Calculus – 5. Series, Solutions

1. Using the limit laws compute $\lim\limits_{n\to\infty} a_n$ for the following sequences

$$
\begin{aligned}
&\text{(a)} \quad a_n = \frac{999n}{n^2+1}\\
&\text{(b)} \quad a_n = \frac{2n^3+5}{\sqrt{n}(n^2-6)}\\
&\text{(c)} \quad a_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2}\\
&\text{(d)} \quad a_n = \frac{3\cdot 4^n + 4\cdot 5^{n+1}}{5^n - 8\cdot 4^n}\\
&\text{(e)} \quad a_n = \sqrt[n]{2n^6+10}.
\end{aligned}
$$

Solution. (a) We have

$$a_n = \frac{999n}{n^2+1} = \frac{\frac{999}{n}}{1+\frac{1}{n^2}}.$$

Since $\lim 1/n = 0$, we obtain from Proposition 3 (c), $\lim 1/n^2 = 0$; by Proposition 3 (b) then $\lim 999/n = 0$. Using Proposition 3 (b) we have $\lim(1+1/n^2) = 1+0 = 1$. Using Proposition 3 (d) we finally have

$$\lim_{n\to\infty} a_n = \frac{\lim\limits_{n\to\infty} \dfrac{999}{n}}{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n^2}\right)} = \frac{0}{1} = 0.$$

(b) We have

$$a_n = \frac{2n^3+5}{\sqrt{n}(n^2-6)} = \frac{2\sqrt{n}+\frac{5}{n^{5/2}}}{1-\frac{6}{n^2}} > 2\sqrt{n}$$

Since $(2\sqrt{n})$ is divergent we *cannot apply* Proposition 3. We will show that $\lim a_n = +\infty$. For, let $E > 0$ and choose n_0 such that $n_0 > E^2$. Then $n \ge n_0$ implies

$$a_n > 2\sqrt{n} > \sqrt{n_0} > E$$

which shows $\lim a_n = +\infty$.

(c) Using induction it is easy to prove that $1+2+3+\cdots+n-1 = n(n-1)/2$. Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n(n-1)/2}{n^2} = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2n}\right) = \frac{1}{2} - \lim_{n\to\infty}\frac{1}{2n} = \frac{1}{2} - 0 = \frac{1}{2}.$$

Note that Proposition 3 (a) applies only to *finitely many* summands.

(d) Dividing both the numerator and the denominator of the quotient by 5^n we obtain

$$a_n = \frac{3\cdot\left(\frac{4}{5}\right)^n + 20}{1-8\cdot\left(\frac{4}{5}\right)^n}.$$

Since $\lim_{n\to\infty} (4/5)^n = 0$ by Example 2 (d) we have, using Proposition 3

$$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} \left(3\cdot\left(\frac{4}{5}\right)^n + 20\right)}{\lim_{n\to\infty} \left(1 - 8\cdot\left(\frac{4}{5}\right)^n\right)}$$

$$= \frac{3\cdot \lim_{n\to\infty} \left(\frac{4}{5}\right)^n + 20}{1 - 8\cdot \lim_{n\to\infty}\left(\frac{4}{5}\right)^n} = \frac{3\cdot 0 + 20}{1 - 8\cdot 0} = 20.$$

(e) By Proposition 5 (b) and (c) we have $\lim \sqrt[n]{3} = \lim \sqrt[n]{n} = 1$ such that

$$\lim_{n\to\infty} \sqrt[n]{3n^6} = \lim_{n\to\infty} \sqrt[n]{3}\cdot\left(\lim_{n\to\infty} \sqrt[n]{n}\right)^6 = 1.$$

For $n \geq 2$ we have $10 < n^6$ and therefore $1 < a_n < \sqrt[n]{2n^6 + n^6}$. The sandwich theorem now applies:

$$1 = \lim_{n\to\infty} 1 \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} \sqrt[n]{3n^6} = 1;$$

hence $\lim a_n = 1$.

2. Let (a_n) be recursively defined by

$$a_1 = \sqrt{6} \quad \text{and} \quad a_{n+1} = \sqrt{6 + a_n} \quad (n \in \mathbb{N}). \tag{1}$$

Prove that (a_n) is monotonically increasing and bounded above. Compute $\lim_{n\to\infty} a_n$.

Proof. Using induction one shows that a_n is positive real for every n. We skip this very part. Next we show by induction that a_n is bounded above by 3, $a_n \leq 3$. Since $a_1 = \sqrt{6} < 3$ the induction start is fulfilled. Suppose $a_n \leq 3$ for some fixed n. We are going to show $a_{n+1} \leq 3$, too. Indeed, $a_n \leq 3$ implies $6 + a_n \leq 6 + 3 = 9$ and homework 4.1.(a) yields $a_{n+1} = \sqrt{6 + a_n} \leq \sqrt{9} = 3$ which completes the induction proof.

We give a direct proof that $a_{n+1} \geq a_n$ for all positive integers n. Since $a_n > 0$ and $a_n \leq 3$ we have $(a_n - 3)(a_n + 2) \leq 0$. This gives

$$(a_n - 3)(a_n + 2) \leq 0 \iff a_n^2 - a_n - 6 \leq 0 \iff a_n^2 \leq a_n + 6 = a_{n+1}^2 \iff a_n \leq a_{n+1}.$$

Hence a_n is monotonically increasing and bounded above, so that by Proposition 6, $\lim_{n\to\infty} a_n = a$ exists. We conclude by taking the limit $n \to \infty$ in the square of the recurrence relation (1), $a_{n+1}^2 = 6 + a_n$

$$a^2 = \lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty} (6 + a_n) = 6 + a.$$

This implies $(a-3)(a+2) = 0$. Since $a_n > 0$, $a \geq 0$ and we obtain $a = 3$.

3. Find the limit points of the sequence (x_n).

$$\begin{aligned}
&\text{(a)} \quad x_n = 1 + 2(-1)^{n+1} + 3(-1)^{\frac{n(n-1)}{2}}\\
&\text{(b)} \quad x_n = \frac{n-1}{n+1} \cos \frac{2\pi n}{3}\\
&\text{(c)} \quad x_n = \cos^n \frac{2\pi n}{3}.
\end{aligned}$$

Solution. (a) We have

$$\begin{aligned}
x_{4n} &= 1 + 2(-1)^{4n+1} + 3(-1)^{2n(4n-1)} = 1 - 2 + 3 = 2\\
x_{4n+1} &= 1 + 2(-1)^{4n+2} + 3(-1)^{(4n+1)2n} = 1 + 2 + 3 = 6\\
x_{4n+2} &= 1 + 2(-1)^{4n+3} + 3(-1)^{(2n+1)(4n+1)} = 1 - 2 - 3 = -4\\
x_{4n+3} &= 1 + 2(-1)^{4n+4} + 3(-1)^{(4n+3)(2n+1)} = 1 + 2 - 3 = 0.
\end{aligned}$$

(a_n) is the disjoint union of four constant subsequences. Hence, the limit points of (a_n) are exactly the limits of the four subsequences, that is -4, 0, 2, and 6.
(b) We have $x_{3n} = \frac{3n-1}{3n+1} \cos 2\pi n = \frac{3n-1}{3n+1}$. Since $\cos 2\pi/3 = \cos 4\pi/3 = -\frac{1}{2}$ we conclude

$$x_{3n+1} = \frac{3n}{3n+2} \cdot \left(-\frac{1}{2}\right) \quad \text{and} \quad x_{3n+2} = \frac{3n+1}{3n+3} \cdot \left(-\frac{1}{2}\right).$$

Since $\lim_{n\to\infty} x_{3n} = 1$ and $\lim_{n\to\infty} x_{3n+1} = \lim_{n\to\infty} x_{3n+2} = -\frac{1}{2}$, the points 1 and $-\frac{1}{2}$ are limit points of (x_n). We show that (x_n) has no other limit points. Suppose to the contrary that $a \neq 1$, $a \neq -\frac{1}{2}$ is another limit point. Choose

$$\varepsilon := \min\{|\,a-1\,|, \left|\,a+\frac{1}{2}\,\right|\}. \tag{2}$$

Since $a \neq 1$ and $a \neq -\frac{1}{2}$, $\varepsilon > 0$. Moreover, the neighborhoods $U_{\varepsilon/2}(1)$ and $U_{\varepsilon/2}(a)$ are disjoint: $U_{\varepsilon/2}(1) \cap U_{\varepsilon/2}(a) = \varnothing$. For, suppose $x \in U_{\varepsilon/2}(1) \cap U_{\varepsilon/2}(a)$; this gives

$$|\,x-1\,| < \varepsilon/2 \quad \text{and} \quad |\,x-a\,| < \varepsilon/2.$$

By the triangle inequality we have

$$|\,a-1\,| = |\,x-1-(x-a)\,| \le |\,x-1\,| + |\,x-a\,| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This contradicts our choice of ε in (2). Smilarly, $U_{\varepsilon/2}(-1/2) \cap U_{\varepsilon/2}(a) = \varnothing$. Since (x_{3n}) tends to 1 and both (x_{3n+1}) and (x_{3n+2}) tend to $-\frac{1}{2}$ we find $n_0 \in \mathbb{N}$ such that $n \ge n_0$ implies $x_{3n} \in U_{\varepsilon/2}(1)$ and $x_{3n+1}, x_{3n+2} \in U_{\varepsilon/2}(-\frac{1}{2})$. Hence, for $n \ge n_0$ all members of the sequence are either in $U_{\varepsilon/2}(1)$ or in $U_{\varepsilon/2}(-\frac{1}{2})$. Therefore, $U_{\varepsilon/2}(a)$ contains only finitely many elements of the sequence; a is not a limit point. ∎

(c) We have $x_{3n} = \cos^{3n} 2\pi n = 1$, $x_{3n+1} = \cos^{3n+1} 2\pi/3 = \left(-\frac{1}{2}\right)^{3n+1}$, and $x_{3n+2} = \cos^{3n+2} \cos 4\pi/3 = \left(-\frac{1}{2}\right)^{3n+2}$. Since $\lim_{n\to\infty} x_{3n} = 1$ and $\lim_{n\to\infty} x_{3n+1} = \lim_{n\to\infty} x_{3n+2} = \lim_{k\to\infty}(-1/2)^k = 0$, 1 and 0 are limit points of (x_n). Since (x_{3n}), (x_{3n+1}), and (x_{3n+2}) cover the whole sequence (x_n), there are no other limit points besides 0 and 1.

The more detailed prove of this uniqueness statement is quite similar to the proof in (b): Suppose there is another limit point a, $a \neq 0, 1$. Choose $\varepsilon = \max\{|\, a\,|, |\, a - 1\,|\}$ then $U_{\varepsilon/2}(a)$ contains only finitely many members of the sequence; hence a is not a limit point.

4. Prove or disprove:

   (a) A sequence is convergent if and only if it has a convergent subsequence.

   (b) A monotonic sequence is convergent if and only if it has a convergent subsequence.

Solution. (a) is false. A counter example is Example 1 (b) $x_n = (-1)^n + 1$. The subsequence $x_{2n} = (-1)^{2n} + 1 = 2$ is constant and therefore convergent, $\lim x_{2n} = 2$. However, (x_n) is divergent.

(b) is true. *Proof.* One direction is clear by Proposition 7. Suppose now, the sequence (x_n) is monotonically increasing and has a convergent subsequence (x_{n_k}), say $\lim_{k\to\infty} x_{n_k} = a$. We will show that (x_n) converges (with the same limit $\lim_{n\to\infty} x_n = a$).

By Proposition 6 it is sufficient to show that (x_n) is bounded above. We already know that a is an upper bound of (x_{n_k}), that is $x_{n_k} \leq a$ for all $k \in \mathbb{N}$. Since n_k is strictly increasing, for every $m \in \mathbb{N}$ there is some $p \in \mathbb{N}$ with $m < n_p$. Since (x_n) is increasing this gives

$$x_m \leq x_{n_p} \leq a;$$

hence a is an upper bound for (x_n). This shows that (x_n) converges; Proposition 7 gives $\lim x_n = a$.

The proof for monotonically decreasing sequences is analogous. ■

5. Prove or disprove: If (x_n) is unbounded then there exists a subsequence of (x_n) which has an improper limit.

Solution. The statement is true. *Proof.* Suppose (x_n) is not bounded above. In particular, for every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that $x_{k_n} > n$. We will show that $\lim_{n\to\infty} x_{k_n} = +\infty$.

For, let $E > 0$ and choose $n_0 > E$. Then $n \geq n_0$ implies

$$x_{k_n} > n \geq n_0 > E.$$

This proves the claim for a sequence which is not bounded above. The proof in case (x_n) is not bounded below is similar with some subsequence (x_{k_n}) such that $\lim\limits_{n\to\infty} x_{k_n} = -\infty$. ■