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Inquiry-based learning in academic teaching compared to traditional teaching: an example

Introduction

In academic teaching, traditional lecturing style sometimes results in the situation that the content of the lecture appears difficult to the students while at the same time the lecturer believes that the content is easy to understand for the students. This can lead to several difficulties:

- The course material is not properly learned by the students.
- The resultant knowledge gaps cause difficulty for the students in future lessons in the same class which require the preceding knowledge.
- Students can get frustrated and discouraged, which also negatively impacts future academic performance.

In this article we present an example from a mathematics course in academia that shows how students can discover mathematical content via inquiry-based learning. As a result of using inquiry-based learning, the aforementioned difficulties are avoided. Moreover, the content learned through inquiry-based learning is remembered substantially better by the students. The students are also better able to apply their new knowledge, acquire better learning strategies and show higher levels of motivation. These effects appear immediately but appear to last for a long time. These results are based on a study done at the PH Vorarlberg, Austria.

For a background on inquiry-based learning (German: *Forschendes Lernen*) see the following recent literature: A theoretical overview, model, goals and key elements of inquiry-based learning in mathematics education is presented by Roth and Weigand (2014a, 2014b). Ulm (2009, 2011) also describes a theoretical framework and identifies six different phases in inquiry-based learning. Messner (2009) treats various types of inquiry-based learning and similar activities; there are also several names for these concepts. Lutz-Westphal (2014) lists some key characteristics and types of questions in inquiry-based learning. See also Dewey (1910, 1938).

Case study 1: The triangle inequality

One of the most basic and most fundamental facts in mathematics, and certainly in any course of real analysis, is the triangle inequality:

For all real numbers x, y, it is true that:

$$|x+y| \leq |x|+|y|.$$

In H. Linneweber-Lammerskitten (Hrsg.), *Beiträge zum Mathematikunterricht 2015* (S. x–y). Münster: WTM-Verlag

Here $|a|$ denotes the absolute value of the number a.

The triangle inequality requires only the definition of absolute value and some of the axioms of the real numbers. This and the fact that the proof is "easy" make it a good early example for demonstrating methods of proof.

This inequality is used extremely often in analysis courses; it is contained in a large fraction of all estimates and proofs. It occurs so frequently that it is often not even explicitly cited by name (except in the lecture where it is first taught, and a few subsequent lectures). Hence students who do not understand the triangle inequality or who are not able to apply it run into problems very often during a course on real analysis, leading to frustration and an increased likelihood of failure. On the other hand, students who master this inequality will experience personal success in doing so; later they will have many opportunities to remind them of their success, leading to increased confidence and motivation.

Useful variations of the triangle inequality

The following inequalities (all direct consequences of the triangle inequality) are useful but seem more difficult for students.

- $|x-y| \geq |x| - |y|$. (Or, more generally, $|x-y| \geq ||x|-|y||$.)
- $|x+y| = |x| + |y|$ if the signs of x and y are the same.
- $|x-y| = |x| - |y|$ if the signs of x and y are the same and $|x| \geq |y|$.
- Equality holds in the triangle inequality if x=0 or y=0.

For motivated students, these variations present only a small challenge. But discouraged students find these equations/inequalities difficult.

Typical difficulties

Students often show the following problems: confusing "≥" and "≤" (the "direction" of the inequality); trouble with all the variations of the triangle inequality; and trouble understanding the proof of the triangle inequality.

Difficulty with the proof

Proofs of the triangle inequality, while neither long nor really difficult, can nonetheless be intimidating to students. As an example, we analyze the proof given in the well-explaining and popular textbook by Barner & Flohr (1987). The proof uses just 3 arguments, making it appear to be easy:

> For all real-valued numbers x, y, it is true that $-|x| \leq x \leq |x|$, and similarly it is true that $-|y| \leq y \leq |y|$.
>
> By adding these two inequalities, we obtain $-|x|-|y| \leq x+y \leq |x+y|$.

From this we deduce $|x+y| \leq |x|+|y|$, as desired. Q.E.D.

Students may not find this proof quite so easy. It uses several facts: $x \leq |x|$, $-|x| \leq x$, and the fact that $|a| \leq b$ if and only if both $a \leq b$ and $-a \leq b$ are true. These facts are all elementary, and students will be able to prove them by themselves if asked; yet students who have not seen them before will not easily understand them *in the middle of this proof*.

A study was done with courses in (undergraduate) Analysis for students about to become secondary school teachers in Austria, which consisted of weekly lectures followed by recitation/exercise sessions. In previous years, when the proof of the triangle inequality was taught in the textbook manner as illustrated above, some students found this approach too theoretical and were not able to understand and correctly remember the triangle inequality. Hence the following inquiry-based learning method was used.

An inquiry-based teaching and learning approach

Students were given a large table whose columns are labelled x, y, $x+y$, $x-y$, $|x|$, $|y|$, $|x+y|$, $|x|+|y|$, $|x-y|$, $|x|-|y|$, plus space for extra columns. The columns for x and y were already completed with pairs of numbers; the other columns were empty. The students were asked to do the following steps:

1. Complete the table (calculate the values in the empty columns).
2. Find relations – such as inequalities – between the columns.
3. Document the findings by writing down precise inequalities between the quantities involved. Present the results to the other students.

Step 2 (finding inequalities) is of course the relevant part in this learning activity. Step 1 (forcing the students to complete the table themselves) serves the purpose of activating the students and giving them a possibility to present something to the rest of the class; this works particularly well in a small classroom setting with 30 students or less. Step 1 may appear too elementary, but in actuality it takes only a small amount of time and actively engages the students in the problem solving activity, probably better enabling them to deal with step 2. Step 3 (documentation) is useful because it asks the students to come up with precise mathematical statements, forcing them to think clearly. It may also enhance students' understanding of their own findings so far, for example due to realizing that some assumptions (e.g. "x is positive") can be dropped and hence the statement they are about to write down can be made simpler and/or more general. Step 3, in particular presenting the results to other students, helps students to memorize their findings. The documentation step is an important part of inquiry-based learning (Ulm 2009, Roth & Weigand 2014a, Roth & Weigand 2014b).

Summary of results and conclusion

The students discovered the triangle inequality and many variations (including all of those mentioned earlier) entirely by themselves, participated very actively, appeared to find this exercise enjoyable, became acutely aware of the direction of the inequalities, showed a working knowledge of how to apply the inequalities, and showed substantially increased interest to learn the abstract proof of the triangle inequality later in the lecture part of the course. When the triangle inequality was used later in the course (during more theoretical arguments), students displayed no difficulty following. In the written final examination for this course, students' answers showed no difficulties with the triangle inequality; in particular, the "typical difficulties" mentioned earlier were not seen. These results show clearly that inquiry-based learning is a useful teaching approach for this topic, especially when compared to the "traditional" method. It appears reasonable to assume that a similar approach could work well with a variety of other mathematical teaching topics. An interesting task for the future will be to create a "catalog" of topics particularly suitable for inquiry-based learning, or even to find some general criteria to decide whether inquiry-based learning should be attempted for topics where this has not been done before.

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