Calculus – 6. Series, Solutions

1. Find $\overline{\lim\limits_{n\to\infty}}\, x_n$ and $\underline{\lim\limits_{n\to\infty}}\, x_n$ for the following sequences

$$\begin{aligned}
&\text{(a)}\quad x_n &&= \frac{n}{n+1}\sin^2\frac{n\pi}{4}\\
&\text{(b)}\quad x_n &&= \sqrt[n]{1+2^{n(-1)^n}}\\
&\text{(c)}\quad (x_n) &&= \left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots\right).
\end{aligned}$$

(*Hint.* 'Find' includes to justify your claim.)

Solution. (a) Since $\sin^2\alpha = (1-\cos 2\alpha)/2$ we find

$$\begin{aligned}
\sin^2(\pi n + \frac{\pi}{4}) &= \frac{1}{2} - \frac{1}{2}\cos(2n\pi + \pi/2) = \frac{1}{2},\\
\sin^2(\pi n + \frac{3\pi}{4}) &= \frac{1}{2} - \frac{1}{2}\cos(2n\pi + 3\pi/2) = \frac{1}{2},\\
\sin^2(\pi n + \frac{\pi}{2}) &= \frac{1}{2} - \frac{1}{2}\cos(2n\pi + 2\pi) = 1
\end{aligned}$$

for all integers n. Hence,

$$\begin{aligned}
x_{4n} &= \frac{4n}{4n+1}\sin^2\pi n = 0 \underset{n\to\infty}{\longrightarrow} 0\\
x_{4n+1} &= \frac{4n+1}{4n+2}\sin^2\left(\pi n + \frac{\pi}{4}\right) = \frac{1}{2}\,\frac{4n+1}{4n+2} \underset{n\to\infty}{\longrightarrow} \frac{1}{2};\\
x_{4n+2} &= \frac{4n+2}{4n+3}\sin^2\left(\pi n + \frac{\pi}{2}\right) = \frac{4n+2}{4n+3} \underset{n\to\infty}{\longrightarrow} 1;\\
x_{4n+3} &= \frac{4n+3}{4n+4}\sin^2\left(\pi n + \frac{3\pi}{4}\right) = \frac{1}{2}\,\frac{4n+3}{4n+4} \underset{n\to\infty}{\longrightarrow} \frac{1}{2}.
\end{aligned}$$

We have found the limit points $0, 1/2, 1$ of (x_n). Since the four subsequences (x_{4n}), (x_{4n+1}), (x_{4n+2}), and (x_{4n+3}) cover the entire sequence (x_n), these are the only limit points. We conclude $\underline{\lim}\, x_n = 0$ and $\overline{\lim}\, x_n = 1$.

(b) We find

$$x_{2n} = \sqrt[2n]{1+2^{2n}} \quad \text{and} \quad x_{2n+1} = \sqrt[2n+1]{1+\frac{1}{2^{2n+1}}}.$$

Since

$$2^{2n} < 1 + 2^{2n} < 2^{2n+1}$$

for all positive integers n, applying Lemma 1.16 yields

$$2 < \sqrt[2n]{1+2^{2n}} < 2\sqrt[2n]{2}.$$

Since $\lim 2 = \lim(2\sqrt[2n]{2}) = 2\cdot 1 = 2$, the sandwich theorem gives $\lim x_{2n} = 2$.

Similarly,

$$1 < 1 + \frac{1}{2^{2n+1}} < 2 \underset{\text{Lemma 1.16}}{\Longrightarrow} 1 < \sqrt[2n+1]{1+\frac{1}{2^{2n+1}}} < \sqrt[2n+1]{2}.$$

Since both the left hand side and the right hand side of the above inequality tend to 1 as $n \to \infty$, the sandwich theorem applies and $\lim x_{2n+1} = 1$. Since the odd and the even positive integers cover the entire set $\mathbb{N}$, almost all elements of (x_n) are in union of the two neighborhoods $U_\varepsilon(1)$ and $U_\varepsilon(2)$ for every $\varepsilon > 0$. Hence 1 and 2 are the only limit points of (x_n); and therefore $\underline{\lim}\, x_n = 1$ and $\overline{\lim}\, x_n = 2$.

(c) The subsequence

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$$

tends to 0; hence 0 is a limit point of (x_n). The subsequence

$$\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right)$$

tends to 1; therefore 1 is a limit point of (x_n).

(Idea by Mathias Aust ans Jan Henning Peters) Since $x_n > 0$ for all positive integers n, Proposition 12 implies $\underline{\lim}\, x_n \geq 0$. This gives $\underline{\lim}\, x_n = 0$ since 0 is indeed a limit point of (x_n).

Since $x_n < 1$ for all n, Proposition 12 implies $\overline{\lim}\, x_n \leq 1$. This gives $\overline{\lim}\, x_n = 1$ since 1 is a limit point of (x_n).

2. Prove: If the sequence (y_n), $y_n \neq 0$ for all n, has an improper limit and the sequence (x_n) is bounded, then (x_n/y_n) tends to 0.

Proof. We consider the case $\lim y_n = -\infty$ only. The proof for $\lim y_n = +\infty$ is similar. Since (x_n) is bounded, there is a positive $C > 0$ such that $|\, x_n \,| \leq C$ for every $n \in \mathbb{N}$. Given $\varepsilon > 0$ choose n_0 in such a way that

$$y_n \leq -\frac{C}{\varepsilon} \tag{1}$$

for every $n \geq n_0$. This choice of n_0 is possible since $\lim y_n = -\infty$. From (1) we conclude $|\, y_n \,| = -y_n \geq C/\varepsilon$ and therefore

$$\frac{1}{|\, y_n \,|} \leq \frac{1}{C/\varepsilon} \quad \text{which implies} \quad \left| \frac{x_n}{y_n} \right| = \frac{|\, x_n \,|}{|\, y_n \,|} \leq \frac{C}{C/\varepsilon} = \varepsilon$$

if $n \geq n_0$. This proves the assertion. ■

3. Let (x_n) be a sequence which is bounded below, $t = \underline{\lim}_{n\to\infty}\, x_n$, and $\varepsilon > 0$ fixed.

(a) Prove that $t - \varepsilon$ is a lower bound for all but finitely many x_n, i. e. there exists a positive integer n_0 such $t - \varepsilon \leq x_n$ if $n \geq n_0$.

(b) Prove that, in general, (a) fails if $\varepsilon = 0$.

Proof. (a) Let s be a lower bound of x_n. Suppose to the contrary that $t - \varepsilon$ is not a lower bound for almost all x_n. That is, infinitely many elements of the sequence, say

(x_{n_k}), $k \in \mathbb{N}$, are smaller than $t - \varepsilon$ (and still greater than s). By Proposition 10, (x_{n_k}) has a limit point r which is in turn a limit point of (x_n). However,

$$r \leq t - \varepsilon$$

contradicts our assumption that t is the lower limit of (x_n). Hence, $t - \varepsilon$ is a lower bound for almost all x_n. ■

(b) The sequence $x_n = \dfrac{(-1)^{n+1}}{n}$ tends to 0 such that $\underline{\lim}\, x_n = \overline{\lim}\, x_n = \lim x_n = 0$. Since $x_{2n} < 0$ for all $n \in \mathbb{N}$, 0 is not a lower bound for all but finitely many x_n.

4. Let (x_n) and (y_n) be sequences which are both bounded above.

(a) Prove or disprove:

$$\overline{\lim_{n\to\infty}}\, (x_n y_n) = \overline{\lim_{n\to\infty}}\, x_n \cdot \overline{\lim_{n\to\infty}}\, y_n.$$

(b) Prove or disprove:

$$\overline{\lim_{n\to\infty}}\, (x_n + y_n) \leq \overline{\lim_{n\to\infty}}\, x_n + \overline{\lim_{n\to\infty}}\, y_n$$

(*Hint.* Use Proposition 12.)

(a) The statement is false. Consider $(x_n) = (1, 2, 1, 2, \dots)$ and $(y_n) = (2, 1, 2, 1, \dots)$. Then $\overline{\lim}\, x_n = \overline{\lim}\, y_n = 2$ but $(x_n y_n) = (2, 2, 2, \dots)$ has also upper limit 2. Note that this example disproves $\overline{\lim}\, (x_n + y_n) = \overline{\lim}\, x_n + \overline{\lim}\, y_n$ as well.

(b) The statement is true. *Proof.* Set $x = \overline{\lim}\, x_n$ and $y = \overline{\lim}\, y_n$. Let $\varepsilon > 0$. We use the analogous statement to homework 6.3 (a) for upper limits: $x + \varepsilon/2$ is an upper bound for all but finitely many x_n and similarly $y + \varepsilon/2$ is an upper bound for all but finitely many y_n. That is

$$x_n \leq x + \varepsilon/2 \quad \text{and} \quad y_n \leq y + \varepsilon/2$$

for all but finitely many n. Taking the sum we have

$$x_n + y_n \leq x + y + \varepsilon$$

for all but finitely many n. Proposition 12 gives

$$\overline{\lim}\, (x_n + y_n) \leq x + y + \varepsilon.$$

Taking the infimum of the right hand side (over all $\varepsilon > 0$) proves the assertion. ■

5. Using the **definition** of a Cauchy sequence, prove that

$$x_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

is a Cauchy sequence.
(*Hint.* $m^{-2} < (m-1)^{-1} - m^{-1}$ for all integers $m \geq 2$.)

Proof. Using the hint, we find the following estimate by a *telescope sum*

$$|\, a_{n+m} - a_n \,| = \sum_{k=n+1}^{n+m} \frac{1}{k^2} < \sum_{k=n+1}^{n+m} \left(\frac{1}{k-1} - \frac{1}{k} \right) = \frac{1}{n} - \frac{1}{n+m} < \frac{1}{n}$$

Given $\varepsilon > 0$ choose $n_0 \geq 1/\varepsilon$. Then $n \geq n_0$ and $m \in \mathbb{N}$ implies

$$|\, a_{n+m} - a_n \,| < \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon.$$

This shows that $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is a Cauchy sequence (and therefore, by the Cauchy criterion, Proposition 18, the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges). ∎