PRECISE VOLUME ESTIMATES IN NONPOSITIVE CURVATURE

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ABSTRACT. We show that on a compact manifold of nonpositive curvature the volume of spheres (hence also that of balls) has an exact asymptotic; it is purely exponential, and the growth rate equals the topological entropy.

The resulting formula is the sharpest one which is known. It generalizes results of G.A. Margulis to the nonuniformly hyperbolic case. It improves the multiplicative asymptotic bound by G. Knieper.

1. INTRODUCTION

Let M be a compact smooth Riemannian manifold whose sectional curvature is nonpositive. We assume the *rank* of M to equal 1; that is, there exists a geodesic which has no parallel Jacobi field except multiples of its velocity vector. (For the geometric background, see [Jos], [BuKa], [BuSp], [Esch], [Bal], [Ebe], [Gro], [BGS] and [KaHa].)

Let $b_r(x) := \mathrm{vol}\, B_r(x)$ be the Riemannian volume of the ball of radius r around x in $\tilde{M}$ (the universal cover of M). Let h be the topological entropy of the geodesic flow on M. We show that

$$b_r(x) \sim c(x)e^{hr}$$

for a continuous function $c : M \to \mathbb{R}$.

This result was obtained by G.A. Margulis in the special case that the curvature is strictly negative everywhere; in that case the geodesic flow is uniformly hyperbolic. His result was published ([Mar2]), but unfortunately the proofs (which were part of his doctoral dissertation [Mar1]) were not.

In our situation, the problem is somewhat more difficult since we are dealing with a *non-uniformly* hyperbolic system. In particular, in our setup one has to deal with the *singular set* where the product structure of stable and unstable manifolds breaks down. We show how to overcome this problem.

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It is known ([Man]) that in nonpositive curvature the exponential growth rate of volume equals h. The best known result so far in this setting is the estimate in [Kni2] and [Kni3] provided by G. Knieper which says that there exist a constant C such that asymptotically $1/C < e^{-ht}b_t(x) < C$. However, the upper and lower bound in his estimates cannot be made to be asymptotically close with the methods he provides. Our methods presented in this article give upper and lower bounds which are asymptotically the same.

2. Construction of (un-)stable measures on fibers which are not in the (un-)stable foliation

Let SM be the unit sphere bundle of M and let $(g^t)_{t\in\mathbb{R}}$ be the geodesic flow. Recall that the regular set $\mathbf{Reg} := \{v \in SM : \text{rank}(v) = 1\}$ is open and dense in SM. Recall that the regular set has a local product structure with respect to the foliations W^s (stable manifolds), W^u (unstable manifolds), and flow lines of the geodesic flow. Denote by W^{0u} the weakly unstable leaves (integral manifolds of W^u and flow lines). The set $\mathbf{Reg}$ has full measure with respect to the measure m of maximal entropy, and m is supported on $\mathbf{Reg}$.

The articles [Gun1] and [Gun2] give independent proofs of the following: On $\mathbf{Reg}$, the measure of maximal entropy has conditional measures m^{0u}, m^s, supported on the weakly unstable and on the stable foliation, respectively. They have the property of being uniformly expanding and contracting, i.e. $m^{0u} \circ g^t = e^{ht}m^{0u}$ and $m^s \circ g^t = e^{-ht}m^s$. Moreover, they are holonomy invariant, i.e. two nearby sets in W^{0u} which are pointwise uniquely connected by short W^s-fibers have the same m^{0u}-measure. Those two articles also provide different constructions of the conditionals.

Let $\mathcal{K} \subset SM$ be a compact submanifold of dimension $\dim M - 1$ which is transversal to W^{0u}. Let $\mathcal{L} \subset SM$ be a compact submanifold of dimension $\dim M$ which is transversal to W^s. Let D be an open subset of the regular set which has diameter at most ε, which has such a product structure and which is topologically a ball. We use the notation $B^s_D(p) := B^s_\varepsilon(p) \cap D$.

Definition 2.1. For a point $p \in D$ define the projection $\pi^s_{D,p} : D \to B^s_D(p)$ by

$$\pi^s_{D,p}(x) := B^{0u}_\varepsilon(x) \cap B^s_D(p).$$

For a set $K \subset \mathcal{K} \cap D$ define the function

$$\text{preimg}_{K,D,p}(x) := \#\{y \in K : \pi^s_{D,p}(y) = x\}.$$

This function is integer-valued and semicontinuous from below, hence integrable.

Definition 2.2. Define

$$m^s_{D,p}(K) := \int_{B^s_D(p)} \mathrm{preimg}_{K,D,p}(x)\,dm^s(x).$$

In the following, we will often deal with pairs of quantities whose proximity we want to quantify. For f_1 and f_2 which are such that f_1/f_2 is close to 1 we define the **logarithmic difference** by

$$\mathrm{diff}(f_1, f_2) := \left|\ln \frac{f_1}{f_2}\right|.$$

This quantifies the proximity of f_1 and f_2. Evidently for $f_1/f_2 \in [0.9, 1.1]$, the expressions $|f_1/f_2 - 1|$ and $\mathrm{diff}(f_1, f_2)$ differ by at most a factor 2. However, error estimates are easier using diff since

$$\mathrm{diff}(f_1, f_3) \le \mathrm{diff}(f_1, f_2) + \mathrm{diff}(f_2, f_3).$$

Lemma 2.3. *Let $\mathcal{K}$, ε and D be as above, let $D' \subset D$ be open and have a product structure. For $p \in D$, $p' \in D$ and $K \subset D \cap D'$ we have*

$$\mathrm{diff}\left(m^s_{D,p}(K), m^s_{D',p'}(K)\right) < h\varepsilon.$$

Proof. Recall that the measure m^s contracts uniformly with exponent h in the time direction, i.e. $m^s \circ g^t = e^{-ht} m^s$, and is invariant under holonomy in u-direction. Using the product structures in D and D' there is a bijective $0u$-holonomy from $\pi^s_{D,p}(K)$ to $\pi^s_{D',p'}(K)$ that moves points by at most ε. $\square$

Definition 2.4. A **regular partition-cover** of $\mathcal{K}$ of size ε is a triple $(\mathbf{D}, \mathbf{K}, \mathbf{p})$ where $\mathbf{D} = (D_i)_{i\in\mathbb{N}}$ is an open cover of $\mathbf{Reg}$ so that all D_i have a product structure and are of diameter at most ε, where $\mathbf{p} = (p_i)_{i\in\mathbb{N}}$ with $p_i \in D_i$ for all i, and where $\mathbf{K} = (K_i)_{i\in\mathbb{N}}$ is a (disjoint) partition of $\mathcal{K} \cap \mathbf{Reg}$ such that $K_i \subset D_i$ for all i.

For a regular partition-cover $(\mathbf{D}, \mathbf{K}, \mathbf{p})$ we define a measure on $\mathcal{K}$ by declaring that for $K \subset \mathcal{K} \cap \mathbf{Reg}$ the measure is

$$m^s_{\mathbf{D},\mathbf{K},\mathbf{p}}(K) := \sum_{i\in\mathbb{N}} m^s_{D_i,p_i}(K \cap K_i)$$

and declaring that $m_{\mathbf{D},\mathbf{K},\mathbf{p}}(\mathcal{K} \cap \mathbf{Sing}) := 0$. Hence $m^s_{\mathbf{D},\mathbf{K},\mathbf{p}}$ is defined on all of $\mathcal{K}$, including the singular part.

Lemma 2.5. *Let $(\mathbf{D}, \mathbf{K}, \mathbf{p})$ and $(\mathbf{D}', \mathbf{K}', \mathbf{p}')$ be regular partition-covers of $\mathcal{K}$ of size ε. Then*

$$\mathrm{diff}\left(m^s_{\mathbf{D},\mathbf{K},\mathbf{p}}(K), m^s_{\mathbf{D}',\mathbf{K}',\mathbf{p}'}(K)\right) < 2h\varepsilon.$$

Proof. Use lemma 2.3 for a common refinement of $\mathbf{D}$ and $\mathbf{D}'$. Note that the common refinement contains only sets which are ε-small and ε-close to a set in each $\mathbf{D}$ and $\mathbf{D}'$. Thus each holonomy from $\pi^s_{D_i\cap D'_j,p}(K)$ to $\pi^s_{D_i\cap D'_j,p'}(K)$ moves points by distance at most 2ε. $\square$

Definition 2.6. Choose a sequence $(\mathbf{D}_i, \mathbf{K}_i, \mathbf{p}_i)_{i\in\mathbb{N}}$ of regular partition-covers of $\mathcal{K}$ of size $1/i$. Let

$$m^s_{\mathcal{K}}(K) := \lim_{i\to\infty} m^s_{\mathbf{D}_i,\mathbf{K}_i,\mathbf{p}_i}(K).$$

By the previous lemma, this does not depend on the sequence $(\mathbf{D}_i, \mathbf{K}_i, \mathbf{p}_i)_{i\in\mathbb{N}}$ chosen. Note that for all $K \subset \mathcal{K}$ and for all $\varepsilon > 0$ there is $N(\varepsilon)$ so that for all $i > N(\varepsilon)$ we have

$$\operatorname{diff}(m^s_{\mathcal{K}}(K), m^s_{\mathbf{D}_i,\mathbf{K}_i,\mathbf{p}_i}(K)) < h\varepsilon.$$

Hence $m^s_{\mathcal{K}}$ is an additive measure.

Similarly, for $\mathcal{L}$ compact and transversal to W^s we construct a measure $m^{0u}_{\mathcal{L}}$ by repeating the construction with s and $0u$ exchanged.Once again, we declare that $m^{0u}_{\mathcal{L}}(L)$ is zero for $L \subset$ **Sing**, which makes $m^{0u}_{\mathcal{L}}$ defined on all of $\mathcal{L}$, including the singular part.

It is interesting to note that for the construction of $m^{0u}_{\mathcal{L}}$, we do not even need the limit of $i \to \infty$ since the holonomy along s-fibers (as opposed to $0u$-fibers) leaves the measure in the $0u$-direction strictly invariant.

Similarly, for Λ compact and transversal to W^{0s} we get a measure m^u.

3. Fiberwise ergodic theorems

Definition 3.1. A family $\mathbf{F}$ of functions $SM \to \mathbb{R}$ is called **uniformly equicontinuous in the $0u$-direction** iff $\forall\varepsilon > 0\,\exists\delta > 0\,\forall f \in \mathbf{F}, \forall p, q \in SM$ the condition $d^{0u}(p,q) < \delta$ implies $|f(p) - f(q)| < \varepsilon$.

We apply this later to nonnegative functions in $C^0(SM, [0,\infty))$.

Lemma 3.2. *For any f which is continuous in the $0u$-direction, the family $\mathbf{F} = \{f \circ g^t : t \le 0\}$ is uniformly equicontinuous in the $0u$-direction.*

Proof. Note that by compactness of SM,the function f is automatically uniformly continuous in $0u$-direction, i.e. $\forall\varepsilon > 0\,\exists\delta > 0\,\forall p, q \in SM$: $d^{0u}(p,q) < \delta$ implies $|f(p) - f(q)| < \varepsilon$. By nonpositivity of the curvature, d^{0u} is nondecreasing with the flow (in positive time direction), i.e. $d^{0u}(p,q) < \delta$ implies $d^{0u}(g^t p, g^t q) < \delta$ for all $t \le 0$, thus $|f(g^t p) - f(g^t q)| < \varepsilon$. ∎

If the set D has a product structure, then for all p in D, the measure $m^{0u}(B^{0u}_D(p))$ is the same (because of strict holonomy invariance along W^s-fibers). We call this number $S(D)$.

Lemma 3.3. *Assume $K \subset W^s$, $D \subset SM$ open and with product structure, for all $i \in \mathbb{N}$ let D_i be open in D, with $B_D(p) \subset D_i$ for $p \in D_i$, with*

$K = \cap_{i\in\mathbb{N}} D_i$ and $D_i \subset B^{0u}_{1/i}(K)$. Let $\mathbf{F}$ be uniformly equicontinuous in $0u$-direction. Then, for all $f \in \mathbf{F}$ we have

$$\frac{1}{S(D_i)} \int_{D_i} f\, dm \sim \int_K f\, dm^s;$$

in fact,

$$\operatorname{diff}\left(\frac{1}{S(D_i)} \int_{D_i} f\, dm, \int_K f\, dm^s \right) < \frac{h}{i}.$$

Proof. It suffices to note that for all $p \in D$, holonomy invariance and uniform expansion gives

$$\operatorname{diff}\left(\int_{B^s_{D_i}(p)} f\, dm, \int_K f\, dm^s \right) < \frac{h}{i}.$$

This, averaged over $0u$-fibers, gives the claim. ☐

Proposition 3.4. *Assume $K \subset W^s$, $D \subset SM$ open and with product structure, for all $i \in \mathbb{N}$ let D_i be open in D, with $B_D(p) \subset D_i$ for $p \in D_i$, with $K = \cap_{i\in\mathbb{N}} D_i$ and $D_i \subset B^{0u}_{1/i}(K)$. Let $f : D \to [0, \infty)$ be uniformly continuous in $0u$-direction. Then*

$$\int_{g^t K} f\, dm^s \sim e^{-ht} m^s(K) \int_{SM} f\, dm$$

for $t \to \infty$. Indeed,

$$\operatorname{diff}\left(\int_{g^t K} f\, dm^s, e^{-ht} m^s(K) \int_{SM} f\, dm \right) < c_1(\mathit{diam}(D_i)) + c_2(t).$$

Proof. Using the previous two lemmata, for all $\varepsilon > 0$,for i large enough and for all $t \leq 0$,

$$\operatorname{diff}\left(\frac{1}{S(D_i)} \int_{D_i} f \circ g^t\, dm, \int_K f \circ g^t\, dm^s \right) < \frac{h}{i}.$$

Using the mixing property (see [Bab]),

$$\int_{D_i} f \circ g^t\, dm = \int_{SM} \chi_{D_i} \cdot (f \circ g^t) dm \to m(D_i) \int_{SM} f\, dm$$

for $t \to -\infty$. (Here χ_{D_i} is the characteristic function of D_i.) In other words,

$$\operatorname{diff}\left(\int_{D_i} f \circ g^t\, dm, m(D_i) \int_{SM} f\, dm \right) < c_2(t).$$

Hence

$$\operatorname{diff}\left(\int_K f \circ g^t dm^s, \frac{m(D_i)}{S(D_i)} \int_{SM} f\, dm \right) < \frac{h}{i} + c_2(t).$$

Note that
$$\frac{m(D_i)}{S(D_i)} \to m^s(K).$$
Thus, using the uniform expansion property
$$\int_{g^t K} f\, dm^s = e^{-ht} \int_K f \circ g^t\, dm^s,$$
we get the claim. □

Lemma 3.5. *For $D \subset SM$ open and $K \subset W^s$:*
$$m^s(D \cap g^t K) \sim e^{ht} m^s(K) m(D).$$

Proof. First assume that D has a product structure. Choose a decreasing nested sequence $(D_i)_{i\in\mathbb{N}}$ of open sets with $D = \bigcap_{i\in\mathbb{N}} D_i$. Choose a pointwise nonincreasing sequence of continuous functions f_i which are 1 on D and 0 outside D_i. Then the previous proposition states that
$$\int_{g^t K} f_i\, dm^s \sim e^{-ht} m^s(K) \int_{SM} f_i\, dm,$$
and letting $i \to \infty$ shows the claim (for this D). Next, note that both sides of the claimed equation are additive in D. Any open subset of $\mathbf{Reg}$ is the union of product cubes, thus the claim is proven for regular D. Finally, recall that $m(\mathbf{Sing}) = 0$,and hence the claim is true for arbitrary open D. □

4. HOLONOMY CONTINUITY AND REGULAR NEIGHBORHOODS

The counting argument in section 7 requires a certain function to be continuous. That property is easily established in the uniformly hyperbolic case; however, for the nonuniform case that we are dealing with in this article, it is quite nontrivial. This section is devoted entirely to that point. We use several fairly new results about the measure of maximal entropy for the geodesic flow, in particular existence of conditional measures, holonomy invariance and uniform expansion for those. The holonomy continuity discussed here differs from the holonomy invariance proved in [Gun1] and [Gun2]: Instead of taking a set and its holonomic counterpart and showing that the conditional measure is preserved, we show that nearby sets of given geometry have similar conditional measure.

Definition 4.1. For $x, y \in SM$ let $d^s(x, y)$ be the distance of x and y along stable leaves; if $x \notin W^s(y)$ then $d^s(x, y) = \infty$. For $r \in [0, \infty)$ define $\vartheta_r : [0, \infty) \to [0, \infty)$ by
$$\vartheta_r(t) := \max(0, r - t).$$

For $r \in [0, \infty)$ and $v, w \in SM$ define

$$\sigma_r(v, w) := \vartheta_r(d^s(v, w)).$$

Finally, define

$$\psi_r(v) := \int_{z \in B^s_r(v)} \sigma_r(v, z)\, dm^s(z).$$

Evidently $\sigma_r(.,.)$ is symmetric. Note that it is also Lipschitz with Lipschitz constant 1 along W^s-leaves, i.e. for all z contained in at least one of the leaves $W^s(x)$ and $W^s(y)$ we have

$$|\sigma_r(v, z) - \sigma_r(w, z)| \leq d^s(v, w).$$

This is so because ϑ is 1-Lipschitz.

Theorem 4.2. *$\psi_r(v)$ is continuous in r. It is continuous in v along any W^s-leaf. If $\overline{B^s_r}(v) \subset \mathbf{Reg}$ then $\psi_r(v)$ is continuous in all variables at v.*

Proof. Continuity int r easily follows from the fact that $\sigma_r(v, w)$ is continuous in r for any v, w.

To show continuity of ψ_r in s-direction, let v, w be such that $d^s(v, w) < \delta$. Write $A_1 := B^s_r(v) \cap B^s_r(w)$ and $A_2 := B(v) \bigtriangleup B^s_r(w)$; then

$$\begin{aligned}|\psi_r(v) - \psi_r(w)| \leq m^s(A_1) \sup_{z \in A_1} |\sigma_r(v, z) - \sigma_r(w, z)| \\ + m^s(A_2) \cdot \left(\sup_{z \in A_2} |\sigma_r(w, z)| + \sup_{z \in A_2} |\sigma_r(v, z)| \right).\end{aligned}$$

Note that if $z \in A_2$ then $d^s(v, w) \in [0, \delta)$, thus $\sigma_r(v, z) < \delta$ and $\sigma_r(w, z) < \delta$. Thus the first summand on the right hand side is at most $\delta m^s(A_1)$ and the second at most $2\delta m^s(A_2)$. Thus $\psi_r(v)$ and $\psi_r(w)$ are arbitrarily close for v, w sufficiently close. Thus ψ_r is continuous along any W^s-leaf.

Now let $\overline{B^s_r}(v) \subset \mathbf{Reg}$; we want to show continuity in v. Continuity in the s-direction is shown above. Next we deal with the 0-direction. Let $w = g^\delta v$. Then

$$\begin{aligned}\psi_r(w) &= \int_{z \in B^s_r(g^\delta v)} \sigma_r(g^\delta v, z)\, dm^s(z) \\ &= \int_{z' \in g^{-\delta} B^s_r(g^\delta v)} \sigma_r(g^\delta v, g^\delta z')\, dm^s(g^\delta z'). \qquad (4.1)\end{aligned}$$

Recall that on a manifold of nonpositive curvature, any stable Jacobi field is nonincreasing in length; hence the map $F : t \mapsto d^s(g^t v, g^t w)$ is nonincreasing, thus for $t \geq 0$ its values are bounded by $d^s(v, w)$. Thus

$$g^\delta B^s_r(v) \subset B^s_r(g^\delta v).$$

On the other hand, the decrease of F is bounded by the derivative of the unstable Jacobi field, which is bounded by compactness of M. Hence for all $\varepsilon > 0$ there is $\delta_0 > 0$ so that for all $\delta < \delta_0$ we have $B^s_\varepsilon(g^\delta B^s_r(v)) \supset B^s_r(g^\delta v)$.

First note that $\delta \mapsto \sigma_r(g^\delta v, g^\delta z')$ is continuous in δ because the foliation W^s is continuous. Next note that $dm^s(g^\delta z')$ is continuous in δ because it is uniformly expanding in δ and $e^{h\delta}$ is arbitrarily close to 1 for δ sufficiently small. Finally note that $\forall v \in SM\, \forall \varepsilon > 0\, \exists \delta_0 > 0\, \forall \delta < \delta_0$ we have

$$B^s_r(g^\delta v) \supset g^\delta B^s_r(v) \supset B^s_{r-\varepsilon}(g^\delta v),$$

hence the value of $\sigma_r$is at most ε on the set $B^s_r(g^\delta v) \setminus gB^s_r(v)$. This shows that $\psi_r(w)$ and $\psi_r(v)$ are close because in the last line of equation (4.1) all terms are continous with respect to δ.

Finally, we show that ψ_r is continuous in the u-direction. Assume that $w \in B^u_\delta(v)$. Recall that the measure m^{0s} is invariant under holonomy along W^u-fibers. Hence if H is a holonomy map along W^{0u} from some set $A \subset W^s$ to some set $B \subset W^s$ so that for all $v_1 \in A$ the points v_1 and Hv_1 are ε_1-close, then diff $(m^s(A), m^s(B)) < h\varepsilon_1$. We are interested in the case $A = B^s_r(v),\ w \in B$.

Note that due to the condition that $\overline{B^s_r}(v) \subset \mathbf{Reg}$, there is some open neighborhood of $\overline{B^s_r}(v)$ which lies inside $\mathbf{Reg}$ and on which W^s, W^{0u} are uniformly transversal.

Note that for H as above we have $|\sigma_r(Hv_1, Hv_2) - \sigma_r(v_1, v_2)| \leq 2\varepsilon_1$ by 1-Lipschitzness of σ_r.Thus

$$\begin{aligned}
\psi_r(w) &= \int_{z \in B^s_r(w)} \sigma_r(z, w)\, dm^s(z) \\
&\leq \int_{z' \in H^{-1}B^s_r(w)} (\sigma_r(z', v) + 2\varepsilon_1)\ dm^s(Hz') \\
&\leq \int_{z' \in B^s_{r+\varepsilon_2}(v)} (\sigma_r(z', v) + 2\varepsilon_1)\ dm^s(z') \\
&\leq \psi_r(v) + 3\varepsilon_1 m^s(B^s_{r+\varepsilon_1}(v)).
\end{aligned}$$

Letting $\varepsilon_1 \to 0$ shows that $\psi_r(w)$ and $\psi_r(v)$ are arbitrarily close for v, w close enough. □

5. Measuring Riemannian volume by counting intersections

Let $x, y \in M$. Define

$$a_r(x, y) := \#(B_r(x) \cap (\pi_1(M) \cdot y))$$

to be the number of copies of y under Deck transformations that are inside the ball of radius r around x.

Lemma 5.1.
$$b_r(x) = \int_{y\in M} a_r(x,y)d\,\mathrm{vol}(y),$$
where vol *is the Riemannian volume on* M.

Proof. Let F be a fundamental domain of M. Denote the characteristic function of B by χ_B. Then

$$\begin{aligned} b_r(x) &= \sum_{\gamma\in\pi_1(M)} \mathrm{vol}(\gamma F\cap B_r(x)) \\ &= \sum_{\gamma\in\pi_1(M)} \int_{y\in\gamma F} \chi_{B_r(x)} d\,\mathrm{vol}(y) \\ &= \int_{y\in\gamma F} \sum_{\gamma\in\pi_1(M)} \chi_{B_r(x)} d\,\mathrm{vol}(\gamma^{-1}y) \\ &= \int_{y\in M} a_r(x,y)d\,\mathrm{vol}(y). \end{aligned}$$

□

For $x, y \in M$ assume $\mathcal{K} := S_yM$ to be transversal to W^{0u} and $\Lambda_0 := S_xM$ to be transversal to W^s. Let $\mathcal{L} := g^{[0,a]}\Lambda_0$. Let $\mathcal{K}, \Lambda_0$ be the disjoint unions $\mathcal{K} = \bigcup_j K_j$, $\Lambda_0 = \bigcup_i L_i$. Then

$$\begin{aligned} a_t(x,y) &= \#((\pi_1(M)\cdot y)\cap B_t(x)) \\ &= \#(S_yM\cap g^{[0,t]}S_xM) \\ &= \#(\mathcal{K}\cap g^{[0,t]}\Lambda_0) \\ &= \sum_{i,j} \#(K_j\cap g^{[0,t]}L_i). \end{aligned}$$

Therefore it suffices to be able to count these intersections in order to find b_r. Note that these intersections are always finite, even though the components of intersection of stable and unstable manifolds can be uncountable.

6. Product neighborhoods

Let D be open with $\overline{D} \subset \mathbf{Reg}$. Hence it has a product structure and transversality is uniform on D. Let $L \subset D$ be transversal to W^s. Note that for each $v \in L$ we have

$$\lim_{r\to 0} \psi_r(v) = 0.$$

By compactness of $\overline{D}$ and continuity of $\psi_r(v)$ in r, the convergence $\psi_r(v) \to 0$ as $r \to 0$ is uniform with respect to v. Hence there exists $c_0 > 0$ so that for all $c < c_0$ and all $v \in \overline{L}$ there exists $r_c = r_c(v)$ so that $\psi_{r_c(v)}(v) = c$. From the product structure of D follows that for c sufficiently small, for each $v \in L$, the intersection of $B^s_{r_c(v)}(v)$ contains exactly one point of L.

Definition 6.1. Write

$$Z = Z(L,c) := B^s_{r_c}(L) := \bigcup_{v \in L} B^s_{r_c(v)}(v).$$

Let $\pi_L : Z \to L$ be the projection defined by $\pi_L(z) = l$ for $z \in B^s_{r_c(l)}(l)$. Define the function $f_{L,c}$ supported on $\overline{Z}$ by

$$f_{L,c}(z) := \vartheta_r(\pi_L(z))$$

for $z \in Z$ and $f_{L,c}(z) := 0$ otherwise.

It is easy to see that

$$\int_Z f_{L,c} dm = c m^{0u}_{\mathcal{L}}(L)$$

since $dm = dm^s dm^{0u}$ on D and since the stable measure of each s-fiber equals c.

For two subspaces E_1, E_2 of a vector space let

$$\mathbf{d}(E_1, E_2) := d_H(E_1 \cap S^1, E_2 \cap S^1)$$

be the distance in the Grassmannian bundle induced by the Hausdorff distance on unit spheres.

Lemma 6.2. *Let $\mathcal{K}$ be compact and transversal to W^{0u}. Then for all $\varepsilon > 0$ there exists t_0 so that for all $t > t_0$ we have*

$$\mathbf{d}(Tg^{-t}\mathcal{K}, TW^s) < \varepsilon.$$

Proof. Each $\xi \in Tg^{-t}\mathcal{K}$ can be written as $\xi = \xi^{||} + \xi^{\perp}$ with $\xi^{||} \in TW^s$, $\xi^{\perp} \perp TW^s$ (perpendicular with respect to the Sasaki metric on SM). Recall that for $t \to -\infty$, any unstable Jacobi field is bounded and any stable Jacobi field is unbounded. Since $(dg^t\xi)^{||}$ is unbounded for $t \to -\infty$ and $(dg^t\xi)^{\perp}$ is bounded, it follows that $\angle((dg^t\xi), TW^s) \to 0$ for $t \to -\infty$. □

7. Intersection estimate

Definition 7.1. For $\mathcal{K}$ compact and transversal to W^{0u} and $\mathcal{L}$ compact and transversal to W^s, $K \subset \mathcal{K}$, $L \subset \mathcal{L}$ define

$$Q(K, L, t) := \#(L \cap g^{-t}K),$$

define $\Phi(K, L, c, t)$ to be the set of components φ of $Z \cap g^{-t}\mathcal{K}$ such that if $p \in \varphi$ and $p \in B^s_{r_c}(l)$ for $l \in L$ then $B^s_{r_c}(l) = \varphi$, and finally define

$$N(K, L, c, t) := \#\Phi(K, L, c, t).$$

For a set K as above with the property that $K \subset D$ where D is open and $\overline{D}$ carries a product structure, we abbreviate the notation $\pi_{D,p}(K)$ by K'. This means that the choice of p is suppressed in the notation. K' may be disconnected even if K is connected.

For a set $K' \subset W^s$ with $B_\alpha K' \subset D$ for $\alpha > 0$ we write

$$\hat{B}^s_\alpha K' := \pi_{D,p} B_\alpha K',$$

i.e. the set K' is (arbitrarily) extended by distance α in the s-direction. Similarly, for $\alpha < 0$ we write

$$\hat{B}^s_\alpha K' := (K \setminus B_{-\alpha}(\partial K))'$$

for the opposite, namely shrinking K' by distance α from its boundary.

Note that by the previous lemma, for each $\mathcal{K}, \mathcal{L}$ there exists t_0 such that for $t > t_0$ the number $Q(K, L, t)$ is finite.

Clearly $\forall K, L$ as before, $c > 0,\, t \geq 0$:

$$N(K, L, c, t) \leq Q(K, L, t).$$

On the other hand, unboundedness of stable Jacobi fields for $t \to -\infty$ and hence unboundedness of the diameter of the image of the annulus $(\hat{B}^s_\alpha K') \setminus K'$ under g^{-t} gives $\forall K, L, c > 0\, \forall \alpha > 0\, \exists T = T(\alpha)\, \forall t > T$:

$$Q(K', L, t) \leq N(\hat{B}^s_\alpha K', L, t).$$

Moreover, $\forall \varphi \in \Phi(K', L, c, t)$:

$$\int_\varphi f_{L,c}\, dm^s = c;$$

hence $\forall K, L, c, t$:

$$N(K', L, c, t) \leq \frac{1}{c} \int_{SM} f_{L,c} \cdot (\chi_{K'} \circ g^t)\, dm^s.$$

On the other hand, $\forall \alpha > 0\, \exists T = T(\alpha)\, \forall t > T$:

$$N(\hat{B}^s_\alpha K', L, c, t) \geq \frac{1}{c} \int_{SM} f_{L,c} \cdot (\chi_{K'} \circ g^t)\, dm^s.$$

Note that $m^s_{\mathcal{K}}(\partial K') = 0 = m^{0u}_{\mathcal{L}}(\partial L)$.

In the following, the only type of L we need to consider is $L = g^{[0,t_0]}\Lambda$ for some Λ contained in some D_i with a product structure (which makes Λ transversal to W^{0s}). This L is without loss of generality the disjoint union $L = \bigcup_{i=0}^{n-1} L_i$ with $L_i \subset g^{[it_0/n,(i+1)t_0/n)}\Lambda$, and we can further assume without loss of generality that that $L_i \subset D_i$ for $i < n$ (by renumbering the D_i appropriately).

Lemma 7.2. *For all $\alpha > 0$ there exists T such that for all $t > T$:*

$$\sum_{i=1}^{k} Q(K', \hat{B}^u_{-\alpha} L_i, t-\varepsilon) \leq \sum_{i=1}^{k} Q(K, L_i, t) \leq \sum_{i=1}^{k} Q(K', \hat{B}^u_{\alpha} L_i, t+\varepsilon).$$

Proof. Note that once more the nonpositivity of the curvature gives unboundedness of the boundary annulus as $t \to -\infty$. Moreover $g^{-t}K'$ is arbitrarily close to W^s as t becomes large. Note also that each point $p \in K$ which lies on $L_i$gets moved by at most ε in the flow direction under the map $K \mapsto K'$ and hence gets mapped back to L_i or gets mapped to $L_j$with $|i-j|\, a/k < \varepsilon$. Note that, for $i \neq j$, increasing the term $Q(K', \hat{B}^u_{\alpha} L_j, t-\varepsilon)$ by 1 and simultaneously decreasing the term $Q(K', \hat{B}^u_{\alpha} L_i, t-\varepsilon)$ by 1 does not change the sum in the statement of the lemma. ☐

Theorem 7.3.

$$\begin{aligned}
\limsup_{t\to\infty} e^{-ht} \sum_{i=1}^{k} Q(K, L_i, t) &\leq e^{2ht} m^s_{\mathcal{K}}(K) \sum_{i=1}^{k} m^{0u}_{\mathcal{L}}(L_i), \\
\liminf_{t\to\infty} e^{-ht} \sum_{i=1}^{k} Q(K, L_i, t) &\geq e^{-2ht} m^s_{\mathcal{K}}(K) \sum_{i=1}^{k} m^{0u}_{\mathcal{L}}(L_i).
\end{aligned}$$

Proof. For the first inequality, we see that for all $\alpha > 0$

$$\limsup_{t\to\infty} e^{-ht} \sum_{i=1}^{k} Q(K, L_i, t) \leq \limsup_{t\to\infty} e^{-ht} \sum_{i=1}^{k} Q(K', \hat{B}^u_{\alpha} L_i, t+\varepsilon).$$

Hence

$$\begin{aligned}
\limsup_{t\to\infty} e^{-ht} \sum_{i=1}^{k} Q(K, L_i, t) &\leq \lim_{\alpha\to 0} \limsup_{t\to\infty} e^{-ht} \sum_{i=1}^{k} Q(K', \hat{B}^u_{\alpha} L_i, t+\varepsilon) \\
&\leq \lim_{\beta\to 0} \lim_{\alpha\to 0} \limsup_{t\to\infty} e^{-ht} \sum_{i=1}^{k} N(\hat{B}^s_{\beta} K', \hat{B}^u_{\alpha} L_i, t+\varepsilon) \\
&\leq \frac{1}{c} \lim_{\beta\to 0} \lim_{\alpha\to 0} \limsup_{t\to\infty} e^{-ht} \sum_{i=1}^{k} \int_{SM} f_{\hat{B}^u_{\alpha} L_i, c} \cdot (\chi_{\hat{B}^s_{\beta} K'} \circ g^{t+\varepsilon}) dm^s \\
&\leq e^{2ht} m^s_{\mathcal{K}}(K) \sum_{i=1}^{k} m^{0u}_{\mathcal{L}}(L_i).
\end{aligned}$$

Here we have used lemma 3.4. Note that the necessary continuity of $f_{\hat{B}^u_{\alpha} L_i, c}$ is provided by theorem 4.2.

The second inequality is proven the same way, using the opposite estimates for exchanging Q with N and N with the integral.

Using the fact that $m_{\mathcal{K}}^s(\partial K') = 0 = m_{\mathcal{L}}^{0u}(\partial L)$, we reach the following conclusion: $\square$

Corollary 7.4. *For L regular and K arbitrary (or for K regular and L arbitrary), we get*

$$\lim_{t\to\infty} e^{-ht}\sum_{i=1}^{k} Q(K, L_i, t) = m_{\mathcal{K}}^s(K)\sum_{i=1}^{k} m_{\mathcal{L}}^{0u}(L_i).$$

Note that regularity is invariant under the flow. Hence if L is regular, any intersection of L with $g^{-t}K$ can only occur at regular points. It is therefore sufficient if just one of the two sets L, K is regular. Hence the case we have treated before suffices.

If neither K nor L is regular, then we split it as $K = \bigcup_j K_j$ and $L = \bigcup_i \tilde{L}_i$ where each K_j and L_i is a subset of $\mathbf{Reg}$ or $\mathbf{Sing}$. We have dealt with the former and are going to show that the latter does not distort the count. Without loss of generality $\tilde{L}_i = \bigcup_{k=0}^{n-1} g^{kt/n}L_i$ and $L_i = g^{[0,t_0]}\Lambda_i$.

Proposition 7.5. *There exists $\gamma > 0$ such that for $K_j \subset \mathbf{Sing}$ or $L_i \subset \mathbf{Sing}$ there exists $T \in \mathbb{R}$ such that for $t > T$ we have*

$$0 \le \sum_{i,j,k} \#\left(K_j \cap g^{kt/n}L_i\right) \le e^{(h-\gamma)t}.$$

Proof. Let $v, w \in K \subset S_yM$ be such that $g^{-t}v, g^{-t+a_1}w \in \Lambda_i \subset S_xM$ for $t \ge 0$ and $0 \le a_1 \le t_0$. Then the geodesic segments $g^{[-t,0]}v$ and $g^{[-t+a_1,0]}w$ either form a geodesic biangle (i.e. 2-gon, bounding a topological 2-disc) or form a topologically nontrivial loop.

Note that in a space of nonpositive curvature, any geodesic biangle is degenerate, i.e. subset of a single geodesic. This is so because any biangle is in particular a triangle with one side of zero length, and by triangle comparison with flat 2-space the nonpositivity of the curvature shows that both angles of the biangle are 0.

Hence either $v = w$ or the orbits of v, w are (t, l_0)-separated, where $2l_0 > 0$ is the length of the shortest closed geodesic in M.

Since the growth rate of any separated set in $\mathbf{Sing}$ is less or equal to the topological entropy of $\mathbf{Sing}$ and since $h_{\mathbf{Sing}} < h$, the claim follows. $\square$

This proves:

Theorem 7.6. *For each $x \in M$ there exists $c(x)$ so that*

$$\operatorname{vol} B_t(x) \sim c(x)e^{ht}.$$

The function $c : M \to \mathbb{R}$ is continuous. It satisfies

$$c(x) = \frac{1}{h}\int_{y\in M} a(x, y)d\operatorname{vol}(y)$$

where

$$a(x,y) = m^s_{\mathcal{K}}(S_xM)m^u_{\Lambda_0}(S_yM).$$

Proof. For x as above, the preceding arguments show that

$$a_t(x,y) \sim e^{ht}m^s_{\mathcal{K}}(S_xM)m^u_{\Lambda_0}(S_yM).$$

In particular, $a_t(x,y) \sim e^{ht}a(x,y)$ where $a(x,y) = m^s_{\mathcal{K}}(S_xM)m^u_{\Lambda_0}(S_yM)$. The function $a(x,y)$ is evidently independent of t.

For continuity, simply note that if y is another point of M with $d(x,y) \leq \varepsilon$ then $b_t(x) = \operatorname{vol} B_t(x)$ satisfies

$$b_{t-\varepsilon}(x) \leq b_t(y) \leq b_{t+\varepsilon}(x).$$

Thus

$$e^{-h\varepsilon} \leq \frac{b_t(y)}{c(x)e^{ht}} \leq e^{h\varepsilon}.$$

Thus $b_t(y)$ and $b_t(x)$ are arbitrarily close for x and y sufficiently close. □

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