Two characterisations of accessible quasi-transitive graphs

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Abstract

We prove two characterisations of accessibility of locally finite quasi-transitive connected graphs. First, we prove that any such graph G is accessible if and only if its set of separations of finite order is an $\mathrm{Aut}(G)$-finitely generated semiring. The second characterisation says that G is accessible if and only if every process of splittings in terms of tree amalgamations stops after finitely many steps.

1 Introduction

Tree amalgamations are a graph product that offers a way to construct graphs that are, in general, multi-ended. (We refer to Section 2 for its definition.) On the other hand, every suitable multi-ended graph can be written as a non-trivial tree amalgamation, see Theorem 1.1. Note that tree amalgamations are a graph theoretic analogue of the following two group products: free products with amalgamation and HNN-extensions. Also, Theorem 1.1 is a graph theoretic version of Stallings'splitting theorem of finitely generated groups [5].

Theorem 1.1. [4, Theorem 5.5] *Every multi-ended quasi-transitive locally finite connected graph is a non-trivial tree amalgamation of two quasi-transitive locally finite connected graphs of finite adhesion and finite identification, distinguishing ends and respecting the action of the involved groups.*

*Supported by the Heisenberg-Programme of the Deutsche Forschungsgemeinschaft (DFG Grant HA 8257/1-1).

When G is a tree amalgamation of G_1 and G_2 with all properties as in Theorem 1.1, then we say that (G_1, G_2) is a *factorisation* of G and G *splits* into G_1 and G_2. More generally, a tuple $(G_1, \ldots, G_i)$ of quasi-transitive locally finite connected graphs is a *factorisation* of G if G is obtained by iterated non-trivial tree amalgamations of finite adhesion, finite identification and respecting the actions of the involved groups of all these graphs G_i. A factorisation is *terminal* if all its graphs have at most one end. We call a graph *accessible* if it has a terminal factorisation.

The question arises which quasi-transitive locally finite connected graphs are accessible. A result of [4] says that such graphs are accessible if and only if they are accessible in the sense of Thomassen and Woess [7]. (We refer to Section 2 for their definition of accessibility.) By examples of Dunwoody [2, 3], it is known that there are inaccessible quasi-transitive locally finite connected graphs. We are looking for characterisation results for accessibility and in this paper we are going to prove two such results.

The first result deals with the set $\mathcal{S}(G)$ of all separations of finite order of quasi-transitive locally finite connected graphs G. (For the definition of separations and related notions, we refer to Section 3.) This set equipped with two natural operations is a semiring and the automorphisms of G induce an action on $\mathcal{S}(G)$. We prove that G is accessible if and only if there are finitely many separations in $\mathcal{S}(G)$ that generate together with their $\mathrm{Aut}(G)$-images the whole semiring $\mathcal{S}(G)$. We then say that $\mathcal{S}(G)$ is $\mathrm{Aut}(G)$-*finitely generated*. This characterisation can be considered as a result analogous to [1, Corollary IV.7.6] for the set of separations instead of the cut space.

Before we explain the second characterisation, let us look at factorisations once more. If (G_1, G_2) is a factorisation of G, we may ask if one of these factors has again more than one end. If so, we can apply Theorem 1.1 to that factor (and the stabiliser of that factor in $\mathrm{Aut}(G)$ as group acting quasi-transitively on it) and obtain a factorisation of it. We can repeat this *process of splittings* as long as there are factors with more than one end. It is clear from the definition that some process of splittings stops if and only if the graph has a terminal factorisation and thus is accessible. It was conjectured in [4] that the property of stopping of the process of splittings does not depend on the particular splittings. To be precise, it was conjectured that one process of splittings stops after finitely many steps if and only if every process does this. Our second characterisation is the confirmation of this conjecture. So we are going to prove the following theorem.

Theorem 1.2. *Let G be a quasi-transitive locally finite connected graph. Then the following statements are equivalent:*

(i) *G is accessible.*

(ii) *$\mathcal{S}(G)$ is an $\mathrm{Aut}(G)$-finitely generated semiring.*

(iii) *Every process of splittings of G must end after finitely many steps.*

Our paper is structured as follows. In Section 2 we are going to define tree amalgamations and all related notions. In Section 3, we are going to prove that (i) implies (iii) of Theorem 1.2. In Section 4, we investigate the semiring $\mathcal{S}(G)$ and prove a major step for the equivalence of (i) and (ii) of Theorem 1.2. In Section 5, we fill in the remaining gaps of the proof of Theorem 1.2.

2 Tree amalgamations

In this section, we will state all notations and results that are needed in the context of tree amalgamations.

A tree T with the canonical bipartition $\{V_1, V_2\}$ of its vertex set is called (p_1, p_2)-*semiregular* if all vertices in V_i have degree p_i for $i = 1, 2$.

Let $e = xy$ be an edge of a graph G and let v_e be a new vertex. Let G' be the graph with vertex set $(V(G) \setminus \{x, y\}) \cup \{v_e\}$ and edges between $u, v \in V(G') \setminus \{v_e\}$ if and only if $uv \in E(G)$ and between $u \in V(G') \setminus \{v_e\}$ and v_e if and only if u is adjacent to either x or y in G. Then G' is the graph obtained by *contracting* the edge e. If E is a subset of $E(G)$, then we denote by G/E the graph obtained by contracting all edges in E.

Let G_i be a graph for $i = 1, 2$. Let I_1 and I_2 be disjoint sets. Let every $V(G_i)$ have a family $(S^i_k)_{k \in I_i}$ of subsets such that all these subsets have the same cardinality. For all $k \in I_1$ and $\ell \in I_2$, let $\phi_{k\ell}$ be a bijective map from S^1_k to S^2_ℓ. We set $\phi_{\ell k} := \phi_{k\ell}^{-1}$ and call the maps $\phi_{k\ell}$ and $\phi_{\ell k}$ *bonding maps*.

Let T be a $(|I_1|, |I_2|)$-*semiregular* tree with the canonical bipartition $\{V_1, V_2\}$ such that the vertices in V_i have degree $|I_i|$. Let $D(T)$ be the set of oriented edges of $E(T)$, i.e. $D(T) = \{\vec{uv} \mid uv \in E(T)\}$. If $\vec{e} = \vec{uv} \in D(T)$, then we denote by $\overleftarrow{e} := \overleftarrow{vu}$ its reverse. Let $v \in V_i$ and let E_v be the set of all edges in $D(T)$ starting at v. Let $f \colon D(T) \to I_1 \cup I_2$ be a labelling such that its restriction to E_v is a bijection to I_i.

For every $i \in \{1, 2\}$ and every $v \in V_i$, let G_v be a copy of G_i. Denote by S^v_k the corresponding copies of S^i_k in $V(G_v)$. Let $G_1 + G_2$ be the graph obtained from the disjoint union of the graphs G_v for all $v \in V(T)$ by adding new edges between each $x \in S^u_k$ and $\phi_{k\ell}(x) \in S^v_\ell$ for every edge $\vec{e} = \vec{uv}$ with $f(\vec{e}) = k$ and $f(\overleftarrow{e}) = \ell$. The new edges do not depend on the orientation of $\vec{e}$ because of $\phi_{\ell k} = \phi_{k\ell}^{-1}$. Let F be the set of these new edges of $G_1 + G_2$. The *tree amalgamation* $G_1 *_T G_2$, or just $G_1 * G_2$, of the graphs G_1 and G_2 over the *connecting tree* T is defined as $(G_1 + G_2)/F$. Let $\pi \colon V(G_1 + G_2) \to V(G_1 * G_2)$ be the canonical map that maps each $x \in V(G_1 + G_2)$ to the vertex obtained from x after all the contractions.

The sets S^i_k and their canonical images in $G_1 * G_2$ are the *adhesion sets* of the tree amalgamation. The tree amalgamation has *finite adhesion* if one (and hence all) of its adhesion sets are finite. We call a tree amalgamation $G_1 *_T G_2$ *trivial* if for some $v \in V(T)$ the restriction of π to G_v is a bijection. Note that if the tree amalgamation has finite adhesion, then it is trivial if, for some $i \in \{1, 2\}$, the set $V(G_i)$ is the only adhesion set of G_i and $|I_i| = 1$.

For a vertex $x \in V(G_1 * G_2)$ let T_x be the maximal subtree of T such that every node of T_x contains a vertex y with $\pi(y) = x$. The *identification size* of a vertex $x \in V(G_1 * G_2)$ is the cardinality of $V(T_x)$. The tree amalgamation has *finite identification* if the identification sizes of its vertices are bounded.

So far, the tree amalgamation do not interact with any group action. In the following, we define some notions that ensure that tree amalgamations of quasi-transitive graphs that satisfy this notion are again quasi-transitive, see [4, Lemma 5.3].

For $i = 1, 2$, let Γ_i be a group that acts on G_i. Let $\{i, j\} = \{1, 2\}$. The tree amalgamation *respects* $\gamma \in \Gamma_i$ if there is a permutation π of I_i such that for every $k \in I_i$ there exists $\ell \in I_j$ and τ in the setwise stabiliser of S_ℓ in Γ_j such that

$$\phi_{k\ell} = \tau \circ \phi_{\pi(k)\ell} \circ \gamma \mid_{S_k} .$$

The tree amalgamation *respects* Γ_i if it respects every $\gamma \in \Gamma_i$.

Let $k \in I_i$ and let $\ell, \ell' \in I_j$. The bonding maps from k to ℓ and ℓ' are *consistent* if there exists $\gamma \in \Gamma_j$ such that

$$\phi_{k\ell} = \gamma \circ \phi_{k\ell'}.$$

The bonding maps between $J_i \subseteq I_i$ and $J_j \subseteq I_j$ are *consistent* if they are consistent for all $k \in J_i$ and $\ell, \ell' \in J_j$.

The tree amalgamation $G_1 * G_2$ is of *Type 1 respecting the actions of* Γ_1 *and* Γ_2 if the following holds:

(i) the tree amalgamation respects Γ_1 and Γ_2;

(ii) the bonding maps between I_1 and I_2 are consistent.

The tree amalgamation $G_1 * G_2$ is of *Type 2 respecting the actions of* Γ_1 *and* Γ_2 if the following holds:

(o) $G_1 = G_2 =: G$, $\Gamma_1 = \Gamma_2 =: \Gamma$, $I_1 = I_2 =: I$,[1] and there exists $J \subseteq I$ such that $f(\vec{e}) \in J$ if and only if $f(\overleftarrow{e}) \notin J$;

(i) the tree amalgamation respects Γ;

(ii) the bonding maps between J and $I \setminus J$ are consistent.

The tree amalgamation $G_1 * G_2$ *respects the actions (of* Γ_1 *and* Γ_2*)* if it is of either Type 1 or Type 2 respecting the actions Γ_1 and Γ_2.

A *ray* is a one-way infinite path. Two rays are *equivalent* if for every finite $S \subseteq V(G)$ both rays have all but finitely many vertices in the same component of $G - S$. This is an equivalence relation whose equivalence classes are the *ends* of G An end is *thick* is is contains infinitely many pairwise disjoint rays. A *double ray* is a two-way infinite path.

The tree amalgamation $G = G_1 * G_2$ *distinguishes* ends if there is some adhesion set $S_k^u = S_\ell^v$ for adjacent $u, v \in V(T)$ such that for every component C of $T - uv$ the graph induced by $\bigcup_{w \in C} G_i^w$ contains an end.

[1] Formally we asked I_1 and I_2 to be disjoint. We can guarantee this easily by using appropriate bijections.

A graph G is *accessible in the sense of Thomassen and Woess* if there is an $n \in \mathbb{N}$ such that any two distinct ends of G are separable by at most n edges, that is, there are n edges such that every double ray between these two ends contains one of those n edges.

Let Γ be a group acting on a tree T. The action is *inversion-free* if there is no edge uv of T and no $\gamma \in \Gamma$ such that $\gamma(u) = v$ and $\gamma(v) = u$. We then also say that Γ acts on T *without inversion*.

Proposition 2.1. *Let G and G_1 be quasi-transitive locally finite connected graphs such that $G = G_1 *_T G_1$ is a tree amalgamation of Type 1 respecting the group actions such that the induced action of* $\mathrm{Aut}(G)$ *on the connecting tree T is with inversion of the edges. Then there exists a finite connected graph $G_2 \ncong G_1$ such that $G = G_1 * G_2$.*

Furthermore, every process of splittings of G that starts with (G_1, G_1) stops if and only if every process of splittings of G that starts with (G_1, G_2) stops.

Proof. Let S be an adhesion set of the tree amalgamation in some G_u with $u \in V(T)$ and let $S' \subseteq V(G_u)$ be connected and finite with $S \subseteq S'$. Let $\varphi \in \mathrm{Aut}(G)$ such that it reverses the edge $uv \in E(T)$ whose adhesion set is S. Then we have $S = \varphi(S)$. If $S' \neq V(G_u)$, let G_2 be the subgraph of G induced by $\pi(S') \cup \varphi(\pi(S'))$. If $S' = V(G_u)$, let G_2 be the subgraph of G induced by $\pi(G_u)$ and $\pi(G_v)$. In both cases, the graph G_2 is finite and connected and the tree amalgamation $G_1 * G_2$ is G with adhesion sets the copies of S' in the first case and of $V(G_1)$ in the second case.

The additional assertion holds since both processes stop if and only if every process of splittings of G_1 stops. □

3 Iterated splittings

Let G be a graph. A *tree-decomposition* of G is a pair $(T, \mathcal{V})$ of a tree T and a set of vertex sets V_t of G, one for every node of T, such that the following hold:

(T1) $V(G) = \bigcup_{t \in V(T)} V_t$;

(T2) for every $e \in E(T)$ there exists $t \in V(T)$ with $e \subseteq V_t$;

(T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ for all $t_1, t_2, t_3 \in V(T)$ such that t_2 separates t_1 and t_3.

The elements of $\mathcal{V}$ are the *parts* of the tree-decomposition. The sets $V_t \cap V_{t'}$ for edges $tt' \in E(T)$ are the *adhesion sets* of $(T, \mathcal{V})$. If all adhesion sets are finite we say that $(T, \mathcal{V})$ has *finite adhesion*.

A *separation* of G is an ordered pair (A, B) such that $G[A] \cup G[B] = G$, that is, $V(G) = A \cup B$ and there is no edge with one end vertex in $A \smallsetminus B$ and the other in $B \smallsetminus A$. The *order* of (A, B) is $|A \cap B|$. A separation is *elementary* if it is of the form $(\{x\} \cup N(x), V(G) \smallsetminus \{x\})$ for some $x \in V(G)$. A separation (A, B) is *tight* if there are components C_A of $A \smallsetminus B$ and C_B of $B \smallsetminus A$ such that every $x \in A \cap B$ has neighbours in C_A and in C_B.

If $(T, \mathcal{V})$ is a tree-decomposition of G, then the separations *induced by* $(T, \mathcal{V})$ are those of the form $(\bigcup_{t \in T_1} V_t, \bigcup_{t \in T_2} V_t)$ for edges $t_1 t_2 \in E(T)$. It follows from (T3) that these are indeed separations and its separator is $V_{t_1} \cap V_{t_2}$.

If a group Γ acts on G, a tree-decomposition $(T, \mathcal{V})$ is Γ-*invariant* if the induced action of Γ on $\mathcal{V}$ induces an action on T.

Let $(T, \mathcal{V})$ and $(T', \mathcal{V}')$ be tree-decompositions of G. We call $(T', \mathcal{V}')$ a *refinement* of $(T, \mathcal{V})$ if there is a family of disjoint subtrees $(T_i)_{i \in I}$ of T' covering $V(T')$ such that the following holds:

(R1) $T = T' / \bigcup_{i \in I} E(T_i)$;

(R2) $\bigcup_{s \in T_i} V'_s = V_t$, where t is the node of T obtained from the contraction of $E(T_i)$.

Let Γ be a group acting on sets X and Y. A map $f \colon X \to Y$ is a Γ-*map* if f commutes with the action of Γ on X and Y, i. e. if for all $x \in X$ and all $\gamma \in \Gamma$ we have $f(\gamma(x)) = \gamma(f(x))$. By $\Gamma \setminus X$ we denote the set of orbits in X under Γ. We denote by Γ_x the stabiliser of x in Γ and by Γx the orbit of x under Γ.

Let Γ be a group acting on a tree T. Then T is Γ-*incompressible* if for every $u, v \in V(T)$ the fact $\Gamma_u \leq \Gamma_v$ implies $\Gamma_u = \Gamma_v$ and $\Gamma u = \Gamma v$. Furthermore, an edge $e = uv \in E(T)$ is Γ-*compressible* if $\Gamma u \neq \Gamma v$ and either $\Gamma_v = \Gamma_e$ or $\Gamma_u = \Gamma_e$. Dicks and Dunwoody [1] proved the following connection between Γ-incompressible trees and Γ-compressible edges of trees.

Lemma 3.1. [1, III.7.2] *Let Γ be a group acting on a tree T with finite edge stabilizers. Then the following statements are equivalent.*

1. *T is a Γ-incompressible Γ-tree.*
2. *T has no Γ-compressible edges.* □

Proposition 3.2. *Let (G_1, G_2) be a factorisation of a locally finite quasi-transitive connected graph G. Then the connecting tree of the tree amalgamation $G = G_1 * G_2$ is $\mathrm{Aut}(G)$-incompressible.*

Proof. Let $G = G_1 *_T G_2$. Let us suppose that T is not $\mathrm{Aut}(G)$-incompressible. Then it has a compressible edge uv by Lemma 3.1. Since $\mathrm{Aut}(G)u \neq \mathrm{Aut}(G)v$, we conclude that the tree amalgamation is not of Type 2 respecting the group actions. So it is of Type 1. Let us assume $\mathrm{Aut}(G)_{uv} = \mathrm{Aut}(G)_u$. Since the induced action of $\mathrm{Aut}(G)_u$ on G_u is quasi-transitive, it follows that G_u is finite and has a unique adhesion set. Thus, the factorisation (G_1, G_2) does not distinguish any ends. This contradiction to the definition of a factorisation shows the assertion. □

Let Γ be a group acting quasi-transitively on a tree T with finite edge stabilizers. Then the *size sequence* of T is defined as

$$\mathsf{size}(T) = (|\Gamma \setminus E(T)| - |\Gamma \setminus V(T)|, |\Gamma \setminus E_1|, \ldots),$$

where $E_n = \{e \in E(T) \mid |\Gamma_e| = n\}$ for every $n > 1$. We compare size sequences lexicographically, that is,

$$(m_0, m_1, m_2, \ldots) > (n_0, n_1, n_2, \ldots)$$

if there exists some i such that $m_j = n_j$ for all $0 \leq j < i$ and $m_i > n_i$.

Our proof of the main result of this section is based on the following lemma by Dicks and Dunwoody [1].

Lemma 3.3. [1, III.7.5] *Let Γ be a group acting quasi-transitively on two Γ-incompressible trees T_1, T_2 with all edge stabilizers finite. If $\phi\colon V(T_1) \to V(T_2)$ is a Γ-map, then $|\Gamma \setminus E(T_1)| + |\Gamma \setminus V(T_1)| \geq |\Gamma \setminus V(T_2)|$ and $\mathsf{size}(T_1) \geq \mathsf{size}(T_2)$ with equality of the size sequences if and only if ϕ is bijective.* □

Another key idea of the proof of Theorem 3.6 is to consider tree amalgamations as tree-decompositions and combine the tree-decompositions obtained during the process of splittings. The way how tree amalgamations $G := G_1 *_T G_2$ induce tree-decompositions was discussed in [4, Remark 5.1]: the pair

$$(T, \{\pi(V(G_u)) \mid u \in V(T)\})$$

is a tree-decomposition *corresponding to* the factorisation (G_1, G_2) of G all of whose parts induce connected graphs.

For a factorisation (G_1, G_2) of a quasi-transitive locally finite connected graph G, we define the following property.

> *Every process of splittings of G that starts with (G_1, G_2) ends after finitely many steps.* $(*)$

Lemma 3.4. *Let (G_1, G_2) be a factorisation of a quasi-transitive locally finite connected graph G and let $S \subseteq V(G)$ be finite. Then there is a factorisation (H_1, H_2) of G that satisfies $(*)$ if and only if (G_1, G_2) satisfies $(*)$ and such that some part of the tree-decomposition corresponding to (H_1, H_2) contains S.*

Proof. Let $(T, \mathcal{V})$ be the tree-decomposition corresponding to (G_1, G_2). Let $S' \subseteq V(G)$ be finite and connected and such that $S \subseteq S'$. Let $T_{S'}$ be the minimal subtree of T such that for every $t \in V(T - T_{S'})$ we have $V_t \cap S' = \emptyset$. This subtree is finite since $G_1 * G_2$ is of finite identification. For every $t \in V(T)$, we set

$$V'_t := V_t \cup \bigcup \{\alpha(S') \mid \alpha \in \mathrm{Aut}(G), t \in \alpha(T_{S'})\}$$

and

$$\mathcal{V}' := \{V'_t \mid t \in V(T)\}.$$

To see that $(T, \mathcal{V}')$ is a tree-decompositionit suffices to prove (T3). For this, it suffices to see that, for every $v \in V(G)$, the subgraph of T that contains v is a tree. But this follows immediately from the definition of the subtrees $T_{S'}$ and the parts $V_{t'}$. By construction, $(T, \mathcal{V}')$ corresponds to a factorisation (H_1, H_2) of G such that for some edge $t_1 t_2 \in E(T)$ and every $i \in \{1, 2\}$ the graph H_i is isomorphic to the subgraph of G induced by V'_{t_i}. □

Let us now define recursively, what is means for a tree-decomposition to correspond to a factorisation of more than two factors.

A tree-decomposition $(T, \mathcal{V})$ of a graph G *corresponds to* a factorisation $(G_1, \ldots, G_n)$ of G if there is a factorisation $(H_1, \ldots, H_{n-1})$ of G such that (G_i, G_j) is a factorisation of some H_m, such that

$$\{G_k \mid 1 \leq k \leq n, k \neq i, k \neq j\} = \{H_\ell \mid 1 \leq \ell \leq n-1, \ell \neq m\}$$

and such that for some tree-decomposition $(T', \mathcal{V}')$ corresponding to the factorisation $(H_1, \ldots, H_{n-1})$ and some tree-decomposition $(T'', \mathcal{V}'')$ corresponding to (G_i, G_j), we have that $(T, \mathcal{V})$ is a refinement of $(T', \mathcal{V}')$ where the only non-trivial subtrees in the covering of T are those that get contracted to nodes whose parts correspond to H_m and the tree-decompositions induced by those trees are isomorphic to $(T'', \mathcal{V}'')$ in a canonical way.

Proposition 3.5. *Let $(G_1, \ldots, G_k)$ be a factorisation of a quasi-transitive locally finite connected graph G and let $(T, \mathcal{V})$ be an* $\mathrm{Aut}(G)$*-invariant inversion-free tree-decomposition corresponding to $(G_1, \ldots, G_k)$ with* $\mathrm{Aut}(G) \setminus E(T)$ *being finite. Assume that G_i has more than one end and let (H_1, H_2) be a factorisation of G_i. Then there is a factorisation (H_1', H_2') of G_i that satisfies $(*)$ if and only if (H_1, H_2) satisfies $(*)$ and an* $\mathrm{Aut}(G)$*-invariant inversion-free tree-decomposition $(T', \mathcal{V}')$ with* $\mathrm{Aut}(G) \setminus E(T')$ *finite such that $(T', \mathcal{V}')$ is a refinement of $(T, \mathcal{V})$ that corresponds to $(G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_k, H_1', H_2')$.*

Proof. Let $t \in V(T)$ such that $G[V_t]$, the graph induced by V_t, corresponds to the factor G_i. By iterated applications of Lemma 3.4 and since there are only finitely many $\mathrm{Aut}(G)$-orbits on $E(T)$, we obtain a factorisation (H_1', H_2') of G_i satisfying $(*)$ if and only if (H_1, H_2) satisfies $(*)$ such that for every adhesion set of $(T, \mathcal{V})$ in $G[V_t]$ there is some part of the tree-decomposition $(\tilde{T}, \tilde{\mathcal{V}})$ corresponding to (H_1', H_2') that contains its image under the canonical map $G[V_t] \to G_i$. If the setwise stabiliser Γ of G_i in $\mathrm{Aut}(G)$ acts on $\tilde{T}$ without inversion of the edges[2], set $H_2'' := H_2'$. Otherwise, H_1' is isomorphic to H_2'. We apply Proposition 2.1 to obtain another factorisation (H_1', H_2'') of G_k, where H_2'' is finite and $H_1' \not\cong H_2''$. Let $(T^\circ, \mathcal{V}^\circ)$ be the tree-decomposition that corresponds to $H_1' * H_2''$. Note that Γ acts without inversion on T°.

For an adhesion set S of $(T, \mathcal{V})$ that lies in V_t, let T_S be the maximal subtree of T° such that for all $v \in V(T_S)$ the part V_v° contains S. This is a finite tree since our tree amalgamations are of finite identification and it is non-empty by construction. Thus, it has either a central vertex or a central edge. If it is a central vertex, let v_S be this vertex. If it is a central edge, choose a vertex v_S that is incident with that edge so that for every adhesion set $S' = \alpha(S)$ with $\alpha \in \mathrm{Aut}(G)$ we have $v_{S'} = \alpha(v_S)$. This is possible since Γ acts on T° without inversion. Let $(T', \mathcal{V}')$ be the tree-decomposition that is a refinement of $(T, \mathcal{V})$ where only the trees for vertices in the $\mathrm{Aut}(G)$-orbits of t are non-trivial and for these we take the tree-decomposition $(T^\circ, \mathcal{V}^\circ)$ and its $\mathrm{Aut}(G)$-images.

[2]Technically, we would have to use an injective map of the setwise stabiliser of $G[V_t]$ in $\mathrm{Aut}(G)$ to $\mathrm{Aut}(G_i)$. For the sake of simplicity, we omit this map.

Note that $\mathrm{Aut}(G)$ acts on $\tilde{T}$ without inversion since this is true for the action of $\mathrm{Aut}(G)$ on T and of Γ on T°. □

Let Γ be a group acting on a tree T with finite edge stabilizers. Set $T_0 := T$. For $i \geq 1$, let E_i be an orbit of Γ-compressible edges of T_{i-1} and let T_i be the tree obtained from T_{i-1} by contracting E_i. If $\Gamma \setminus E(T)$ is finite, then there is some $i \geq 0$ such that T_i has no Γ-compressible edge. Set $\mathcal{C}(T) := T_i$ and let $c\colon V(T) \to V(\mathcal{C}(T))$ be the canonical map defined by all contractions, i. e. a vertex is mapped to the vertex it ends up as after doing all contractions. Note that, in general, $\mathcal{C}(T)$ is not uniquely defined but relies on the choices of the edge sets E_i. By Lemma 3.1, the tree $\mathcal{C}(T)$ is Γ-incompressible.

If $(T, \mathcal{V})$ is a Γ-invariant tree-decomposition of a graph G, then the pair

$$\Big(\mathcal{C}(T), \mathcal{V}^{\mathcal{C}(T)} := \Big\{\bigcup\{V_s \in \mathcal{V} \mid c(s) = t\} \mid t \in V(\mathcal{C}(T))\Big\}\Big) \tag{1}$$

is a Γ-invariant tree-decomposition of G.

Now we are ready to prove the main result of this section.

Theorem 3.6. *Let G be a locally finite quasi-transitive connected graph. If some process of splittings stops after finitely many steps, then every process of splittings stops after finitely many steps.*

Proof. Let $(G_1, \ldots, G_n)$ be a terminal factorisation of G that is the result of a process of splittings and let $(T, \mathcal{V})$ be the tree-decomposition corresponding to that factorisation. Let us suppose that there exists an infinite process of splittings. Let $(G_{i,1}, \ldots, G_{i,i+1})$ be the factorisation of G obtained in the i-th step of that modified process and let $(T_i, \mathcal{V}_i)$ be the tree-decomposition corresponding to that factorisation such that $(T_{i+1}, \mathcal{V}_{i+1})$ is a refinement of $(T_i, \mathcal{V}_i)$ as in Proposition 3.5. By Proposition 3.5, we may also assume that $\mathrm{Aut}(G)$ acts on all trees T_i without inversion since it does so on the trivial tree on one vertex and we may assume that $\mathrm{Aut}(G) \setminus E(T_i)$ is finite. Furthermore, we may assume that $\mathcal{C}(T_{i+1})$ is obtained from $\mathcal{C}(T_i)$ by possibly further contractions: every compressible edge of T_i has an incident vertex t whose part is finite with its stabiliser being the finite stabiliser of the edge; this part does not get factorised any further and hence the edge is also compressible in T_{i+1}.

Let us consider the construction of $\mathcal{C}(T_i)$. We claim the following.

> *In each step j of the construction of $\mathcal{C}(T_i)$ the graph H induced by E_j, the set of edges that get contracted, is a disjoint union of stars each of which contains at most one node t whose part in the corresponding tree-decomposition is infinite.* (2)

Let uv be an $\mathrm{Aut}(G)$-compressible edge in some step. We may assume that the stabiliser of uv is the stabiliser of u. Let w be a neighbour of u such that uv and uw lie in the same $\mathrm{Aut}(G)$-orbit, i. e. there is some $\varphi \in \mathrm{Aut}(G)$ with $\varphi(uv) = uw$. By definition of compressible edges, we have $\varphi(u) = u$, so φ lie in $\mathrm{Aut}(G)_u = \mathrm{Aut}(G)_{uv}$. Thus, φ fixes v and hence u has degree 1 in H. The leaves of H are finite as otherwise the stabiliser of them cannot stabilise their incident edges.This proves (2).

Let us prove the following.

$$\textit{There are } \mathrm{Aut}(G)\textit{-maps } \phi_i\colon V(\mathcal{C}(T)) \to V(\mathcal{C}(T_i)) \textit{ for every } i \in \mathbb{N}. \tag{3}$$

Let $t \in V(\mathcal{C}(T))$. If $V_t^{\mathcal{C}(T)}$ is an infinite part, then $V_t^{\mathcal{C}(T)}$ must contain an end ω. Since $(T, \mathcal{V})$ corresponds to a terminal factorisation it follows inductively from (2) that ω is the only end in $V_t^{\mathcal{C}(T)}$. Since $V_t^{\mathcal{C}(T)}$ induces a one-ended locally finite quasi-transitive connected subgraph of G, its unique end ω is thick by Thomassen [6, Proposition 5.6]. Since ω is thick, there must be a unique part V_s of the tree-decomposition $(T_i, \mathcal{V}_i)$ containing ω, cf. [4, Proposition 4.8 (ii)]. We set $\phi_i(t) = c(s)$.

If $V_t^{\mathcal{C}(T)}$ is finite, let T_i° be the maximal subtree of T_i such that for every $s \in V(T_i^\circ)$ we have $V_t^{\mathcal{C}(T)} \cap V_s \neq \emptyset$. Note that since G is locally finite and each part of $(T_i, \mathcal{V}_i)$ is connected, we deduce that T_i° is a finite tree. So T_i° has a central vertex s or a central edge. By our choice of $(T_i, \mathcal{V}_i)$, the action of $\mathrm{Aut}(G)$ on T_i is inversion-free. So if T_i° has a central edge, let s be one of its incident vertices. We set $\phi_i(t) = c(s)$. Since $\mathrm{Aut}(G)$ acts on T_i without inversion, we can make the choices for s commute with the action of $\mathrm{Aut}(G)$, i. e. if $t' \in V(\mathcal{C}(T))$ and $\varphi \in \mathrm{Aut}(G)$ with $\varphi(t) = t'$, we set $s' := \varphi(s)$. This finishes the proof of (3).

Lemma 3.3 together with (3) implies

$$|\mathrm{Aut}(G) \setminus E(\mathcal{C}(T))| + |\mathrm{Aut}(G) \setminus V(\mathcal{C}(T))| \geq |\mathrm{Aut}(G) \setminus V(\mathcal{C}(T_i))| \tag{4}$$

and

$$\mathsf{size}(\mathcal{C}(T)) \geq \mathsf{size}(\mathcal{C}(T_i)) \tag{5}$$

for every $i \in \mathbb{N}$.

Since T_{i+1} is a refinement of T_i and hence, by our construction, $\mathcal{C}(T_{i+1})$ is a refinement of $\mathcal{C}(T_i)$, we directly obtain the existence of $\mathrm{Aut}(G)$-maps

$$f_{i+1}\colon V(\mathcal{C}(T_{i+1})) \to V(\mathcal{C}(T_i))$$

for all $i \in \mathbb{N}$, which are the identity on those vertices that appear as trees with only one vertex in the refinement. This together with (4) and Lemma 3.3 implies

$$\mathsf{size}(\mathcal{C}(T)) \geq \mathsf{size}(\mathcal{C}(T_{i+1})) \geq \mathsf{size}(\mathcal{C}(T_i)) \tag{6}$$

for all $i \in \mathbb{N}$.

Let $\mathcal{T}$ be the set of non-trivial trees when considering T_{i+1} as refinement of T_i. The edges of T_{i+1} that lie in some $T' \in \mathcal{T}$ have a strictly smaller stabiliser than any of their incident vertices by Proposition 3.2. Note that every edge of T_{i+1} that does not lie in any $T' \in \mathcal{T}$ has at most one incident vertex that lies in elements of $\mathcal{T}$. Every compressible edge e of T_{i+1} that gets contracted while constructing $\mathcal{C}(T_{i+1})$ either gets contracted while constructing $\mathcal{C}(T_i)$ or is incident with a vertex u that lies in some $T' \in \mathcal{T}$. Let $v \neq u$ be the other vertex that is incident with e. Since v lies in no $T' \in \mathcal{T}$ and since the edge e is not compressible in $\mathcal{C}(T_i)$, the stabiliser of v is not the stabiliser of e. Thus, the stabiliser of e equals the stabiliser of u and hence is larger than the stabiliser of the edges in any $T' \in \mathcal{T}$. Thus, we have shown

$$\mathsf{size}(\mathcal{C}(T_{i+1})) > \mathsf{size}(\mathcal{C}(T_i)) \tag{7}$$

for every $i \in \mathbb{N}$.

By (6) and (4) we have

$$\begin{aligned} & |\mathrm{Aut}(G) \setminus E(\mathcal{C}(T_i))| \\ \leq & |\mathrm{Aut}(G) \setminus E(\mathcal{C}(T))| - |\mathrm{Aut}(G) \setminus V(\mathcal{C}(T))| + |\mathrm{Aut}(G) \setminus V(\mathcal{C}(T_i))| \\ \leq & 2|\mathrm{Aut}(G) \setminus E(\mathcal{C}(T))|. \end{aligned}$$

So for every $i \in \mathbb{N}$, the sum of all but the first entries of $\mathsf{size}(\mathcal{C}(T_i))$, which is $|\mathrm{Aut}(G) \setminus E(\mathcal{C}(T_i))|$, is bounded by $2|\mathrm{Aut}(G) \setminus E(\mathcal{C}(T))|$ and the first entry is bounded by the same number. Thus, the sequence $(\mathsf{size}(\mathcal{C}(T_i)))_{i\in\mathbb{N}}$ of size sequences will be constant, i. e.

$$\textit{there is a } j_0 \in \mathbb{N} \textit{ such that } \mathsf{size}(\mathcal{C}(T_i)) = \mathsf{size}(\mathcal{C}(T_j)) \textit{ for all } i, j \geq j_0. \tag{8}$$

This contradiction to (7) shows that any process of splittings must stop after finitely many steps. □

4 The semiring $\mathcal{S}(G)$

A *semiring* is a triple $(R, +, \times)$ such that $(R, +)$ is an abelian monoid, $(R, \times)$ is a monoid and $\times$ is distributive over $+$. A semiring $(R, +, \times)$ is *commutative* if $(R, \times)$ is commutative. A set $S \subseteq R$ *generates* R if every $r \in R$ is obtained by finitely many additions and multiplications of elements of S.

An immediate corollary of a result by Thomassen and Woess [7, Proposition 4.2] is the following.

Proposition 4.1. *Let G be a locally finite graph, let $v \in V(G)$ and let $k \in \mathbb{N}$. Then there are only finitely many tight separations of order k with v in their separator.* □

Let $\mathcal{S}(G)$ be the set of all separations of finite order of G.

We define for $(A, B), (C, D) \in \mathcal{S}(G)$ the following operations:

$$(A, B) + (C, D) := (A \cap C, B \cup D),$$

$$(A, B) \times (C, D) := (A \cup C, B \cap D).$$

Simple calculations show that $(\mathcal{S}(G), +, \times)$ is a commutative semiring, where $(V(G), \emptyset)$ is the neutral element with respect to $+$ and $(\emptyset, V(G))$ is the neutral element with respect to $\times$.

Let $\mathcal{S}_n(G)$ be the subsemiring of $\mathcal{S}(G)$ that is generated by the tight separations of order at most n.

Proposition 4.2. *Let G be a locally finite graph. Every separation of order n is generated by tight separations of order at most n.*

Proof. We prove the assertion by induction on the order of the separation. Let (A,B) be a separation of order n that is not tight. Since (A,B) is not tight, either $A \setminus B$ or $B \setminus A$ has no component C with $A \cap B \subseteq N(C)$.

If $A \setminus B$ has no such component, let $\mathcal{K}_A$ be the set of all components of $A \setminus B$. Then

$$(A,B) = (X, V(G)) \times \prod_{C \in \mathcal{K}_A} (C \cup N(C), V(G) \setminus C),$$

where $X = (A \cap B) \setminus \bigcup_{C \in \mathcal{K}_A} N(C)$. Every separation of that product has order less than n. By induction, (A,B) can be generated by tight separations of order less than n.

If $B \setminus A$ has no component C with $A \cap B \subseteq C$, let $\mathcal{K}_B$ be the set of components of $B \setminus A$. Then

$$(A,B) = (V(G), Y) + \sum_{C \in \mathcal{K}_B} (V(G) \setminus C, C \cup N(C)),$$

where $Y = (A \cap B) \setminus \bigcup_{C \in \mathcal{K}_B} N(C)$. Every summand of that sum has order less than n, so by induction, (A,B) is generated by tight separations of order at most n. □

A tree-decomposition $(T, \mathcal{V})$ *distinguishes* two ends *(efficiently)* if for some separation induced by $(T, \mathcal{V})$ its separator separates those ends *(minimally)*. We need the following result for the proof of Proposition 4.4.

Theorem 4.3. [4, Theorem 6.4] *Let G be a connected locally finite graph that is accessible in the sense of Thomassen and Woess and let Γ be a group acting quasi-transitively on G. Then there exists a Γ-invariant tree-decomposition $(T, \mathcal{V})$ of G of finite adhesion such that $(T, \mathcal{V})$ distinguishes all ends of G efficiently and such that there are only finitely many Γ-orbits on $E(T)$.* □

The proof of the following result is based on the idea of a proof of Thomassen and Woess [7, Theorem 7.6].

Proposition 4.4. *Let G be a quasi-transitive locally finite connected graph that is accessible in the sense of Thomassen and Woess. Then there exists $n \in \mathbb{N}$ such that $\mathcal{S}(G) = \mathcal{S}_n(G)$.*

Proof. Let $(T, \mathcal{V})$ be a tree-decomposition as in Theorem 4.3. In particular, there are only finitely many $\mathrm{Aut}(G)$-orbits on $E(T)$. Let $\mathcal{T}$ be the set of separations that are induced by $(T, \mathcal{V})$. Then there are only finitely many $\mathrm{Aut}(G)$-orbits on $\mathcal{T}$ as well. Let n_1 be the maximum order of separations in $\mathcal{T}$ and let n_2 be the maximum degree of G. Set $n := \max\{n_1, n_2\}$. We will show $\mathcal{S}(G) = \mathcal{S}_n(G)$.

Let $(A,B) \in \mathcal{S}(G)$ and let Ω_A, Ω_B be the set of ends of G that live in A, in B, respectively. We claim the following.

There is a finite $\mathcal{F} \subseteq \mathcal{T}$ such that for every $\omega_A \in \Omega_A$ and every $\omega_B \in \Omega_B$ there exists $(C,D) \in \mathcal{F}$ such that ω_A lives in C and ω_B lives in D. (9)

Suppose (9) does not hold. Let $\{(A_1,B_1),(A_2,B_2),\dots\} = \mathcal{T}$. For every $i \in \mathbb{N}$ let $\omega_i^A \in \Omega_A$ and $\omega_i^B \in \Omega_B$ such that ω_i^A and ω_i^B are not distinguished by any (A_j,B_j) with $j \le i$. These ends exist as (9) does not hold. For every $i \in \mathbb{N}$ let P_i be a double ray between ω_i^A and ω_i^B none of whose vertices are separated by any (A_j,B_j) with $j \le i$. Every double ray P_i meets the finite vertex set $A \cap B$. Thus the sequence $(P_i)_{i\in\mathbb{N}}$ has a subsequence that converges to a double ray P: infinitely many P_i share an edge incident with some vertex of $A \cap B$, among which we find an infinite subsequence whose edges adjacent to the first one coincide on each side and so on. By construction, one tail of P lies in A and another tail lies in B. In particular it has tails in distinct ends of G. By the choice of the double rays P_i, no (A_i,B_i) separates tails of P, that is, the two ends of G that contain tails of P are not distinguished by any (A_i,B_i) and thus are not distinguished by $(T,\mathcal{V})$. This contradiction to the choice of $(T,\mathcal{V})$ shows (9).

Let F be the set of edges of T that corresponds to the finite set $\mathcal{F}$. Let V_A be the set of nodes of T that lie in components C of $T-F$ such that some end of Ω_A lives in $\bigcup_{t\in C} V_t$. Set $V_B := V(T) \setminus V_A$. We consider the separation

$$(C,D) := \left(\bigcup_{t\in V_A} V_t, \bigcup_{t\in V_B} V_t \right).$$

Then (C,D) is generated by $\mathcal{F}$. By construction, Ω_A is the set of ends of G that live in C and Ω_B is the set of ends of G that live in D. We shall prove the following.

The sets $A \setminus C$ and $C \setminus A$ are finite. (10)

For every vertex $x \in A \setminus C$ and every neighbour y of x outside of $A \setminus C$, we have either $y \in C \cap D$ or $x \in A \cap B$ and $y \in N(A)$. Since $X := N(A) \cup (C \cap D)$ is finite, since G is locally finite and since each component of $A \setminus C$ is a component of $G - X$, the vertex set $A \setminus C$ induces only finitely many components in G. If one of these components is infinite, there would be an end living in $A \setminus C$ which is impossible as we already saw that this set is empty. Thus $A \setminus C$ is finite. An analogous argument shows that $C \setminus A$ is finite. This completes the proof of (10).

Since (10) holds, (A,B) and (C,D) differ only by addition and multiplication of elementary separations. So (A,B) is generated by separations of order at most n. Proposition 4.2 implies that $(A,B) \in \mathcal{S}_n(G)$. □

5 The main theorem

In this section, we are going to prove our main result, Theorem 5.1. Theorem 1.2 follows immediately from Theorem 5.1.

Theorem 5.1. *Let G be a quasi-transitive locally finite connected graph. Then the following statements are equivalent.*

(i) *G is accessible.*

(ii) *G is accessible in the sense of Thomassen and Woess.*

(iii) *$\mathcal{S}(G)$ is an $\mathrm{Aut}(G)$-finitely generated semiring.*

(iv) *There is an $n \in \mathbb{N}$ such that $\mathcal{S}_n(G) = \mathcal{S}(G)$.*

(v) *Every process of splittings of G must end after finitely many steps.*

Proof. The equivalence of (i) and (ii) is [4, Theorem 6.3]. The implication (ii) to (iv) holds by Proposition 4.4. Theorem 3.6 implies the implication (i) to (v) and its reverse is trivial.

To prove (iv) to (ii), let $n \in \mathbb{N}$ such that $\mathcal{S}_n(G) = \mathcal{S}(G)$. Let ω_1 and ω_2 be ends of G. We are going to show that there is a separation of order at most n distinguishing ω_1 and ω_2. Let (A, B) be a separation of finite order separating ω_1 and ω_2. Since $\mathcal{S}_n(G) = \mathcal{S}(G)$, there are separations $(A_1, B_1), \ldots, (A_m, B_m)$ of order at most n such that (A, B) is generated by these separations. Note that neither the sum nor the product of two separations distinguishes two ends if none of the summands or factors does so. Thus, there is some $i \in \{1, \ldots, m\}$ such that (A_i, B_i) distinguishes ω_1 and ω_2. Hence, (ii) holds.

It remains to prove the equivalence of (iii) and (iv). If (iii) holds, let $\mathcal{X}$ be an $\mathrm{Aut}(G)$-invariant generating set that consists of finitely many orbits. Let n be the maximum order of separations in $\mathcal{X}$. Then $\mathcal{S}(G) = \mathcal{S}_n(G)$ by Proposition 4.2 and (iv) holds.

Assume that there is some $n \in \mathbb{N}$ such that $\mathcal{S}(G) = \mathcal{S}_n(G)$. By Proposition 4.1 and as G is quasi-transitive, there are only finitely many orbits of tight separations of order at most n. Thus, $s_n(G)$ and hence $\mathcal{S}(G)$ is $\mathrm{Aut}(G)$-finitely generated. □

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