Canonical trees of tree-decompositions

Johannes Carmesin
University of Birmingham
Birmingham, UK

Matthias Hamann*
Mathematics Institute, University of Warwick
Coventry, UK

Babak Miraftab
Department of Mathematics, University of Hamburg
Hamburg, Germany

October 31, 2021

Abstract

We prove that every graph has a canonical tree of tree-decompositions that distinguishes all principal tangles (these include the ends and various kinds of large finite dense structures) efficiently.

Here 'trees of tree-decompositions' are a slightly weaker notion than 'tree-decompositions' but much more well-behaved than 'tree-like metric spaces'. This theorem is best possible in the sense that we give an example that 'trees of tree-decompositions' cannot be strengthened to 'tree-decompositions' in the above theorem.

This implies results of Dunwoody and Krön as well as of Carmesin, Diestel, Hundertmark and Stein. Beyond that for locally finite graphs our result gives for each $k \in \mathbb{N}$ canonical tree-decompositions that distinguish all k-distinguishable ends efficiently.

1 Introduction

Automorphisms-group invariant tree-decompositions[1] of infinite graphs are applied to study groups via their Cayley graphs (e. g. Krön [13], see also Dunwoody and Krön [6], for the proof of Stallings' Theorem) or other highly symmetric structures; such as [12, 14]. For applications in structural graph theory or matroid theory where canonicity does not play a role, we refer the reader to [1].

Often it is measured how well a tree-decomposition displays the rough structure of a graph by the highly connected substructures it separates. In our context, the most important highly connected substructures are the ends (or more

*Supported by the Heisenberg-Programme of the Deutsche Forschungsgemeinschaft (DFG Grant HA 8257/1-1).

[1]See Section 2 for a definition.

generally the tangles, see below). Consequently we are interested in tree-decompositions separating lots of ends.

The aim of this paper is to determine when we can find such tree-decompositions which are canonical, that is, invariant under graph automorphisms.

On one hand, we prove a general decomposition result that implies the existence of such tree-decompositions in special cases. On the other hand, we complement this with many counterexamples to various related conjectures, explaining why we believe that this theorem answers this question in a best possible way.

Given a graph G, let $\mu(G)$ be the minimal size of a separator of G separating two ends. Dunwoody and Krön [6] showed under 'mild' additional assumptions that every graph has a canonical tree-decomposition separating any two ends separable by at most $\mu(G)$ vertices. And they also provided an example of a graph that does not have a canonical tree-decomposition separating all ends, see Figure 1. We remark that this graph is not locally finite. There are such examples for locally finite graphs, see [1, Example 3.7], but they are more exotic in the sense that they need to have for every number k two ends that cannot be separated by at most k vertices (compare Corollary 1.2).

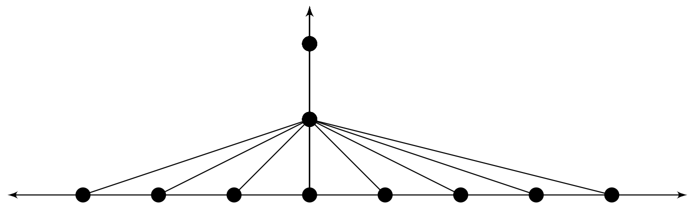

Figure 1: The depicted graph is obtained from the disjoint union of a ray with a double ray by joining the starting vertex of the ray to all vertices on the double ray. This graph has no canonical tree-decomposition distinguishing all its ends.

These two examples may seem counter-intuitive as one might expect to be able to obtain a single canonical tree-decomposition by just iterating the result of Dunwoody and Krön as follows. Starting with a tree-decomposition of Dunwoody and Krön, we apply the theorem again to each part of that tree-decomposition, and then to each part of that and so on. There are some technical aspects to consider when doing this approach, for example one has to consider the torsos of the parts and not the parts themselves and one has to take extra care when constructing the previous tree-decompositions to not spoil a later one. All this put aside for now, the above examples show that this plan cannot work.

We have the following perspective on this. Intuitively speaking, we define a *tree of tree-decompositions* to be a collection of all these iterative tree-decompositions – before any sticking together takes place, see Figure 2. The main result of this paper is that trees of tree-decompositions separating all the ends always can be constructed canonically.

Hence the above-mentioned obstructions can only occur in the gluing process from trees of tree-decompositions to a single tree-decomposition. More precisely, the graph in Figure 1 shows that we cannot always stick two tree-decomposi-

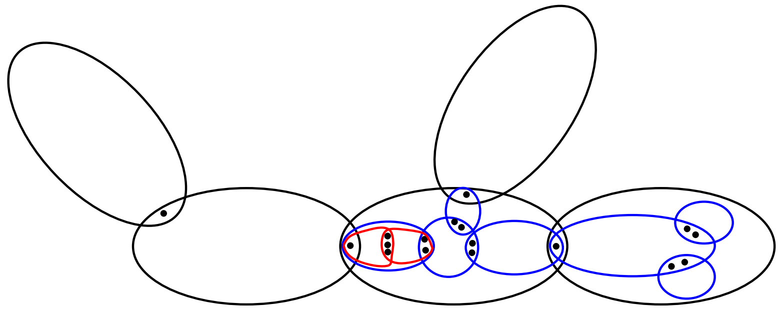

Figure 2: A tree of tree-decompositions. Trees of tree-decompositions can be thought of as a family of tree-decompositions, where each part of one of these tree-decompositions is refined by another tree-decompositions. For finite graphs, these can be canonically combined into a single tree-decomposition. The main result of this paper informally says that the tangle tree theorem of Robertson and Seymour can be extended to infinite graphs, in that we get a canonical tree of tree-decompositions (that is, a tree-decomposition that is 'well-defined up to gluing') distinguishing the principal tangles.

tions together in a canonical way; [1, Example 3.7] shows that we cannot always stick together infinitely many tree-decompositions at once.

Our main result is the following. A tree-decomposition, or more generally a tree of tree-decompositions, *distinguishes the set of ends efficiently* if for any two ends it contains a separator of minimal size separating the two ends.

Theorem 1.1. *Every graph has a canonical tree of tree-decompositions that distinguishes the set of ends efficiently.*

While the above mentioned result of Dunwoody and Krön is immediately implied by Theorem 1.1, our theorem also has the following consequence.

Corollary 1.2. *For every $k \in \mathbb{N}$, every locally finite graph has a canonical tree-decomposition that distinguishes any two ends distinguishable by at most k vertices efficiently.*

Tangles were invented to describe dense substructures in finite graphs and play a key role in the Graph Minor Theory of Robertson and Seymour [15]. It is a simple observation that ends of infinite graphs are also tangles, see Section 6 for details. In the context of our proof it is technically slightly easier to work with the more general notion of principal tangles than ends. In fact we prove the following generalisation of Theorem 1.1.

Theorem 1.3. *Every graph has a canonical tree of tree-decompositions that distinguishes the set of principal tangles (or even more generally, the set of robust profiles[2]) efficiently.*

A side-effect of our general approach is that this theorem also implies recent works concerning tree-decompositions of finite graphs by Carmesin, Diestel, Hundertmark and Stein [2] and by Diestel, Hundertmark and Lemanczyk [5].

[2]See Section 2 for a definition of robust profiles.

The remainder of this paper is structured as follows. In Section 2, we state some main definitions (such as profiles or separations) and prove some basic results on separations of graphs. In Section 3 we explore the relationship between nested sets of separations and tree-decompositions. Its content is based mainly on Carmesin, Diestel, Hundertmark and Stein [2], other parts of the paper are more similar to [1]. Section 4 focuses on how profiles of graphs induce profiles in the torsos of its tree-decompositions. In Section 5 we prove our key auxiliary result, which is the existence of nested sets of separations distinguishing certain profiles under some mild assumption on the graph. In Section 6 we prove our main theorem. After that we discuss the connections between profiles and ends, k-blocks and tangles. In Section 7 we deduce our aforementioned results on locally finite graphs. We conclude the paper with some remarks in Section 8.

2 Sets of separations

For basic notations and terminology for graphs, we refer readers to [4]. Let G be a graph. A *separation* of G is an ordered pair (A, B) of vertex sets such that $G[A] \cup G[B] = G$; that is, there is no edge between $A \setminus B$ and $B \setminus A$. The set of all separations of G is partially ordered by

$$(A, B) \leq (C, D) \qquad :\Leftrightarrow \qquad A \subseteq C \textit{ and } B \supseteq D.$$

A separation (A, B) is *proper* if neither $(A, B) \leq (B, A)$ nor $(B, A) \leq (A, B)$, that is, if $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$. The *order* of the separation (A, B) is the size of its *separator* $A \cap B$.

It is easy to see that this defines a partial order on the set of all separations. A separation (A, B) is *nested* with a separation (C, D), denoted by $(A, B) \parallel (C, D)$, if (A, B) is comparable with (C, D) or (D, C). If (A, B) is not nested with (C, D) then these two separations *cross*. The *centre* of two separations (A, B) and (C, D) is $A \cap B \cap C \cap D$, their four *corners* are the sets $A \cap C$, $B \cap C$, $B \cap D$, and $A \cap D$. The corners $A \cap C$ and $B \cap D$ are *opposite* as are the corner $B \cap C$ and $A \cap D$. Corners are *adjacent* if they are not opposite. The *link* between two adjacent corners is their intersection without the centre. For a corner $E \cap F$ with $\{E, E'\} = \{A, B\}$ and $\{F, F'\} = \{C, D\}$, its *interior* is $(E \cap F) \setminus (E' \cup F')$ and its *corner separation* is the separation $(E \cap F, E' \cup F')$. We call the corner separations $(E \cap F, E' \cup F')$ and $(E \cup F, E' \cap F')$ *opposite*.

A set P of separations of a graph G is a *profile* if it satisfies the following conditions.

(P1) P is *consistent*, i. e. if for every two separations (A, B) and (C, D) with $(C, D) \leq (A, B)$ and $(A, B) \in P$ we have $(D, C) \notin P$;

(P2) if $(A, B), (C, D) \in P$, then $(B \cap D, A \cup C) \notin P$.

If two separations do not form a consistent set we say that they *point away* from each other. A profile P is *principal* if for every family $((A_i, B_i))_{i \in I}$ in P with $A_i \cap B_i = A_j \cap B_j$ for all $i, j \in I$, we have $\bigcap_{i \in I}(B_i \setminus A_i) \neq \emptyset$. A profile P of G is a *k-profile* if all separations in P have order less than k and if for every separation (A, B) of G of order less than k, either $(A, B) \in P$ or $(B, A) \in P$. Note that in a principal k-profile P there is for every vertex set S of size less than k some component C of $G - S$ such that $(V(G) \setminus C, C \cup S) \in P$.

In finite graphs, (P2) ensures that profiles "cannot hide in small separators". But this fails for infinite graphs if we ask only (P2) but not that the profiles are principal as the following example shows.

Example 2.1. Let G be the countably infinite star with vertex set $\{c, v_1, v_2, \ldots\}$, where c is the center of the star. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$. Let P be the set consisting of the separations (A_F, B_F), where $B_F := \{c\} \cup \{v_i \mid i \in F\}$ and $A_F := (V(G) \setminus B_F) \cup \{c\}$, together with the separations $(A, V(G))$ with $|A| \leq 1$. We note that we take separations (A_F, B_F) only for $F \in \mathcal{F}$, not for all $\mathcal{F}$. It follows from the definition of ultrafilters that P is a profile but not principal.

A separation (A, B) *distinguishes* two profiles P, P' if $(A, B) \in P$ and $(B, A) \in P'$ or vice versa and it distinguishes the profiles *efficiently* if there is no separation of smaller order distinguishing P and P'. Two profiles are ℓ-*distinguishable* if a separation (A, B) of order at most ℓ distinguishes them. For a set $\mathcal{P}$ of profiles with at least two distinguishable profiles, let $\kappa(\mathcal{P}, G)$ denote the minimum order of separations separating two profiles of $\mathcal{P}$ i.e. there are two profiles in $\mathcal{P}$ that can be separated by a separation of order $\kappa(\mathcal{P}, G)$.

A separation of a graph G is k-*relevant* where $k \in \mathbb{N} \cup \{\infty\}$ and with respect to $\mathcal{P}$ if it has finite order of at most k and distinguishes two profiles in $\mathcal{P}$. It is *relevant* if it is k-relevant for some $k \in \mathbb{N} \cup \{\infty\}$. It is easy to see that relevant separations are proper. We denote by $\mathcal{R}(k, \mathcal{P}, G)$ the set of k-relevant separations of G with respect to $\mathcal{P}$. We use $\mathcal{R}(k, \mathcal{P})$ or $\mathcal{R}(k)$ if G and $\mathcal{P}$ are obvious from the context. Let $\mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$ be the set of all separations of order $\kappa(\mathcal{P}, G)$ distinguishing two profiles of $\mathcal{P}$ efficiently and for two profiles $P, P' \in \mathcal{P}$ set

$$\mathcal{R}_{\text{eff}}(P, P') := \{(A, B) \in \mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G) \mid (A, B) \text{ distinguishes } P, P' \text{ efficiently}\}.$$

A component C of $G - (A \cap B)$ for a separation (A, B) of G is *degenerated* if $N(C)$ is a proper subset of $A \cap B$. We call a separation *degenerated* relative to (A, B) if it is of the form $(C \cup N(C), V(G) \setminus C)$ where C is a degenerated component of $G - (A \cap B)$. The *degenerator* of a set $\mathcal{S}$ of separations is the set of separations that are degenerated relative to some separation in $\mathcal{S}$. We denote the degenerator of $\mathcal{R}(k, \mathcal{P}, G)$ by $\mathcal{S}(k, \mathcal{P}, G)$ and write $\mathcal{S}(k, \mathcal{P})$ or $\mathcal{S}(k)$ if G and $\mathcal{P}$ are obvious from the context. We call G *well-separable* (with respect to a set $\mathcal{P}$ of profiles) if for every relevant separation (A, B) of order $\kappa(\mathcal{P}, G)$ no component of $G - (A \cap B)$ is degenerated, that is, if $\mathcal{S}(\kappa(\mathcal{P}, G), \mathcal{P}, G) = \emptyset$. A separation (A, B) is *left-connected* if $A \setminus B$ is connected.

For $n \in \mathbb{N}$, a profile P is n-*robust* if for every $(A, B) \in P$ and every separation (C, D) of order at most n the following holds: if $(B \cap C, A \cup D)$ and $(B \cap D, A \cup C)$ both have order less than $|A \cap B|$, then one of those corner separations does not lie in P. It is *robust* if it n-robust for every $n \in \mathbb{N}$.

A separator separates two vertices *minimally* if no other separator of smaller size separates them, too. The following result is by Halin [8].

Lemma 2.2. [8, 2.4] *Let G be a graph, $u, v \in V(G)$ and $k \in \mathbb{N}$. Then there are only finitely many separators of size at most k separating u and v minimally.* □

For a set $\mathcal{S}$ of separations we define the property:

For all $(A, B), (C, D) \in \mathcal{S}$, there are only finitely many $(E, F) \in \mathcal{S}$ with $(A, B) < (E, F) < (C, D)$. $(*)$

Lemma 2.3. *Let G be a graph and let $k \in \mathbb{N}$. Then any nested set of left-connected proper separations of order at most k has property $(*)$.*

Proof. Let $\mathcal{S}$ be a nested set of left-connected separations of order at most k and let $(A,B),(C,D) \in \mathcal{S}$ with $(A,B) \le (C,D)$. Let us suppose that $\mathcal{S}$ does not satisfy property $(*)$. Then there are infinitely many distinct $(E,F) \in \mathcal{S}$ with $(A,B) < (E,F) < (C,D)$. We note that all of them are left-connected, and all their separators are different. In addition assume that (E,F) is one of these separations. Let $\{C_i\}_{i\in I}$ be the set of all components of $G - (E \cap F)$. Each $N(C_i)$ separates any vertex in $C \setminus D$ from any vertex in $B \setminus A$ minimally and $E \cap F = \cup_{i \in I} N(C_i)$. This contradicts Lemma 2.2 and shows $(*)$. □

Lemma 2.4. *Let G be a graph and let (A,B), (C,D) and (E,F) be three separations such that $(A,B) \nparallel (C,D)$. Then the following statements hold:*

(i) [2, Lemma 2.2] *If (E,F) is nested with (A,B) and with (C,D), then every corner of (A,B) and (C,D) is nested with (E,F);*

(ii) *if (E,F) is nested with (A,B), then there are two adjacent corners of (A,B) and (C,D) which are nested with (E,F).*

Proof. For the proof of (i) we refer readers to [2, Lemma 2.2].

To prove (ii), let us assume that (E,F) is nested with (A,B). We may assume $(E,F) \le (A,B)$. In particular, we have $E \subseteq A \cup C$ and $E \subseteq A \cup D$ as well as $B \cap C \subseteq F$ and $B \cap D \subseteq F$. Hence, (E,F) is nested with $(A \cap C, B \cup D)$ and $(A \cap D, B \cup C)$. □

Let (A,B) be a separation and let $\mathcal{S}$ be a set of separations of a graph G. We set

$$\mathcal{C}_{\mathcal{S}}(A,B) := \{(C,D) \in \mathcal{S} \mid (A,B) \nparallel (C,D)\},$$

i.e. $\mathcal{C}_{\mathcal{S}}(A,B)$ is the set of all separations in $\mathcal{S}$ that cross (A,B). We set $c_{\mathcal{S}}(A,B) := |\mathcal{C}_{\mathcal{S}}(A,B)|$. If $c_{\mathcal{S}}(A,B)$ is finite, then we say that (A,B) has *finite crossing number* (with respect to $\mathcal{S}$). Otherwise we say that the *crossing number* of (A,B) is *infinite*.

The following lemma shows that corner separations of crossing separations are crossing with fewer separations than the crossing ones. This is an adaption of a result of Dunwoody and Krön [6, Lemma 5.1] to our situation.

Lemma 2.5. *Let (A,B) and (C,D) be two crossing separations of a graph G and let $\mathcal{S}$ be a set of separations. If (X_1,Y_1) and (X_2,Y_2) are two opposite corner separations of (A,B) and (C,D), then the following holds:*

(i) $\mathcal{C}_{\mathcal{S}}(X_1,Y_1) \cap \mathcal{C}_{\mathcal{S}}(X_2,Y_2) \subseteq \mathcal{C}_{\mathcal{S}}(A,B) \cap \mathcal{C}_{\mathcal{S}}(C,D)$;

(ii) $\mathcal{C}_{\mathcal{S}}(X_1,Y_1) \cup \mathcal{C}_{\mathcal{S}}(X_2,Y_2) \subsetneq \mathcal{C}_{\mathcal{S}}(A,B) \cup \mathcal{C}_{\mathcal{S}}(C,D)$.

In particular, if all sets are finite, then we have

$$c_{\mathcal{S}}(X_1,Y_1) + c_{\mathcal{S}}(X_2,Y_2) < c_{\mathcal{S}}(A,B) + c_{\mathcal{S}}(C,D).$$

Proof. For (i), let (E,F) be a separation that is nested with either (A,B) or (C,D). By Lemma 2.4 (ii), there are two adjacent corners of (A,B) and (C,D) which are nested with (E,F). Thus, (E,F) is nested with either (X_1,Y_1) or (X_2,Y_2).

To prove (ii), let $(E,F) \in \mathcal{C}_\mathcal{S}(X_1,Y_1) \cup \mathcal{C}_\mathcal{S}(X_2,Y_2)$. By Lemma 2.4 (i), the separation (E,F) belongs to $\mathcal{C}_\mathcal{S}(A,B) \cup \mathcal{C}_\mathcal{S}(C,D)$. The inclusion is strict, as (A,B) belongs to $\mathcal{C}_\mathcal{S}(A,B) \cup \mathcal{C}_\mathcal{S}(C,D)$ but not to $\mathcal{C}_\mathcal{S}(X_1,Y_1) \cup \mathcal{C}_\mathcal{S}(X_2,Y_2)$.

The additional assertion follows directly from (i) and (ii). □

Lemma 2.6. *Let G be a graph and $\mathcal{P}$ be a set of distinguishable $(k+1)$-profiles in G with $k = \kappa(\mathcal{P},G)$. Let $(A_1,A_2),(B_1,B_2) \in \mathcal{R}(k,\mathcal{P})$. Then there are two opposite corner separations of (A_1,A_2) and (B_1,B_2) that lie in $\mathcal{R}(k,\mathcal{P})$.*

Proof. For $i \in \{1,2\}$, let P_{A_i} and P_{B_i} be profiles in $\mathcal{P}$ such that $(A_{3-i},A_i) \in P_{A_i}$ and $(B_{3-i},B_i) \in P_{B_i}$. Let $\mathcal{P}' := \{P_{A_1},P_{A_2},P_{B_1},P_{B_2}\}$. First, we will show the following.

There are $i,j \in \{1,2\}$ and $P,P' \in \mathcal{P}'$ such that $(A_{3-i},A_i),(B_{3-j},B_j) \in P$ and $(A_i,A_{3-i}),(B_j,B_{3-j}) \in P'$. (†)

If (B_1,B_2) distinguishes P_{A_1} and P_{A_2}, then (†) holds for those two profiles. Let us assume that (B_1,B_2) does not distinguish P_{A_1} and P_{A_2}. We may assume that $(B_1,B_2) \in P_{A_1} \cap P_{A_2}$. Without loss of generality, we may assume that $(A_1,A_2) \in P_{B_1}$. Then (†) holds for the profiles P_{A_1} and P_{B_1}.

Let us consider the corner separations $(X_1,X_2) := (A_{3-i} \cup B_{3-j}, A_i \cap B_j)$ and $(Y_1,Y_2) := (A_i \cup B_j, A_{3-i} \cap B_{3-j})$. By definition of a profile, we have $(X_2,X_1) \notin P$ and $(Y_2,Y_1) \notin P'$. It is easy to see that the orders of (X_1,X_2) and (Y_1,Y_2) sum to the sum of the orders of (A_1,A_2) and (B_1,B_2), which is $2k$. So if one of those corner separations has order at most k, it follows that it distinguishes P and P', and if its order is less than k, this would contradict $k = \kappa(\mathcal{P},G)$. Thus, both corner separations have order at least k and hence exactly k. So they lie in $\mathcal{R}(k,\mathcal{P})$. □

Lemma 2.7. *Let G be a graph, let $\mathcal{P}$ be a set of principal k-profiles, let $P,P' \in \mathcal{P}$ and let $(A,B) \in \mathcal{R}_{\mathit{eff}}(P,P')$. Then there is a component X of $A \setminus B$ such that $(X \cup N(X), V(G) \setminus X)$ lies in $\mathcal{R}_{\mathit{eff}}(P,P')$.*

Proof. Let us assume that $(A,B) \in P$ and $(B,A) \in P'$. Let $\{C_i\}_{i\in I}$ be the components of $A \setminus B$. Since $(C_i \cup N(C_i), V(G) \setminus C_i) \leq (A,B)$ and $(A,B) \in P$, we conclude that $(C_i \cup N(C_i), V(G) \setminus C_i) \in P$ for all $i \in I$. Since $(D \cup N(D), V(G) \setminus D) \leq (B,A)$ for all components D of $B \setminus A$, it follows by (P1) and as the profiles are principal that $(V(G) \setminus C_i, C_i \cup N(C_i)) \in P'$ for some $i \in I$. Thus we have $(C_i \cup N(C_i), V(G) \setminus C_i) \in \mathcal{R}_{\mathrm{eff}}(P,P')$. □

Let G be a graph and $\mathcal{P}$ be a set of profiles of G each of which is an ℓ-profile for some $\ell > k$. Note that for every left-connected separation (A,B) in $\mathcal{R}(k,\mathcal{P})$ its separator separates two vertices minimally: by Lemma 2.7, we find a component C of $B \setminus A$ such that $(C \cup N(C), V(G) \setminus C)$ lies in $\mathcal{R}(k,\mathcal{P})$ as well and hence we may choose any vertex a in $A \setminus B$ and any vertex b in C; these two vertices are separated by $A \cap B$ minimally. Thus, every left-connected separation in $\mathcal{R}(k,\mathcal{P})$ has a finite crossing number with respect to the subset $\mathcal{R}_{\mathrm{lc}}(k,\mathcal{P})$ of left-connected separations in $\mathcal{R}(k,\mathcal{P})$ by Lemma 2.2. Let $\mathcal{O}(k,\mathcal{P})$ be the set of left-connected separations in $\mathcal{R}(k,\mathcal{P})$ with minimum crossing number in $\mathcal{R}_{\mathrm{lc}}(k,\mathcal{P})$.

The following lemma is essentially already proved in Dunwoody and Krön [6]. But their result requires the existence of 'cut systems' as they call it. Instead of showing that we can define a cut system in our case, we briefly prove Lemma 2.8 directly. To state Lemma 2.8, we define canonicity slightly stronger than in the introduction.

Let G and G' be graph and $\varphi\colon G \to G'$ an isomorphism. Let $\mathcal{P}$ be a set of profiles and let $\mathcal{S}$ be a set of separations. We set

$$\mathcal{S}_\varphi := \{(\varphi(A), \varphi(B)) \mid (A, B) \in \mathcal{S}\}$$

and $\mathcal{P}_\varphi := \{P_\varphi \mid P \in \mathcal{P}\}$. Then $\mathcal{S}_\varphi$ is a set of separations of G' and $\mathcal{P}_\varphi$ is a set of profiles of G'. Let us assume that we obtained $\mathcal{S}$ by a construction starting with G and $\mathcal{P}$, e. g. in a constructive proof. Then we denote $\mathcal{S}$ by $\psi(G, \mathcal{P})$ and call ψ the map *associated with* that construction. We call $\mathcal{S}$ *canonical (with respect to $\mathcal{P}$)* if for a fixed construction and for all choices of G, G', φ and $\mathcal{P}$ we have $\psi(G', \mathcal{P}_\varphi) = (\psi(G, \mathcal{P}))_\varphi$, where ψ is the map associated with that construction. So roughly speaking, $\mathcal{S}$ is canonical if its construction commutes with isomorphisms.

Lemma 2.8. *Let G be a well-separable graph with respect to a set $\mathcal{P}$ of robust principal $(k+1)$-profiles of G with $k = \kappa(\mathcal{P}, G)$ such that $\mathcal{R}(k, \mathcal{P}) \neq \emptyset$. Then the set $\mathcal{O}(k, \mathcal{P})$ is nested, canonical with respect to $\mathcal{P}$ and not empty.*

Proof. Let $(A, B) \in \mathcal{O}(k, \mathcal{P})$. Let us suppose that (A, B) is not nested with an another left-connected separation (C, D) in $\mathcal{R}(k, \mathcal{P})$. Then there are two opposite corners whose separations lie in $\mathcal{R}(k, \mathcal{P})$ by Lemma 2.6 and one of these corner separations, call it (E, F), is crossing with a smaller number of separations in $\mathcal{R}(k, \mathcal{P})$ than (A, B) by Lemma 2.5. If (E, F) is not left-connected, then there is a component K of $E \setminus F$ such that $(E', F') := (K \cup N(K), V(G) \setminus K)$ is a left-connected separation in $\mathcal{R}(k, \mathcal{P})$ by Lemma 2.7. It is easy to see that (E', F') is nested with all separations that are nested with (E, F) and thus is crossing with less separations in $\mathcal{R}_{\mathrm{lc}}(k, \mathcal{P})$ than (A, B), a contradiction to the choice of (A, B). □

3 Tree-decompositions

Carmesin, Diestel, Hundertmark and Stein [2] presented a method for finite graphs how to build a tree-decomposition from a nested set of separations. Essentially, their method carries over to infinite graphs but the proofs need a small adjustment to deal with an additional assumption that we need. In this section, we will recap their definitions and results. We will omit the proof where appropriate and highlight the differences from the finite to the infinite case.

A *tree-decomposition* of a graph G is a pair $(T, \mathcal{V})$ of a tree T, called the *decomposition tree*, and a family $\mathcal{V} = (V_t)_{t \in V(T)}$ of vertex sets $V_t \subseteq V(G)$, one for every node of T, such that:

(T1) $V(G) = \bigcup_{t \in V(T)} V_t$,

(T2) for every edge $e \in E(G)$, there exists a $t \in V(T)$ with $e \subseteq V_t$,

(T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ for all t_2 that lie on the t_1-t_3 path in T.

The elements of $\mathcal{V}$ are the *parts* of the tree-decomposition. The sets $V_s \cap V_t$ with $st \in E(T)$ are the *adhesion sets*. The *adhesion* is the supremum of the cardinalities of the adhesion sets. A tree-decomposition is *k-balanced* if all its adhesion sets have size k and it is *balanced* if it is k-balanced for some $k \in \mathbb{N}$.

Let $\mathcal{N}$ be a nested set of separations such that $(B, A) \in \mathcal{N}$ for all $(A, B) \in \mathcal{N}$. We construct a tree decomposition $(T(\mathcal{N}), \mathcal{V}(\mathcal{N}))$ via $\mathcal{N}$. We define a relation $\sim$ on $\mathcal{N}$:

$$(A, B) \sim (C, D) \;\Leftrightarrow \begin{cases} (A, B) = (C, D) \text{ or} \\ (B, A) \text{ is a predecessor of } (C, D) \text{ in } (\mathcal{N}, \leq), \end{cases}$$

where a *predecessor* of an element z of a partial order $(P, \leq)$ is an element $x < z$ of P such that there is no $y \in P$ with $x < y < z$. By the same argument as in [2, Lemma 3.1], one can show that $\sim$ is an equivalence relation on $\mathcal{N}$. The nodes of $T(\mathcal{N})$ are the equivalence classes of $\sim$ on $\mathcal{N}$. We define the set of edges of $T(\mathcal{N})$ as

$$E(T(\mathcal{N})) := \big\{\{(A, B), (B, A)\} \mid (A, B) \in \mathcal{N}\big\}$$

and an edge is incident with the two equivalence classes of its elements. Let $\mathcal{V}(\mathcal{N})$ consists of all

$$V_t := \bigcap \{A \mid (A, B) \in t\}$$

with $t \in V(T(\mathcal{N}))$.

Proposition 3.1. *Let G be a graph and $\mathcal{N}$ a nested set of separations of G satisfying $(*)$. Then $T(\mathcal{N})$ is a tree.*

Proof. The proof of connectedness and lack of having cycles follows the proof of the analogous result for finite graphs of Carmesin, Diestel, Hundertmark and Stein [2, Theorem 3.4] almost verbatim. We just have to apply $(*)$ at the according place to verify that $T(\mathcal{N})$ is connected. □

If $(T, \mathcal{V})$ is a tree-decomposition of a graph G, $e \in E(T)$ and T_1, T_2 the components of $T - e$, then $(\bigcup_{t \in V(T_1)} V_t, \bigcup_{t \in V(T_2)} V_t)$ is a separation of G and its order is the size of the adhesion set corresponding to e. The separation is the separation *induced by* the edge e.

With Proposition 3.1, the proof of [2, Theorem 4.8] carries over to our situation.

Theorem 3.2. [2, Theorem 4.8] *Let G be a graph and $\mathcal{N}$ a nested set of separations satisfying $(*)$. Then $(T(\mathcal{N}), \mathcal{V}(\mathcal{N}))$ is a tree-decomposition. The separations induced by $(T(\mathcal{N}), \mathcal{V}(\mathcal{N}))$ are precisely those in $\mathcal{N}$.* □

We say that a profile P *lives in* a part V_t of a tree-decomposition $(T, \mathcal{V})$ if for every separation (A, B) that is induced by $(T, \mathcal{V})$ we have $(A, B) \in P$ if $V_t \subseteq B$. Note that consistency of P implies that if P lives in a part V_t, then V_t is no adhesion set.

4 Profiles and parts of tree-decompositions

Let $(T, \mathcal{V})$ be a tree-decomposition of a graph G. For $t \in V(T)$, the *torso* H of V_t is the subgraph of G induced by V_t with additional edges xy for all $x, y \in V_t$ that lie in a common adhesion set in V_t. A separation (A, B) of G induces a separation

$$(A_H, B_H) := (A \cap V_t, B \cap V_t)$$

of H if and only if (A, B) does not separate any adhesion set in H. Note that proper separations of G need not induce proper separations of H. For a set $\mathcal{S}$ of separations of G let

$$\mathcal{S}_H := \{(A_H, B_H) \mid (A, B) \in \mathcal{S}, (A_H, B_H) \text{ is a separation of } H\}$$

be the set of separations induced by $\mathcal{S}$ on H. That way, a profile of G induces a set of separations of H. In the following proposition, we will see that this induced set of separations is indeed a profile in the cases we are interested in. Furthermore, we will prove that the induced separations on a torso separate the induced profiles in a best possible way.

Proposition 4.1. *Let G be a graph and let $k \in \mathbb{N}$. Let $\mathcal{P}$ be a set of robust principal profiles of G such that for every $P \in \mathcal{P}$ there is some $\ell > k$ such that P is an ℓ-profile. Let $(T, \mathcal{V})$ be a tree-decomposition of adhesion at most $\kappa(\mathcal{P}, G)$ such that $N(C) = S$ for every adhesion set S of $(T, \mathcal{V})$ and every component C of $G - S$. Let V_t be a part of $(T, \mathcal{V})$ and H be its torso. Assume that all separations of G induced by edges of the decomposition tree are proper. Then the following hold.*

(i) *For every (robust) principal ℓ-profile P that lives in V_t, where ℓ is larger than the adhesion of $(T, \mathcal{V})$, the set P_H is a (robust) principal ℓ-profile of H;*

(ii) *for all separations (A, B) of G with $A \cap B \subseteq V_t$ that distinguish profiles in $\mathcal{P}$ that live in V_t, the pair $(A \cap V_t, B \cap V_t)$ is a separation of H of the same order and distinguishes the induced profiles;*

(iii) *for all distinguishable profiles $P_1, P_2 \in \mathcal{P}_H$ and all distinguishable profiles $Q_1, Q_2 \in \mathcal{P}$ such that Q_i induces P_i for $i = 1, 2$, there is a separation (A, B) of G that distinguishes Q_1 and Q_2 efficiently such that $(A \cap V_t, B \cap V_t)$ is a separation of order $|A \cap B|$ that distinguishes P_1, P_2 efficiently;*

(iv) *if the subset $\mathcal{P}'$ of $\mathcal{P}$ consisting of all profiles that live in V_t contains at least two elements, then $\kappa(\mathcal{P}_H, H) = \kappa(\mathcal{P}', G)$.*

Proof. Let (A, B) be a separation of the graph G whose separator lies in V_t. Set $(A_H, B_H) := (A \cap V_t, B \cap V_t)$. Since the separator $A \cap B$ is included in the vertex set $V(H)$ of the torso H, we have $|A_H \cap B_H| = |A \cap B|$.

Suppose for a contradiction (A_H, B_H) is not a separation of the torso H. Then there is an edge ab in H with $a \in A \setminus B$ and $b \in B \setminus A$. This edge does not lie in G and hence both its end vertices lie in a common adhesion set. Let x be an edge of the decomposition tree whose adhesion set contains the vertices a and b. Let (X, Y) be the separation corresponding to the edge x. Either the vertex set $X \setminus Y$ or $Y \setminus X$ is disjoint from the part V_t. By symmetry, we may assume that $X \setminus Y$ is disjoint from V_t. As (X, Y) is proper by assumption, the set $X \setminus Y$ is nonempty. Let C be a component of $G - (X \cap Y)$ included in $X \setminus Y$. By the assumptions on $(T, \mathcal{V})$ and its adhesion sets, we have $X \cap Y = N(C)$. So C includes a path whose endvertices are adjacent to the vertices a and b. As this path is disjoint from the separator $A \cap B$, this violates our assumption that (A, B) is a separation of the graph G. Hence it must be that (A_H, B_H) is a separation of the torso H.

Let (A, B) be a separation of H. By definition, all adhesion sets of $(T, \mathcal{V})$ induce complete graphs in H. Thus, every adhesion set lies either in A or in B. We define a separation of G by adding all components C of $G - V_t$ to A if $N(C) \subseteq A$ and to B otherwise. It is easy to see that the resulting pair (A^G, B^G) is a separation of G. Its order is $|A \cap B|$ since we did not add anything to A and B simultaneously. So if (A^G, B^G) lies in a profile $P \in \mathcal{P}$, then $(A, B) \in P_H$.

Let P be an ℓ-profile that lives in V_t such that ℓ is larger than the adhesion of $(T, \mathcal{V})$. First, we will show that P_H is consistent. Let (A_H, B_H) and (C_H, D_H) be separations of H such that $(C_H, D_H) \leq (A_H, B_H)$ and $(A_H, B_H) \in P_H$. Suppose that (D_H, C_H) lies in P_H. So we may assume that there are $(A, B), (D, C) \in P$ whose induced separations in H are $(A_H, B_H), (D_H, C_H)$, respectively. Note that we have $A \cap B \cap V(H) = A_H \cap B_H$ and $C \cap D \cap V(H) = C_H \cap D_H$ by definition of induced separations. Since P is a principal ℓ-profile, there is a component K of $C \setminus D$ with $(V(G) \setminus K, K \cup N(K)) \in P$. If $K \cap V_t \neq \emptyset$, then there is a vertex in $K \cap C_H \subseteq K \cap (A_H \setminus B_H)$. Since $A \cap B = A_H \cap B_H \subseteq D_H$, we conclude $K \subseteq A$. As $(K \cup N(K), V(G) \setminus K) \leq (A, B) \in P$, the separation $(K \cup N(K), V(G) \setminus K)$ lies in P by (P1). This is a contradiction to (P1) for P. So K contains no vertex of C_H. In particular $K \cap V_t = \emptyset$. Let e be an edge of T that is incident with t and such that for the separation (A_e, B_e) induced by e we have $V_t \subseteq A_e$ and $K \subseteq B_e$. Then we have $(A_e, B_e) \leq (V(G) \setminus K, K \cup N(K)) \in P$. So (P1) implies $(A_e, B_e) \in P$ and hence P does not live in t, a contradiction to its choice. This shows that P_H is consistent.

Let $(A_H, B_H), (C_H, D_H) \in P_H$ and let $(A, B) := (A_H^G, B_H^G)$ and $(C, D) := (C_H^G, D_H^G)$. Then $(A, B), (C, D) \in P$ since P_H is consistent and $(A, B), (C, D)$ induce $(A_H, B_H), (C_H, D_H)$, respectively. By (P2), we have $(E, F) := (B \cap D, A \cup C) \notin P$. If the order of (E, F) is less than ℓ, then $(F, E) \in P$ and hence $(F_H, E_H) \in P_H$. So consistency of P_H implies $(E_H, F_H) \notin P_H$. If the order of (E, F) is at least ℓ, then the same holds for the order of (E_H, F_H) and hence it does not lie in P_H. Thus, (P2) follows for P_H and P_H is a profile.

If a separation $(A_H, B_H) \in P_H$ is induced by $(A, B) \in P$, then the order of (A_H, B_H) is at most the order of (A, B). Thus, all elements of P_H have order at most ℓ. Let (A, B) be a separation of H of order at most ℓ. Then the order of (A^G, B^G) is at most ℓ and hence either (A^G, B^G) or (B^G, A^G) lies in P. Thus, either (A, B) or (B, A) lies in P_H.

As P is a principal ℓ-profile, there is for every vertex set S of size less than k a component K of $G - S$ such that $(A, B) := (V(G) \setminus K, K \cup N(K)) \in P$. As in the proof of (P1) for P_H, it follows that (A_H, B_H) is a proper separation of H. Then each separation (C, D) with $S = C \cap D$ lies in P_H if and only if D contains B_H. It follows that P_H is principal.

Now let P be n-robust for some $n \in \mathbb{N}$. Let $(A_H, B_H) \in P_H$ and let (C_H, D_H) be a separation of H of order at most n. Then there are $(A, B) \in P$ and a separation (C, D) of order at most n that induce (A_H, B_H) and (C_H, D_H) in H, respectively. Suppose that $(B_H \cap C_H, A_H \cup D_H)$ and $(B_H \cap D_H, A_H \cup C_H)$ are in P_H and both have order less than $|A \cap B|$. As P is n-robust, we may assume that $(B \cap C, A \cup D) \notin P$. As the orders of $(B \cap C, A \cup D)$ and $(B_H \cap C_H, A_H \cup D_H)$ coincide, we conclude $(A \cup D, B \cap C) \in P$. So by definition, $(A_H \cup D_H, B_H \cap C_H) \in P_H$ which contradicts consistency of P_H as $(B_H \cap C_H, A_H \cup D_H) \in P_H$. Thus, (i) holds.

We have already seen that separations (A, B) whose separators lie in V_t induce separations in H and it is obvious from the definitions that, if (A, B)

distinguishes two profiles, the induced separation distinguishes the induced profiles. Thus, (ii) holds.

To prove (iii), let $P_1, P_2 \in \mathcal{P}_H$ and $Q_1, Q_2 \in \mathcal{P}$ such that Q_i induces P_i for $i = 1, 2$. Let (A, B) be a separation of G such that (A_H, B_H) distinguishes P_1 and P_2 efficiently. Since any separation (C, D) of G that separates Q_1 and Q_2 efficiently, also induces a separation (C_H, D_H) that distinguishes P_1 and P_2, we conclude $|A \cap B| \leq |C \cap D|$ and thus we may assume $(A, B) \in Q_1$ and either (A, B) or (B, A) lies in Q_2. If $(A, B) \in Q_2$, then we conclude $(A_H, B_H) \in P_1 \cap P_2$. But as it distinguishes P_1 and P_2, we also have $(B_H, A_H) \in P_2$, a contradiction to consistency of P_2. Thus, (A, B) distinguishes Q_1 and Q_2. Any separation of smaller order than (A, B) that distinguishes Q_1 and Q_2 induces a separation of smaller order than (A_H, B_H) distinguishing P_1 and P_2. Thus, (A, B) distinguishes Q_1 and Q_2 efficiently. This shows (iii).

Finally, (iv) follows immediately from (ii) and (iii). □

Lemma 4.2. *Let G be a well-separable graph with respect to a set $\mathcal{P}$ of robust principal profiles of G of order $k+1$ with $k = \kappa(\mathcal{P}, G)$. Let $(T, \mathcal{V})$ be a k-balanced tree-decomposition such that all separations of G induced by edges of T are proper. Let X be a torso of $(T, \mathcal{V})$ and let $\mathcal{P}'$ be the set of profiles of X that are induced by profiles in $\mathcal{P}$. Then $\mathcal{S}(k, \mathcal{P}', X) = \emptyset$ and $\mathcal{R}(k-1, \mathcal{P}', X) = \emptyset$.*

Proof. Proposition 4.1 implies $\mathcal{R}(k-1, \mathcal{P}', X) = \emptyset$. Let $(A, B) \in \mathcal{R}(k, \mathcal{P}', X)$ and let C be a component of $X - (A \cap B)$. By Proposition 4.1 (iii), there is a separation (A^G, B^G) that induces (A, B) on X and has the same order as (A, B). In particular, we have $A \cap B = A^G \cap B^G$.

Let us suppose that $N(C) \subsetneq A \cap B$. Then C lies in a component K of $G - (A \cap B) = G - (A^G \cap B^G)$. Let us show that $N(K) \subsetneq A \cap B$, since this leads to an immediate contradiction to $\mathcal{S}(k, \mathcal{P}) = \emptyset$. Let $x \in K \setminus C$ have a neighbour y in $(A \cap B) \setminus N(C)$. Then y lies in an adhesion set S of $(T, \mathcal{V})$. Thus, any path in K from a vertex in C to x must contain a vertex z of S. Since z lies in S, it lies in X and hence in C. So by definition of a torso, y has z as a neighbour and hence $y \in N(C)$. So we have $N(K) = N(C) \subsetneq A \cap B$. □

Now we are going to construct a tree-decomposition of an arbitrary graph with a set of profiles such that the tree-decomposition has a unique part in which all profiles live and whose torso is well-separable with respect to the induced profiles. We call a tree-decomposition *canonical* if the set of separations induced by the edges of the tree-decomposition is canonical.

Proposition 4.3. *Let G be a graph and let $k \in \mathbb{N}$. Let $\mathcal{P}$ be a set of robust principal profiles of G that are pairwise $\kappa(\mathcal{P}, G)$-distinguishable and such that for every $P \in \mathcal{P}$ there is some $\ell > \kappa(\mathcal{P}, G)$ such that P is an ℓ-profile. Then there exists a tree-decomposition $(T, \mathcal{V})$ of adhesion less than $\kappa(\mathcal{P}, G)$ that is canonical with respect to $\mathcal{P}$ such that there exists a unique part V_t of $(T, \mathcal{V})$ in which all profiles of $\mathcal{P}$ live and such that the torso H of V_t is well-separable with respect to $\mathcal{P}_H$. Moreover, all separations corresponding to edges of the decomposition tree are proper.*

Proof. Let $\mathcal{C}$ be the set of all degenerated components of separations $(A, B) \in \mathcal{R}^{\kappa}_{\mathrm{eff}}(\mathcal{P}, G)$. Let T be a star with $|\mathcal{C}|$ leaves. Let x be the central vertex of T and let φ be a bijection from the set of leaves of T to $\mathcal{C}$. For a leaf y, we set

$V_y := \varphi(y) \cup N(\varphi(y))$ and we set $V_x := V(G) \setminus \bigcup \mathcal{C}$. We claim that $(T, \mathcal{V})$ with $\mathcal{V} := \{V_z \mid z \in V(T)\}$ is a tree-decomposition of adhesion less than $\kappa(\mathcal{P}, G)$.

Since the adhesion set of an edge xy has size at most $N(\varphi(y))$, it follows from the choice of $\mathcal{C}$ that the adhesion is less than $\kappa(\mathcal{P}, G)$. To show that $(T, \mathcal{V})$ is a tree-decomposition, it suffices to show (T3). This follows immediately once we showed the following two properties.

(a) Distinct elements of $\mathcal{C}$ are disjoint;

(b) elements of $\mathcal{C}$ are disjoint from separators of separations in $\mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$.

Let (A, B) and (C, D) be two separations in $\mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$ and let X be a degenerated component of $G - (A \cap B)$. If there is also a degenerated component Y of $G - (C \cap D)$ with $X \cap Y \neq \emptyset$ that is distinct from X, then either X intersects $C \cap D$ or Y intersects $A \cap B$. Thus, (b) implies (a) and it remains to prove (b).

Let us suppose $X \cap (C \cap D) \neq \emptyset$ for some $(C, D) \in \mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$. Without loss of generality, we may assume $X \subseteq B$. Let $P, P', Q, Q' \in \mathcal{P}$ such that (A, B) distinguishes P and P' efficiently and (C, D) distinguishes Q and Q' efficiently. We may assume $(A, B) \in P$ and $(C, D) \in Q$. We will show that one corner separation, either $(B \cap C, A \cup D)$ or $(B \cap D, A \cup C)$, has order at most $|A \cap B|$. Let us suppose that both have order at least $|A \cap B| + 1$. As the orders of opposite corner separations sum to $2|A \cap B|$, the orders of $(A \cap C, B \cup D)$ and $(A \cap D, B \cup C)$ are less than $|A \cap B|$. Since these two corner separations are less than (A, B), they lie in P. But neither $(B \cup D, A \cap C)$ nor $(B \cup C, A \cap D)$ can lie in P' as their orders are less than $|A \cap B|$ but $(A, B) \in \mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$. So $(A \cap C, B \cup D)$ and $(A \cap D, B \cup C)$ lie in P'. Then robustness implies $(B, A) \notin P'$. This contradiction shows that either $(B \cap C, A \cup D)$ or $(B \cap D, A \cup C)$ has order at most $|A \cap B|$.

Next, we will show that either $(B \cap C, A \cup D)$ or $(B \cap D, A \cup C)$ lies in $\mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$. Let us assume that the order of $(B \cap C, A \cup D)$ is at most $|A \cap B|$. We are done if this separation lies in $\mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$. So let us assume that it does not lie in $\mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$. In particular, we have $(B \cap C, A \cup D) \in P' \cap Q$. Since the orders of opposite corner separations sum to $2|A \cap B|$, either $(A \cap C, B \cup D)$ or $(B \cap D, A \cup C)$ has order at most $|A \cap B|$. In the first case, $(A \cap C, B \cup D)$ must lie in $\mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$ since otherwise $(A \cap C, B \cup D) \in Q$ and hence we get a contradiction to robustness of Q by the three separations $(A \cap C, B \cup D)$, $(B \cap C, A \cup D)$ and (C, D). So in each case, the order of $(B \cap D, A \cup C)$ is at most $|A \cap B|$. Using robustness of P', we obtain that $(B \cap D, A \cup C)$ lies in $\mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$. The case that the order of $(B \cap D, A \cup C)$ is at most $|A \cap B|$ is analogous. Thus, either $(B \cap C, A \cup D)$ or $(B \cap D, A \cup C)$ lies in $\mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$. Let us denote this separation by (E, F). Note that $X \cap (C \cap D)$ lies in the separator of (E, F). By Lemma 2.7 there is a component K of $E \setminus F$ such that $(K \cup N(K), V(G) \setminus K)$ lies in $\mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$. Then we have $X \cap (C \cap D) \subseteq N(K)$. This implies $K \subseteq X$. But then $(X \cup N(X), V(G) \setminus X)$ distinguishes two profiles in $\mathcal{P}$ while its order is less than $\kappa(\mathcal{P}, G)$, which is a contradiction. This shows (b). So we have verified that $(T, \mathcal{V})$ is a tree-decomposition. It is canonical as the set of separations induced by its edges is

$$\{(C \cup N(C), V(G) \setminus C) \mid C \in \mathcal{C}\} \cup \{(V(G) \setminus C, C \cup N(C)) \mid C \in \mathcal{C}\}.$$

Since the adhesion of $(T, \mathcal{V})$ is less than $\kappa(G, \mathcal{P})$, it only remains to show that the torso H of V_x is well-separable with respect to $\mathcal{P}_H$.

Proposition 4.1 (iv) implies $\kappa(\mathcal{P}, G) = \kappa(\mathcal{P}_H, H)$. Let $(A, B) \in \mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}_H, H)$. Let (A^G, B^G) be a separation of G of order $|A \cap B|$ that induces (A, B) on H as constructed in the proof of Proposition 4.1: if $N(C) \subseteq A$ for a component C of $G - V_x$, add C to A and otherwise to B. Then $(A^G, B^G) \in \mathcal{R}^{\kappa}_{\text{eff}}(\mathcal{P}, G)$. Let us suppose that there is a component C of $H - (A \cap B)$ with $N(C) \subsetneq A \cap B$. Let C^G be the component of $G - (A \cap B)$ that contains C. Since $C^G \cap V_x \neq \emptyset$, the construction of $(T, \mathcal{V})$ implies $N(C^G) = A \cap B$. Let $u \in N(C^G) \setminus N(C)$ and $v \in C^G$ a neighbour of u. Let $u' \in N(C)$ and v' be a neighbour of u' in C. Let P be a u-u' path all of whose inner vertices lie in C^G and whose first and last edges are uv, $v'u'$, respectively. We construct a u-u' path P' in H: whenever P leaves V_x, it does so through an adhesion set and must reenter V_x through the same adhesion set by (T3). We then replace this subpath by the edge between its two end vertices, which lies in H. The resulting path has no vertices outside of P and thus no vertex of $A \cap B$. But it contains all vertices of P that lie in V_x. So it contains v' and, thus, it contains a vertex of C. Hence, all inner vertices of P' lies in C. This contradicts $u \notin N(C)$ and shows that H is well-separable with respect to $\mathcal{P}_H$.

The 'Moreover-part', directly follows from (a) and (b). □

5 The case: fixed k

Let $\mathcal{N}$ be a set of separations and $\mathcal{P}$ be a set of profiles of a graph G. We call $\mathcal{N}$ *nice* (for $\mathcal{P}$) if it is a nested set of left-connected separations of order k in $\mathcal{R}(k, \mathcal{P})$ and we call $\mathcal{N}$ *distinguishing* if it distinguishes all k-distinguishable pairs of profiles in $\mathcal{P}$. An $\mathcal{N}$*-block* X is a maximal subset of $V(G)$ such that for every $(A, B) \in \mathcal{N}$ we have either $X \subseteq A$ or $X \subseteq B$ but not both. Note that X is the intersection of all sides A for $(A, B) \in \mathcal{N}$ that contain X. For an $\mathcal{N}$-block X, its *torso* is the graph induced by X in G with additional edges xy whenever x and y lie in a separator $A \cap B$ of a separation $(A, B) \in \mathcal{N}$ with $A \cap B \subseteq X$.

We call $\mathcal{N}$ *extendable (for $\mathcal{P}$)* if for any two (distinct) robust profiles in $\mathcal{P}$ of the same order, there is some separation of G distinguishing these two profiles efficiently that is nested with $\mathcal{N}$.

Theorem 5.1. *Let G be a well-separable graph with respect to a set $\mathcal{P}$ of robust principal $(k+1)$-profiles of G with $k = \kappa(\mathcal{P}, G)$. Then G has a tree-decomposition $(T, \mathcal{V})$ satisfying the following.*

1. *$(T, \mathcal{V})$ distinguishes any two robust profiles in $\mathcal{P}$;*
2. *$(T, \mathcal{V})$ is canonical with respect to $\mathcal{P}$;*
3. *$(T, \mathcal{V})$ is k-balanced.*

Proof. Our first aim is to construct a canonical set $\mathcal{N}$ that is nice and distinguishing.[3] We construct the set $\mathcal{N}$ by transfinite recursion. We set $\mathcal{N}_0 := \emptyset$. Assume we already constructed all sets $\mathcal{N}_\alpha$ for $\alpha < \beta$ such that each $\mathcal{N}_\alpha$ is nice and canonical and $\mathcal{N}_{\alpha_1} \subseteq \mathcal{N}_{\alpha_2}$ whenever $\alpha_1 < \alpha_2 < \beta$. If β is a limit ordinal we set $\mathcal{N}_\beta := \bigcup_{\alpha<\beta} \mathcal{N}_\alpha$. This set is nice and canonical as so are all sets $\mathcal{N}_\alpha$.

[3] The existence of a non-canonical set $\mathcal{N}$ follows from [1, Theorem 5.9]. Here we show how the proof of that theorem can be modified to give a canonical set $\mathcal{N}$.

Now assume that $\beta = \gamma + 1$ is a successor ordinal. If $\mathcal{N}_\gamma$ is distinguishing, we stop and set $\mathcal{N} := \mathcal{N}_\gamma$. Otherwise there are robust profiles $P, Q \in \mathcal{P}$ that are distinguished by a separation of order k in G but not by $\mathcal{N}_\gamma$. Hence the profiles P and Q have to live in the same $\mathcal{N}_\gamma$-block X. Let X' be the torso of X. As G is well-separable and $k = \kappa(\mathcal{P}, G)$, the assumptions of [1, Theorem 5.9] are satisfied. So [1, Theorem 5.9][4] says that the set $\mathcal{N}_\gamma$ is extendable for $\mathcal{P}$. Let $\mathcal{P}'$ be the set of profiles induced by $\mathcal{P}$ on X'. By Proposition 4.1 (i), the robust profiles P and Q induce robust profiles in the set $\mathcal{P}'$; in particular, $\mathcal{P}'$ is not empty. By Proposition 4.1 (iii), the set $R(k, \mathcal{P}', X')$ of k-relevant separations is not empty. By Lemma 4.2 $S(k, \mathcal{P}') = \emptyset$ and $R(k-1, \mathcal{P}') = \emptyset$. So by Lemma 2.8 the set $\mathcal{M}(X)$ of left-connected separations in $\mathcal{R}(k, \mathcal{P}', X')$ with minimum crossing number in $\mathcal{R}_{\text{lc}}(k, \mathcal{P}', X')$ is nested and not empty.

Similarly, as in the proof of Proposition 4.1, we extend the set $\mathcal{M}(X)$ of separations of the torso X' to a set of separations in the graph G: given $(A, B) \in \mathcal{M}(X)$, we obtain A^G from A by adding all components of $G - X$ that have a neighbour in $A \setminus B$ and we obtain B^G from B by adding all other components. By construction (A^G, B^G) is a left-connected separation of G of order k. Let $\mathcal{M}^G(X)$ be the set of all separations (A^G, B^G) for all (A, B) in $\mathcal{M}(X)$. We set $\mathcal{N}_\beta := \mathcal{N}_\gamma \cup \bigcup_Y \mathcal{M}^G(Y)$, where the union ranges over all $\mathcal{N}_\gamma$-blocks Y such that there are at least two profiles of $\mathcal{P}$ living in Y. This definition ensures that $\mathcal{N}_\beta$ is canonical. Let us prove

$$\mathcal{N}_\beta \text{ is nice.} \tag{‡}$$

The separation (A^G, B^G) is in $\mathcal{R}(k, \mathcal{P}, G)$ as it distinguishes the two robust profiles whose induced profiles in $\mathcal{P}'$ are distinguished by (A, B) in X'. It remains to prove that $\mathcal{N}_\beta$ is nested.

Any separation in $\mathcal{M}^G(X)$ is nested with any separation in $\mathcal{N}_\gamma$ by [1, Observation 4.22] (recall that this Observation[5] says that given a set $\mathcal{N}$ of nested separations, any separation of the $\mathcal{N}$-torso extended in the natural way to a separation of G is nested with $\mathcal{N}$). The same result [1, Observation 4.22] also implies that every separation in $\mathcal{M}^G(Y_1)$ is nested with any separation in $\mathcal{M}^G(Y_2)$ if $Y_1 \neq Y_2$.

Thus it suffices to show that any two separations (A_1^G, B_1^G) and (A_2^G, B_2^G) in $\mathcal{M}^G(X)$ are nested. The separations (A_1, B_1) and (A_2, B_2) are nested as they lie in $\mathcal{M}(X)$.

Let us consider the case $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$ first. Let K be a component of $G - X$ with a neighbour $z \in A_1 \setminus B_1$. If $z \notin A_2 \setminus B_2$, then we have $z \in B_2 \subseteq B_1$, which is a contradiction. Thus, every component of $G - X$ that has a neighbour in $A_1 \setminus B_1$ also has a neighbour in $A_2 \setminus B_2$. So we have $A_1^G \subseteq A_2^G$ and $B_2^G \subseteq B_1^G$.

The case $A_2 \subseteq A_1$ and $B_1 \subseteq B_2$ is analogous to the previous case.

Let us now consider the case $A_1 \subseteq B_2$ and $A_2 \subseteq B_1$. Then for every component K of $G - X$ that has a neighbour in $A_1 \setminus B_1$ we have $N(K) \subseteq A_1 \subseteq B_2$ as (A_1, B_1) is a separation of the torso X'; in particular $N(K) \cap (A_2 \setminus B_2) = \emptyset$.

[4] The 'In particular'part of that theorem is not applied here

[5] Recall that in that paper there are further technicalities related to [1, Observation 4.22]. These are only important to prove a lemma further below in that paper (where it is shown that extensions of separations of $\mathcal{M}(X)$ are nested). Here we make use of the assumption that the separations in the set $\mathcal{M}(X)$ are left connected and have the same order as the separations in the set $\mathcal{N}_\gamma$ and prove an analogue of that later lemma in the other paper 'by foot'.

So K does not lie in A_2^G but in B_2^G. This shows $A_1^G \subseteq B_2^G$. An analogous argument shows $A_2^G \subseteq B_1^G$.

The last case to consider is $B_1 \subseteq A_2$ and $B_2 \subseteq A_1$. If $(A_1, B_1) = (B_2, A_2)$, then we are in the above case $A_1 \subseteq B_2$ and $A_2 \subseteq B_1$. So we may assume that those two separations are distinct; in particular $B_1 \subsetneq A_2$ and $B_2 \subsetneq A_1$. First we show that there is a vertex v in $(A_2 \cap B_2) \setminus B_1$, see Figure 3. As the separation (A_2, B_2) is proper, there is a vertex w in $B_2 \setminus A_2$. This vertex w must be contained in $A_1 \setminus B_1$ as $B_1 \subseteq A_2$. As the connected set $A_1 \setminus B_1$ is not a subset of B_2, there must be a vertex v in the link $(A_2 \cap B_2) \setminus B_1$.

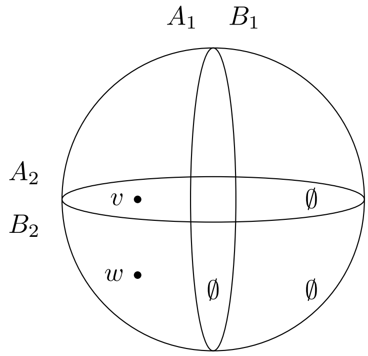

Figure 3: The corner diagram of (A_1, B_1) and (A_2, B_2). By nestedness the bottom right corner and its two adjacent links are empty. The vertex w exists as (A_2, B_2) is proper. The vertex v exists as the connected set $A_1 \setminus B_1$ is not a subset of B_2.

Our aim is to show $B_2^G \subseteq A_1^G$. So let K be a component of $G - X$ that is a subset of B_2^G. If K has a neighbour in $B_2 \setminus A_2$, then it also has a neighbour in the superset $A_1 \setminus B_1$, and thus X is included in A_1^G. Thus we may assume that K has no neighbour in $B_2 \setminus A_2$. By its choice K has no neighbour in $A_2 \setminus B_2$. So all its neighbours are in the separator $A_2 \cap B_2$. As (A_2, B_2) is in $\mathcal{R}(k, \mathcal{P})$ and $\mathcal{S}(k, \mathcal{P})$ is empty, the component K has the whole separator $A_2 \cap B_2$ in its neighbourhood. Thus the vertex v lies in the neighbourhood of K but also in $A_1 \setminus B_1$. So K is included in A_1^G. We have shown $B_2^G \subseteq A_1^G$. An analogous argument shows $B_1^G \subseteq A_2^G$. Thus (A_1^G, B_1^G) and (A_2^G, B_2^G) are nested.

All cases combined show that $\mathcal{N}_\beta$ is nested, which proves (‡).

Every separation (A^G, B^G) in $\mathcal{M}^G(X)$ distinguishes two robust profiles in $\mathcal{P}$ that are not distinguished by $\mathcal{N}_\gamma$. Indeed, there are profiles in $\mathcal{P}$ inducing profiles in the torso X' such that (A, B) distinguishes these induced profiles. So $\mathcal{M}^G(X)$ contains a separation distinguishing two profiles not distinguished by $\mathcal{N}_\gamma$. So $\mathcal{N}_\beta$ is strictly larger than $\mathcal{N}_\gamma$. As the sequence of the $\mathcal{N}_\alpha$ is strictly increasing but the number of separations of order k is bounded (by a function of the cardinality of the graph G), this recursions has to stop eventually. So there is some canonical, nice and distinguishing set $\mathcal{N}$.

Note that all separations in $\mathcal{N}$ are proper since they are all relevant. So by Lemma 2.3 the set $\mathcal{N}$ has property $(*)$ and thus induces by Theorem 3.2 a canonical tree-decomposition that distinguishes all robust profiles in $\mathcal{P}$. It is k-balanced by construction. □

6 Trees of tree-decompositions

In this section, we will prove the main theorem of this paper. Before we do that, we give the formal definition of trees of tree-decompositions.

A *rooted* tree is a pair (T, r) of a tree T and a vertex $r \in V(T)$, called the *root* of T. The *level* of a vertex $t \in V(T)$ is $d(t, r) + 1$. A *tree of tree-decompositions* of a graph G is a triple $((T, r), (G_t)_{t \in V(T)}, (T_t, \mathcal{V}_t)_{t \in V(T)})$ of a rooted tree (T, r), a family $(G_t)_{t \in V(T)}$ of graphs, one for every node of T with $G_r = G$, and a family $(T_t, \mathcal{V}_t)_{t \in V(T)}$ such that for every node $t \in V(T)$ the pair $(T_t, \mathcal{V}_t)$ is a tree-decomposition of G_t such that for every node $t \in V(T)$ the graphs assigned to its neighbours on the next level are distinct torsos of $(T_t, \mathcal{V}_t)$. The tree of tree-decompositions *distinguishes* two profiles (*efficiently*) if there exists a node $t \in V(T)$ such that some separation of G_t induced by $(T_t, \mathcal{V}_t)$ distinguishes the induced profiles (efficiently) and is induced by a separation of G that distinguishes the profiles (efficiently).

Remark 6.1. It was proved first in [3] that any tree of tree-decompositions of a finite graph can be stuck together into a single tree-decomposition in a canonical way. Our proof of Proposition 7.2 is inspired by the arguments of that paper (yet with somewhat different notation).

A tree of tree-decompositions $((T, r), (G_t)_{t \in V(T)}, (T_t, \mathcal{V}_t)_{t \in V(T)})$ is *canonical* if every $(T_t, \mathcal{V}_t)$ for $t \in V(T)$ is canonical and if the construction of the tree-decompositions $(T_s, \mathcal{V}_s)$ and $(T_t, \mathcal{V}_t)$ where s and t have the same level is the same, e. g. the tree-decompositions are obtained by the same constructive proof.

Theorem 6.2. *Let G be a graph and let $\mathcal{P}$ a set of distinguishable robust principal profiles such that for every $P \in \mathcal{P}$ there is some $\ell \in \mathbb{N} \cup \{\aleph_0\}$ such that P is an ℓ-profile. Then there exists a tree of tree-decompositions $((T, r), (G_t)_{t \in V(T)}, (T_t, \mathcal{V}_t)_{t \in V(T)})$ that is canonical with respect to $\mathcal{P}$ such that the following hold.*

(1) *The tree of tree-decompositions distinguishes $\mathcal{P}$ efficiently;*

(2) *if $t \in V(T)$ is on level $2k$, then $(T_t, \mathcal{V}_t)$ is k-balanced;*

(3) *nodes t on level $2k$ have $|V(T_t)|$ neighbours on level $2k + 1$ and the graphs assigned to them are all torsos of $(T_t, \mathcal{V}_t)$;*

(4) *if $t \in V(T)$ is on level $2k + 1$, the tree-decomposition $(T_t, \mathcal{V}_t)$ has adhesion at most k;*

(5) *nodes on level $2k + 1$ have at most one neighbour on level $2k + 2$; their associated graphs are well-separable.*

Note that Theorem 6.2 (1) implies Theorem 1.3 and hence Theorem 1.1.

Proof of Theorem 6.2. We construct the tree of tree-decompositions recursively. More precisely, we have one step for every node of the rooted tree (T, r), which is constructed recursively during the process. Starting with the root r, for each node of T in its step we define a tree-decomposition of its associated graph and define its neighbours at the next level of T and their associated graphs.

We start by assigning the graph G to the root; that is, we set $G_r := G$. Assume that a node t of T is defined and we already assigned a graph G_t to this node. First we consider the case that the node t is on an odd level. We denote its level by $2k+1$. Let $\mathcal{P}_t$ be the set of $(k+1)$-profiles induced by $\mathcal{P}$ on G_t. If $\mathcal{P}_t$ is empty, let $(T_t, \mathcal{V}_t)$ be the trivial tree-decomposition with a unique node and let t have no neighbour on the next level. If $\mathcal{P}_t$ is not empty but $k+1 \neq \kappa(\mathcal{P}_t, G_t)$ (we note that this always includes the case that the set $\mathcal{P}_t$ consists of a single profile), let $(T_t, \mathcal{V}_t)$ be the trivial tree-decomposition of G_t and let t have a unique neighbour on the next level whose associated graph is G_t. If $\mathcal{P}_t$ is not empty and $k+1 = \kappa(\mathcal{P}_t, G_t)$, let $(T_t, \mathcal{V}_t)$ be the canonical tree-decomposition of Proposition 4.3. It has adhesion at most k by that proposition. Only the unique node of $(T_t, \mathcal{V}_t)$ whose torso contains the induced profiles of $\mathcal{P}_t$ we add a neighbour on the next level whose associated graph is that torso. Note that this torso is well-separable by Proposition 4.3. By construction, (4) and (5) hold.

If $t \in V(T)$ is on level $2k$, let $\mathcal{P}_t$ be the set of $(k+1)$-profiles induced by $\mathcal{P}$ on G_t. If $k \neq \kappa(\mathcal{P}, G)$, let $(T_t, \mathcal{V}_t)$ be the trivial tree-decomposition of G_t, i. e. let T_t be a tree with one vertex and $\mathcal{V}_t = \{V(G_t)\}$. If $k = \kappa(\mathcal{P}, G)$, then G_t is well-separable by construction. Let $(T_t, \mathcal{V}_t)$ be the canonical tree-decomposition from Theorem 5.1 for G_t and $\mathcal{P}_t$. Then t gets $|V(T_t)|$ neighbours on the next level whose associated graphs are the torsos of $(T_t, \mathcal{V}_t)$. Then (2) and (3) hold for t by construction. This completes the construction of the tree of tree-decompositions $((T, r), (G_t)_{t \in V(T)}, (T_t, \mathcal{V}_t)_{t \in V(T)})$. By Proposition 4.1 (iv), it distinguishes $\mathcal{P}$ efficiently.

Note that the tree of tree-decompositions is canonical since we chose for nodes on even level the same theorem and for nodes on odd levels the same proposition to obtain canonical tree-decompositions. □

As we already mentioned in the introduction, profiles are a generalisation of ends, k-blocks and tangles in finite graphs. As a corollary, it will follow that Theorem 6.2 holds for the set of profiles induced by ends, the set of distinguishable profiles induced by robust k-blocks and the set of profiles induced by principal tangles of order k. Here we will give brief definitions of all these concepts and discuss how they induce profiles.

Let G be a graph. A *ray* is a one-way infinite path and two rays in G are *equivalent* if for every finite $S \subseteq V(G)$ there is a component of $G - S$ such that both rays have all but finitely many vertices in that component. This is an equivalence relation whose equivalence classes are the *ends* of G. Let ω be an end of G. We say that ω *lies* in a component of $G - S$ if for every ray $R \in \omega$ all but finitely many vertices of ω lie in $G - S$. Let P_ω be the set of separations (A, B) of finite order such that the component of $G - (A \cap B)$ that contains ω lies in B. Then (P1) and (P2) are true by definition. Obviously, P_ω is robust and principal. Thus, sets of ends define sets of robust profiles.

Corollary 6.3. *Let G be a graph and let Ω be a set of ends of G. Let $\mathcal{P}$ be the set of profiles defined by Ω. Then there exists a tree of tree-decompositions distinguishing all ends in Ω.* □

A *k-block* of a graph G is a maximal set b of at least k vertices such that no two of its vertices can be separated in G by fewer than k vertices. Then for every separation (A, B) of order at most $k-1$ we have either $b \subseteq A$ or $b \subseteq B$. Let P_b be the set of separations (A, B) of order at most $k-1$ with $b \subseteq B$. It is easy to see that P_b is a principal profile. We call b *robust* if P_b is robust and two k-blocks are *distinguishable* if their profiles are distinguishable.

Corollary 6.4. *Let G be a graph and let $\mathcal{B}$ be a set of distinguishable robust k-blocks. Let $\mathcal{P}$ be the set of profiles defined by $\mathcal{B}$. Then there exists a tree of tree-decompositions distinguishing all profiles in $\mathcal{P}$.* □

A *principal tangle of order k* in a graph G is a set θ of separations of order at most $k-1$ satisfying the following conditions.

(θ1) For all $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \theta$, we have

$$G \neq G[A_1] \cup G[A_2] \cup G[A_3],$$

where $G[A_i]$ is the graph induced by the vertex set A_i for $i \in \{1, 2, 3\}$;

(θ2) if X is a set of at most k vertices, there is a component C of $G - X$ such that $(G - C, C \cup X) \in \theta$;

(θ3) for all separations (A, B) of order at most $k-1$ we have either $(A, B) \in \theta$ or $(B, A) \in \theta$.

Recall that two tangles θ_1, θ_2 are *distinguishable* if there is a separation $(A, B) \in \theta_1$ with $(B, A) \in \theta_2$.

Proposition 6.5. *Let G be a graph. Every principal tangle of order k is a robust principal k-profile.*

Proof. Let θ be a principal tangle and let $(C, D) \leq (A, B)$ with $(A, B) \in \theta$. If $(D, C) \in \theta$, then $G = G[A] \cup G[D]$ which violates (θ1). So θ is consistent.

To see (P2), let $(A, B), (C, D) \in \theta$. Suppose for a contradiction that $(B \cap D, A \cup C)$ is in θ. Then the three small sides A, C and $B \cup D$ together cover the whole graph G, a contradiction to property (θ1). Thus ($P2$) is satisfied.

By assumptions tangles of order k only contain separations of order at most $k-1$. So by property (θ3), the principal tangle θ is a k-profile.

Next we show that θ is principal as a profile. So let $((A_i, B_i))_{i \in I}$ be a family of separations in θ with $A_i \cap B_i = A_j \cap B_j$ for all $i, j \in I$. Denote the common separator $A_i \cap B_i$ of these separations by X. By property (θ2), there is a component C of $G - X$ such that $(G - C, C \cup X) \in \theta$. By consistency (which is proved above), we deduce that $X \subseteq B_i \setminus A_i$ for all $i \in I$. Hence θ is principal as a profile.

To prove robustness of θ, let $(A, B) \in \theta$ and (C, D) be a separation such that $(B \cap C, A \cup D)$ and $(B \cap D, A \cup C)$ have order less than k. If both $(B \cap C, A \cup D)$ and $(B \cap D, A \cup C)$ belong to θ, then we have

$$G = G[A] \cup [B \cap C] \cup G[B \cap D]$$

which is impossible by (θ1). Thus, θ is robust. □

Corollary 6.6. *Let G be a graph and let $\mathcal{P}$ be a set of distinguishable principal tangles of finite order. Then there exists a tree of tree-decompositions distinguishing all elements in $\mathcal{P}$.* □

7 Locally finite graphs

In this section, we apply Theorem 6.2 to the special case of locally finite graphs. While we will show that for fixed $k \in \mathbb{N}$ there is a canonical tree-decomposition distinguishing all k-distinguishable profiles efficiently (Theorem 7.3), it is not possible to extend this to all distinguishable profiles as shown directly below that theorem. But as a further positive result, Theorem 7.4 shows that we can at least find a nested set of separations distinguishing the distinguishable profiles. This nested set does not define a tree-decomposition as it does not satisfy ($*$). Note that for locally finite graphs, all profiles are principal. For a good introduction to tree-like properties that go beyond tree-decompositions we refer the reader to [7].

This whole section is a straightforward extension of the ideas of [3] from finite to locally finite graphs. A more detailed analysis of this procedure can be found in there.

A separation (A, B) of a graph is *tight* if there are components C_A of $A \setminus B$ and C_B of $B \setminus A$ such that every vertex in $A \cap B$ has neighbours in C_A and in C_B. By applying Lemma 2.2 to the pairs of vertices in the neighbourhood of a vertex v, we get the following corollary.

Proposition 7.1. *Let G be a locally finite graph, let $v \in V(G)$ and let $k \in \mathbb{N}$. Then there are only finitely many tight separations of order k with v in their separator.* □

For a tree T and be a subset E of $E(T)$, we denote by T/E the tree obtained by contracting all edges of E. A tree-decomposition $(T', \mathcal{V}')$ of G is a *refinement* of a tree-decomposition $(T, \mathcal{V})$ of G if there is a family of disjoint subtrees $(T_i)_{i \in I}$ of T' covering $V(T')$ such that the following holds:

(R1) $T = T' / \bigcup_{i \in I} E(T_i)$;

(R2) $\bigcup_{s \in T_i} V'_s = V_t$, where t is the node of T obtained from the contraction of $E(T_i)$.

If T is a finite tree, then it is well-known that there is either a unique vertex or unique edge that lies in the middle of every path of maximum length in T. We call this vertex or edge the *central* vertex or edge of T. It is preserved by all automorphisms of T.

Proposition 7.2. *Let G be a locally finite graph and $(T, \mathcal{V})$ be a canonical tree-decomposition of G of finite adhesion. For every torso H_t of $(T, \mathcal{V})$ let $(T^t, \mathcal{V}^t)$ be a canonical tree-decomposition of H_t of finite adhesion such that every separation (A, B) induced by $(T^t, \mathcal{V}^t)$ is tight and such that no two of these induced separations have the same separators. Assume that all tree-decomposition s $(T^t, \mathcal{V}^t)$ are obtained by the same canonical construction. Then there is a canonical tree-decomposition $(T', \mathcal{V}')$ that is a refinement of $(T, \mathcal{V})$ with respect to a family $(R^t)_{t \in V(T)}$, where R^t is a subdivision of T^t, such that every adhesion set of $(T', \mathcal{V}')$ is an adhesion set of either $(T, \mathcal{V})$ or one of the tree-decompositions $(T^t, \mathcal{V}^t)$.*

Proof. We are going to construct a new tree-decomposition (T', V') of G by gluing together the tree-decompositions $(T^t, \mathcal{V}^t)$ along the tree T in a canonical way. Let $tt' \in E(T)$. Let $S^{tt'}$ be the maximal subtree of T^t such that all V^t_s

with $s \in V(S^{tt'})$ contain $V_t \cap V_{t'}$. Then also all adhesion sets corresponding to edges with both its incident vertices in $S^{tt'}$ contain $V_t \cap V_{t'}$. As the induced separations of these edges are all distinct and tight, Proposition 7.1 implies that $S^{tt'}$ is finite. As we mentioned above, $S^{tt'}$ has a unique central vertex or edge, which is fixed by all automorphisms of $S^{tt'}$.

Let E^t, U^t be the set of edges, of vertices of T^t that are a central edge, a central vertex, for some tree $S^{tt''}$ with $tt'' \in E(T)$, respectively. We subdivide all edges in E^t once and obtain a new tree R^t. Let $\mathcal{W}^t$ be a set of vertex sets, one for every $s \in V(R^t)$ such that $W_s = V^t_s$ if s is a vertex of T^t and such that W_s is the adhesion set corresponding to the edge $e \in E(T^t)$ if s is the vertex that subdivided e. It directly follows from the fact that $(T^t, \mathcal{V}^t)$ is a tree-decomposition that also $(R^t, \mathcal{W}^t)$ is one.

Let T' be the graph obtained from the disjoint union of all trees R^t for $t \in V(T)$ by adding for every edge $tt' \in E(T)$ an edge between the central vertex or vertex on the subdivided central edge of $S^{tt'}$ and that of $S^{t't}$. It is easy to see that contracting the subgraphs R^t of T' results in T and hence T' is a tree. Let $\mathcal{V}'$ be the union of the sets $\mathcal{W}^t$ for all $t \in V(T)$.

That $(T, \mathcal{V})$ and all $(R^t, \mathcal{W}_t)$ are tree-decompositions implies that $(T', \mathcal{V}')$ is one, too. The tree-decomposition $(T', \mathcal{V}')$ is canonical as the same is true for $(T, \mathcal{V})$ and all $(R^t, \mathcal{W}^t)$ and by construction of T'. As the properties (R1) and (R2) hold by construction, the assertion follows. □

Now we are able to prove the following, which implies Corollary 1.2 already mentioned in the Introduction.

Theorem 7.3. *Let G be a locally finite graph and let $k \in \mathbb{N}$. Let $\mathcal{P}$ be a set of k-distinguishable robust profiles each of which is an ℓ-profile for some $\ell \in \mathbb{N} \cup \{\infty\}$. Then there is a tree-decomposition that is canonical with respect to $\mathcal{P}$ and that distinguishes $\mathcal{P}$ efficiently.*

Proof. Let $((T, r), (G_t)_{t \in V(T)}, (T_t, \mathcal{V}_t)_{t \in V(T)})$ be a tree of tree-decompositions of G with the properties of Theorem 6.2. Since $\mathcal{P}$ is k-distinguishable, the maximum level of (T, r) is $2k+1$ by construction. So $2k+1$ iterated applications of Proposition 7.2, where in each step we use all tree-decomposition of the next level of T, lead to a tree-decomposition $(T', \mathcal{V})$ of G. Since the tree of tree-decompositions distinguishes all k-distinguishable profiles of $\mathcal{P}$ efficiently, so does $(T', \mathcal{V})$. □

We cannot omit the condition 'k-distinguishable' in Theorem 7.3 as [1, Example 3.7] shows.

Theorem 7.4. *Let G be a locally finite graph and let $\mathcal{P}$ be a set of distinguishable robust profiles such that for every $P \in \mathcal{P}$ there is some $\ell \in \mathbb{N} \cup \{\infty\}$ such that P is an ℓ-profile. Then there is a nested set of separations that is canonical with respect to $\mathcal{P}$ and that distinguishes $\mathcal{P}$ efficiently.*

Proof. Let $((T, r), (G_t)_{t \in V(T)}, (T_t, \mathcal{V}_t)_{t \in V(T)})$ be a tree of tree-decompositions of G with the properties of Theorem 6.2. For $k \in \mathbb{N}$, let $(T^k, \mathcal{V}^k)$ be the tree-decomposition obtained by applying Proposition 7.2 iteratively for the the subtree of (T, r) consisting of all vertices on the first $2k$ levels. By construction, $(T^{k+1}, \mathcal{V}^{k+1})$ is a refinement of $(T^k, \mathcal{V}^k)$. Let $\mathcal{N}^k$ be the nested set of separations induced by $(T^k, \mathcal{V}^k)$. Then $\mathcal{N}^k \subseteq \mathcal{N}^{k+1}$. Thus, $\bigcup_{n \in \mathbb{N}} \mathcal{N}^n$ is a nested set of separations. It distinguishes $\mathcal{P}$ efficiently as the tree of tree-decompositions does so and it is canonical, as all steps in this proof keep this property and the tree of tree-decompositions we started with is canonical. $\square$

Remark 7.5. *While it is fairly straightforward to show that any nested set of separations can be turned into a tree of tree-decompositions, we highlight that canonical nested sets of separations distinguishing all ends do not always exist (an example is provided in Figure 1).*

8 Concluding remarks

In the exposition in the Introduction we focused on main ideas and gave some theorems in a more concrete formulation. Here we summarise some results that are slightly stronger in details than those stated in the Introduction.

Remark 8.1. *Theorem 7.3 is also true if we relax 'locally finite' to the property that the removal of finitely many vertices only leaves finitely many components. The proof is essentially the same.*

Remark 8.2. *In this paper we proved the strengthening of Theorem 1.3 for arbitrary subsets of the set of robust principal profiles, and the tree of tree-decompositions we obtain is canonical with respect to that subset, compare Theorem 6.2.*

Remark 8.3. *Theorem 6.2 easily implies the following variant. (To see this one has to simply 'move separations of low order at higher levels down to lower levels'. We leave the details to the reader.)*

Let G be a graph and $\mathcal{P}$ a set of distinguishable robust principal profiles each of which is an ℓ-profile for some $\ell \in \mathbb{N} \cup \{\aleph_0\}$. Then there exists a tree of tree-decompositions $((T, r), (G_t)_{t\in V(T)}, (T_t, \mathcal{V}_t)_{t\in V(T)})$ that is canonical with respect to $\mathcal{P}$ such that the following hold.

(1) *The tree of tree-decompositions distinguishes $\mathcal{P}$ efficiently;*

(2) *if $t \in V(T)$ is on level k, then $(T_t, \mathcal{V}_t)$ is k-balanced;*

(3) *nodes t at all levels have $|V(T_t)|$ neighbours on the next level and the graphs assigned to them are all torsos of $(T_t, \mathcal{V}_t)$.*

Further related work. If we consider the class of quasi-transitive graphs, then Hamann, Lehner, Miraftab and Rühmann [12] proved with the aid of our main result that those graphs that admit a canonical tree-decomposition distinguishing all their ends are the accessible graphs, that is, the graphs that are obtained from finite or one-ended quasi-transitive graphs by tree amalgamations of finite adhesion and finite identification respecting the group actions. This result in turn is used in [9, 10, 11] to investigate quasi-isometry types, homeomorphism types of hyperbolic boundaries and asymptotic dimension of quasi-transitive locally finite graphs. Also Miraftab and Stavropoulos [14] used canonical tree-decompositions to classify all infinite groups which admit cubic Cayley graphs of connectivity 2 in terms of splittings over a subgroup.

Acknowledgement

We thank C. Elbracht and J. Kneip for pointing out an error in an earlier version of this paper.

References

[1] J. Carmesin, *All graphs have tree-decompositions displaying their topological ends*, Combinatorica **39** (2019), no. 3, 545–596.

[2] J. Carmesin, R. Diestel, F. Hundertmark, and M. Stein, *Connectivity and tree structure in finite graphs*, Combinatorica **34** (2014), no. 1, 1–35.

[3] J. Carmesin and P. Gollin, *Canonical tree-decompositions of a graph that display its k-blocks*, J. Combin. Theory Ser. B **122** (2017), 1–20.

[4] R Diestel, *Graph Theory* (5th edition), Springer-Verlag, 2017.

[5] R. Diestel, F. Hundertmark, and S. Lemanczyk, *Profiles of separations: in graphs, matroids and beyond*, Combinatorica **39** (2019), 37–75.

[6] M.J. Dunwoody and B. Krön, *Vertex cuts*, J. Graph Theory **80** (2015), 136–171.

[7] J Pascal Gollin and Jakob Kneip, *Representations of infinite tree sets*, Order (2020), 1–18.

[8] R. Halin, *Lattices of cuts in graphs*, Abh. Math. Sem. Univ. Hamburg **61** (1991), 217–230.

[9] M. Hamann, *Tree amalgamations and quasi-isometries*, arXiv:1812.04987, 2018.

[10] ———, *Asymptotic dimension of multi-ended quasi-transitive graphs*, arXiv:1912.02683, 2019.

[11] ———, *Tree amalgamations and hyperbolic boundaries*, arXiv:1909.13749, 2019.

[12] M. Hamann, F. Lehner, B. Miraftab, and T. Rühmann, *A Stallings' type theorem for quasi-transitive graphs*, arXiv:1812.06312, 2018.

[13] B. Krön, *Cutting up graphs revisited – a short proof of Stallings' structure theorem*, Groups Complex Cryptol **2** (2010), 213–221.

[14] B. Miraftab and K. Stavropoulos, *Splitting groups with cubic Cayley graphs of connectivity two*, arXiv:1902.02307, 2019.

[15] N. Robertson and P.D. Seymour, *Graph minors. X. Obstructions to tree-decomposition*, J. Combin. Theory (Series B) **52** (1991), 153–190.