Calculus – 4. Series, Solutions

1. Prove the following statements.

   (a) For $a, b > 0$ and $r \in \mathbb{Q}$ we have

$$a < b \iff a^r < b^r \quad \text{if } r > 0,$$
$$a < b \iff a^r > b^r \quad \text{if } r < 0.$$

   (b) For $a > 0$ and $r, s \in \mathbb{Q}$ we have

$$r < s \iff a^r < a^s \quad \text{if } a > 1,$$
$$r < s \iff a^r > a^s \quad \text{if } a < 1.$$

   Hint: Use Lemma 16.

   Proof. Suppose that $r > 0$, $r = m/n$ with integers $m, n \in \mathbb{Z}$, $n > 0$. Using Lemma 16 (a) twice we get

$$a < b \iff a^m < b^m \iff (a^m)^{\frac{1}{n}} < (b^m)^{\frac{1}{n}},$$

   which proves the first claim. The second part $r < 0$ can be obtained by setting $-r$ in place of r in the first part and using Proposition 9 (e):

$$a < b \iff a^{-r} < b^{-r} \iff \frac{1}{a^r} < \frac{1}{b^r} \iff b^r < a^r.$$

   (b) Suppose first $a > 1$. Suppose further that $s > r$. Put $x = s - r$, then $x \in \mathbb{Q}$ and $x > 0$. By (a), $1 < a$ implies $1^x < a^x$. Hence $1 < a^{s-r} = a^s/a^r$, and therefore, since $a^r > 0$, $a^r < a^s$.
   If $s < r$, then $x = s - r < 0$ and, by (a), $1 < a$ implies $1 > a^x = a^s/a^r$. Hence, $a^r > a^s$.
   Suppose now $0 < a < 1$. Then $1/a > 1$ and (a) applies with $1/a$ in place of a:

$$r < s \iff \left(\frac{1}{a}\right)^r < \left(\frac{1}{a}\right)^s \iff \frac{1}{a^r} < \frac{1}{a^s} \iff a^s < a^r.$$

   In the last step we used Proposition 9 (e). ∎

2. Using the arithmetic-geometric mean inequality prove that the cube has the greatest volume among all cuboids with a fixed area of the surface.

   Solution. The volume V of a cuboid with edges of lengths a, b, and c is $V = abc$; its surface area is $A = 2(ab + bc + ca)$. By the arithmetic-geometric mean inequality applied to $n = 3$, $x_1 = ab$, $x_2 = bc$, and $x_3 = ca$ we find

$$\frac{A}{6} = \frac{ab + bc + ca}{3} \geq \sqrt[3]{ab \cdot bc \cdot ac} = \sqrt[3]{a^2b^2c^2} = V^{\frac{2}{3}},$$

   so that $V \leq (A/6)^{3/2}$. If A is fixed, $(A/6)^{3/2}$ is an upper bound for the volume. The maximum is attained if all estimates are equalities, that is $ab = bc = ca$; hence $a = b = c$.

3. Let x, y be real and $n \geq 0$ an integer. Prove that

$$2 \sin y \sum_{k=0}^{n} \sin(x + 2ky) = \cos(x - y) - \cos(x + (2n+1)y). \tag{1}$$

Proof. First we will show that

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}, \qquad a, b \in \mathbb{R}. \tag{2}$$

Set $x = (a+b)/2$ and $y = (a-b)/2$, then $a = x + y$ and $b = x - y$. By the addition law for cosine we have

$$\begin{aligned} \cos a - \cos b = \cos(x+y) - \cos(x-y) &= (\cos x \cos y - \sin x \sin y) \\ &\quad - (\cos x \cos y + \sin x \sin y) \\ = -2 \sin x \sin y &= -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}. \end{aligned}$$

We use induction over n to prove the assertion. In case $n = 0$ the left hand side reads as $2 \sin y \sin x$ and the right hand side is $\cos(x - y) - \cos(x + y)$. Both terms are equal since by (2)

$$\cos(x - y) - \cos(x + y) = -2 \sin x \sin(-y) = 2 \sin x \sin y.$$

Suppose now that the claim is true for some fixed n. We want to show that (1) is true for $n + 1$, i.e.

$$2 \sin y \sum_{k=0}^{n+1} \sin(x + 2ky) = \cos(x - y) - \cos(x + (2n+3)y) \quad \text{(induction assertion)}.$$

Using the induction hypothesis, (2) with $a = x + (2n+1)y$ and $b = x + (2n+3)y$, and $\sin(-y) = -\sin y$ we find

$$\begin{aligned} \cos(x-y) - \cos(x + (2n+3)y) &= \cos(x-y) - \cos(x + (2n+1)y) \\ &\quad + \cos(x + (2n+1)y) - \cos(x + (2n+3)y) \\ &= \cos(x-y) - \cos(x + (2n+1)y) - 2 \sin(x + (2n+2)y) \sin(-y) \\ &\underset{\text{ind.hyp.}}{=} 2 \sin y \sum_{k=0}^{n} \sin(x + 2ky) + 2 \sin y \sin(x + 2(n+1)y) \\ &= 2 \sin y \sum_{k=0}^{n+1} \sin(x + 2ky). \end{aligned}$$

This completes the induction proof. ∎

4. (a) Using only the *definition* of the limit of a sequence show that

$$\lim_{n\to\infty} \frac{4n^3+2n}{n^3+1} = 4.$$

(b) Prove that the sequence $a_n = n^{(-1)^n}$ is unbounded but $\lim\limits_{n\to\infty} a_n \neq +\infty$.

Solution. (a) Set $a_n = \frac{4n^3+2n}{n^3+1}$. For $n \geq 2$ we have $|\,2n-4\,| = 2n-4$ and $\frac{2}{n^2} \leq \frac{1}{n}$ so that

$$|\,a_n - 4\,| = \left| \frac{4n^3+2n}{n^3+1} - \frac{4n^3+4}{n^3+1} \right| =$$
$$= \frac{|\,2n-4\,|}{n^3+1} = \frac{2n-4}{n^3+1} < \frac{2n}{n^3} = \frac{2}{n^2} < \frac{1}{n}.$$

Given $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $n_0 \geq \frac{1}{\varepsilon}$ and $n_0 \geq 2$. Then $n \geq n_0$ implies

$$|\,a_n - 4\,| < \frac{1}{n} < \frac{1}{n_0} < \varepsilon;$$

hence $a_n \to 4$.

(b) Suppose (a_n) is bounded, say $|\,a_n\,| \leq C$ for some fixed $C > 0$ and all positive integers n. Choose $m \in \mathbb{N}$, such that $m > C$. Then

$$a_{2m} = (2m)^{(-1)^{2m}} = (2m)^1 = 2m > 2C > C$$

which contradicts $|\,a_n\,| \leq C$. Hence (a_n) is unbounded.
Let $E > 0$ be given. Choose $m \in \mathbb{N}$ such that $m > 1/E$, then $n \geq m$ implies

$$a_{2n-1} = (2n-1)^{(-1)^{2n-1}} = (2n-1)^{-1} = \frac{1}{2n-1} \leq \frac{1}{n} \leq \frac{1}{m} < E.$$

This contradicts $\lim_{n\to\infty} a_n = +\infty$.

5. Prove that
(a) $\lim\limits_{n\to\infty} \dfrac{n}{2^n} = 0$; (b) $\lim\limits_{n\to\infty} \dfrac{2^n}{n!} = 0$.

Solution. (a) In Chapter I, Example 1 (b) we have shown that $2^n > 3n^2$ for $n \geq 8$. This is equivalent to $\frac{n}{2^n} < \frac{1}{3n}$. Given $\varepsilon > 0$ choose $n_0 \geq \max\{8, 1/\varepsilon\}$. Then $n \geq n_0$ implies

$$\left| \frac{n}{2^n} - 0 \right| = \frac{n}{2^n} < \frac{1}{3n} < \frac{1}{n} < \frac{1}{n_0} < \varepsilon.$$

This proves $\lim_{n\to\infty} \frac{n}{2^n} = 0$.

(b) We have the following (very rough) estimate

$$0 < \frac{2^n}{n!} = \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdot \cdots \cdot \frac{2}{n} \leq 2 \cdot 1 \cdot 1 \cdots \cdot 1 \cdot \frac{2}{n} = \frac{4}{n}.$$

Given $\varepsilon > 0$ choose $n_0 \in \mathbb{N}$ such that $n_0 > \frac{4}{\varepsilon}$. Then $n \geq n_0$ implies

$$\left| \frac{2^n}{n!} - 0 \right| = \frac{2^n}{n!} \leq \frac{4}{n} \leq \frac{4}{n_0} < \varepsilon;$$

hence $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.