges* to x if it converges to x with respect to $\mathcal{O}_b$. The sequence is called *forward Cauchy*, or *f-Cauchy*, if for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n \geq N$. It is called *backward Cauchy*, or *b-Cauchy*, if for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m \geq n \geq N$. We call X *f-complete* if every f-Cauchy sequence b-converges to a point in X and we call X *b-complete* if every b-Cauchy sequence f-converges to a point in X. Note that there are different notions of completeness for semimetric spaces, see e. g. [9, 18, 23]; some of them differ from ours, e. g. in that they ask for f-completeness that f-Cauchy sequences f-converge.

We call X *sequentially f-compact* if for every sequence $(x_i)_{i \in \mathbb{N}}$ in X that satisfies $d(x_i, x_j) < \infty$ for all $i < j$ has a b-convergent subsequence. We call X *sequentially b-compact* if for every sequence $(x_i)_{i \in \mathbb{N}}$ in X that satisfies $d(x_j, x_i) < \infty$ for all $i < j$ has an f-convergent subsequence.

Proposition 2.1. *Let X be a semimetric space. Then the following hold.*

(i) *If X is sequentially f-compact, then it is f-complete.*
(ii) *If X is sequentially b-compact, then it is b-complete.*

Proof. Let X be sequentially f-compact. Let $(x_i)_{i \in \mathbb{N}}$ be an f-Cauchy sequence in X. Then there is a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ that b-converges to a point $x \in X$. For $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_i, x_j) < \varepsilon/2$ for all $j > i > N$ and, for every $i > N$, there exists $k(i) \in \mathbb{N}$ with $d(x_{n_{k(i)}}, x) < \varepsilon/2$. Thus, we have

$$d(x_i, x) < d(x_i, x_{n_{k(i)}}) + d(x_{n_{k(i)}}, x) < \varepsilon$$

for all $i > N$. So $(x_i)_{i \in \mathbb{N}}$ b-converges to x and X is f-complete.

We obtain (ii) by an analogous argument. □

2.2. **Digraphs.** Here, a digraph $D = (V(D), E(D))$ is an oriented multigraph. In particular, we allow our digraphs to have loops and multiple edges between the same two vertices. For a subset X of $V(D)$, we denote by $D[X]$ the digraph *induced by* X, i. e. the digraph with vertex set X and all edges of D both of whose incident vertices lie in X.

A *directed path* is a sequence $x_0 \dots x_n$ of vertices such that $x_i x_{i+1} \in E(D)$ for all $0 \leq i < n$. A *proper directed path* P is a sequence $x_0 \dots x_n$ of pairwise distinct vertices such that $x_i x_{i+1} \in E(D)$ for all $0 \leq i < n$.[1] The *length* $\ell(P)$ of P is n, the number of edges of the path.

If $x_0, x_1, \dots$ are distinct vertices in D with $x_i x_{i+1} \in E(D)$ we call $x_0 x_1 \dots$ a *ray*. If we have $x_{i+1} x_i \in E(D)$ instead we say that $x_0 x_1 \dots$ is an *anti-ray*.

For $x, y \in V(D)$, let $d(x, y)$ be the length of a shortest directed path from x to y or ∞ if no such path exists. Note that in contrast to graphs, this distance function is not a metric but a semimetric. The *out-degree* of a vertex $x \in V(D)$ is the number of vertices $y \in V(D)$ with $d(x, y) = 1$ and the *in-degree* of x is the number of vertices $y \in V(D)$ with $d(y, x) = 1$. A digraph is *locally finite* if its in- and out-degrees are all finite.

We consider the the following semimetric space associated to a digraph D: we consider the 1-complex of the underlying undirected (multi-)graph of D. The distance between two points x, y corresponding to vertices u, v, respectively, is set as follows: $d(x, y) := d(u, v)$. For an inner point x of an edge uv, its distance to a point y corresponding to a vertex w or an inner point of an edge ww' is set as $d(x, y) := d(x, v) + d(v, w) + d(w, y)$, where we consider the edges e to be directed paths of length 1 in the semimetric space.

3. Thin triangles

Let X be a geodesic semimetric space. A *triangle* consists of three points of X and three directed paths, one between every two of those vertices. We call these paths the *sides* and the three point the *end points* of the triangle. The triangle is *geodesic* if all three sides are geodesics and it is *transitive* if two of its sides are composable and the resulting directed path is parallel to the third side.

We consider thin triangles as defined by Gray and Kambites in [16]. Let $\delta \geq 0$. A geodesic triangle is *δ-thin* if the following holds:

> *if P, Q, R are the sides of the triangle and the starting point of P is either the starting or the end point of Q and the last point of P is either the starting or the end point of R, then P is contained in $\mathcal{B}^+_\delta(Q) \cup \mathcal{B}^-_\delta(R)$.*

If all geodesic triangles in X are δ-thin then X is *δ-hyperbolic*. We call X *hyperbolic* if it is δ'-hyperbolic for some $\delta' \geq 0$.

We note that this definition differs slightly from the definition of Gray and Kambites in that they only ask transitive geodesic triangles to be δ-thin. However, our first result is that – up to the constant δ – both definitions are equivalent.

Proposition 3.1. *Let X be a geodesic semimetric space and $\delta \geq 0$. If all transitive geodesic triangles are δ-thin, then all geodesic triangles are 3δ-thin.*

[1] With our terminology, we differ from the usual one for digraphs: what we call directed paths are usually directed walks and our proper directed paths are directed paths. We chose this different terminology because now directed paths in digraphs are the same regardless whether we consider it as a digraph or as a semimetric space.

Proof. Let $x, y \in X$ such that there are geodesics P and Q from x to y and from y to x, respectively. Let R be the trivial directed path with image x. Then P, Q, R form the sides of a transitive geodesic triangle. So P lies in $\mathcal{B}^+_\delta(R) \cup \mathcal{B}^-_\delta(Q)$. Since P is a geodesic, only its directed subpath with starting point x and end point $P(\delta)$ is contained in $\mathcal{B}^+_\delta(R)$. The remaining part of P lies in $\mathcal{B}^-_\delta(Q)$ and thus all of P lies in $\mathcal{B}^-_{2\delta}(Q)$. Similarly, if we take the third side as trivial directed path y, we obtain that P lies in $\mathcal{B}^+_{2\delta}(Q)$.

Now let us consider an arbitrary geodesic triangle that is not transitive and let P, Q, R be its sides and x, y, z be its end points such that P is an x-y geodesic, Q is a y-z geodesic and R is a z-x geodesic. Then P and Q are composable and hence there is an x-z geodesic S. By our previous argumentation, we know that R lies in $\mathcal{B}^-_{2\delta}(S) \cap \mathcal{B}^+_{2\delta}(S)$. Since the geodesic triangle with sides P, Q, S is δ-thin, we obtain that R lies in $\mathcal{B}^-_{3\delta}(P) \cap \mathcal{B}^+_{3\delta}(Q)$. □

Contrary to metric spaces, in a semimetric space X, the lengths of geodesics with starting and end point in $\mathcal{B}^+_r(x)$ for $x \in X$ and $r \in R$ need not be bounded. (In metric spaces, this is bounded by $2r$.) However, we will often restrict ourselves to situations, where this is satisfied. We define the following two properties.

(B1) *There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $x \in X$, for every $r \geq 0$ and for all $y, z \in \mathcal{B}^+_r(x)$ the distance $d(y, z)$ is either ∞ or bounded by $f(r)$.*

(B2) *There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $x \in X$, for every $r \geq 0$ and for all $y, z \in \mathcal{B}^-_r(x)$ the distance $d(y, z)$ is either ∞ or bounded by $f(r)$.*

These properties are satisfied in several applications as we will see now. As mentioned above, if X is also a metric space, then it satisfies (B1) and (B2) for the function $f(r) = 2r$.

Lemma 3.2. *Every hyperbolic digraph of bounded in- and bounded out-degree satisfies (B1) and (B2).*

Proof. Let D be a δ-hyperbolic digraph of in- and out-degree at most ρ and let $x \in V(D)$. Then the number of vertices in $\mathcal{B}^+_k(x)$ and the number of vertices in $\mathcal{B}^-_k(x)$ is bounded by $f_k := \sum_{i=0}^k (\rho - 1)^i$. Let P be a geodesic with starting and end vertex in $\mathcal{B}^+_k(x)$. Let Q be a geodesic from x to the starting vertex of P and let R be a geodesic from x to the end vertex of P. So P, Q, R form the sides of a geodesic triangles. Since this is δ-thin, each vertex of P lies either in a ball $\mathcal{B}^+_\delta(y)$ for some vertex y on Q or in $\mathcal{B}^-_\delta(z)$ for some vertex z on R. In these balls, there are at most $kf_\delta + kf_\delta$ many vertices. Thus, this number is also an upper bound on the length of P. □

It follows from Lemma 3.2 that locally finite transitive[2] hyperbolic digraphs satisfy (B1) and (B2). Another class of digraphs satisfying (B1) and (B2) are Cayley digraphs of finitely generated right cancellative hyperbolic semigroups, see Section 16 for more details.

[2]Recall that a digraph is *transitive* if for all $x, y \in V(D)$ there is an automorphism α of D that maps x to y.

A major property that follows from (B1) and (B2) is formulated in our next propositions. Note that, by Lemma 3.2, that proposition is a generalization of a result of Gray and Kambites [16, Lemma 3.1].

Proposition 3.3. *Let $\delta \geq 0$ and let X be a δ-hyperbolic geodesic semimetric space that satisfies (B1) for the function $f\colon \mathbb{R} \to \mathbb{R}$ and (B2) for the function $g\colon \mathbb{R} \to \mathbb{R}$.*

(i) *If P, Q, R are the sides of a geodesic triangle such that the starting point of P is either the starting or the end point of Q and the end point of P is either the starting or the end point of R, then we have*
$$\ell(P) \leq (\ell(Q)/\varepsilon) f(\delta + \varepsilon) + (\ell(R)/\varepsilon) g(\delta + \varepsilon).$$
for all $\varepsilon > 0$.

(ii) *If $x, y \in X$ with $d(x, y) \neq \infty$ and $d(y, x) \neq \infty$, then we have*
$$d(x, y) \leq (d(y, x)/\varepsilon) f(\delta + \varepsilon) + g(\delta)$$
and
$$d(x, y) \leq (d(y, x)/\varepsilon) g(\delta + \varepsilon) + f(\delta)$$
for all $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$. By assumption, P lies in $\mathcal{B}^+_\delta(Q) \cup \mathcal{B}^-_\delta(R)$. Let a be a point on Q and let $u = P(r)$, $v = P(R)$ in $\mathcal{B}^+_{\delta+\varepsilon}(a)$ such that $r < R$. By (B1), we have $d(u, v) \leq f(\delta + \varepsilon)$. All points on P that lie in $\mathcal{B}^+_\delta(Q)$ lie in the union of all $\mathcal{B}^+_{\delta+\varepsilon}(Q(i\varepsilon))$ with $i \in \mathbb{N}$ and $i\varepsilon \leq \ell(Q)$. For each $i \leq \ell(Q)/\varepsilon$, let P_i be a smallest subpath of P containing all points of P that lie in $\mathcal{B}^+_{\delta+\varepsilon}(Q(i\varepsilon))$. The lengths of these paths sum up to at most $(\ell(Q)/\varepsilon) f(\delta + \varepsilon)$. Analogously, the lengths of the smallest subpaths of P that contain all point on P that lie in $\mathcal{B}^-_{\delta+\varepsilon}(R(i\varepsilon))$ sum up to at most $(\ell(R)/\varepsilon) g(\delta + \varepsilon)$. This proves (i).

Finally, (ii) follows directly from (i): we just choose P to be an x-y geodesic and in the first case R to be the trivial path y and Q a y-x geodesic and in the second case R to be a y-x geodesic and Q the trivial path x. Note that it suffices to take $g(\delta)$ or $f(\delta)$ for the trivial path. □

In our last result in this section, we prove that we can control that side in a transitive triangle that is parallel to the composition of the other two even further than the definition of a δ-thin triangle indicates.

Lemma 3.4. *Let $\delta \geq 0$ and let X be a δ-hyperbolic geodesic semimetric space that satisfies (B1) and (B2) for the function $f\colon \mathbb{R} \to \mathbb{R}$. Let P, Q, R be the sides of a geodesic triangle such that P and Q are composable and their composition is parallel to R. Then R lies in the out-ball of radius $6\delta + 2\delta f(\delta + 1)$ around $P \cup Q$ and in the in-ball of the same radius around $P \cup Q$.*

Proof. Set $k^* := 4\delta + 2\delta f(\delta + 1)$ and $k := k^* + 2\delta$. Let a be the starting and b be the end point of R. Let x be a point on R. Then $x \in \mathcal{B}^+_\delta(P) \cup \mathcal{B}^-_\delta(Q)$. If $x \in \mathcal{B}^+_\delta(P)$, then $d(P, x) \leq \delta \leq k^* \leq k$. So we may assume that there is a point y on Q with $d(x, y) \leq \delta$. Then $y \in \mathcal{B}^+_\delta(P) \cup \mathcal{B}^-_\delta(R)$.

Let us first assume that there is a point z on P with $d(z, y) \leq \delta$. Let P' be a z-y geodesic, Q' an a-y geodesic and R' an x-y geodesic. Then every point of Q' that lies in $\mathcal{B}^-_\delta(P')$ lies in $\mathcal{B}^-_{2\delta}(y)$. Thus, Q' lies completely in $\mathcal{B}^+_{3\delta}(P)$. Using Proposition 3.3 with $\varepsilon = 1$, at most the subpath of length $2\delta f(\delta + 1)$ with end vertex x of aRx lies in $\mathcal{B}_\delta^-(R')$. Thus, there exists u on Q' with $d(u,x) \leq \delta + 2\delta f(\delta+1)$ and hence we have

$$d(P,x) \leq d(P,u) + d(u,x) \leq 4\delta + 2\delta f(\delta+1) = k^*.$$

Let us now assume that there is a point z on R with $d(y,z) \leq \delta$. If z lies on aRx, then Proposition 3.3 implies $d(z,x) \leq 2\delta f(\delta+1)$. So we have

$$d(y,x) \leq \delta + 2\delta f(\delta+1) \leq k^*.$$

Hence, we assume that z lies on xRb. If $d(a,x) \leq 2\delta$, we immediately obtain $d(P,x) \leq 2\delta \leq k$ Otherwise, let x' be on aRx with $d(x',x) = 2\delta$. By an analogous situation for x' as for x either we have $d(P \cup Q, x') \leq k^*$ or there exists a point v on $x'Rb$ with $d(Q,v) \leq \delta$ and $d(x',v) \leq 2\delta$. Thus, we obtain $d(P \cup Q, x) \leq k$.

By a symmetric argument, we also obtain that $d(x, P \cup Q) \leq k$. □

4. Examples of hyperbolic semimetric spaces

We recall that for geodesic metric spaces X, the definition of hyperbolicity is as follows: X is *hyperbolic* if there exists $\delta \geq 0$ such that for all $x,y,z \in X$ and all x-y, y-z and x-z geodesics each of these geodesics lies in the δ-neighbourhood of the other two geodesics. It directly follows from the definitions that hyperbolic geodesic metric spaces are also hyperbolic when considering the space as a semimetric space and using the definition from Section 3. This is similar to the observation of Gray and Kambites [16, Section 2] that starting with a graph and replacing each edge by two directed edges that are oppositely oriented leads to a digraph that is hyperbolic if and only if the graph is hyperbolic when viewed as a metric space.

Two points x,y in a semimetric space X are *equivalent*, if $d(x,y) < \infty$ and $d(y,x) < \infty$. This is an equivalence relation whose classes are the strongly connected components of X. Gray and Kambites [16, Proposition 2.5] showed that in hyperbolic digraphs, the strongly connected components are hyperbolic, too. Their proof immediately carries over to semimetric spaces, so we obtain the following.

Proposition 4.1. *Let X be a δ-hyperbolic geodesic semimetric space. Then every strongly connected component of X is δ-hyperbolic.* □

In the case of graphs, the 0-hyperbolic graphs are the trees. For digraphs, this is no longer the case: while Gray and Kambites [16, Proposition 2.4] noted that digraphs whose underlying undirected graphs are trees are 0-hyperbolic, there are more 0-hyperbolic digraphs. We will characterise the 0-hyperbolic digraphs by using one property of trees: they have uniquely determined paths between every two vertices.

Proposition 4.2. *Let D be a digraph. Then D is 0-hyperbolic if and only if there are no $x,y \in V(D)$ with two distinct directed x-y paths (which may be trivial).*

Proof. First, let us assume that D is 0-hyperbolic. Let us suppose that there are $x,y \in V(D)$ with two distinct directed x-y paths P_1 and P_2. First, let us assume that $x = y$. Then we may assume that P_1 is trivial and that P_2 contains a vertex distinct from x. Let z be an out-neighbour of x on P_2 and let a be a point on an edge $e = xz$ distinct from x and z. Let P be a z-x geodesic. Then P does not contain the edge e. In particular, a lies neither on P nor on the trivial directed path x. This is a contradiction to 0-hyperbolicity.

So let us now assume that $x \neq y$ and that all vertices of P_1 are distinct and all vertices of P_2 are distinct. Let P be an x-y geodesic. We may assume that $P_1 \neq P$. Let $e = uv$ be an edge on P_1 that does not lie on P. Let Q_u^1 and Q_v^1 be x-u and x-v geodesics, respectively, and let Q_u^2 and Q_v^2 be u-y and v-y geodesics, respectively. If e does not lie on Q_v^1, then the geodesic triangle with end vertices x, u, v and sides Q_u^1, Q_v^1 and e contradicts 0-hyperbolicity. Thus, e lies on Q_v^1 and the geodesic triangle with end vertices x, v, y and sides Q_v^1, Q_v^2 and P contradicts 0-hyperbolicity. Thus, there exists a uniquely determined directed x-y path.

Let us now assume that for all $x, y \in V(D)$ there exists a unique directed x-y path. If a geodesic triangle is not transitive, then there are two distinct directed z-z paths for every end vertex of the triangle: one trivial one and the other one following all three sides of the triangle. Thus, all geodesic triangles are transitive. But then the composition of the composable sides must coincide with the third side and thus the triangle is 0-thin. □

5. Slim triangles

In this section, we discuss another condition on geodesic triangles that is similar to the thin triangles of the previous section but that is generally much stronger. The reason to discuss this stronger condition is that – in particular when looking at boundaries – we obtain other natural and interesting results, see Sections 8, 9, 10 and 13.

Let $\delta \geq 0$ and let X be a geodesic semimetric space. A geodesic triangle in X is *δ-slim* if the following holds:

> *if P, Q, R are the sides of the triangle, then P is contained in $\mathcal{B}_\delta^+(Q) \cup \mathcal{B}_\delta^+(R)$ and in $\mathcal{B}_\delta^-(Q) \cup \mathcal{B}_\delta^-(R)$.*

If all geodesic triangles in X are δ-slim then X is *strongly δ-hyperbolic*. We call X *strongly hyperbolic* if it is strongly δ'-hyperbolic for some $\delta' \geq 0$.

Completely analogous to Proposition 3.1, we also get in the case of strong hyperbolicity that it suffices to ask slimness for transitive geodesic triangles, see Proposition 5.1. We omit its proof, since it does not differ much from the proof of Proposition 3.1.

Proposition 5.1. *Let X be a geodesic semimetric space and let $\delta \geq 0$. If all transitive geodesic triangles are δ-slim, then all geodesic triangles are 3δ-slim.* □

Our first aim is to prove that strongly hyperbolic geodesic semimetric spaces satisfy (B1) if and only they satisfy (B2). But before we prove that, we need the following lemma.

Lemma 5.2. *Let X be a strongly δ-hyperbolic geodesic semimetric space. Let $x \in X$ and let $k \geq 0$.*

(i) *Let $y, z \in \mathcal{B}_k^+(x)$ such that $d(y, z) < \infty$ and let P be a y-z geodesic. Let u be on P with $d(u, z) > k + \delta$ if it exists and $u = y$ otherwise. Then P lies in $\mathcal{B}_{k+\delta}^+(x)$ and its directed subpath yPu lies in $\mathcal{B}_{k+\delta}^-(y)$.*
(ii) *Let $y, z \in \mathcal{B}_k^-(x)$ such that $d(y, z) < \infty$ and let P be a y-z geodesic. Let u be on P with $d(y, u) > k + \delta$ if it exists and $u = z$ otherwise. Then P lies in $\mathcal{B}_{k+\delta}^-(x)$ and its directed subpath uPz lies in $\mathcal{B}_{k+\delta}^+(z)$.*

Proof. By strong hyperbolicity, P lies in $\mathcal{B}^+_\delta(Q \cup R)$, where Q and R are geodesics from x to y and to z, respectively. The geodesics Q and R have length at most k and thus, P lies in $\mathcal{B}^+_{k+\delta}(x)$.

Since P lies in $\mathcal{B}^-_\delta(Q \cup R)$ and it is a geodesic, all points on P of distance more than $k+\delta$ to z cannot lie in $\mathcal{B}^-_\delta(R)$. Thus, they lie in $\mathcal{B}^-_\delta(Q)$ and hence in $\mathcal{B}^-_{k+\delta}(y)$. This shows (i).

A completely symmetric argument shows (ii). □

Lemma 5.3. *Let X be a strongly hyperbolic geodesic semimetric space. Then X satisfies (B1) if and only if it satisfies (B2).*

Proof. The arguments of both directions are symmetric. Thus, we only prove the forward one. So let us assume that X satisfies (B1) for the function $f\colon \mathbb{R} \to \mathbb{R}$. Let $x \in X$ and let $y, z \in \mathcal{B}^-_k(x)$ such that $d(y,z) \neq \infty$. Let us assume that $d(y,z) > k+\delta$ and let a be on a y-z geodesic P with $d(y,a) = k+\delta+\varepsilon$ for some $\varepsilon > 0$. By Lemma 5.2 (ii), the directed subpath aPz lies in $\mathcal{B}^+_{k+\delta}(z)$, so its length is bounded by $f(k+\delta)$. Thus, $d(y,z)$ is bounded by $f(k+\delta)+k+\delta+\varepsilon$. □

Analogously to Proposition 3.3, we obtain Proposition 5.4. Since also in this case the proof follows almost verbatim the proof of Proposition 3.3, we omit it here.

Proposition 5.4. *Let X be a strongly hyperbolic geodesic semimetric space that satisfies (B1) for the function $f\colon \mathbb{R} \to \mathbb{R}$.*

(i) *If P, Q, R are the sides of a geodesic triangle, then we have*
$$\ell(P) \leq (\ell(Q)/\varepsilon + \ell(R)/\varepsilon) f(\delta+\varepsilon)$$
for all $\varepsilon > 0$.

(ii) *If $x, y \in V(D)$ with $d(x,y) \neq \infty$ and $d(y,x) \neq \infty$, then we have*
$$d(x,y) \leq (d(y,x)/\varepsilon) f(\delta+\varepsilon) + f(\delta)$$
for all $\varepsilon > 0$. □

Now we are able to prove that strong hyperbolicity implies hyperbolicity if (B1) is satisfied.

Lemma 5.5. *Every strongly hyperbolic geodesic semimetric space that satisfies (B1) or (B2) is hyperbolic.*

Proof. Let $\delta \geq 0$ and let X be a strongly δ-hyperbolic geodesic semimetric space. Recall that X satisfies (B2) and (B1) by Lemma 5.3. Let X satisfy (B1) for the function $f\colon \mathbb{R} \to \mathbb{R}$.

Set $\delta' := \max\{2\delta, 2\delta f(\delta+1) + f(\delta) + \delta\}$. By Proposition 3.1, it suffices to prove that transitive triangles are δ'-thin. Let x, y, z be the end points of a transitive geodesic triangle with sides P, Q, R such that P is a directed x-y path and such that Q, R, has y, x, as a starting or end point, respectively. Let us assume that P does not lie in $\mathcal{B}^+_\delta(R) \cup \mathcal{B}^-_\delta(Q)$. Let a be on P but outside of $\mathcal{B}^+_\delta(R) \cup \mathcal{B}^-_\delta(Q)$. Then there are points u on Q and v on R with $d(u,a) \leq \delta$ and $d(a,v) \leq \delta$. In particular, we have $d(u,v) \leq 2\delta$. There is a point u' on P or R with $d(u',u) \leq \delta$ and a point v' on P or Q with $d(v,v') \leq \delta$. If u' lies on R, then we have $a \in \mathcal{B}^+_{2\delta}(R)$ and if v' lies on Q, then we have $a \in \mathcal{B}^-_{2\delta}(Q)$. So we assume that u' and v' lie on P.

Let us show that we find a directed path from v to u. If R is directed towards x, then we take the composition of $vRxPu'$ and a u'-u geodesic. If R is directed away from x and Q is directed away from y, then we take the composition of a v-v' geodesic and $v'PyQu$. If R is directed away from x and Q is directed towards y, then we take $vRzQu$. Thus, we obtained in each case a directed v-u path. Since $d(u,v) \leq 2\delta$, we conclude $d(v,u) \leq 2\delta f(\delta+1) + f(\delta)$ by Proposition 5.4. It follows that a lies in $\mathcal{B}^+_{\delta'}(R) \cup \mathcal{B}^-_{\delta'}(Q)$. Thus, this triangle is δ'-thin. □

We will focus on hyperbolicity in the next sections. All results that we obtain there hold for strong hyperbolicity as well, if (B1) is satisfied. But we will not explicitly state it except if we get stronger results for strong hyperbolicity than for hyperbolicity.

6. Divergence of geodesics

For geodesic metric spaces, hyperbolicity is equivalent to divergence of geodesics. For semimetric spaces, the analogue is not true. But we will see that at least one direction holds under the assumptions of (B1) and (B2): hyperbolicity implies (exponential) divergence of geodesics.

Let us start by giving the definition of divergence of geodesics in the context of semimetric spaces. Let X be a geodesic semimetric space. Let P be an x-y geodesic in X and let $0 \leq R \leq d(x,y)$. We denote by $P^x(R)$ be the point u with $d(x,u) = R$ and we denote by $P^y(R)$ be the point v with $d(v,y) = R$.

A function $e\colon \mathbb{R} \to \mathbb{R}$ is a *divergence function* if for all $x \in X$, all geodesics P_1, P_2 that start or end at x and all $r, R \in \mathbb{R}$ the following holds: if $d(P_1^x(R), P_2) > e(0)$, then every directed P_1-P_2 path that lies outside of $\mathcal{B}^+_{R+r}(x) \cup \mathcal{B}^-_{R+r}(x)$ has length more than $e(r)$. We say that geodesics *diverge (exponentially)* in X if there exists an (exponential) divergence function. They diverge *properly* if $e(r) \to \infty$ for $r \to \infty$.

Proposition 6.1. *Let X be a hyperbolic geodesic semimetric space that satisfies (B1) and (B2). Then the geodesics diverge exponentially in X.*

Proof. Let $\delta \geq 0$ such that X is δ-hyperbolic. Let $x \in X$, let P_1, P_2 be two geodesics that have x as their starting or end point and let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that X satisfies (B1) and (B2) for f. Let $\varepsilon > 0$. Set

$$e(0) := (2\delta + \varepsilon + 1) f(\delta+1) + f(\delta) + \delta$$

and, for $r > 0$ and $k := 6\delta + 2\delta f(\delta+1)$, set

$$e(r) := 2^{\frac{r-2\delta-1}{k}} - 1.$$

Let $R, r \in \mathbb{R}$ with $d(P_1^x(R), P_2) > e(0)$. Let P be a directed path from P_1 to P_2 that lies completely outside of $\mathcal{B}^+_{R+r}(x) \cup \mathcal{B}^-_{R+r}(x)$ and let Q be a geodesic from the starting point a of P to its end point b.

We are going to show the following.

(i) If x is the starting point of P_1, then $d(x,Q) \leq R + 2\delta$.
(ii) If x is the end point of P_1, then $d(Q,x) \leq R + \delta$.

If P_1 ends at x, then there is a point v either on Q with $d(v, P_1^x(R)) \leq \delta$ or on P_2 with $d(P_1^x(R), v) \leq \delta$. By the choice of R, we know that v must lie on Q and this shows (ii). So let P_1 start at x and let v be a point either on Q with $d(P_1^x(R), v) \leq \delta$ or on P_2 with $d(v, P_1^x(R)) \leq \delta$. If v lies on Q, then we have (i). So let us assume that v lies on P_2. If x is the end point of P_2, then the directed path $P_1^x(R)P_1aQbP_2v$ shows that we find a $P_1^x(R)$-v geodesic, whose length is at most $\delta f(\delta+1) + f(\delta)$ by Proposition 3.3 (ii), a contradiction to the choice of R. So let us assume that x is the starting point of P_2. Let us suppose that (i) is false. Since $d(P_2^x(R+\delta),Q) > \delta$, there is a point u on xP_1a with $d(u,P_2^x(R+\delta)) \leq \delta$. This must lie on $P_1^x(R)P_1a$ as P_2 is a geodesic. Now let v be on $xP_2P_2^x(R+\delta)$ such that there is a point v' on $P_1^x(R)P_1a$ with $d(v',v) \leq \delta$ but for no point w on xP_2v with $d(w,v) \geq \varepsilon$ there exists a point w' on $P_1^x(R)P_1a$ with $d(w',w) \leq \delta$. Note that we also have $d(v,Q) > \delta$ since (i) is false. Let w be on xP_2v with $d(w,v) = \varepsilon$ if it exists and $w = x$ otherwise. Then there is a point w' on P_1 with $d(w',w) \leq \delta$ since (i) is false. This point must lie on $xP_1P_1^x(R)$. Considering a geodesic triangle with end points w',v,v' where the side between w' and v' is $w'P_1v'$ and the other two sides are directed towards v, we obtain $d(w',v') \leq (2\delta+\varepsilon+1)f(\delta+1)$ by Proposition 3.3 (i). Since $P_1^x(R)$ lies on the side between w' and v', this is a contradiction to the choice of R. This finishes the proof of (i) and (ii).

We consider Q as being indexed with the empty word. We define a set of directed paths Q_σ with starting and end points on P and points u_σ on P indexed by finite words over $\{0,1\}$ such that the following holds.

(I) If Q_σ is a directed path of length more than one, then u_i is a point on P such that u_i halves the length of the directed subpath of P between the starting and the end points of Q_σ.
(II) $Q_{\sigma 0}$ is a geodesic with the same starting point as Q_σ and with end point u_σ.
(III) $Q_{\sigma 1}$ is a geodesic with u_σ as starting point and the same end point as Q_σ.

It follows from the choice of the points u_σ that the length of any $\{0,1\}$-word used as index is at most $\ln(\ell(P))$. By applying Lemma 3.4 recursively, we obtain for every $n \geq 0$ a point v_n on some Q_σ, where σ has length n, such that $d(x,v_n) \leq R+2\delta+nk$ if x is the starting point of P_1 or $d(v_n,x) \leq R+\delta+nk$ if x is the end point of P_1. Note that for $n = \ln(\ell(P))$ the point v_n has distance at most 1 to and from P. So we found a point y on P with either $d(x,y) \leq R+2\delta+nk+1$ or $d(y,x) \leq R+2\delta+nk+1$. Since P lies outside of $\mathcal{B}^+_{R+r}(x) \cap \mathcal{B}^-_{R+r}(x)$, we have

$$R+r \leq R+2\delta+nk+1 \leq R+2\delta+k\ln(\ell(P))+1.$$

Thus, we have

$$e(r) < 2^{\frac{r-2\delta-1}{k}} \leq \ell(P),$$

which shows the assertion. □

As mentioned earlier, the reverse implication of Proposition 6.1 does not hold as the following example shows.

Example 6.2. Let D be the digraph whose vertex set is given by two directed rays $x_0x_1\ldots$ and $y_1y_2\ldots$ with additional edges x_0y_1 and x_iy_i for all $i \in \mathbb{N}$. Then there is no $\delta \geq 0$ such that all geodesic triangles with end vertices x_0,x_i,y_i are δ-thin. Furthermore, every function $e\colon \mathbb{R} \to \mathbb{R}$ with $e(r) > 2$ for all $r \in \mathbb{R}$ is a divergence function for D.

A property for geodesic spaces is that divergence of geodesics that is faster than linear already implies that it is exponential. For geodesic semimetric spaces, we pose it as an open problem.

Problem 6.3. *Do there exist geodesic semimetric spaces in which geodesics diverge exponentially but not superlinearly?*

7. Geodesic stability

Another property in geodesic spaces that is equivalent to hyperbolicity is *geodesic stability*, that is, for every two points, all (γ, c)-quasi-geodesics between those two points lie close to each other. This is the main step in showing that hyperbolicity is preserved under quasi-isometries.

A geodesic semimetric space X satisfies *geodesic stability* if for all $\gamma \geq 1$ and $c \geq 0$ there exists a $\kappa \geq 0$ such that, for all $x, y \in X$ and all (γ, c)-quasi-geodesics P and Q from x to y, every point of P lies in $\mathcal{B}^+_\kappa(Q) \cap \mathcal{B}^-_\kappa(Q)$.

As a first step, we prove that geodesics lie close to and from quasi-geodesics with the same starting and end points (Proposition 7.1) and then that quasi-geodesics lie close to and from geodesics with the same starting and end points (Proposition 7.2).

Proposition 7.1. *Let X be a δ-hyperbolic geodesic semimetric space that satisfies (B1) and (B2) for the function $f\colon \mathbb{R} \to \mathbb{R}$. Let $\gamma \geq 1$ and $c \geq 0$. Then there is a constant $\kappa = \kappa(\delta, \gamma, c, f)$ depending only on δ, γ, c and f such that for all $x, y \in X$ every x-y geodesic lies in the κ-out-ball and in the κ-in-ball of every (γ, c)-quasi-geodesic from x to y.*

Proof. By Proposition 6.1, we find a superlinear divergence function $e\colon \mathbb{R} \to \mathbb{R}$ for X.[3] First, we will show the following.

(1) *There exists κ_1 such that for all $x, y \in X$ and for all x-y geodesics P and (γ, c)-quasi-geodesics Q from x to y we have $d^{\leftrightarrow}(z, Q) \leq \kappa_1$ for all z on P.*

Let us suppose that (1) does not hold. Then there are $x, y \in X$, an x-y geodesic P and a (γ, c)-quasi-geodesic Q from x to y such that there exists z on P with $2d^{\leftrightarrow}(z, Q) > e(0)f(\delta + 1) + f(\delta)$. Let $\varepsilon > 0$. We choose z on P such that for no z' on P we have $d^{\leftrightarrow}(z', Q) \geq D := d^{\leftrightarrow}(z, Q) + \varepsilon$. Set

$$K := (D + \delta)f(\delta + 1) + \delta + \varepsilon$$

and

$$L := (K + D)(2 + f(\delta + 1)) + \varepsilon.$$

In order to show (1) we will first show the following

(2) *There are points a^+, a^- on xPz and a_Q on Q with $D - \varepsilon \leq d(a^+, z) \leq L$ and $D - \varepsilon \leq d(a^-, z) \leq L$, with $d(a^+, a_Q) \leq K$ and $d(a_Q, a^-) \leq K$ and such that all a^+-a_Q and a_Q-a^- geodesics lie outside of $B := \mathcal{B}^+_D(z) \cup \mathcal{B}^-_D(z)$.*

This obviously holds if $d(x, z) \leq L$ by choosing $a^+ = a^- = a_Q = x$. Otherwise, let a on xPz with $d(a, z) = K + D + (K + D)f(\delta + 1) + \varepsilon$. By the choice of z, we have $d^{\leftrightarrow}(a, Q) \leq D$.

First, let us consider the case that $d(Q, a) < D$. Let a_Q be on Q with $d(Q, a) \leq d(a_Q, a) + \varepsilon$. We consider a geodesic triangle with end points x, a, a_Q such that the side P' between x and a is a directed subpath of P. Let Q_1 be the side from x to a_Q and let Q_2 be the side from a_Q to a. Then there are u, v on Q_1 with $d(u, v) \leq \varepsilon$ such that u lies in $\mathcal{B}^+_\delta(P')$ and v lies in $\mathcal{B}^-_\delta(Q_2)$. Let w be on P' with $d(w, u) \leq \delta$

[3] Proposition 6.1 implies that e is exponential, but in this proof it suffices to know that e is superlinear.

and let w' be on Q_2 with $d(v,w') \le \delta$. Applying Proposition 3.3 to the geodesic triangle with end vertices v, a_Q, w', we obtain $d(v,a_Q) \le (D+\delta)f(\delta+1)$ and thus

$$d(w,a_Q) \le d(w,u) + d(u,v) + d(v,a_Q) \le \delta + \varepsilon + (D+\delta)f(\delta+1) = K.$$

It follows that

$$d(w,a) \le d(w,a_Q) + d(a_Q,a) \le K + D.$$

We set $a^+ := w$ and $a^- := a$. Then we have

$$d(a^+,z) = d(a^+,a^-) + d(a^-,z) \le (K+D)(2+f(\delta+1)) + \varepsilon = L$$

and $d(a^+,a_Q) = d(w,a_Q) \le K$. If there were a point on an a^+-a_Q or a_Q-a^- geodesic inside B, then we apply Proposition 3.3 and obtain $d(a^+,z) \le (K+D)f(\delta+1)$ or $d(a^-,z) \le 2Df(\delta+1)$, which is impossible.

If $d(a,Q) \le D$, then let a_Q be on Q with $d(a,Q) \le d(a,a_Q) + \varepsilon$. Similarly as in the first case, we use the geodesic triangle with end points a, a_Q, y to find a point w on aPy with $d(a_Q,w) \le K$ and $d(a,w) \le K + D$. We set $a^+ := a$ and $a^- := w$. Again by Proposition 3.3, the a^+-a_Q and a_Q-a^- geodesics lie outside of B. Thus, we have proved (2).

A completely symmetric argument shows the following.

(3) *There are points b^+, b^- on zPy and b_Q on Q with $D-\varepsilon \le d(z,b^+) \le L$ and $D-\varepsilon \le d(z,b^-) \le L$, with $d(b^+,b_Q) \le K$ and $d(b_Q,b^-) \le K$ and such that all b^+-b_Q and b_Q-b^- geodesics lie outside of B.*

Let z_a be on xPz and let z_b be on zPy with $d(z_a,z) = D = d(z,z_b)$. Then we have $d(z_a,z_b) = 2D > e(0)$ and

$$d(z_b,z_a) \ge \frac{d(z_a,z_b) - f(\delta)}{f(\delta+1)} > e(0)$$

by Proposition 3.3. If a_Q lies on xQb_Q, then let R be the composition of an a^+-a_Q geodesic with a_QQb_Q and with a b_Q-b^- geodesic. Then R is a directed a^+-b^- path outside of B. The composition of an a_Q-a^- geodesic with a^-Pb^+ and a b^+-b_Q geodesic shows that $d(a_Q,b_Q) \le 2(K+L)$. So we have

$$\ell(R) \le 2K + 2\gamma(K+L) + c,$$

which is linear in D.

Let us now assume that b_Q lies on xQa_Q. Let R be the composition of a b^+-b_Q geodesic with b_QQa_Q and with an a_Q-a^- geodesic. Then R is a b^+-a^- path outside of B. The composition of an a_Q-a^- geodesic with a^-Pb^+ and a b^+-b_Q geodesic shows $d(a_Q,b_Q) \le 2(K+L)$. Since b_Q lies on the x-a_Q subpath of Q, there is a b_Q-a_Q geodesic, which has length at most $2(K+L)f(\delta+1) + f(\delta)$ by Proposition 3.3. Thus, we have

$$\ell(R) \le 2K + 2\gamma(K+L)f(\delta+1) + \gamma f(\delta) + c,$$

which is also linear in D.

So in each case we obtain a contradiction to e being a superlinear divergence function. We thus proved (1).

Let

$$\kappa_2 > 2(\kappa_1 + 7\delta + (2\delta + \kappa_1)f(\delta+1))$$

and set

$$\begin{aligned}\lambda_1 :=& ((\kappa_2+\kappa_1)f(\delta+1)+\kappa_1)f(\delta+1),\\ \lambda_2 :=& \gamma\lambda_1 + c,\\ \kappa_3 :=& \lambda_2 + 7\delta + 2\delta f(\delta+1) + \kappa_2/2.\end{aligned}$$

We will prove that the assertion holds for $\kappa = \kappa_2 + \kappa_3$.

Let $x, y \in X$, let P be an x-y geodesic and let Q be a (γ, c)-quasi-geodesic from x to y. For all $0 \leq i$ with $i\kappa_2 \leq d(x,y)$ let x_i be the point on P with $d(x,x_i) = i\kappa_2$ and y_i be a point on Q with $d^{\leftrightarrow}(x_i, y_i) \leq \kappa_1$. We assume $x_0 = y_0$. We will show the following.

(4) *For all $i \in \mathbb{N}$ such that $i\kappa_2 \leq d(x,y)$, we have $d(x_i, Q) \leq \kappa_3$ and $d(Q, x_i) \leq \kappa_3$.*

Note that the assertion follows for the constant κ from (4) directly.

Let $N \in \mathbb{N}$ be maximum such that $N\kappa_2 \leq d(x,y)$ and set $x_{N+1} := y_{N+1} := y$. For all $i \in \{0, \ldots, N-1\}$ such that y_i lies on xQy_{i+1}, there is either a directed y_i-x_{i+1} path or a directed x_i-y_{i+1} path depending on the direction of the shortest path between x_i and y_i. Applying Proposition 3.3 twice to geodesic triangles with end points either x_i, y_i, x_{i+1} and y_i, x_{i+1}, y_{i+1} or x_i, x_{i+1}, y_{i+1} and x_i, y_i, y_{i+1} we obtain $d(y_i, y_{i+1}) \leq \lambda_1$.

Let $n \in \{1, \ldots, N\}$ such that $d(x_n, y_n) \leq \kappa_2$ and let $i \in \{0, \ldots, n-1\}$ be largest such that y_i lies on xQy_n. Then y_i lies on xQy_{i+1} and y_n lies on y_iQy_{i+1}. Since Q is a (γ, c)-quasi-geodesic, the length of the path y_iQy_{i+1} is at most λ_2 and thus we have $d(y_i, y_n) \leq \lambda_2$.

We will show $d(y_i, x_n) \leq \kappa_3$. This holds trivially, if $d(y_i, x_i) \leq \kappa_1$. So let us assume that $d(x_i, y_i) \leq \kappa_1$. Note that there is a directed x_i-y_n path, so we find an x_i-y_n geodesic R_1. Let P' the subpath of P between x_i and x_n and let Q' be a y_i-y_n geodesic. Let R_2 be a x_n-y_n geodesic and let R_3 be a geodesic between x_i and y_i. We consider two geodesic triangles: one with x_i, x_n, y_n as its end points and P', R_1, R_2 as its sides and the other with x_i, y_i, y_n as its end points and Q', R_1, R_3 as its sides. Let u be on P' with $d(u, x_n) = \kappa_2/2$. By hyperbolicity, there is either a point v on R_1 with $d(v,u) \leq \delta$ or a point v' on R_2 with $d(u,v') \leq \delta$. The latter case leads to a contradiction since then Proposition 3.3 implies $d(u,x_n) \leq (\delta+\kappa_1)f(\delta+1) < \kappa_2/2$. So we find a point v on R_1 with $d(v,u) \leq \delta$.

Applying Lemma 3.4, we find a point w on either Q' or R_3 with $d(w,v) \leq 6\delta + 2\delta f(\delta+1)$. If w is on R_3, we find a path from x_i via w and v to u of length at most $\kappa_1 + 7\delta + 2\delta f(\delta+1)$, which contradicts $d(x_i, u) = \kappa_2/2$. Thus, w lies on Q' and hence, the composition of $y_iQ'w$ with a w-v geodesic, a v-u geodesic and uPx_n shows

$$d(y_i, x_n) \leq \lambda_2 + 7\delta + 2\delta f(\delta+1) + \kappa_2/2 = \kappa_3.$$

Now let us consider the case that for $n \in \{1, \ldots, N\}$ we have $d(y_n, x_n) \leq \kappa_1$. We follow a completely symmetric argument as in the previous case with reversed directions. Most importantly, in this case x_i is chosen so that $i \in \{n+1, \ldots, N+1\}$ is smallest such that y_i lies on y_nQy.

This finishes the proof of (4) and thus the proof of our assertion as mentioned earlier. □

Proposition 7.2. *Let X be a δ-hyperbolic geodesic semimetric space satisfying (B1) and (B2) for the function $f\colon \mathbb{R} \to \mathbb{R}$. Let $\gamma \geq 1$ and $c \geq 0$. Then there is a* *constant $\lambda = \lambda(\delta, \gamma, c, f)$ depending only on δ, γ, c and f such that for all $x, y \in X$ every (γ, c)-quasi-geodesic from x to y lies in the λ-out-ball and in the λ-in-ball of every x-y geodesic.*

Proof. Let κ be the constant from Proposition 7.1. Let $x, y \in X$ and let P be an x-y geodesic and Q be a (γ, c)-quasi-geodesic from x to y. Let z be on Q such that either $d(z, P) > \kappa$ or $d(P, z) > \kappa$. Set $Q_1 := xQz$ and $Q_2 := zQy$. Let $\varepsilon > 0$ and let u be on P such that there exists w on Q_1 with $d(u, w) \leq \kappa$ and such that for no point u' on uPy with $d(u, u') \geq \varepsilon$ there is w' on Q_1 with $d(u', w') \leq \kappa$.

If there is w' on Q_2 with $d(w', u) \leq \kappa$, then $d(w', w) \leq 2\kappa$ and since there is a directed w-w' path, we have $d(w, w') \leq 2\kappa f(\delta + 1) + f(\delta)$ by Proposition 3.3. So the distance from w to w' on Q is at most $2\gamma\kappa f(\delta + 1) + \gamma f(\delta) + c$ and hence z and u lie on a closed directed walk of length at most $2\kappa + 2\gamma\kappa f(\delta + 1) + \gamma f(\delta) + c$.

Let us now assume that there is no w' on Q_2 with $d(w', u) \leq \kappa$. In particular, we have $u \neq y$. By Proposition 7.1, there is a point w' on Q_1 with $d(w', u) \leq \kappa$. Let v be on uPy with $d(u, v) = \varepsilon$. By the choice of u, there is a point v' on Q_2 with $d(v, v') \leq \kappa$. So we have $d(w', v') \leq 2\kappa + \varepsilon$ and hence the length of the directed w'-v' subpath of Q is at most $\gamma(2\kappa + \varepsilon) + c$. Note that this directed subpath contains z. We consider the geodesic triangle with end points v, v', y such that the side P' between v and y is vPy, the side R_1 is a v'-y geodesic and the side R_2 is a v-v' geodesic. Then there a, b on R_1 with a on $v'R_1b$ and $d(a, b) \leq \varepsilon$ such that $d(R_2, a) \leq \delta$ and $d(b, P') \leq \delta$. By Proposition 3.3, we conclude $d(v', a) \leq (\kappa + \delta)f(\delta + 1)$. So we have

$$d(v', P) \leq d(v', a) + d(a, b) + d(b, P) \leq (\kappa + \delta)f(\delta + 1) + \varepsilon + \delta$$

and hence

$$d(z, P) \leq d(z, v') + d(v', P) \leq (\kappa + \delta)f(\delta + 1) + \varepsilon + \delta + \gamma(2\kappa + \varepsilon) + c.$$

With a symmetric argument by considering a geodesic triangle with end vertices x, w', u, we conclude

$$d(P, z) \leq (\kappa + \delta)f(\delta + 1) + \varepsilon + \delta + \gamma(2\kappa + \varepsilon) + c.$$

So the assertion follows for

$$\lambda := \max\{2\kappa + 2\gamma\kappa f(\delta + 1) + \gamma f(\delta) + c, (\kappa + \delta)f(\delta + 1) + 1 + \delta + \gamma(2\kappa + 1) + c\}. \quad \square$$

We obtain the following corollary directly from Propositions 7.1 and 7.2 the following.

Corollary 7.3. *Every hyperbolic geodesic semimetric space that satisfies (B1) and (B2) satisfies geodesic stability.* □

To compare geodesic stability with divergence of geodesics, we note that exponential divergence of geodesics does not imply geodesic stability as Example 6.2 shows. The reverse is also false as our next example shows. That example is a digraph that satisfies geodesic stability but whose geodesics do not diverge properly and that is not hyperbolic.

Example 7.4. Let $x_0, x_1 \dots$ and $y_0, y_1, \dots$ be two disjoint sequences of distinct vertices. For every i, we add an edge $x_i y_i$, a path from x_{i+1} to x_i of length $i + 1$ and a path from y_i to y_{i+1} of length $i + 1$ such that all these new paths are pairwise disjoint and meet the two sequences only in its end vertices. For $n \in \mathbb{N}$, we take two geodesics: one that ends at x_0 and passed through all x_i for $i \leq n$ and one that starts at x_0 and passes through all y_i for $i \leq n$. There are vertices on the first geodesic that have distance at least $n/2$ to the other geodesic, but the edge $x_n y_n$ shows that these two geodesics do not diverge properly. The geodesic triangles with end vertices x_n, x_{n+1} and y_{n+1} are not $(n/2-1)$-thin. So D is not hyperbolic.

Let R_1 be the geodesic anti-ray that passes through all x_i and ends at x_0 and let R_2 be the geodesic ray that starts at y_0 and passes through all y_i. Then every directed path that is not a subpath of R_1 or R_2 can be obtained by composing a subpath of R_1 with a path consisting of an edge $x_i y_i$ and with a subpath of R_2. Hence, if a geodesic and a (γ, c)-quasi-geodesic have the same first and the same last vertex, then they differ in that an edge $x_i y_i$ of the geodesic is replaced by $P = x_i R_1 x_j y_j R_2 y_i$ for some $j < i$. Note that we have $\ell(P) \leq \gamma + c$ as $d(x_i, y_i) = 1$. If follows that the distance of points on P from and to the geodesic is at most $\ell(P)$ and hence bounded by $\gamma + c$. Trivially, all points on the geodesic lie within 1 from and to the quasi-geodesic. It follows that the digraph satisfies geodesic stability.

8. Quasi-isometries

In this section, we look at quasi-isometries of geodesic semimetric spaces and ask which of our properties are preserved by quasi-isometries. This is motivated by the fact that hyperbolicity of geodesic metric spaces is preserved by quasi-isometries, see e. g. [2, Theorem III.H.1.9]. We will prove that hyperbolicity (with the additional assumptions of (B1) and (B2)) and geodesic stability are preserved under quasi-isometries and pose it as open problem what happens for (superlinear/exponential) divergence of geodesics.

Proposition 8.1. *Geodesic stability is preserved by quasi-isometries.*

Proof. Let X and Y be geodesic semimetric spaces such that Y satisfies geodesic stability and let $f\colon X \to Y$ be a (λ, c)-quasi-isometry for some $\gamma \geq 1$ and $c \geq 0$. Let $x_1, x_2 \in X$, let P be an x_1-x_2 geodesic and let Q be a (λ, ℓ)-quasi-geodesic from x_1 to x_2 for some $\lambda \geq 1$ and $\ell \geq 0$. Then there are constants $\gamma' \geq 1$ and $c' \geq 0$ such that the images of P and Q under f are (γ', c')-quasi-geodesics, see Gray and Kambites [15, Proposition 1]. By geodesic stability, there is a constant $\kappa \geq 0$ such that

$$f(P) \subseteq \mathcal{B}^+_\kappa(f(Q)) \cap \mathcal{B}^-_\kappa(f(Q))$$

and

$$f(Q) \subseteq \mathcal{B}^+_\kappa(f(P)) \cap \mathcal{B}^-_\kappa(f(P)).$$

Since f is a quasi-isometry, there is a constant $\kappa' \geq 0$ such that

$$P \subseteq \mathcal{B}^+_{\kappa'}(Q) \cap \mathcal{B}^-_{\kappa'}(Q)$$

and

$$Q \subseteq \mathcal{B}^+_{\kappa'}(P) \cap \mathcal{B}^-_{\kappa'}(P). \qquad \square$$

The proof that hyperbolicity is preserved under quasi-isometries relies also on their geodesic stability.

Proposition 8.2. *Let X and Y be two geodesic semimetric spaces such that X is hyperbolic and satisfies (B1) and (B2). If X is quasi-isometric to Y, then Y is hyperbolic.*

Proof. Let X be δ-hyperbolic for some $\delta \geq 0$. Let $f\colon Y \to X$ be a (γ, c)-quasi-isometry for some $\gamma \geq 1$ and $c \geq 0$. We consider a geodesic triangle in Y with end points x, y, z and sides P_1, P_2, P_3, where P_1 is an x-y geodesic, P_2 a geodesic between y and z and P_3 are geodesic between x and z. Then the images of the sides are (γ, c)-quasi-geodesics in X. Let κ be a constant such that geodesic stability holds for κ in X with respect to (γ, c)-quasi-geodesics, see Corollary 7.3.

Let u be on P_1. By geodesic stability, we find u_1, u_2 on some $f(x)$-$f(y)$ geodesic in X with $d(u, u_1) \leq \kappa$ and $d(u_2, u) \leq \kappa$. As geodesic triangles are δ-thin, there are v_1, v_2 on geodesics either from $f(x)$ to $f(z)$ or from $f(y)$ to $f(z)$ with $d(u_1, v_1) \leq \delta$ and $d(v_2, u_2) \leq \delta$. Applying geodesic stability once more, there are w_1, w_2 on the images under f of P_3 and P_2 with $d(v_1, w_1) \leq \kappa$ and $d(w_2, v_2) \leq \kappa$. So we have $d(u, w_1) \leq 2\kappa + \delta$ and $d(w_2, u) \leq 2\kappa + \delta$. Since f is a quasi-isometry, it follows that there exists $\delta' \geq 0$, depending only on κ, δ, γ and c such that the geodesic triangle is δ'-thin. □

By an analogous proof, we also obtain the following result.

Proposition 8.3. *Let X and Y be two geodesic semimetric spaces such that X is strongly hyperbolic and satisfies (B1). If X is quasi-isometric to Y, then Y is strongly hyperbolic.* □

We end this section by a stating the corresponding problem for divergence of geodesics.

Problem 8.4. *Is proper divergence (or superlinar/exponential divergence) of geodesics preserved under quasi-isometries?*

9. Geodesic boundary

One possibility to obtain the hyperbolic boundary in case of metric spaces is to consider an equivalence relation of geodesic rays, where two geodesic rays are equivalent if they are eventually close together. We mimic this construction for semimetric spaces that satisfy (B1) and (B2) and obtain a quasiorder. Then we use this quasiorder to define the geodesic boundary. In the case of locally finite digraphs, we will see that this boundary is preserved by quasi-isometries, see Section 10, and that it refines the ends, see Section 11.

Let X be a hyperbolic geodesic semimetric space. A *ray* is a map $R\colon [0, \infty) \to X$ that is continuous with respect to the forward and backward topologies and such that for every $x \in X$ and $r \geq 0$ there exists $p \geq 0$ such that $R([p, \infty)) \cap B_r^+ = \emptyset$, i. e. R leaves every out-ball of finite radius eventually. An *anti-ray* is a map $R\colon (-\infty, 0] \to X$ that is continuous with respect to the forward and backward topologies and such that for every $x \in X$ and $r \geq 0$ there exists $p \leq 0$ such that $R((-\infty, p]) \cap B_r^- = \emptyset$. For the sake of simplicity, we also denote by $R(i)$ the point $R(-i)$ for an anti-ray R and $i > 0$. Note that this definition of rays and anti-rays in the case of digraphs canonically corresponds to the one of Section 2.2.

For geodesic rays or anti-rays R_1 and R_2, we write $R_1 \leq R_2$ if there exists some $M \geq 0$ such that for every $r \geq 0$ and every $x \in X$ there is a directed R_1-R_2 path of length at most M outside of $\mathcal{B}_r^+(x) \cup \mathcal{B}_r^-(x)$.

The following lemma is straight forward to see.

Lemma 9.1. *Let X be a geodesic semimetric space.*

(i) *If for all geodesic rays R_1 and R_2 in X with $R_1 \leq R_2$ there exists $m \geq 0$ such that for every $r \geq 0$ and every $x \in X$ there is a directed R_2-R_1 paths of length at most m outside of $\mathcal{B}_r^+(x) \cup \mathcal{B}_r^-(x)$, then $\leq$ is symmetric.*

(ii) *If for all geodesic rays and anti-rays R_1 and R_2 in X with $R_1 \leq R_2$ there exist $M \geq 0$ and a directed subpath P of R_1 such that $d(x, R_2) \leq M$ for all x on R_1 that do not lie on P, then $\leq$ is transitive.* □

Lemma 9.2. *Let X be a δ-hyperbolic geodesic semimetric space for some $\delta \geq 0$ that satisfies (B1) and (B2) for the function $f\colon \mathbb{R} \to \mathbb{R}$. Let $M \geq 0$ and let $x_1, x_2, y_1, y_2 \in X$ such that*

(1) *$d(x_1, y_1) \leq M$ and $d(x_2, y_2) \leq M$;*
(2) *$d(x_1, x_2) < \infty$;*
(3) *either $d(y_1, y_2) < \infty$ or $d(y_2, y_1) < \infty$.*

Then the following hold.

(i) *Every y_1-y_2 geodesic and every y_2-y_1 geodesic lie in the out-ball of radius $(2M + 5\delta) + (2M + 2\delta + 1)f(\delta + 1)$ around any x_1-x_2 geodesic.*

(ii) *If $d(y_1, y_2) < \infty$, then all points a on every x_1-x_2 geodesic but those with*
$$d(x_1, a) \leq ((M + 6\delta + 2\delta f(\delta + 1))f(\delta + 1) + \delta)f(\delta + 1)$$
or $d(a, x_2) \leq (M + \delta)f(\delta + 1)$ lie in the out-ball of radius $7\delta + 2\delta f(\delta + 1)$ around any y_1-y_2 geodesic.

(iii) *If X is strongly δ-hyperbolic, then all points a on every x_1-x_2 geodesic but those with $d(x_1, a) \leq M + 2\delta$ or $d(a, x_2) \leq (M + \delta)f(\delta + 1)$ lie in the out-ball of radius 2δ around any geodesic between y_1 and y_2.*

Proof. By the assumptions, there is a the geodesic triangle with end points x_1, x_2 and y_2 and sides P_2, Q and R such that P_2 is an x_2-y_2 geodesic, Q is an x_1-y_2 geodesic and R is an x_1-x_2 geodesic. Let P_1 be an x_1-y_1 geodesic, which has length at most M by our assumption, and let S be a geodesic between y_1 and y_2.

In order to prove (i), set $K := (2M + 5\delta) + (2M + 2\delta + 1)f(\delta + 1)$. If $d(x_1, y_2) \leq M + \delta$, then Proposition 3.3 implies
$$\ell(S) \leq Mf(\delta + 1) + (M + \delta)f(\delta + 1) = (2M + \delta)f(\delta + 1).$$
Hence, S lies in the out-ball of radius $((2M + \delta)f(\delta + 1) + M + \delta) \leq K$ around x_1.

Let us now assume that $d(x_1, y_2) > M + \delta$. Considering the geodesic triangle with sides R, P_2 and Q, we obtain that Q lies in $\mathcal{B}_\delta^+(R) \cup \mathcal{B}_\delta^-(P_2)$. Since every point in $\mathcal{B}_\delta^-(P_2)$ has distance at most $\delta + M$ to y_2, we obtain that Q lies in the $(2\delta + M)$-out-ball of R.

Let $v_1, \ldots, v_n$ be points on Q such that $d(x_1, v_1) = M + \delta + 1$, such that $d(v_1, y_2) = (n - 1)\delta + j$ for some $0 \leq j < \delta$, such that $d(v_i, v_{i+1}) = \delta$ for all $i < n - 1$ and such that $v_n = y_2$. Since Q lies in $\mathcal{B}_\delta^+(P_1) \cup \mathcal{B}_\delta^-(S)$ and the length of P_1 is at most M, there is for every $i \leq n$ a point w_i on S with $d(v_i, w_i) \leq \delta$. We may assume that $w_n = y_2$. For every $i \leq n$, let A_i be a v_i-w_i geodesic and, for every $i < n$, let B_i be a v_i-w_{i+1} geodesic, which exists as the composition of v_iQv_{i+1} and A_{i+1} is a directed v_i-w_{i+1} path. Note that $\ell(B_i) \leq 2\delta$. If w_i lies on S before w_{i+1}, i. e. the preimage of w_i is smaller than that of w_{i+1}, then every point on w_iSw_{i+1} that lies in $\mathcal{B}_\delta^-(B_i)$ has distance at most 3δ to w_{i+1}. By hyperbolicity, all other points on w_iSw_{i+1} lie in $\mathcal{B}_\delta^+(A_i)$. In particular, w_iSw_{i+1} lies in $\mathcal{B}_{5\delta}^+(v_i)$. If w_{i+1} lies on S before w_i, then every point on $w_{i+1}Sw_i$ that lies in $\mathcal{B}_\delta^-(A_i)$ has distance at most 2δ

to w_i. By hyperbolicity, all other points on $w_{i+1}Sw_i$ lie in $\mathcal{B}^+_\delta(B_i)$. So we also have in this case that $w_{i+1}Sw_i$ lies in $\mathcal{B}^+_{5\delta}(v_i)$. Thus, the directed subpath of S between w_1 and y_2 lies in $\mathcal{B}^+_{5\delta}(Q)$.

Let us consider the geodesic triangle with end points x_1, y_1 and w_1 with P_1 as one side, the directed subpath of S between y_1 and w_1 as another side and an x_1-w_1 geodesic as third side. Since $d(x_1, w_1) \leq M+2\delta+1$, we conclude by Proposition 3.3 that the side between y_1 and w_1 has length at most $(2M + 2\delta + 1)f(\delta + 1)$. Thus, S lies in the out-ball of radius

$$\begin{aligned}&\max\{5\delta, M + (2M + 2\delta + 1)f(\delta + 1), \delta + (2M + 2\delta + 1)f(\delta + 1)\}\\ \leq&(M + 3\delta) + (2M + 2\delta + 1)f(\delta + 1)\end{aligned}$$

around Q. Since we already saw that Q lies in the out-ball of radius $2\delta + M$ around R in this case, we obtain that S lies in the out-ball of radius

$$(2M + 5\delta) + (2M + 2\delta + 1)f(\delta + 1) \leq K$$

around R. Together with the first case, this proves (i).

Now let us assume that $d(y_1, y_2) < \infty$. Then Q lies in $\mathcal{B}^+_c(P_1 \cup S)$ for $c := 6\delta+2\delta f(\delta+1)$ by Lemma 3.4. By Proposition 3.3, all points b on Q with $d(x_1, b) > (M + c)f(\delta + 1)$ lie in $\mathcal{B}^+_c(S)$. By hyperbolicity, R lies in $\mathcal{B}^+_\delta(Q) \cup \mathcal{B}^-_\delta(P_2)$ and by Proposition 3.3 all points a on R but those with $d(a, x_2) \leq (M + \delta)f(\delta + 1)$ lie in $\mathcal{B}^+_\delta(Q)$. So all points a on R but those with $d(a, x_2) \leq (M + \delta)f(\delta + 1)$ or $d(x_1, a) \leq ((M + c)f(\delta + 1) + \delta)f(\delta + 1)$ lie in $\mathcal{B}^+_{c+\delta}(S)$. This shows (ii).

Let us now assume that X is strongly δ-hyperbolic. Since the geodesic triangle with sides P_1, Q and S is δ-slim, Q lies in $\mathcal{B}^+_\delta(P_1 \cup S)$. Any point on Q that lies in $\mathcal{B}^+_\delta(P_1)$ lies in $\mathcal{B}^+_{\delta+M}(x_1)$. So all points a with $d(x_1, a) > M + \delta$ lie in $\mathcal{B}^+_\delta(S)$. Now we take the geodesic triangle with sides R, Q and P_2. Then R lies in $\mathcal{B}^+(Q \cup P_2)$. Any vertex a on R that lies in $\mathcal{B}^+_\delta(P_2)$ satisfies $d(x_2, a) \leq M+\delta$. Since R is directed towards x_2, we conclude $d(a, x_2) \leq (M + \delta)f(\delta + 1)$ by Proposition 5.4. All points on R that lie within the out-ball of radius δ around points b on Q with $d(P_1, b) \leq \delta$, have distance at most $M + 2\delta$ from x_1. This proves (iii). □

Proposition 9.3. *Let X be a hyperbolic geodesic semimetric space that satisfies (B1) and (B2). Then $\leq$ is a quasiorder.*

Proof. Let R_1, R_2 be geodesic rays or anti-rays in X with $R_1 \leq R_2$ and let $M \geq 0$ be such that $R_1 \leq R_2$ holds for this M. Let $x_0, x_1, \ldots$ be infinitely many points on R_1 such that x_{i+1} lies between x_i and x_{i+2} with $d(x_i, x_{i+1}) \geq 1$, such that there is, for every $i \in \mathbb{N}$, a directed path from x_i to some point y_i on R_2 with $d(x_i, y_i) \leq M$ and such that all of these paths are pairwise disjoint. We may assume that x_0 and y_0 are the starting or end points of R_1 and R_2, respectively. Set

$$K := (2M + 5\delta) + (2M + 2\delta + 1)f(\delta + 1).$$

If R_1 is directed away from x_1, then we apply Lemma 9.2 (i) with (x_0, x_i, y_0, y_i) as (x_1, x_2, y_1, y_2) for every $i \in \mathbb{N}$. If R_1 is directed towards x_1, then we apply the same lemma with (x_i, x_1, y_i, y_1) as (x_1, x_2, y_1, y_2) for every $i \in \mathbb{N}$.

In both situations, Lemma 9.2 (i) implies that R_2 lies in the out-ball of radius K around R_1 and hence Lemma 9.1 (ii) implies the assertion. □

Let X be a hyperbolic geodesic semimetric space that satisfies (B1) and (B2). If $\leq$ is a quasiorder on the set of geodesic rays and anti-rays of X, then we write $R_1 \approx$

R_2 if $R_1 \leq R_2$ and $R_2 \leq R_1$. This new relation is an equivalence relation whose equivalence classes form the *geodesic boundary* $\partial_{geo}X$ of X. We define two related boundaries: the *geodesic f-boundary* $\partial^f_{geo}X$ consists of the equivalence classes of $\approx$ restricted to the rays and the *geodesic b-boundary* $\partial^b_{geo}X$ consists of the equivalence classes of $\approx$ restricted to the anti-rays. Every geodesic boundary point is the union of at most one geodesic f-boundary point and at most one geodesic b-boundary point.

Note that $\leq$ extends to an order on the three sets $\partial_{geo}X$, ∂^f_{geo} and ∂^b_{geo}. Lemma 9.2 enables us to prove some order-theoretic results on the geodesic boundary.

Proposition 9.4. *Let X be a hyperbolic geodesic semimetric space that satisfies (B1) and (B2). Let $\eta, \mu \in \partial_{geo}X$ with $\eta < \mu$. Then either η or μ contains no ray and the other one contains no anti-ray.*

In particular, there are no chains of length at least 3 in $\partial_{geo}X$.

Proof. Let $\eta, \mu \in \partial_{geo}X$ with $\eta \leq \mu$ and let $R \in \eta$ and $Q \in \mu$. Assume that either both, R and Q, are rays or that both are anti-rays. Then we use the method of the proof of Proposition 9.3 and apply Lemma 9.2 (ii) to conclude $Q \leq R$. So we have $Q \approx R$ and hence $\eta = \mu$. □

If X is strongly hyperbolic, then $\leq$ is already an equivalence relation as the following result shows.

Proposition 9.5. *Let X be a strongly hyperbolic geodesic semimetric space that satisfies (B1) or (B2). Then $\leq$ is an equivalence relation.*

Proof. We obtain by Proposition 9.3 that $\leq$ is a quasiorder. If we use the method in the proof of Proposition 9.3 but apply it with Proposition 9.2 (iii) instead of (i) and use Lemma 9.1 (i), we directly obtain the assertion. □

Since $\leq$ is an equivalence relation on the geodesic rays and anti-rays in the case of strong hyperbolicity, we obtain that $\leq$ and $\approx$ coincide in this case.

It would be interesting to know whether divergence of geodesics or geodesic stability is strong enough to give rise to a geodesic boundary.

10. Quasi-geodesic boundary

In this section, we define a different boundary that builds upon a quasiorder on the set of quasi-geodesic rays and anti-rays. For hyperbolic digraphs satisfying (B1) and (B2) this new boundary will coincide with the geodesic boundary, see Proposition 10.4. As a corollary we obtain that quasi-isometries preserve the structure of the geodesic boundary, see Theorem 10.5.

We extend $\leq$ to the class of quasi-geodesic rays and anti-rays: for quasi-geodesic rays or anti-rays R_1 and R_2 in a hyperbolic geodesic semimetric space X, we write $R_1 \leq R_2$ if there exists some $M \geq 0$ such that for every $r \geq 0$ and every $x \in X$ there is a directed R_1-R_2 path of length at most M outside of $\mathcal{B}^+_r(x) \cup \mathcal{B}^-_r(x)$.

Proposition 10.1. *For every hyperbolic geodesic semimetric space X that satisfies (B1) and (B2), the relation $\leq$ is a quasiorder on the set of quasi-geodesic rays and anti-rays.*

Proof. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that D satisfies (B1) and (B2) for f. Let $\gamma \geq 1$ and $c \geq 0$. Let R_1 and R_2 be (γ, c)-quasi-geodesic rays or anti-rays in D

such that $R_1 \leq R_2$. Let x be the starting or end point of R_1 and y be the starting or end point of R_2. Let $M \geq 0$ and let $x_0, x_1, \ldots$ be points on R_1 and $y_0, y_1, \ldots$ be points on R_2 such that $d(x_i, y_i) \leq M$ and $d^{\leftrightarrow}(x, x_i) \geq i$ and $d^{\leftrightarrow}(y, y_i) \geq i$ for all $i \geq 0$. We may assume $x = x_0$ and $y = y_0$. Let $\kappa \geq 0$ such that geodesic stability holds for (γ, c)-quasi-geodesics with respect to the value κ, cp. Corollary 7.3.

Let R_1^i be the subpath of R_1 between x and x_i and let R_2^i be the subpath of R_2 between y and y_i. Let P_i be a geodesic with the same starting point as R_1^i and the same end point as R_1^i and let Q_i be a geodesic with the same starting point as R_2^i and the same end point as R_2^i. We apply Lemma 9.2 (i) for the four points x, x_i, y, y_i for every $i \geq 1$ and use geodesic stability to conclude that R_2 lies in the ball of radius

$$2\kappa + (2M + 5\delta) + (2M + 2\delta + 1) f(\delta + 1)$$

around R_1. This shows with an analogue to Lemma 9.1 (ii) for the relation $\leq$ on quasi-geodesic rays and anti-rays instead of geodesic ones that $\leq$ is transitive. Since the relation is obviously reflexive, the assertion follows. □

Using the same method as in the proof of Proposition 10.1, but also applying Lemma 9.2 (i) and an analogue of Lemma 9.1 (i), we obtain the following result for strong hyperbolicity.

Proposition 10.2. *For every strongly hyperbolic geodesic semimetric space X that satisfies (B1) or (B2), the relation $\leq$ is an equivalence relation on the set of quasi-geodesic rays and anti-rays.* □

Similar to the case of geodesic rays and anti-rays, if $\leq$ is a quasiorder on the set of quasi-geodesic rays and anti-rays, we write $R_1 \approx R_2$ if $R_1 \leq R_2$ and $R_2 \leq R_1$, where R_1 and R_2 are quasi-geodesic rays or anti-rays in a hyperbolic geodesic semimetric space X that satisfies (B1) and (B2). Then $\approx$ is an equivalence relation whose equivalence classes form the *quasi-geodesic boundary* ∂X of the semimetric space. The *quasi-geodesic f-boundary* $\partial^f X$ are the equivalence classes of $\approx$ restricted to the quasi-geodesic rays and the *quasi-geodesic b-boundary* $\partial^b X$ are the equivalence classes of $\approx$ restricted to the quasi-geodesic anti-rays. We also have in this case that every quasi-geodesic boundary point is the union of at most one quasi-geodesic f-boundary point with at most one quasi-geodesic b-boundary point. Note that we can extend the quasiorder $\leq$ to an order on ∂X, on $\partial^f X$ and on $\partial^b X$. Analogous to Proposition 9.4, we obtain the following result.

Proposition 10.3. *Let X be a hyperbolic geodesic semimetric space that satisfies (B1) and (B2). Let $\eta, \mu \in \partial X$ with $\eta < \mu$. Then either η or μ contains no ray and the other one contains no anti-ray.*

In particular, there are no chains of length at least 3 in ∂X. □

In hyperbolic geodesic spaces, we can apply the Arzelà-Ascoli theorem to prove that every quasi-geodesic ray lies close to and from some geodesic ray, see e. g. [2, Lemma I.8.28]. Generally, in the case of semimetric spaces an analogue of the Arzelà-Ascoli theorem is false, but gets true with additional strong requirements, see [9]. Since we do not satisfy these additional requirements in general, we prove the desired result on quasi-geodesic and geodesic rays and anti-rays only in the case of digraphs, where we can apply elementary arguments instead of the Arzelà-Ascoli theorem.

Proposition 10.4. *Let D be a hyperbolic digraph satisfying (B1) and (B2). Then the following hold.*

(i) *If all vertices of D have finite out-degree, then every quasi-geodesic f-boundary point contains a geodesic f-boundary point.*
(ii) *If all vertices of D have finite in-degree, then every quasi-geodesic b-boundary point contains a geodesic b-boundary point.*
(iii) *If D is locally finite, then every quasi-geodesic boundary point contains a geodesic boundary point.*

Proof. Let us assume that every vertex of D has finite out-degree and let $Q = x_0x_1\dots$ be a (γ, c)-quasi-geodesic ray in D for some $\gamma \geq 1$ and $c \geq 0$. For every $i \in \mathbb{N}$, let P_i be an x_0-x_i geodesic. Since x_0 has finite out-degree, infinitely many P_i have a common first edge. Similarly, among those there are again infinitely many with a common second edge and so on. This way we obtain a ray R with starting vertex x_0 and such that every finite directed subpath is geodesic. Hence, R is geodesic as well.

By geodesic stability, see Corollary 7.3, there is some $\kappa \geq 0$ such that every P_i lies in the out-ball and in-ball of radius κ around x_0Qx_i. Thus, R lies in the out-ball and in-ball of radius κ around Q. Thus, we have $Q \leq R$ and $R \leq Q$. This shows (i).

By a symmetric argument with all directions of the edges reversed, we obtain (ii) and (iii) follows immediately from (i) and (ii). □

The advantage of the quasi-geodesic boundary is that it is preserved by quasi-isometries since quasi-isometries preserve the set of quasi-geodesic rays and anti-rays. In the case of locally finite digraphs, Proposition 10.4 implies that quasi-isometries also preserve the geodesic boundary. Thus, we immediately have the following results.

Theorem 10.5. *Let $f\colon X_1 \to X_2$ be a quasi-isometry between hyperbolic geodesic semimetric spaces X_1 and X_2 that satisfy (B1) and (B2). Then f canonically defines three order-preserving bijective maps: one between the quasi-geodesic f-boundaries, one between the quasi-geodesic b-boundaries and one between the quasi-geodesic boundaries.* □

Theorem 10.6. *Let $f\colon D_1 \to D_2$ be a quasi-isometry between hyperbolic digraphs D_1 and D_2 that satisfy (B1) and (B2). Then the following hold.*

(i) *If every vertex of D_1 and D_2 has finite out-degree, then f canonically defines an order-preserving bijective map between the geodesic f-boundaries of the digraphs.*
(ii) *If every vertex of D_1 and D_2 has finite in-degree, then f canonically defines an order-preserving bijective map between the geodesic b-boundaries of the digraphs.*
(iii) *If D_1 and D_2 are locally finite, then f canonically defines an order-preserving bijective map between the geodesic boundaries of the digraphs.* □

11. Ends of digraphs

In this section, we will briefly introduce the notion of ends of digraphs as introduced by Zuther [25] and then show that the geodesic boundary of hyperbolic locally finite digraphs is a refinement of the set of ends. We note that Bürger and Melcher [3, 4, 5] recently investigated a different notion of ends of digraphs. Roughly speaking, their ends are those ends in Zuther's sense that contain rays and anti-rays. In general, the geodesic boundary is not a refinement of the ends in sense of Bürger and Melcher: while each of their ends still contains a geodesic boundary point, there may be geodesic boundary points no belonging to any of their ends. Jackson and Kilibarda [20] used a different notion for ends of semigroups that is based on the ends of the underlying undirected graph of their Cayley digraphs. Gray and Kambites [15] proved that the ends in the sense of Jackson and Kilibarda are invariant under quasi-isometries. In this section, we will also show that Zuther's notion of ends of digraphs is preserved by quasi-isometries in the case of locally finite digraphs.

Our main interest in this section is to prove that the geodesic boundary of hyperbolic locally finite digraphs is a refinement of their ends. By reasons addressed in the previous section, we do not obtain this result for semimetric spaces: apart from a notion of ends for semimetric spaces, we would need a suitable notion of the Arzelà-Ascoli theorem.

In order to define the ends of digraphs, we first define a relation on the set $\mathcal{R}$ of all rays and anti-rays in a digraph D. For $R_1, R_2 \in \mathcal{R}$, we write $R_1 \preccurlyeq R_2$ if there are infinitely many pairwise disjoint R_1-R_2 paths in D. Zuther [25, Proposition 2.2] showed that $\preccurlyeq$ is a quasiorder on $\mathcal{R}$. We write $R_1 \sim R_2$ if $R_1 \preccurlyeq R_2$ and $R_2 \preccurlyeq R_1$. This is an equivalence relation whose classes are the *ends* of D. We denote this set by ΩD. Its restrictions to either the rays or the anti-rays are also equivalence relations whose classes are the *f-ends* or *b-ends*, respectively. Every end is the union of at most one f-end and at most one b-end. Note that $\preccurlyeq$ extends to an order on the set of ends, of f-ends and of b-ends of D.

In the case of a graph G, it is easy to see that two rays have infinitely many pairwise disjoint paths between them if and only if for every finite vertex set S those rays have tails that lie in the same component of $G - S$. Our next result shows proves an analogous result for digraphs.

Proposition 11.1. *Let D be a digraph.*

(i) *Let every vertex of D have finite out-degree and let R_1, R_2 be two rays in D. Then $R_1 \preccurlyeq R_2$ if and only if for every $x \in V(D)$ and every $R \in \mathbb{N}$ there is a directed R_1-R_2 path outside of $\mathcal{B}^+_R(x)$.*

(ii) *Let every vertex of D have finite in-degree and let R_1, R_2 be two anti-rays in D. Then $R_1 \preccurlyeq R_2$ if and only if for every $x \in V(D)$ and every $R \in \mathbb{N}$ there is a directed R_1-R_2 path outside of $\mathcal{B}^-_R(x)$.*

(iii) *Let D be locally finite and let R_1 and R_2 be rays or anti-rays in D. Then $R_1 \preccurlyeq R_2$ if and only if for every $x \in V(D)$ and every $R \in \mathbb{N}$ there is a directed R_1-R_2 path outside of $\mathcal{B}^+_R(x) \cup \mathcal{B}^-_R(x)$.*

Proof. To prove (i), let $x \in V(D)$ and $r \in \mathbb{N}$. If $R_1 \preccurlyeq R_2$, then we have infinitely many pairwise disjoint directed R_1-R_2 paths. All but finitely many of them do not meet $\mathcal{B}^+_r(x)$, since that is a finite set.

To prove the remaining direction of (i), let x be the starting vertex of R_1. Suppose that there are only finitely many pairwise disjoint directed R_1-R_2 paths. Since for each of those finitely many directed paths P there is a directed subpath of R_1 from x to the starting vertex of P, there exists $r \in \mathbb{N}$ such that all of those directed R_1-R_2 paths lie in $\mathcal{B}^+_r(x)$. This is a contradiction to the assumption that there is a directed R_1-R_2 path outside of $\mathcal{B}^+_r(x)$.

With a similar argument but with x being the last vertex of the anti-ray R_2 for the reverse direction, we obtain (ii).

Let us prove (iii). If there are infinitely many pairwise disjoint directed R_1-R_2 paths, then for every $x \in V(D)$ and every $r \in \mathbb{N}$, there is an R_1-R_2 path outside of the finite set $\mathcal{B}_r^+(x) \cup \mathcal{B}_r^-(x)$. To prove the other direction, it suffices to consider the case that R_1 is an anti-ray and R_2 a ray, since we can follow the proof of (i) if R_1 is a ray and the proof of (ii) if R_2 is an anti-ray. So let us assume that for every $x \in V(D)$ and every $r \in \mathbb{N}$ there is a directed R_1-R_2 path outside of $\mathcal{B}_r^+(x) \cup \mathcal{B}_r^-(x)$. Let us suppose that there are only finitely many pairwise disjoint directed R_1-R_2 paths $P_1, \ldots, P_n$. Let x_i be the starting vertex of P_i for all $1 \leq i \leq n$ and let a be the end vertex of R_1. Set $N := \max\{\ell(P_i) \mid 1 \leq i \leq n\}$ and $r := N + \max\{d(x_i, a) \mid 1 \leq i \leq n\}$. Let $x \in \{x_1, \ldots, x_n\}$ with $d(x, a) = r - N$. Then $P_1, \ldots, P_n$ lie in $\mathcal{B}_r^+(x)$. By assumption, we find an R_1-R_2 path P outside of $\mathcal{B}_r^+(x) \cup \mathcal{B}_r^-(x)$. This is disjoint to all paths P_i by its choice. This contradiction shows $R_1 \preccurlyeq R_2$. □

Craik et al. [11, Corollary 2.3] proved that Zuther's definition of ends of digraphs extends to a notion of ends of finitely generated semigroups that is preserved under changing the finite generating set. More precisely, for a finitely generated semigroup every right Cayley digraph has the same isomorphism type (as a partially ordered set) of their ends. We prove a similar result for quasi-isometries of digraphs.

Theorem 11.2.

(i) *Quasi-isometries between digraphs all of whose vertices have finite out-degree extend canonically to bijective maps between their f-ends that preserve the order $\preccurlyeq$.*

(ii) *Quasi-isometries between digraphs all of whose vertices have finite in-degree extend canonically to bijective maps between their b-ends that preserve the order $\preccurlyeq$.*

(iii) *Quasi-isometries between locally finite digraphs extend canonically to bijective maps between their ends that preserve the order $\preccurlyeq$.*

Proof. Let D_1 and D_2 be digraphs and let $f\colon D_1 \to D_2$ be a (γ, c)-quasi-isometry for some $\gamma \geq 1$ and $c \geq 0$. Let $R_1 = x_1x_2\ldots$ be a ray in D_1. Then there is a directed $f(x_i)$-$f(x_{i+1})$ path P_i^1 of length at most $\gamma + c$ in D_2. Concatenating all these directed paths leads to a one-way infinite directed path W_1. This contains a ray Q_1. Note that there are infinitely many pairwise disjoint Q_1-$\{f(x_i) \mid i \in \mathbb{N}\}$ paths and infinitely many pairwise disjoint $\{f(x_i) \mid i \in \mathbb{N}\}$-$Q_1$ paths. So all possible rays obtained the same way as Q_1 lie in the same f-end. In the same way, we obtain for a second ray $R_2 = y_1y_2\ldots$ in D_1 paths P_i^2 from $f(y_i)$ to $f(y_{i+1})$ of length at most $\gamma + c$ and a one-way infinite directed path W_2 and a ray Q_2 in W_2.

Let us assume that $R_1 \preccurlyeq R_2$ and that all vertices of D_1 and D_2 have finite out-degree. Let $x \in V(D_2)$ and $n \in \mathbb{N}$. Then W_1, W_2, Q_1 and Q_2 have all but finitely many of their vertices outside of $\mathcal{B}_n^+(x)$. Let Q_1' and Q_2' be tails of Q_1 and Q_2, respectively, such that Q_1' and Q_2' do not meet $\mathcal{B}_n^+(x)$. Let $N \in \mathbb{N}$ such that for all $i \geq N$ there is a Q_1'-$f(x_i)$ path and an $f(y_i)$-Q_2' path outside of $\mathcal{B}_n^+(x)$. Let $y \in V(D_1)$ such that $d(f(y), x) \leq c$ and $d(x, f(y)) \leq c$. There exists a directed R_1-R_2 path P outside of $\mathcal{B}^+_{(\gamma+c)(n+1)+c}(y)$ by Proposition 11.1. We may assume that it is from x_i to y_j for some $i, j \geq N$. Its f-image induces a directed $f(x_i)$-$f(y_j)$

path outside of $\mathcal{B}_n^+(x)$. By the choice of i and j, we find a Q_1'-Q_2' path outside of $\mathcal{B}_n^+(x)$. This shows $Q_1 \preccurlyeq Q_2$.

Note that there is a quasi-isometry $g\colon D_2 \to D_1$ such that $g \circ f$ is almost the identity: that there exists some $\ell \geq 0$ with $d(u, g(f(u))) \leq \ell$ for all $u \in V(D_1)$ and $d(v, f(g(v))) \leq \ell$ for all $v \in V(D_2)$. This implies that $R_1 \preccurlyeq R_2$ if and only if $Q_1 \preccurlyeq Q_2$ and finishes the proof of (i).

To prove (ii), we essentially follow the proof of (i) with reversed directions of the edges. So e. g. the paths P_i^1 go from $f(x_{i+1})$ to $f(x_i)$ and the distances are measured towards x instead of from x. Also, the proof of (iii) is essentially the same. □

Let us now prove that the geodesic boundary is a refinement of the ends.

Proposition 11.3. *For every hyperbolic digraph D that satifies (B1) and (B2) the following hold.*

- (i) *If every vertex of D has finite out-degree, then the geodesic f-boundary is a refinement of the f-ends.*
- (ii) *If every vertex of D has finite in-degree, then the geodesic b-boundary is a refinement of the b-ends.*
- (iii) *If D is locally finite, then the geodesic boundary is a refinement of the ends.*

Proof. Let us assume that every vertex has finite out-degree and let $f\colon \mathbb{R} \to \mathbb{R}$ such that (B1) and (B2) are satisfied for the function f.

Let $R = x_0x_1\ldots$ be a ray in D. For every $i \in \mathbb{N}$, let P_i be an x_0-x_i geodesic. These paths give rise to a ray in that infinitely many directed paths P_i have a common first edge since x_0 has finite out-degree, among which there are again infinitely many P_i that also share a common second edge and so on. The obtained ray $Q = y_0y_1\ldots$ with $x_0 = y_0$ is geodesic since each finite directed subpath is contained in some P_i. Obviously, we have $Q \preccurlyeq R$.

Let $k \in \mathbb{N}$ and set $i_k := d(x_0, x_k)$. Let $\ell \in \mathbb{N}$ such that $y_0Qy_{i_k+2\delta+1}$ is a subpath of P_ℓ. Let us consider the geodesic triangle with sides P_k, P_ℓ and an x_k-x_ℓ geodesic P. Let v be the last vertex of P in $\mathcal{B}_\delta^+(P_k)$ and let w be its out-neighbour on P. By δ-hyperbolicity, we know that w lies in $\mathcal{B}_\delta^-(u_k)$ for some u_k on P_ℓ. It follows that $d(x_0, u_k) \leq i_k + 2\delta + 1$, so u_k lies on Q. Thus, the concatenation of x_kPw with a w-u_k geodesic shows the existence of an x_k-u_k geodesic Q_k. So we find infinitely many directed R-Q paths. It remains to show that we can find infinitely many pairwise disjoint ones.

Let $\mathcal{P}$ be a maximal set of these directed paths Q_i that are pairwise disjoint. Suppose that $\mathcal{P}$ is finite. Let i be the maximum distance from x_0 to vertices on elements of $\mathcal{P}$ and set $n := (\delta + i)f(\delta + 1) + 1$. Let $k \in \mathbb{N}$ such that y_0Qy_n is a subpath of P_k. Considering the geodesic triangle with sides $y_nP_kx_k$, Q_k and y_nQu_k, we get by δ-hyperbolicity that Q_k lies in $\mathcal{B}_\delta^+(y_nP_kx_k) \cup \mathcal{B}_\delta^-(y_nQu_k)$. Let us suppose that Q_k had a vertex v in $\mathcal{B}_i^+(x_0)$. If $v \in \mathcal{B}_\delta^+(y_nP_kx_k)$, then for $w \in V(y_nP_kx_k)$ with $d(w, v) \leq \delta$, Proposition 3.3 implies $d(x_0, w) \leq (\delta + i)f(\delta + 1)$, which contradicts that P_k is a geodesic and thus $d(x_0, w) \geq (\delta + i)f(\delta + 1) + 1$. So v lies in $\mathcal{B}_\delta^-(u)$ for some $u \in V(y_nQu_k)$. But then $d(x_0, u) \leq i + \delta$, which is also a contradiction. Thus, Q_k is disjoint to every element of $\mathcal{P}$, which contradicts the maximality of $\mathcal{P}$. Thus, there are infinitely many pairwise disjoint R-Q paths and hence we have $R \preccurlyeq Q$. So R and Q lie in the same f-end of D. This shows (i).

Analogously with the directions of the edges reversed, we obtain (ii). Finally, (iii) is an immediate consequence of (i) and (ii). □

We will use the map of Proposition 11.3 (iii) in Section 15 to understand the connection between the ends and the (quasi-)geodesic boundary in more detail.

12. Topologies for the quasi-geodesic boundary

In this section, we extend the two topologies $\mathcal{O}_f$ and $\mathcal{O}_b$ to the boundary of hyperbolic geodesic semimetric spaces and show that quasi-isometries extend to homeomorphisms with respect to both topologies on the boundaries. Let X be a hyperbolic geodesic semimetric space that satisfies (B1) and (B2). Let $x \in X$, let $r \geq 0$ and let $\omega \in \partial^f X$. We set

$$C^-(\omega, x, r) := \{y \in X \mid \exists R \in \omega\ \forall z \text{ on } R\ \exists y\text{-}z \text{ geodesic outside of } \mathcal{B}_r^+(x) \cup \mathcal{B}_r^-(x)\}.$$

Analogously, if $\eta \in \partial^b X$, then we set

$$C^+(\eta, x, r) := \{y \in X \mid \exists R \in \omega\ \forall z \text{ on } R\ \exists z\text{-}y \text{ geodesic outside of } \mathcal{B}_r^+(x) \cup \mathcal{B}_r^-(x)\}.$$

For $\mu \in \partial X$ with $\mu = \mu_1 \cup \mu_2$ for $\mu_1 \in \partial^f X$ and $\mu_2 \in \partial^b X$, we set

$$C^-(\mu, x, r) := C^-(\mu_1, x, r)$$

and

$$C^+(\mu, x, r) := C^+(\mu_2, x, r).$$

We say that an element μ' of $\partial X \cup \partial^f X \cup \partial^b X$ *lives in* $C^-(\mu, x, r)$ or $C^+(\mu, x, r)$ if an element of μ' lies in $C^-(\mu, x, r)$ or $C^+(\mu, x, r)$, respectively. We denote by $C_\partial^-(\mu, x, r)$ and $C_\partial^+(\mu, x, r)$ the sets $C^-(\mu, x, r)$ and $C^+(\mu, x, r)$ together with the quasi-geodesic boundary points living in them.

Generally, not all quasi-geodesic boundary points that live in $C^-(\omega, x, r)$ or $C^+(\omega, x, r)$ have the property that each of their elements contains a subray or anti-subray that is contained in $C^-(\omega, x, r)$ or $C^+(\omega, x, r)$, respectively. However, we shall show that this is true up to small changes on the constant r.

Lemma 12.1. *Let X be a hyperbolic geodesic semimetric spaces that satisfies (B1) and (B2). Let $x \in X$, let $r \geq 0$ and let $\omega \in \partial^f X \cup \partial^b X$. Set $\kappa := 6\delta + 2\delta f(\delta + 1)$. Then the following hold.*

(i) *If $\omega \in \partial^f X$ and R is a quasi-geodesic ray or anti-ray in $C^-(\omega, x, r)$, then every quasi-geodesic ray or anti-ray $Q \approx R$ lies in $C^-(\omega, x, r - \kappa)$* eventually, *i. e. there is at most a directed subpath of Q of finite length outside of $C^-(\omega, x, r - \kappa)$.*

(ii) *If $\omega \in \partial^b X$ and R is a quasi-geodesic ray or anti-ray in $C^+(\omega, x, r)$, then every quasi-geodesic ray or anti-ray $Q \approx R$ lies in $C^+(\omega, x, r - \kappa)$ eventually.*

Proof. By symmetry, it suffices to prove (i). By (B1), we know that at most some directed subpath of finite length of Q has its starting and end point in $\mathcal{B}_r^+(x) \cup \mathcal{B}_r^-(x)$. So we may assume that Q lies outside of $\mathcal{B}_r^+(x) \cup \mathcal{B}_r^-(x)$. Let $M \geq 0$ such that $Q \preccurlyeq R$ holds with respect to the constant M. If a Q-R geodesic P of length at most M contains a point of $\mathcal{B}_r^+(x) \cup \mathcal{B}_r^-(x)$, then either its end point lies in $\mathcal{B}_{r+M}^+(x)$ or its starting point lies in $\mathcal{B}_{r+M}^-(x)$. Since there is at most a directed subpath of R of finite length with its starting and end point in $\mathcal{B}_{r+M}^-(x)$ by (B1), we may replace R by a subray or anti-subray that lies outside of $\mathcal{B}_{r+M}^+(x) \cup \mathcal{B}_{r+M}^-(x)$ and hence the first case does not happen. Analogously, we may replace Q by a subray or anti-subray of Q that avoids $\mathcal{B}^+_{r+M}(x) \cup \mathcal{B}^-_{r+M}(x)$ by (B2). Thus, no Q-R geodesic intersects $\mathcal{B}^+_r(x) \cup \mathcal{B}^-_r(x)$. Applying Lemma 3.4 shows that if two composable geodesics lie outside of $\mathcal{B}^+_r(x) \cup \mathcal{B}^-_r(x)$, then any geodesic that is parallel to the composition lies outside of $\mathcal{B}^+_{r-\kappa}(x) \cup \mathcal{B}^-_{r-\kappa}(x)$. This shows (i). □

Now we are able to define a base for the topologies on $X \cup \partial X$. Let $\omega \in \partial X$. We set

$$C^f_\partial(\omega, x, r) := \bigcup_{\mu \in \partial X, \omega \leq \mu} C^+_\partial(\mu, x, r)$$

for all $x \in X$ and $r \geq 0$. Then we declare all sets $C^f_\partial(\omega, x, r)$ as open. These sets together with the open balls $\mathring{\mathcal{B}}^+_r(x)$ form a base for the topology $\mathcal{O}_f$ of $X \cup \partial X$. Analogously, we set

$$C^b_\partial(\omega, x, r) := \bigcup_{\mu \in \partial X, \mu \leq \omega} C^-_\partial(\mu, x, r)$$

for all $x \in X$ and $r \geq 0$ and obtain a base for the topology $\mathcal{O}_b$ of $X \cup \partial X$ if we declare the sets $C^b_\partial(\omega, x, r)$ as open and take them together with the open balls $\mathring{\mathcal{B}}^-_r(x)$.

We denote by $\partial^+(\omega)$, by $\partial^-(\omega)$, the elements η of ∂X with $\omega \leq \eta$, with $\eta \leq \omega$, respectively. It immediately follows from the above definition that every f-neighbourhood of ω contains $\partial^+(\omega)$ and every b-neighbourhood of ω contains $\partial^-(\omega)$.

Let us illustrate the definition of the topologies via the following example.

Example 12.2. Let D be the digraph with distinct vertices u_i, v_i, w_i, x_i, y_i for all $i \in \mathbb{N}$. The edges are the following:

– for every $i \in \mathbb{N}$ we have the edges $u_i v_i$, $v_i w_i$, $w_i x_i$ and x_i, y_i,
– for every $i \in \mathbb{N}$ there is an edge $v_i v_{i+1}$ and
– for every $i \in \mathbb{N}$ there is an edge $x_{i+1} x_i$.

Then D is hyperbolic with one quasi-geodesic boundary point η being the equivalence class of the ray $v_0 v_1 \ldots$ and the only other quasi-geodesic boundary point μ being the equivalence class of the anti-ray $\ldots x_1 x_0$. Then the typical open b-neighbourhood of η that lies in the defined base consists of η and the vertices u_i and v_i for all $i \geq i_0$ for some $i_0 \in \mathbb{N}$. The (typical) open f-neighbourhoods of η in the base consist of η and μ and the vertices x_i and y_i for all $i \geq i_0$ for some $i_0 \in \mathbb{N}$. The neighbourhoods of μ are obtained symmetrically with u_i swapped with y_i and v_i swapped with x_i. At first, it may be surprising that none of the vertices w_i lie in these typical neighbourhoods, but intuitively, there are no directed w_i-η or μ-w_i paths.

Theorem 12.3. *Let $f\colon X_1 \to X_2$ be a quasi-isometry between hyperbolic geodesic semimetric spaces that satisfy (B1) and (B2). Then f canonically defines a map $\widehat{f}\colon \partial X_1 \to \partial X_2$ that is a homeomorphism with respect to $\mathcal{O}_f$ and $\mathcal{O}_b$.*

Proof. Let $\gamma \geq 1$ and $c \geq 0$ such that f is (γ, c)-quasi-geodesic. Theorem 10.5 implies that f canonically defines an order-preserving bijective map $\widehat{f}\colon \partial X_1 \to \partial X_2$. Let us consider a non-trivial set $C^+(\eta, x, r)$ with $\eta \in \partial X$, $x \in X$ and $r \geq 0$. Then

$$f(C^+(\eta, x, r)) \subseteq C^+(\widehat{f}(\eta), f(x), \frac{r}{\gamma} - c).$$

Similarly, we have

$$f(C^-(\eta,x,r)) \subseteq C^-(\widehat{f}(\eta), f(x), \frac{r}{\gamma} - c)$$

for non-trivial sets $C^-(\eta,x,r)$. Furthermore, every boundary point that lives in $C^+(\eta,x,r)$ or $C^-(\eta,x,r)$ is mapped by $\widehat{f}$ to a boundary point that lives in $C^+(\widehat{f}(\eta), f(x), \frac{r}{\gamma} - c)$ or $C^-(\widehat{f}(\eta), f(x), \frac{r}{\gamma} - c)$, respectively. Thus, $\widehat{f}$ is continuous with respect to both topologies.

Since f is a quasi-isometry, there exists a quasi-isometry $g\colon X_2 \to X_1$. Let $\widehat{g}$ be the bijection $\partial X_2 \to \partial X_1$ that is canonically defined by g, which exists due to Theorem 10.5. It is easy to see that $\widehat{g} \circ \widehat{f}$ is the identity on ∂X_1. So we have $\widehat{g} = \widehat{f}^{-1}$. As $\widehat{g}$ is continuous with respect to both topologies, $\widehat{f}$ is a homeomorphism with respect to both topologies. □

13. A PSEUDO-SEMIMETRIC FOR $X \cup \partial X$

A subset S of a semimetric space X is a *base* of X if for every $x \in X$ there exists $s \in S$ with $d^{\leftrightarrow}(x,s) < \infty$. We call X *finitely based* if it has a finite base. For the following definition, recall that we set $R(i) := R(-i)$ if R is an anti-ray and $i > 0$.

Let (X,d) be a finitely based hyperbolic geodesic semimetric space that satisfies (B1) and (B2). Let S be a finite base of X. Let $\eta, \mu \in X \cup \partial X$ and let $s \in S$. If $\eta, \mu \in \partial X$, set

$$\rho_s(\eta,\mu) := \sup_{R \in \eta, Q \in \mu} \{\liminf\{d^{\leftrightarrow}(s,P) \mid i,j \to \infty,\ P \text{ is an } R(i)\text{-}Q(j) \text{ geodesic}\}\}.$$

If $\eta \in X$ and $\mu \in \partial X$, set

$$\rho_s(\eta,\mu) := \sup_{Q \in \mu}\{\liminf\{d^{\leftrightarrow}(s,P) \mid i \to \infty,\ P \text{ is an } \eta\text{-}Q(i) \text{ geodesic}\}\}$$

and

$$\rho_s(\mu,\eta) := \sup_{Q \in \mu}\{\liminf\{d^{\leftrightarrow}(s,P) \mid i \to \infty,\ P \text{ is a } Q(i)\text{-}\eta \text{ geodesic}\}\}.$$

If $\eta, \mu \in X$, set

$$\rho_s(\eta,\mu) := \liminf\{d^{\leftrightarrow}(s,P) \mid P \text{ is an } \eta\text{-}\mu \text{ geodesic}\}.$$

Set

$$\rho_S(\eta,\mu) := \min\{\rho_s(\eta,\mu) \mid s \in S\}$$

for all $\eta, \mu \in X \cup \partial X$.

We will see later that that we may have $\rho_S(\eta,\mu) = \infty$ for distinct $\eta, \mu \in \partial X$. If we now follow the usual approach for hyperbolic geodesic metric spaces, we would define $\rho_S^{\varepsilon}(\eta,\mu) := e^{-\varepsilon \rho_S(\eta,\mu)}$. However, this is then undefined if $\rho_S(\eta,\mu) = \infty$. Thus, we need the following slightly different definition in that we switch the supremum with the exponent.

Let $\varepsilon > 0$. For all $\eta, \mu \in \partial X$, we set

$$\rho_s^{\varepsilon}(\eta,\mu) := \inf_{R \in \eta, Q \in \mu}\{e^{-\varepsilon \liminf\{d^{\leftrightarrow}(s,P) \mid i,j \to \infty,\ P \text{ is an } R(i)\text{-}Q(j) \text{ geodesic}}\}.$$

For all $\eta \in X$ and $\mu \in \partial X$, we set

$$\rho_s^{\varepsilon}(\eta,\mu) := \inf_{Q \in \mu}\{e^{-\varepsilon \liminf\{d^{\leftrightarrow}(s,P) \mid i \to \infty,\ P \text{ is an } \eta\text{-}Q(i) \text{ geodesic}}\}$$

and
$$\rho^\varepsilon_s(\eta,\mu) := \inf_{Q\in\mu}\{e^{-\varepsilon\liminf\{d^{\leftrightarrow}(s,P)\mid i\to\infty,\ P\text{ is a }Q(i)\text{-}\eta\text{ geodesic}}\}.$$
For all $\eta,\mu\in X$, we set
$$\rho^\varepsilon_s(\eta,\mu) := \inf_{R\in\eta, Q\in\mu}\{e^{-\varepsilon\liminf\{d^{\leftrightarrow}(s,P)\mid P\text{ is an }\eta\text{-}\mu\text{ geodesic}}\}.$$
Set
$$\rho^\varepsilon_S(\eta,\mu) := \max\{\rho^\varepsilon_s(\eta,\mu)\mid s\in S\}$$
for all $\eta,\mu\in X\cup\partial X$.

Finally, we set
$$d_{S,\varepsilon}(\eta,\mu) := \inf\{\sum_{i=0}^{n-1}\rho^\varepsilon_S(\eta_i,\eta_{i+1})\mid n\in\mathbb{N}, \eta=\eta_0,\eta_1,\dots,\eta_n=\mu\in X\cup\partial X\}.$$
for all $\eta,\mu\in X\cup\partial X$.

Lemma 13.1. *Let $\delta\geq 0$ and let X be a δ-hyperbolic geodesic semimetric space with finite base $S\subseteq X$ that satisfies (B1) and (B2) for the function $f\colon\mathbb{R}\to\mathbb{R}$. Let $\varepsilon>0$ and set $\varepsilon' := e^{2\varepsilon(6\delta+2\delta f(\delta+1))}$. Then*
$$\rho^\varepsilon_S(\eta_1,\eta_2)\leq\varepsilon'\max\{\rho^\varepsilon_S(\eta_1,\eta_3),\rho^\varepsilon_S(\eta_3,\eta_2)\}$$
holds for all $\eta_1,\eta_2,\eta_3\in X\cup\partial X$.

Proof. We assume that $\eta_1,\eta_2,\eta_3\in\partial X$: the case that some of them lie in X is dealt with in the same way but a bit simpler. Let $R_1\in\eta_1$, $R_2\in\eta_2$ and $R_3,R_4\in\eta_3$. Let $s\in S$ such that $\rho^\varepsilon_s(\eta_1,\eta_3)=\rho^\varepsilon_S(\eta_1,\eta_3)$. Set $c:=6\delta+2\delta f(\delta+1)$. We consider four points $R_1(i)$, $R_2(j)$, $R_3(i')$ and $R_4(j')$ for $i,j,i',j'\in\mathbb{R}$ such that
$$d(R_1(i),R_3(i'))<\infty,$$
$$d(R_3(i'),R_4(j'))<\infty \text{ and}$$
$$d(R_4(j'),R_2(j))<\infty.$$
Let P_{12} be an $R_1(i)$-$R_2(j)$ geodesic, P_{13} be an $R_1(i)$-$R_3(i')$ geodesic, P_{14} be an $R_1(i)$-$R_4(j')$ geodesic, P_{34} be an $R_3(i')$-$R_4(j')$ geodesic and P_{42} be an $R_4(j')$-$R_2(j)$ geodesic. Applying Lemma 3.4 twice, once to a geodesic triangle with sides P_{12}, P_{14} and P_{42} and once to a geodesic triangle with sides P_{13}, P_{34} and P_{14}, we obtain that P_{12} lies in the out- and in-ball of radius $2c$ around $P_{13}\cup P_{34}\cup P_{42}$. Thus, we obtain
$$d^{\leftrightarrow}(s,P_{12})+2c\geq\min\{d^{\leftrightarrow}(s,P_{13}),d^{\leftrightarrow}(s,P_{34}),d^{\leftrightarrow}(s,P_{42})\}.$$
Since R_3 and R_4 both lie in η_3, we have
$$\liminf\{d^{\leftrightarrow}(s,P)\mid k,\ell\to\infty, P\text{ is an }R_3(k)\text{-}R_4(\ell)\text{ geodesic}\}=\infty.$$
Thus, we obtain
$$\begin{aligned}\rho_s(\eta_1,\eta_2)+2c&\geq\min\{\rho_s(\eta_1,\eta_3),\rho_s(\eta_3,\eta_3),\rho_s(\eta_3,\eta_2)\}\\&=\min\{\rho_s(\eta_1,\eta_3),\rho_s(\eta_3,\eta_2)\}\end{aligned}$$
and hence
$$\begin{aligned}\rho_S(\eta_1,\eta_2)+2c&=\rho_s(\eta_1,\eta_2)+2c\\&\geq\min\{\rho_s(\eta_1,\eta_3),\rho_s(\eta_3,\eta_2)\}\\&\geq\min\{\rho_S(\eta_1,\eta_3),\rho_S(\eta_3,\eta_2)\}.\end{aligned}$$

The assertion now follows from the definition of ρ_S^ε. $\square$

A (pseudo-)semimetric d_a on $X \cup \partial X$ is a *visual (pseudo-)semimetric* with parameter $a > 1$ if there is $C > 0$ such that

$$\frac{1}{C} a^{-\rho_S(\eta,\mu)} \leq d_a(\eta,\mu) \leq C a^{-\rho_S(\eta,\mu)}$$

for all $\eta, \mu \in X \cup \partial X$.

Now we prove that $d_{S,\varepsilon}$ is a visual pseudo-semimetric. This proof is almost verbatim the same as for the case of metric spaces as in e. g. [2, Proposition III.H.3.21].

Theorem 13.2. *Let $\delta \geq 0$ and let X be a δ-hyperbolic geodesic semimetric space with finite base $S \subseteq X$ that satisfies (B1) and (B2). Let $\varepsilon > 0$ such that $\varepsilon' < \sqrt{2}$ holds for $\varepsilon' := e^{2\varepsilon(6\delta + 2\delta f(\delta+1))}$. Then $d_{S,\varepsilon}$ is a visual pseudo-semimetric on $X \cup \partial X$ that satisfies*

$$(3 - 2\varepsilon')\rho_S^\varepsilon(\eta,\mu) \leq d_{S,\varepsilon}(\eta,\mu) \leq \rho_S^\varepsilon(\eta,\mu)$$

for all $\eta, \mu \in X \cup \partial X$.

Furthermore, if X is strongly hyperbolic, then $d_{S,\varepsilon}$ is a visual semimetric on $X \cup \partial X$.

Proof. First, we note that the inequality $d_{S,\varepsilon}(\eta,\mu) \leq \rho_S^\varepsilon(\eta,\mu)$ holds trivially for all $\eta, \mu \in X \cup \partial X$ and that the definition of $d_{S,\varepsilon}$ implies directly that it is a pseudo-semimetric. We prove by induction on the length of chains $(\eta_0, \ldots, \eta_n)$ that

$$(5) \qquad (3 - 2\varepsilon')\rho_S^\varepsilon(\eta_0,\eta_n) \leq \sum_{i=0}^{n-1} \rho_S^\varepsilon(\eta_i,\eta_{i+1}).$$

We set

$$S(m) := \sum_{i=0}^{m-1} \rho_S^\varepsilon(\eta_i,\eta_{i+1}).$$

Since $\varepsilon' > 1$, we note that (5) holds trivially if $n = 1$ or $S(n) \geq 3 - 2\varepsilon'$. So let us assume that $n \geq 2$ and $S(n) < 3 - 2\varepsilon'$. Let $m \leq n$ be largest such that $S(m) \leq S(n)/2$. Then we have

$$\sum_{i=m+1}^{n-1} \rho_S^\varepsilon(\eta_i,\eta_{i+1}) = S(n) - S(m+1) < S(n)/2.$$

By induction we have

$$\rho_S^\varepsilon(\eta_0,\eta_m) \leq \frac{S(n)}{2(3-2\varepsilon')} \qquad \text{and} \qquad \rho_S^\varepsilon(\eta_{m+1},\eta_n) \leq \frac{S(n)}{2(3-2\varepsilon')}.$$

Trivially, we also have $\rho_S^\varepsilon(\eta_m,\eta_{m+1}) \leq S(n)$. Lemma 13.1 applied twice implies

$$\begin{aligned}\rho_S^\varepsilon(\eta_0,\eta_n) &\leq (\varepsilon')^2 \max\{\rho_S^\varepsilon(\eta_0,\eta_m), \rho_S^\varepsilon(\eta_m,\eta_{m+1}), \rho_S^\varepsilon(\eta_{m+1},\eta_n)\} \\ &\leq (\varepsilon')^2 S(n) \max\left\{1, \frac{1}{2(3-2\varepsilon')}\right\}.\end{aligned}$$

Since $(\varepsilon')^2/2 \leq 1$ and $(\varepsilon')^2(3 - 2\varepsilon') \leq 1$ holds for all $1 \leq \varepsilon' \leq \sqrt{2}$, we immediately obtain (5). Thus, we proved the inequality in the assertion and hence $d_{S,\varepsilon}$ is a visual pseudo-semimetric.

Now let X be strongly hyperbolic. It suffices to show $d_{S,\varepsilon}(\eta,\mu) \neq 0$ for all distinct $\eta, \mu \in \partial X$. This follows immediately, if we prove $\rho_S(\eta,\mu) \neq \infty$. So let $R \in \eta$ and $Q \in \mu$. These are (γ, c)-quasi-geodesics for some $\gamma \geq 1$ and $c \geq 0$.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that X satisfies (B1) and (B2) for this function. Let κ be the constant such that geodesic stability holds with respect to this constant for (γ, c)-quasi-geodesics, cp. Corollary 7.3. Let P_1 be an R-Q geodesic. Let x_1 be its starting point and y_1 be its end point. Let P_2 be another R-Q geodesic and let x_2 be its starting point and y_2 be its end point. Let Q' be the directed subpath of Q between y_1 and y_2. Let R_1 be the directed subpath of R between x_1 and x_2 and let R_2 be a geodesic with the same starting and end points. By strong hyperbolicity and geodesic stability, Q' lies in the out-ball and in the in-ball of radius $2\delta + \kappa$ around P_1, P_2 and R_2.

Let $k \geq 0$ such that P_1 lies in $\mathcal{B}^+_k(S) \cup \mathcal{B}^-_k(S)$. Let a be on Q' outside of $\mathcal{B}^+_{k+C}(S) \cup \mathcal{B}^-_{k+C}(S)$ with $C := (2\delta + \kappa)f(\delta + 1) + f(\delta)$. Then a does not lie in $\mathcal{B}^+_{2\delta+\kappa}(P_1) \cup \mathcal{B}^-_{2\delta+\kappa}(P_1)$ by Proposition 5.4 (ii). Set

$$\ell := \max(\{d(a,s) \mid s \in S, d(a,s) < \infty\} \cup \{d(s,a) \mid s \in S, d(s,a) < \infty\}).$$

If P_2 was chosen outside of $\mathcal{B}^+_{\ell+C}(S) \cup \mathcal{B}^-_{\ell+C}(S)$, then a lies neither in $\mathcal{B}^+_{2\delta+\kappa}(P_2)$ nor in $\mathcal{B}^-_{2\delta+\kappa}(P_2)$. So a lies in $\mathcal{B}^+_{2\delta+\kappa}(R_2) \cap \mathcal{B}^-_{2\delta+\kappa}(R_2)$ and hence in $\mathcal{B}^+_{2\delta+2\kappa}(R_1) \cap \mathcal{B}^-_{2\delta+2\kappa}(R_1)$. Thus, we have $Q \approx R$ and $\eta = \mu$. Hence, $d_{S,\varepsilon}$ is a semimetric in this case. □

We note that the digraph of Example 12.2 shows that, in the case of hyperbolicity, we cannot expect $d_{S,\varepsilon}$ to be a semimetric. However, we will prove that in the special case that X is a one-ended locally finite digraph, we will prove that the pseudo-semimetric is in fact a semimetric (Proposition 15.6). We will look more closely to the situation that $d_{S,\varepsilon}(\eta, \mu) = 0$ for distinct quasi-geodesic boundary points η and μ in Proposition 14.1.

Let us now relate the topologies that we obtain from the pseudo-semimetric $d_{S,\varepsilon}$ with the topologies from Section 12 in various situations. For that, we call a subset Y of a semimetric space X *independent* if $d(y, z) = \infty$ for all $y, z \in Z$.

Proposition 13.3. *Let $\delta \geq 0$ and let X be a δ-hyperbolic geodesic semimetric space with finite base $S \subseteq V(D)$ that satisfies (B1) and (B2) such that either*

(i) $|S| = 1$ *or*

(ii) *for no $r \in \mathbb{R}$ and $x \in X$ the balls $\mathcal{B}^+_r(x)$ and $\mathcal{B}^-_r(x)$ contain an infinite independent point set.*

Let $\varepsilon > 0$ such that $\varepsilon' < \sqrt{2}$ holds for $\varepsilon' := e^{2\varepsilon(6\delta+2\delta f(\delta+1))}$. Then the forward and backward topologies induced by $d_{S,\varepsilon}$ coincide with those of Section 12.

Proof. It suffices to prove that the neighbourhoods around quasi-geodesic boundary points ω coincide. Furthermore, both types of topology can be treated analogously; so we just consider the forward topologies.

Let $f\colon \mathbb{R} \to \mathbb{R}$ such that X satisfies (B1) and (B2) for that function. For $\gamma \geq 1$ and $c \geq 0$, let $\kappa(\delta, \gamma, c, f) \geq 0$ be the constant for geodesic stability, cp. Corollary 7.3. Let $D > 0$ and let $r \geq 0$ such that

$$\varepsilon r > \ln \frac{3 - 2\varepsilon'}{D}.$$

Let $y \in C^+(\eta, s, r)$ for some $\eta \in \partial^+(\omega)$ and $s \in S$. Then there is a quasi-geodesic $R \in \omega$ such that for all x on R there is an x-y geodesic outside of $\mathcal{B}^+_r(s) \cup \mathcal{B}^-_r(s)$ and hence $\rho_s(\eta, y) \geq r$. Now if we take the finite intersection

$$C := \bigcap \{C^+(\omega, s, r) \mid s \in S\},$$

then we obtain $\rho_S(\omega, y) \geq r$ for all $y \in C$. So Theorem 13.2 implies

$$(3-2\varepsilon')\rho_S^\varepsilon(\eta, y) \leq (3-2\varepsilon')e^{-\varepsilon r} < D.$$

Hence the out-ball of radius D around ω with respect to the pseudo-semimetric $d_{S,\varepsilon}$ contains C and thus also

$$\bigcap\{C_\partial^+(\omega, s, r) \mid s \in S\}.$$

Since this is true for every ω, this out-ball contains an f-neighbourhood with respect to the topology of Section 12.

To show that every $C_\partial^+(\omega, x, r)$ such that ω contains an anti-ray contains some forward neighbourhood of ω with respect to $d_{S,\varepsilon}$ it suffices to show that there exists some $k \in \mathbb{R}$ with

$$\mathcal{B}_r^+(x) \cup \mathcal{B}_r^-(x) \subseteq \mathcal{B}_k^+(S) \cup \mathcal{B}_k^-(S).$$

Let us suppose that this is false. Since S is a base, there is $s \in S$ with $d^\leftrightarrow(s, x) < \infty$. We may assume $d(s, x) < \infty$; the other case follows with a symmetric argument. So we have $\mathcal{B}_r^+(x) \subseteq \mathcal{B}_{r+d(s,x)}^+(s)$. For every $i \in \mathbb{N}$, let $y_i \in \mathcal{B}_r^-(x)$ with $d^\leftrightarrow(S, y_i) \geq i$. By the pigeonhole principle, there is some $s' \in S$ with either $d(s', y_i) < \infty$ or $d(y_i, s) < \infty$ for infinitely many $i \in \mathbb{N}$. If there were infinitely many $i \in \mathbb{N}$ with $d(s', y_i) < \infty$, then we can consider a geodesic triangle with x, y_i and s' as end points and apply Proposition 3.3 (i) to bound $d(s', y_i)$ in terms of $d(y_i, x)$ and $d(s', x)$. This contradicts $d^\leftrightarrow(S, y_i) \geq i$. Hence, we may assume that $d(y_i, s) < \infty$ for infinitely many $i \in \mathbb{N}$. Throwing all other out of the sequence, we may assume that $d(y_i, s) < \infty$ holds for all $i \in \mathbb{N}$.

In the case that (i) holds, we directly obtain a contradiction to the choice of the points y_i: by applying Proposition 3.3 (i) to geodesic triangles with x, s and y_i as end points we can bound $d(y_i, s)$ in terms of $d(s, x)$, $d(y_i, x)$ and f, but we also know $d^\leftrightarrow(y_i, s) \leq d(y_i, s) \to \infty$ for $i \to \infty$. This is not possible. So let us now assume $|S| > 1$ and hence that (ii) holds.

Applying Proposition 3.3 (i) to geodesic triangles with x, y_i and y_j as end points shows that $d^\leftrightarrow(y_i, y_j)$ is bounded by $2rf(\delta+1)$ if it is not ∞. If $d^\leftrightarrow(y_i, y_j) \neq \infty$, then a geodesic triangle with end points s', y_i and y_j shows that for fixed $i \in \mathbb{N}$ we obtain that $d(s', y_j)$ is bounded in terms of $d(s', y_i)$, $d^\leftrightarrow(y_i, y_j)$ and f by Proposition 3.3 (i). Thus, for fixed $i \in \mathbb{N}$ there are only finitely many $j \in \mathbb{N}$ with $d^\leftrightarrow(y_i, y_j) < \infty$. Now we can easily find an infinite independent subset of $\{y_i \mid i \in \mathbb{N}\}$ which contradicts (ii). □

Note that it follows from Proposition 13.3 that the forward and backward topologies induced by $d_{S,\varepsilon}$ do not depend on the particular base for those semimetric spaces satisfying the assumptions of that proposition. These assumptions are satisfied by some natural classes of semimetric spaces, e. g. hyperbolic digraphs of bounded degree and hence right cancellative hyperbolic semigroups, cp. Section 16: they satisfy (ii) of Proposition 13.3. On the other side, hyperbolic monoids that have a finite generating set whose Cayley digraph satisfies (B2) satisfy (i) of Proposition 13.3.

Let us briefly discuss the situation that the hyperbolic geodesic semimetric space is not finitely based. It is natural if we just replace in the definition of ρ_S the minimum by the infimum over an infinite base. Then it is easy to verify that $d_{S,\varepsilon}$ is a pseudo-semimetric, too. However, the topologies defined by $d_{S,\varepsilon}$ do not coincide with the topologies from Section 12 as the following example shows.

Example 13.4. Let G be a 3-regular tree and let $R = \ldots x_{-1}x_0x_1\ldots$ be a double ray in G, i. e. all x_i are different vertices and x_i and x_{i+1} are adjacent for all $i \in \mathbb{Z}$. We orient the edges on R from x_i to x_{i+1} and the other edges incident with vertices on R away from R, i. e. if y is a neighbour of x_i, we orient the edge from x_i to y. All other edges will be replaced by two oppositely directed edges, i. e. an edge between y, z that do not lie on R will be replaced by one edge directed from y to z and one edge directed from z to y. The resulting digraph D is clearly hyperbolic and it is easy to see that it has no finite base.

The vertex set $S := \{x_i \mid i \leq 0\}$ is an infinite base of D. Let us consider the semimetric $d_{S,\varepsilon}$ where we replaced the definition of ρ_S by

$$\rho_S(\eta, \mu) := \inf\{\rho_s(\eta, \mu) \mid s \in S\}.$$

Let $\omega \in \partial D$ such that $\ldots x_{-1}x_0 \in \omega$. Then it follows that $\rho_S(\omega, \eta) = 0$ for all $\eta \in X \cup \partial X$ with $\eta \neq \omega$. So we have $d_{S,\varepsilon}(\omega, \eta) = 1$. Thus, ω has a trivial f-neighbourhood, while the topology $\mathcal{O}_f$ from Section 12 has no trivial neighbourhood of ω.

14. Properties of $X \cup \partial X$

In the situation of hyperbolic geodesic spaces, the boundaries are complete metric spaces and if the spaces are proper, the boundaries are compact, see e. g. [2, Proposition III.H.3.7]. In the situation of semimetric spaces, we get at least for digraph that local finiteness implies f- and b-completeness, see Theorem 14.4. But before we proceed to that result, we first discuss situations with $d_{S,\varepsilon}(\eta, \mu) = 0$ and then we prove the existence of geodesic rays, anti-rays and double rays with almost prescribed starting and end points.

We have already seen in Example 12.2 that $d_{S,\varepsilon}(\eta, \mu) = 0$ is possible, see the discussion after Theorem 13.2. Let us discuss this situation in a bit more detail.

Proposition 14.1. *Let $\delta \geq 0$ and let X be a δ-hyperbolic geodesic semimetric space with finite base $S \subseteq X$ that satisfies (B1) and (B2). Let $\varepsilon > 0$ such that $\varepsilon' < \sqrt{2}$ holds for $\varepsilon' := e^{2\varepsilon(6\delta + 2\delta f(\delta+1))}$. Let $\eta, \mu, \omega \in X \cup \partial X$. Then the following hold.*

(i) *If $d_{S,\varepsilon}(\eta, \mu) = 0$ and $\eta \neq \mu$, then $\eta, \mu \in \partial X$ and $\eta \leq \mu$.*
(ii) *If $d_{S,\varepsilon}(\eta, \mu) = 0 = d_{S,\varepsilon}(\mu, \eta)$, then $\eta = \mu$.*
(iii) *If $d_{S,\varepsilon}(\eta, \mu) = 0$ and $\eta \neq \mu$, then either η contains only rays and μ contains only anti-rays or η contains only anti-rays and μ contains only rays.*
(iv) *If $d_{S,\varepsilon}(\eta, \mu) = 0$ and $d_{S,\varepsilon}(\mu, \omega) = 0$, then we have either $\eta = \mu$ or $\mu = \omega$.*

Proof. Let $\eta, \mu \in X \cup \partial X$ with $d_{S,\varepsilon}(\eta, \mu) = 0$. By definition of $d_{S,\varepsilon}$, it is only possible to have $d_{S,\varepsilon}(\eta, \mu) = 0$ if η and μ lie in ∂X. Let $R \in \eta$ and $Q \in \mu$. They are (γ, c)-quasi-geodesics for some $\gamma \geq 1$ and $c \geq 0$. Let X satisfy (B1) and (B2) for the function $f : \mathbb{R} \to \mathbb{R}$. Let κ be the constant for geodesic stability with respect to δ, γ, c and f, cp. Corollary 7.3. Let P_1, P_2 be two R-Q geodesics with starting points x_1, x_2 and end points y_1, y_2, respectively. Let R_1 be the directed subpath of R between x_1 and x_2 and let R_2 be geodesic with the same starting and end points. Let Q_1 be the directed subpath of Q between y_1 and y_2 and let Q_2 be geodesic with the same starting and end points. Let $i \neq j \in \{1, 2\}$ such that x_i is the starting point of R_1 and x_j is its end point. Then there exists an x_i-y_j geodesic P. By Lemma 3.4, we know that P lies in the out- and in-balls of radius $K := 6\delta + 2\delta f(\delta + 1)$ around $R_2 \cup P_j$.

Let us first consider the case that y_i is the starting point and y_j is the end point of Q_2. Then P lies in the out- and in-ball of radius K around $P_i \cup Q$ by Lemma 3.4. Since $d_{S,\varepsilon}(\eta,\mu)=0$, we may assume that we have chosen P_2 such that there is some point a on P with $d^{\leftrightarrow}(P_1,a)>K$ and $d^{\leftrightarrow}(P_2,a)>K$. Applying geodesic stability, there exists b_R on R_1 and b_Q on Q_1 with $d(b_R,b_Q)\leq 2\kappa+2K$. Since $d_{S,\varepsilon}(\eta,\mu)=0$, we may choose, for every $k\in\mathbb{N}$, the directed paths R, Q, P_1 and P_2 such that P_1, P_2, R_1 and Q_1 lie outside of $\mathcal{B}^+_k(S)\cup\mathcal{B}^-_k(S)$. Let T_k be an R_1-Q_1 geodesic. Let us suppose that there exists $\ell\in\mathbb{N}$ such that $\mathcal{B}^+_\ell(x)\cup\mathcal{B}^-_\ell(x)$ meets all T_k. Then either $\mathcal{B}^+_{\ell+2K+2\kappa}(x)$ or $\mathcal{B}^-_{\ell+2K+2\kappa}(x)$ contains a point of the directed subpaths Q_1 or R_1 of Q or R that we used for the existence of T_k, which is a contradiction to (B1) or (B2). Thus, we have $R\leq Q$ and hence $\eta\leq\mu$ in this case.

Let us now consider the case that y_j is the starting point and y_i is the end point of Q_2. Then Q_1 lies in

$$\mathcal{B}^+_{K+\delta+\kappa}(R_2\cup P_j)\cup\mathcal{B}^-_{\delta+\kappa}(P_1).$$

So if we choose P_2 and a on Q_1 with $d^{\leftrightarrow}(P_1,a)\geq\kappa+\delta$ and $d(P_2,a)>K+\delta+\kappa$, then there is a directed R_1-Q_1 path of length at most $2\kappa+K+\delta$. As in the previous case, we obtain $R\leq Q$ and $\eta\leq\mu$. This finishes the proof of (i).

Finally, (ii) is an immediate consequence of (i) and (iii) and (iv) directly follow from (i) and Proposition 10.3. □

We note that there may be three distinct quasi-geodesic boundary points η,μ,ω with $d_{S,\varepsilon}(\omega,\eta)=0$ and $d_{S,\varepsilon}(\omega,\mu)=0$ as the following short example shows.

Example 14.2. Let D be the digraph that consists of a ray $R=x_0x_1\ldots$ and two anti-rays $Q_1=\ldots y_{-1}y_0$ and $Q_2\ldots z_{-1}z_0$ with edges from x_i to y_i and z_i for all $i\in\mathbb{N}$. The resulting digraph D is hyperbolic and R, Q_1 and Q_2 lies in distinct quasi-geodesic boundary points ω, η and μ, respectively. The vertex x_0 is a base of D. Since $\omega\leq\eta$ and $\omega\leq\mu$, we have $d_{\{x_0\},\varepsilon}(\omega,\eta)=0$ and $d_{\{x_0\},\varepsilon}(\omega,\mu)=0$.

In a digraph D, we call $\ldots x_{-1}x_0x_1\ldots$ a *double ray* if x_ix_{i+1} is an edge of D. It is *geodesic* if every finite directed subpath is geodesic.

Proposition 14.3. *Let $\delta\geq 0$ and let D be a δ-hyperbolic digraph that satisfies (B1) and (B2) with finite base S. Let $\varepsilon>0$ such that $e^{2\varepsilon(6\delta+2\delta f(\delta+1))}<\sqrt{2}$. Then the following hold.*

(i) *If every vertex of D has finite out-degree, then for every $x\in V(D)$ and $\eta\in\partial D$ with $d_{S,\varepsilon}(x,\eta)<\infty$, then there is a geodesic ray R starting at x such that $R\in\mu$ for some $\mu\in\partial D$ with $\mu\leq\eta$.*
(ii) *If every vertex of D has finite in-degree, then for every $x\in V(D)$ and $\eta\in\partial D$ with $d_{S,\varepsilon}(\eta,x)<\infty$, then there is a geodesic anti-ray R ending at x such that $R\in\mu$ for some $\mu\in\partial D$ with $\eta\leq\mu$.*
(iii) *If D is locally finite, then for all $\eta,\mu\in\partial D$ with $0<d_{S,\varepsilon}(\eta,\mu)<\infty$ there exists $\eta',\mu'\in\partial D$ with $\eta\leq\eta'$ and $\mu'\leq\mu$ such that there is a geodesic η'-μ' double ray.*

Proof. Let us prove (i). By definition of $d_{S,\varepsilon}$, there exists $Q\in\eta$ such that for every y on Q there is an x-y geodesic in D. Let $Q=y_0y_1\ldots$ if Q is a ray and $Q=\ldots y_1y_0$ if Q is an anti-ray. Then there is a geodesic ray R starting at x such that each of its finite directed subpaths starting at x lie in infinitely many of these x-y_i geodesics. Using thin geodesic triangles with end vertices x, y_0 and y_i we obtain for large i

that almost all of the directed subpath of Q between y_0 and y_i lies in the out-ball of radius δ around the x-y_i geodesic. Thus, all but a finite directed subpath of Q lies in the out-ball of radius δ around R. This shows (i).

With a completely symmetric argument, we obtain (ii).

To prove (iii), let $R \in \eta$ and $Q \in \mu$ such that either $R = x_0x_1\ldots$ or $R = \ldots x_1x_0$ and either $Q = y_0y_1\ldots$ or $Q = \ldots y_1y_0$. By Theorem 13.2 and using hyperbolicity, there exists $M \geq 0$ such that for $i, j \to \infty$ all x_i-y_j geodesics have a vertex of $\mathcal{B}^+_M(S) \cup \mathcal{B}^-_M(S)$. Similar as before, we obtain a geodesic double ray P such that each of its inner directed subpaths lies in infinitely many of these x_i-y_j geodesics with the property that neither the involved indices i nor the involved indices j are bounded. Let P_1 be an anti-ray and P_2 be a ray in P. Then hyperbolicity implies $R \leq P_1$ and $P_2 \leq Q$, which implies the assertion. □

Now we are ready to prove the main result of this section.

Theorem 14.4. *Let $\delta \geq 0$ and let D be a δ-hyperbolic digraph that satisfies (B1) and (B2) for the function $f\colon \mathbb{R} \to \mathbb{R}$ with finite base S. Let $\varepsilon > 0$ such that $e^{2\varepsilon(6\delta+2\delta f(\delta+1))} < \sqrt{2}$. Then the following hold.*

(i) *If every vertex of D has finite out-degree, then $D \cup \partial D$ is sequentially f-compact.*
(ii) *If every vertex of D has finite in-degree, then $D \cup \partial D$ is sequentially b-compact.*

Proof. Let every vertex of D have finite out-degree and let $(x_i)_{i\in\mathbb{N}}$ be a sequence in $V(D) \cup \partial D$ with $d_{S,\varepsilon}(x_i, x_j) < \infty$ for all $i < j$. If $x_0 \in \partial D$, then by definition of $d_{S,\varepsilon}$, there exists a vertex x_0' on an element of x_0 with $d_{S,\varepsilon}(x_0', x_1) < \infty$. Thus, we have $d_{S,\varepsilon}(x_0', x_i) < \infty$ for all $i \in \mathbb{N}$ with $i \neq 0$. Since b-convergence of sequences does not depend on the first element of the sequence, we may assume that x_0 is a vertex of D.

For every $i \in \mathbb{N}$, if $x_i \in V(D)$, then let P_i be an x_0-x_i geodesic. If $x_i \in \partial D$, then there exists a geodesic ray P_i starting at x_0 that lies in some $x_i' \in \partial D$ with $x_i' \leq x_i$ by Proposition 14.3 (i). These directed paths and rays define a geodesic ray $R = v_0v_1\ldots$ such that infinitely many P_i share the first edge of R among which infinitely many share the next edge of R and so on. By switching to a subsequence of $(x_i)_{i\in\mathbb{N}}$, if necessary, we may assume that P_i and R have their first i edges in common. Let $\eta \in \partial D$ with $R \in \eta$. We shall show that $(x_i)_{i\in\mathbb{N}}$ b-converges to η.

For each $i \in \mathbb{N}$, let u_i be the first vertex of P_i such that the next vertex on P_i does not lie in $\mathcal{B}^+_\delta(R)$, if it exists, and, if $P_i \subseteq \mathcal{B}^+_\delta(R)$, let $u_i = x_i$ if $x_i \in V(D)$ and let u_i be on P_i with $d(x_0, u_i) \geq i$, otherwise. Let u_i' be on R with $d(u_i', u_i) \leq \delta$. If x_i is a vertex, set $y_i := x_i$. If $x_i \in \partial D$, then let y_i be a vertex on $u_iP_ix_i$. Then we have

$$d_{S,\varepsilon}(y_i, x_j) \leq d_{S,\varepsilon}(y_i, x_i) + d_{S,\varepsilon}(x_i, x_j) < \infty$$

for all $j > i$. Hence there is a y_i-x_j geodesic P_{ij} if x_j is a vertex. If $x_j \in \partial D$, then there is a geodesic ray with starting vertex y_i that lies in a quasi-geodesic boundary point $x_j' \leq x_j$ by Proposition 14.3 and hence we may assume that we have chosen y_j such that there is a y_i-y_j geodesic P_{ij}.

Let $r > 0$. We consider the set $C_\partial^-(\eta, x_0, r)$. Set

$$\begin{aligned} k &:= 6\delta + 2\delta f(\delta+1),\\ \ell_1 &:= f(\delta+1)r + f(\delta),\\ \ell_2 &:= (\ell_1+\delta)f(\delta+1)\\ \ell_3 &:= \ell_2 + k. \end{aligned}$$

Let $i \in \mathbb{N}$ such that $d(x_0, u_i) > \ell_3 + \delta$ and set $m := d(x_0, y_i) + 3\delta + 1$. Let $j \in \mathbb{N}$ such that

$$d(x_0, u_j) > d(x_0, y_i) + \delta + 2$$

and x_0Rv_m lies in $\mathcal{B}_\delta^+(P_j)$. Note that this holds for all but finitely many j by the choice of our sequence and the choices of u_j and y_j. Let z_1 be on P_{ij} with

$$d(x_0, z_1) = d(x_0, y_i) + \delta + 1.$$

By hyperbolicity for the geodesic triangle with end vertices x_0, y_i and y_j and sides P_iy_i, P_{ij} and P_jy_j, there exists a vertex z_2 on P_j with $d(z_1, z_2) \leq \delta$. By the choice of z_1, of j and of m, we know that $d(z_2P_j, v_m) \leq \delta$ and in particular $d(z_2, v_m) < \infty$.

Let x on v_mR. Then there is a y_i-x geodesic P. Let Q_1 be a u_i'-u_i geodesic and Q_2 a u_i'-y_i geodesic. We shall show that P lies in $C_\partial^-(\eta, x_0, r)$.

Since $d(x_0, u_i) > \ell_3 + \delta$, we have $d(x_0, Q_1) > \ell_3$. If $d(x_0, Q_2) \leq \ell_2$, then the vertex verifying this distance lies in $\mathcal{B}_k^-(Q_1 \cup u_iP_iy_i)$ by Lemma 3.4 and thus, we find a vertex on Q_1 or $u_iP_iy_i$ that has distance at most $\ell_2 + k = \ell_3$ from x_0. Since this is impossible, we have $d(x_0, Q_2) > \ell_2$. Let z be on P. Then there exists a vertex y either on Q_2 with $d(y, z) \leq \delta$ or on $u_i'Rx$ with $d(z, y) \leq \delta$. In the latter case, we directly obtain $d(x_0, z) > \ell_1$ and in the first case, we apply Proposition 3.3 (i) and obtain $d(x_0, z) > \ell_1$ as well. So P lies outside of $\mathcal{B}_{\ell_1}^+(x_0)$. Finally, Proposition 3.3 (ii) implies $d(P, x_0) > r$ and thus we have shown that P lies in $C_\partial^-(\eta, x_0, r)$. This implies that $(x_i)_{i\in\mathbb{N}}$ b-converges to η.

By an analogous argument, we obtain (ii). □

If $D \cup \partial D$ in Theorem 14.4 is a semimetric space, then we obtain the following corollary of Theorem 14.4 by using Proposition 2.1.

Corollary 14.5. *Let $\delta \geq 0$ and let D be a δ-hyperbolic digraph that satisfies (B1) and (B2) for the function $f\colon \mathbb{R} \to \mathbb{R}$ and that has a finite base S. Let $\varepsilon > 0$ such that $e^{2\varepsilon(6\delta+2\delta f(\delta+1))} < \sqrt{2}$. Then the following hold.*

(i) *If every vertex of D has finite out-degree and $D \cup \partial D$ is a semimetric space, then $D \cup \partial D$ is f-complete.*
(ii) *If every vertex of D has finite in-degree and $D \cup \partial D$ is a semimetric space, then $D \cup \partial D$ is b-complete.* □

15. The size of the boundary

In the case of hyperbolic spaces, those hyperbolic boundary points that belong to a common end of the space form a connected set, see e. g. [14, Proposition 7.5.17], which immediately implies that an end with at least two hyperbolic boundary points contains continuum many of them. We restrict ourselves to the case of digraphs here, since we defined ends only for them and not for general semimetric spaces. We cannot hope to prove that the quasi-geodesic boundary is connected, since our two topologies make if very hard to ask for this: even a digraph is far from being connected in the topological sense, since e. g. a digraph on two vertices with a unique edge has the following partition into two open sets: one set is the end vertex of the edge and the other set consists of the starting vertex of the edge and the inner points of the edge. Thus, we consider a different notion in our situation, which we will call semiconnectednes and that basically asks that the sets shall satisfy the connectedness condition with respect to both topologies simultaneously; see below for details.

As a first step in understanding the relations between the geodesic boundary and the ends better, we prove the following result.

Proposition 15.1. *Let $\delta \geq 0$ and let D be a locally finite δ-hyperbolic digraph that satisfies (B1) and (B2) and that has a finite base S. Let $\varepsilon > 0$ such that $\varepsilon' < \sqrt{2}$ holds for $\varepsilon' := e^{2\varepsilon(6\delta+2\delta f(\delta+1))}$. Let ω_1 and ω_2 be ends of D with $\omega_1 \preccurlyeq \omega_2$ and let $\eta, \mu \in \partial D$ with $\eta \subseteq \omega_1$ and $\mu \subseteq \omega_2$. Then we have $d_{S,\varepsilon}(\eta, \mu) < \infty$.*

In particular, if $\omega_1 = \omega_2$, then we have $d_{S,\varepsilon}(\eta, \mu) < \infty$ and $d_{S,\varepsilon}(\mu, \eta) < \infty$.

Proof. Let $R \in \eta$ and $Q \in \mu$. For every $n \in \mathbb{N}$ there exists a directed R-Q path P_n outside of $\mathcal{B}_n^+(S) \cup \mathcal{B}_n^-(S)$ by Proposition 11.1. Let P_n' be a geodesic with the same starting and end vertex as P_n. It follows from the definition of $d_{S,\varepsilon}$ that $d_{S,\varepsilon}(\eta, \mu) < \infty$.

The additional statement follows trivially, since $\omega_1 = \omega_2$ implies $\omega_2 \preccurlyeq \omega_1$. □

We call a pseudo-semimetric space X *semiconnected* if there is no partition $\{U, V\}$ of X such that U and V are open with respect to $\mathcal{O}_b$ and $\mathcal{O}_f$. A *semiconnected component* is a maximal semiconnected subset of X. It is easy to see that distinct semiconnected components are disjoint and hence the semiconnected components form a partition of the pseudo-semimetric space.

Lemma 15.2. *Let $\delta \geq 0$ and let D be a locally finite δ-hyperbolic digraph that satisfies (B1) and (B2) and that has a finite base S. Let $\varepsilon > 0$ such that $\varepsilon' < \sqrt{2}$ holds for $\varepsilon' := e^{2\varepsilon(6\delta+2\delta f(\delta+1))}$. Let $\varphi\colon \partial D \to \Omega D$ be the canonical map with $\eta \subseteq \varphi(\eta)$ for all $\eta \in \partial D$. Let $\omega \in \Omega D$ and let A, B be two subsets of ∂D with $\varphi^{-1}(\omega) \subseteq A \cup B$ and $A \cap B \cap \varphi^{-1}(\omega) = \emptyset$ such that A is closed in $\mathcal{O}_f$ and B is closed in $\mathcal{O}_b$. If $(D \cup \partial D, d_{S,\varepsilon})$ is a semimetric space, then*

$$d_{S,\varepsilon}(A \cap \varphi^{-1}(\omega), B \cap \varphi^{-1}(\omega)) > 0.$$

Proof. Let us suppose that $d_{S,\varepsilon}(A \cap \varphi^{-1}(\omega), B \cap \varphi^{-1}(\omega)) = 0$. By Proposition 15.1 we know that $d_{S,\varepsilon}(a_i, a_j) < \infty$ and $d_{S,\varepsilon}(b_i, b_j) < \infty$ for all $i, j \in \mathbb{N}$. Then there are sequences $(a_i)_{i\in\mathbb{N}}$, $(b_i)_{i\in\mathbb{N}}$ in $A \cap \varphi^{-1}(\omega)$, in $B \cap \varphi^{-1}(\omega)$, respectively, such that $d_{S,\varepsilon}(a_i, b_i) \to 0$ for $i \to \infty$. By Theorem 14.4, there exists $a \in \partial D$ such that $(a_i)_{i\in\mathbb{N}}$ has a subsequence that f-converges to a. By replacing $(a_i)_{i\in\mathbb{N}}$ by this subsequence, we may assume that $(a_i)_{i\in\mathbb{N}}$ f-converges to a. But then we also replace $(b_i)_{i\in\mathbb{N}}$ by a subsequence such that $d_{S,\varepsilon}(a_i, b_i) \to 0$ for $i \to \infty$ is still satisfied. Applying Theorem 14.4 once more, there is a subsequence of $(b_i)i \in \mathbb{N}$ that b-converges to some $b \in \partial D$. Again, we switch to subsequences to obtain that $(b_i)_{i\in\mathbb{N}}$ b-converges to b and that $d_{S,\varepsilon}(a_i, b_i) \to 0$ for $i \to \infty$. Thus, we obtain

$$d_{S,\varepsilon}(a, b) \leq d_{S,\varepsilon}(a, a_i) + d_{S,\varepsilon}(a_i, b_i) + d_{S,\varepsilon}(b_i, b) \to 0 \text{ for } i \to \infty.$$

Since $d_{S,\varepsilon}$ is a semimetric, we conclude $a = b$. We have $a \in A$ and $b \in B$ since A is closed in $\mathcal{O}_f$ and B is closed in $\mathcal{O}_b$. Thus, $A \cap B$ is not empty, which contradicts the assumptions. Thus, the assertion follows. □

Theorem 15.3. *Let $\delta \geq 0$ and let D be a locally finite δ-hyperbolic digraph that satisfies (B1) and (B2) and that has a finite base S. Let $\varepsilon > 0$ such that $\varepsilon' < \sqrt{2}$ holds for $\varepsilon' := e^{2\varepsilon(6\delta+2\delta f(\delta+1))}$. Let $\varphi\colon \partial D \to \Omega D$ be the canonical map with $\eta \subseteq \varphi(\eta)$ for all $\eta \in \partial D$. If $(D \cup \partial D, d_{S,\varepsilon})$ is a semimetric space, then there is, for every $\omega \in \Omega_D$, a unique semiconnected component containing $\varphi^{-1}(\omega)$.*

Proof. Let $\omega \in \Omega D$ and suppose that $\varphi^{-1}(\omega)$ does not lie in a unique semiconnected component. Let $X \subseteq \partial D$ be the union of all semiconnected components that meet $\varphi^{-1}(\omega)$. Since X is not semiconnected, there is a partition $\{A', B'\}$ of X such that both A' and B' are open in $\mathcal{O}_f$ and in $\mathcal{O}_b$. Then their complements A and B in ∂D are closed with respect to both topologies, cover $\varphi^{-1}(\omega)$ and are disjoint inside $\varphi^{-1}(\omega)$. Thus, we can apply Lemma 15.2 and obtain

$$d_{S,\varepsilon}(A \cap \varphi^{-1}(\omega), B \cap \varphi^{-1}(\omega)) > 0. \tag{6}$$

Let $\eta \in A \cap \varphi^{-1}(\omega)$ and $\mu \in B \cap \varphi^{-1}(\omega)$. Since $d_{S,\varepsilon}$ is a semimetric, Proposition 14.3 (iii) implies the existence of a geodesic η-μ double ray R. Let R_1 be an anti-subray of R that lies in η and let R_2 be a subray of R that lies in μ. Since η and μ belong to the same end, there exists, for every $n \in \mathbb{N}$, a directed R_1-R_2 path P_n that lies outside of $\mathcal{B}^+_n(S) \cup \mathcal{B}^-_n(S)$. For every $n > N$ there exists a vertex x_n on P_n that lies outside of A' and B'. Moreover, we may choose x_n such that

$$d_{S,\varepsilon}(A \cap \varphi^{-1}(\omega), x_n) > d_{S,\varepsilon}(A \cap \varphi^{-1}(\omega), B \cap \varphi^{-1}(\omega)) \text{ and}$$
$$d_{S,\varepsilon}(x_n, B \cap \varphi^{-1}(\omega)) > d_{S,\varepsilon}(A \cap \varphi^{-1}(\omega), B \cap \varphi^{-1}(\omega)).$$

Since there are directed paths from R_1 to x_i and from x_i to R_2 for every $i \in \mathbb{N}$, we have $d_{S,\varepsilon}(\eta, x_i) < \infty$ and $d_{S,\varepsilon}(x_i, \mu) < \infty$. Thus, Proposition 14.3 (i) and (ii) imply the existence of geodesic η-x_i anti-rays Q_1^i and geodesic x_i-μ rays Q_2^i. Since $d_{S,\varepsilon}$ is a visual semimetric by Theorem 13.2, since D is locally finite and by the choices of the x_n, there exists a vertex that lies on infinitely many of these anti-rays Q_1^i and a vertex that lies on infinitely many of these rays Q_2^i. Hence, there exists a geodesic double ray Q_1 and a subset $I \subseteq \mathbb{N}$ such that some anti-subray of Q_1 lies in all Q_1^i for $i \in \mathbb{N}$ and every other vertex lies on all but finitely many of the anti-rays Q_1^i for $i \in I$. Let $\nu_1 \in \partial D$ such that some subray of Q_1 lies in ν_1. By changing the sequence $(x_i)_{i\in\mathbb{N}}$, we may assume that $I = \mathbb{N}$. Analogously, we use the geodesic rays Q_2^i to define a geodesic double ray Q_2 with similar properties as Q_1 that goes from some $\nu_2 \in \partial D$ to μ.

Let P be a Q_1-Q_2 geodesic. Let x be its starting vertex and x' be its end vertex. For every large enough $i \in \mathbb{N}$, we consider the geodesic triangle with end vertices x, x', x_i and sides $xQ_1^i x_i$, $x_i Q_2^i x'$ and P. Since there are only finitely many vertices close to or from P, we find infinitely many disjoint directed Q_1-Q_2 paths of length at most δ. Thus, we have $d_{S,\varepsilon}(\nu_1, \nu_2) = 0$ and since $d_{S,\varepsilon}$ is a semimetric, we conclude $\nu_1 = \nu_2$.

Let Q_1^+ be a subray of Q_1 and let Q_2^- be an anti-subray of Q_2. For every $n \in \mathbb{N}$ all but finitely many x_i have a geodesic T_i from Q_1^+ to them outside of $\mathcal{B}^+_n(S) \cup \mathcal{B}^-_n(S)$. Thus, for all but finitely many $i \in \mathbb{N}$, the composition of T_i with $x_i P_i$ is a directed path from Q_1^+ to R_2 outside of $\mathcal{B}^+_n(S) \cup \mathcal{B}^-_n(S)$. Similarly, we find for every $n \in \mathbb{N}$ a directed path from R_1 to Q_2^- outside of $\mathcal{B}^+_n(S) \cup \mathcal{B}^-_n(S)$. Since R_1 and R_2 belong to the same end ω, we conclude that ν_1 lies in $\varphi^{-1}(\omega)$, too.

Since $Q_1^+ \leq Q_2^- \leq Q_1^+$, the sequence $(x_i)_{i\in\mathbb{N}}$ f-converges to ν_1: we find for every $n \in \mathbb{N}$ for all but finitely many $i \in \mathbb{N}$ a directed path from x_i to Q_1^+ outside of

$\mathcal{B}_n^+(S) \cup \mathcal{B}_n^-(S)$. Similarly, $(x_i)_{i\in\mathbb{N}}$ b-converges to ν_2. So if ν_1 were in either A or B, some subsequence of $(x_i)_{i\in\mathbb{N}}$ must lie in either A' or B'. Since this is false by the choice of the vertices x_i, we conclude that ν lies in neither A nor B. This is a contradiction to the fact that $A \cup B$ covers $\varphi^{-1}(\omega)$. □

In order to find the preimages of semiconnected components of ∂D in ΩD, we pose Problem 15.4, for which we need the following definition.

The *components* of an order $(X, \leq)$ are the maximal subsets Y of X such that for all $x, y \in Y$ there are $z_1, \dots z_n \in Y$ with $x = z_1$ and $y = z_n$ and such that either $z_i \leq z_{i+1}$ or $z_{i+1} \leq z_i$ for all $1 \leq i < n$.

Problem 15.4. *Let D be a locally finite hyperbolic digraph that satisfies (B1) and (B2) and that has a finite base and let $\varphi\colon \partial D \to \Omega D$ be the canonical map. Are the semiconnected components of ∂D the preimages under φ of the components of the ends with respect to $\preccurlyeq$?*

Now we are turning our attention to the question of how large the geodesic boundary of locally finite digraphs can be. In order to count the geodesic boundary points, we need the following result on the number of elements of semiconnected semimetric spaces.

Proposition 15.5. *Every semiconnected semimetric space with at least two elements contains infinitely many elements.*

Proof. Let us suppose that a semiconnected semimetric space X has more than one but only finitely many elements. Then any partition of its elements into two sets has the property that both of its sets are open in $\mathcal{O}_f$ and in $\mathcal{O}_b$, which is impossible. □

As another preliminary result, we prove that for one-ended locally finite hyperbolic digraphs in our usual setting, the pseudo-semimetric is indeed a semimetric.

Proposition 15.6. *Let $\delta \geq 0$ and let D be a locally finite δ-hyperbolic digraph that satisfies (B1) and (B2) and that has a finite base S. Let $\varepsilon > 0$ such that $\varepsilon' < \sqrt{2}$ holds for $\varepsilon' := e^{2\varepsilon(6\delta+2\delta f(\delta+1))}$. If there are $\eta, \mu \in \partial D$ with $d_{S,\varepsilon}(\eta,\mu) = 0$ and $d_{S,\varepsilon}(\mu,\eta) < \infty$, then $\eta = \mu$.*

In particular, if D is one-ended, then $d_{S,\varepsilon}$ is a semimetric.

Proof. Let $\eta, \mu \in \partial D$ with $d_{S,\varepsilon}(\eta,\mu) = 0$ and $d_{S,\varepsilon}(\mu,\eta) < \infty$ and let us suppose that $\eta \neq \mu$. By Proposition 14.3 (iii) there exists a geodesic double ray R from μ' to η', where $\mu', \eta' \in \partial D$ with $d(\mu,\mu') = 0$ and $d(\eta',\eta) = 0$. Proposition 14.1 (iv) implies $\eta' = \eta$ and $\mu = \mu'$. Let R_1 be a subray of R and let R_2 be a anti-subray of R. Since $d_{S,\varepsilon}(\eta,\mu) = 0$, we have $R_1 \leq R_2$ by Proposition 14.1 (i). Hence, there exists $M \in \mathbb{N}$ such that outside all balls $\mathcal{B}_r^+(S)$ and $\mathcal{B}_r^-(S)$ there exists an R_1-R_2 geodesic of length at most M. This contradicts Proposition 3.3 (ii) as the reverse distance between the end vertices of these geodesics strictly increases. Thus, we have $\eta = \mu$.

The additional statement immediately follows from Proposition 15.1. □

Corollary 15.7. *Let D be a one-ended locally finite hyperbolic digraph that satisfies (B1) and (B2) and that has a finite base. Then ∂D has either a unique or infinitely many elements.*

Proof. Let $\delta \geq 0$ such that D is δ-hyperbolic. Let S be a finite base of D and let $\varepsilon > 0$ such that $\varepsilon' < \sqrt{2}$ holds for $\varepsilon' := e^{\varepsilon(6\delta+2\delta f(\delta+1))}$. Then Proposition 15.6 implies that $d_{S,\varepsilon}$ is a semimetric. So we can apply Theorem 15.3 and obtain that ∂D is semiconnected. The assertion follows from Proposition 15.5. □

16. Hyperbolic semigroups

Let S be a semigroups and let A be a finite generating set of S. The *(right) Cayley digraph* of S (with respect to A) has S as its vertex set and edges from x to xa for all $x \in S$ and $a \in A$. This way, A is a semimetric space. A straight-forward argument shows that different finite generating sets define quasi-isometric Cayley digraphs, see Gray and Kambites [15, Proposition 4].

A finitely generated semigroup is *hyperbolic* if it has a finite generating set such that its Cayley digraph with respect to that generating set is hyperbolic.

Gray and Kambites [16] asked whether every Cayley digraph of a hyperbolic monoid with respect to a finite generating set is hyperbolic. Proposition 8.2 allows us to give at least a partial answer.

Proposition 16.1. *For every hyperbolic right cancellative semigroup, each of its Cayley digraphs with respect to finite generating sets is hyperbolic.*

Proof. Let S be a hyperbolic right cancellative semigroup. Let A be a finite generating set and let D be the Cayley digraph of S with respect to A. Then the out-degree of all vertices of D is $|A|$. Similarly, since S is right cancellative, the in-degree of all vertices is bounded by $|A|$. In particular, D satisfies (B1) and (B2).

Let B be a finite generating set of S such that the Cayley digraph D' of S with respect to B is hyperbolic. We also have the in- and out-degrees of all vertices bounded by $|B|$ in this situation. Thus, Lemma 3.2 implies that D' satisfies (B1) and (B2), too. Changing finite generating sets of semigroups lead to quasi-isometric Cayley digraphs, see e.g. [15, Proposition 4]. So Proposition 8.2 directly implies that D is hyperbolic. □

Moving on to the geodesic boundary of finitely generated semigroups, Theorems 10.5, 10.6 and 12.3 imply that the homeomorphism types of the (quasi-)geodesic f-boundary of finitely generated semigroups whose Cayley digraphs satisfy (B2) does not depend on the particular generating set and, if the semigroup is right cancellative, then the same holds for the (quasi-)geodesic boundary. Thus, we denote by $\partial^f S$, for a finitely generated semigroup S, the quasi-geodesic f-boundary of S and, if S is right cancellative, we denote by ∂S the quasi-geodesic boundary of S.

The results of Section 15 together with results on the number of ends of semigroups by Craik et al. [11] enable us to obtain some results on the size of the quasi-geodesic boundary of hyperbolic semigroups. First, we immediately have the following corollary of Corollary 15.7.

Corollary 16.2. *Let S be a one-ended finitely generated right cancellative hyperbolic semigroup. Then ∂S has either exactly one or infinitely many elements.* □

The possible numbers of ends of left cancellative semigroups were determined by Craik et al. [11, Theorem 3.7] to be in $\{0, 1, 2, \infty\}$. This immediately implies the following corollary for cancellative semigroups. (Note that if the geodesic boundary satisfies the separation axiom T_1 with respect to $\mathcal{O}_f$ or $\mathcal{O}_b$, then it does so for the other topology as well and it is a semimetric space.)

Corollary 16.3. *Let S be a finitely generated cancellative hyperbolic semigroup such that its geodesic boundary is a T_1 space with respect to either $\mathcal{O}_f$ or $\mathcal{O}_b$. Then $|\partial S| \in \{0, 1, 2, \infty\}$.* □

An obvious question arising is whether the assumption of T_1 separability is necessary.

Problem 16.4. *Does there exist a finitely generated right cancellative hyperbolic semigroup whose geodesic boundary is not a semimetric space?*

In the situation of hyperbolic groups, those with few boundary points, i. e. with at most two, are called elementary and their structure can be described pretty easily in that they are either finite or quasi-isometric to $\mathbb{Z}$, cf. [21, Theorem 2.28]. In analogy to the case of groups, we call a finitely generated right cancellative hyperbolic semigroup *elementary* if it has at most two geodesic boundary points.

The right cancellative hyperbolic semigroups without geodesic boundary points have no end as well by Proposition 11.3 (iii). So they are finite.

An example for a finitely generated right cancellative hyperbolic semigroup with a unique geodesic boundary point is $\mathbb{N}$ and, in analogy to the group case, one might think that all other examples are quasi-isometric to $\mathbb{N}$. However, this is not the case as the following example shows.

Example 16.5. Consider the monoid $S := \langle a, b \mid a^2 = b^2, ab = ba \rangle$. It is straightforward to check that this is hyperbolic and that it has a unique geodesic boundary point. To see that S is not quasi-isometric to $\mathbb{N}$, it suffices to note that $d(a, b) = \infty = d(b, a)$ but that there are no two elements of $\mathbb{N}$ with such a property.

Still, the monoid of the previous example has a structure that reminds very much of $\mathbb{N}$ but the connection is weaker than quasi-isometry.

Let us now consider the case of precisely two geodesic boundary points. It follows directly from Corollary 16.2 that finitely generated right cancellative hyperbolic semigroups have exactly two ends. If the semigroup is cancellative, then it is a two-ended group by Craik et al. [11, Corollary 3.8], so it is quasi-isometric to $\mathbb{Z}$. It remains open to look at the case of right cancellative semigroups that are not cancellative: do those still look like $\mathbb{Z}$ in a way as in the case of exactly one geodesic boundary point the semigroups look like $\mathbb{N}$?

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Matthias Hamann, Mathematics Institute, University of Warwick, Coventry, UK