Calculus – 3. Series, Solutions

1. (a) Compute real part, imaginary part, and absolute value of the complex numbers

$$(1-7\mathrm{i})(4+3\mathrm{i}), \quad \frac{2+3\mathrm{i}}{1-4\mathrm{i}}, \quad (1-7\mathrm{i})^2, \quad 5\left(\cos\frac{5\pi}{3}+\mathrm{i}\sin\frac{5\pi}{3}\right).$$

(b) Determine the polar form of the following complex numbers

$$-2\mathrm{i}, \quad 1-\mathrm{i}, \quad -\sqrt{3}-\mathrm{i}.$$

Solution. (a) $z_1 = (1-7\mathrm{i})(4+3\mathrm{i}) = 4+3\mathrm{i}-28\mathrm{i}+21 = 25-25\mathrm{i}$. Hence, $\mathrm{Re}\, z_1 = 25$ and $\mathrm{Im}\, z_1 = -25$, and $|\, z_1\,| = \sqrt{25^2+25^2} = 25\sqrt{2}$.

$$z_2 = \frac{2+3\mathrm{i}}{1-4\mathrm{i}} = \frac{(2+3\mathrm{i})(1+4\mathrm{i})}{(1-4\mathrm{i})(1+4\mathrm{i})} = \frac{2-12+8\mathrm{i}+3\mathrm{i}}{1+16} = -\frac{10}{17}+\frac{11\mathrm{i}}{17}.$$

Hence $\mathrm{Re}\, z_2 = -10/17$, $\mathrm{Im}\, z_2 = 11/17$, and $|\, z_2\,| = \sqrt{100+121}/17 = \sqrt{221}/17$.
Further $z_3 = (1-7\mathrm{i})^2 = 1-14\mathrm{i}-49 = -48-14\mathrm{i}$. We find $\mathrm{Re}\, z_3 = -48$, $\mathrm{Im}\, z_3 = -14$, and $|\, z_3\,| = \sqrt{48^2+14^2} = \sqrt{2304+196} = \sqrt{2500} = 50$. The last statement also follows from $|\, z_3\,| = |\,(1-7\mathrm{i})^2\,| = |\,1-7\mathrm{i}\,|^2 = 1+49 = 50$.
Since $5\pi/3$ corresponds to an angle in the 4th quadrant, $\sin 5\pi/3 = -\sin\pi/3 = -\frac{1}{2}\sqrt{3}$ and $\cos 5\pi/3 = \cos(-\pi/3) = \cos\pi/3 = \frac{1}{2}$. Hence,

$$z_4 = 5\left(\cos\frac{5\pi}{3}+\mathrm{i}\sin\frac{5\pi}{3}\right) = 5\left(\frac{1}{2}-\mathrm{i}\frac{1}{2}\sqrt{3}\right) = \frac{5}{2}-\mathrm{i}\frac{5}{2}\sqrt{3}.$$

Hence, $\mathrm{Re}\, z_4 = \frac{5}{2}$, $\mathrm{Im}\, z_4 = -\frac{5}{2}\sqrt{3}$, and $|\, z_4\,| = 5$. The last statement is clear since the first factor in the polar form gives the absolute value of the complex number.

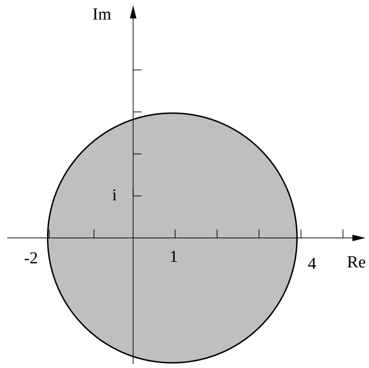
(b) We have $\mathrm{Re}\,(-2\mathrm{i}) = 0$, $\mathrm{Im}\,(-2\mathrm{i}) = -2$, and therefore $|\,-2\mathrm{i}\,| = \sqrt{0^2+(-2)^2} = 2$. For the argument φ of $-2\mathrm{i}$ we find $\cos\varphi = 0$ and $\sin\varphi = -1$; hence $\varphi = 3\pi/2$, and $-2\mathrm{i} = 2(\cos 3\pi/2 + \mathrm{i}\sin 3\pi/2)$.

We have $\mathrm{Re}\,(1-\mathrm{i}) = 1$, $\mathrm{Im}\,(1-\mathrm{i}) = -1$, and therefore $|\,1-\mathrm{i}\,| = \sqrt{1+1} = \sqrt{2}$. For the argument φ of $1-\mathrm{i}$ we find $\cos\varphi = 1/\sqrt{2}$ and $\sin\varphi = -1/\sqrt{2}$; φ is in the 4th quadrant. Hence $\varphi = 7\pi/4$, and $1-\mathrm{i} = \sqrt{2}(\cos 7\pi/4 + \mathrm{i}\sin 7\pi/4)$.
Since sine and cosine are 2π periodic functions, it is also possible to use $7\pi/4 - 2\pi = -\pi/4$ in the polar form, $1-\mathrm{i} = \sqrt{2}(\cos(-\pi/4) + \mathrm{i}\sin(-\pi/4))$.

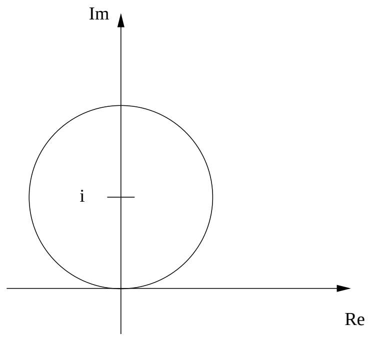
For $z = -\sqrt{3}-\mathrm{i}$ we find $\mathrm{Re}\, z = -\sqrt{3}$, $\mathrm{Im}\, z = -1$, and $|\, z\,| = \sqrt{3+1} = 2$. Hence $\varphi = \arg z$ satisfies $\cos\varphi = -\sqrt{3}/2$ and $\sin\varphi = -1/2$. Since both sine and cosine are negative in the 3rd quadrant, we find using our tabular with special values, $\varphi = 210° = 7\pi/6$. Hence, $z = 2(\cos 7\pi/6 + \mathrm{i}\sin 7\pi/6)$.

2. Which subsets of the complex plane are described by the following inequalities?

$$\text{a) } |\, z-1\,| \le 3, \quad \text{b) } (z-\mathrm{i})(\overline{z}+\mathrm{i}) \ge 1, \quad \text{c) } z+\overline{z} \ge -1.$$



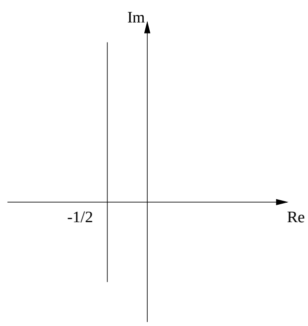
Solution. a) The closed disc midpoint 1 and radius 3.



b) The inequality is equivalent to

$$(z - \mathrm{i})(\overline{z} + \mathrm{i}) = (z - \mathrm{i})\overline{z - \mathrm{i}} = |\, z - \mathrm{i}\,|^2 \geq 1.$$

Taking the square root, this is equivalent to $|\, z - \mathrm{i}\,| \geq 1$. This describes the exterior of a disc with midpoint i and radius 1.



c) $2\,\mathrm{Re}\, z = z + \overline{z} \geq -1$, therefore $\mathrm{Re}\, z \geq -1/2$. This is the right half plane described by the line $x = -1/2$ including the line itself.

3. Use de Moivre formula to express

$$\cos 3\alpha \quad \text{and} \quad \sin 4\alpha$$

in terms of $\cos\alpha$ and $\sin\alpha$.

Solution. The binomial formula gives

$$(\cos\alpha + \mathrm{i}\sin\alpha)^3 = \cos^3\alpha + 3\cos^2\alpha \mathrm{i}\sin\alpha + 3\cos\alpha(\mathrm{i}\sin\alpha)^2 + (\mathrm{i}\sin\alpha)^3.$$

Now De Moivre's formula shows that the real part of the above formula is $\cos(3\alpha)$ whereas the imaginary part is $\sin(3\alpha)$. Hence,

$$\begin{aligned}\cos(3\alpha) &= \cos^3\alpha - 3\cos\alpha\sin^2\alpha = \cos^3\alpha - 3\cos\alpha(1-\cos^2\alpha)\\ &= 4\cos^3\alpha - 3\cos\alpha.\end{aligned}$$

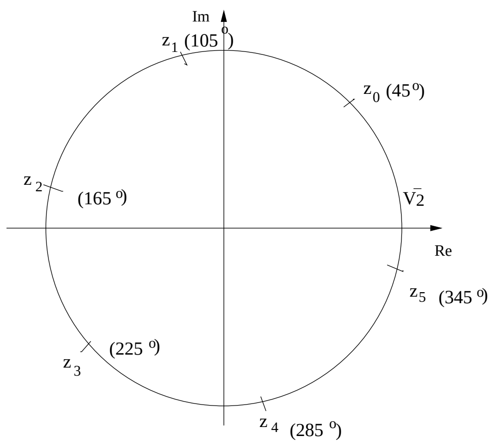
Here we used $\cos^2\alpha + \sin^2\alpha = 1$.
Similarly, the binomial formula gives

$$\begin{aligned}(\cos\alpha + \mathrm{i}\sin\alpha)^4 &= \cos^4\alpha + 4\cos^3\alpha\,\mathrm{i}\sin\alpha + 6\cos^2\alpha(\mathrm{i}\sin\alpha)^2 + 4\cos\alpha(\mathrm{i}\sin\alpha)^3 + (\mathrm{i}\sin\alpha)^4\\ &= \cos^4\alpha + 4\mathrm{i}\cos^3\alpha\sin\alpha - 6\cos^2\alpha\sin^2\alpha - 4\mathrm{i}\cos\alpha\sin^3\alpha + \sin^4\alpha.\end{aligned}$$

Now De Moivre's formula shows that the imaginary part of the above formula is $\sin(4\alpha)$. Hence,

$$\begin{aligned}\sin(4\alpha) &= 4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha = (4\cos^3\alpha - 4\cos\alpha(1-\cos^2\alpha))\sin\alpha\\ &= \sin\alpha(8\cos^3\alpha - 4\cos\alpha).\end{aligned}$$

4. Solve for $z \in \mathbb{C}$. a) $z^6 + 8\mathrm{i} = 0$, b) $z^2 + \mathrm{i} = 0$.



Solution. a) First we have to find the polar form of $w = -8\mathrm{i}$. Since $\operatorname{Re} w = 0$ and $\operatorname{Im} w = -8$ we have $|w| = 8$ and $\cos\varphi = 0$, $\sin\varphi = -1$. Therefore, $\varphi = \arg w = 3\pi/2$ and $w = 8(\cos 3\pi/2 + \mathrm{i}\sin 3\pi/2)$. By the formula for the nth roots with $n = 6$ we find

$$\begin{aligned}z_k &= \sqrt[6]{8}\left(\cos\frac{3\pi/2 + 2k\pi}{6} + \mathrm{i}\sin\frac{3\pi/2 + 2k\pi}{6}\right), \qquad k = 0, \dots, 5\\ &= \left(2^3\right)^{\frac{1}{6}}\left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{4} + \frac{k\pi}{3}\right)\right).\end{aligned}$$

Explicitly,

$$\begin{aligned}z_0 &= \sqrt{2}(\cos\pi/4 + \mathrm{i}\sin\pi/4), & z_1 &= \sqrt{2}(\cos 7\pi/12 + \mathrm{i}\sin 7\pi/12),\\ z_2 &= \sqrt{2}(\cos 11\pi/12 + \mathrm{i}\sin 11\pi/12), & z_3 &= \sqrt{2}(\cos 15\pi/12 + \mathrm{i}\sin 15\pi/12),\\ z_4 &= \sqrt{2}(\cos 19\pi/12 + \mathrm{i}\sin 19\pi/12), & z_5 &= \sqrt{2}(\cos 23\pi/12 + \mathrm{i}\sin 23\pi/12).\end{aligned}$$

Using addition formulas for sine and cosine we find

$$\cos 7\pi/12 = \cos 105^\circ = \cos(45^\circ + 60^\circ) = \cos 45^\circ\cos 60^\circ - \sin 45^\circ\sin 60^\circ = \frac{1}{4}(\sqrt{2} - \sqrt{6}).$$

Similarly one gets, $\sin 7\pi/12 = (\sqrt{2} + \sqrt{6})/4$, $\cos 11\pi/12 = (-\sqrt{2} - \sqrt{6})/4$, and $\sin 11\pi/12 = (\sqrt{6} - \sqrt{2})/4$. The rectangular coordinates of the six roots are

$$\begin{aligned}z_0 &= 1 + \mathrm{i}, & z_1 &= \frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}\mathrm{i} && (1)\\ z_2 &= \frac{-1-\sqrt{3}}{2} + \frac{\sqrt{3}-1}{2}\mathrm{i}, & z_3 &= -z_0, && (2)\\ z_4 &= -z_1, & z_5 &= -z_2. && (3)\end{aligned}$$

b) Since $-\mathrm{i} = 1(\cos 3\pi/2 + \mathrm{i}\sin 3\pi/2)$, the formula for the nth roots with $n = 2$ gives

$$z_0 = 1(\cos 3\pi/4 + \mathrm{i}\sin 3\pi/4), \quad z_1 = 1(\cos 7\pi/4 + \mathrm{i}\sin 7\pi/4).$$

Hence, $z_0 = -\sqrt{2}/2 + \mathrm{i}\sqrt{2}/2$, $z_1 = -z_0$.

5. Find the mistake in the following deduction. Let $a, b \in \mathbb{R}$ with $a > b$. Then

$$\begin{aligned}
\sqrt{a-b} &= \sqrt{(-1)(b-a)} = \sqrt{-1}\sqrt{b-a} \\
\sqrt{a-b} &= \sqrt{-1}\sqrt{(-1)(a-b)} = \sqrt{-1}\sqrt{-1}\sqrt{a-b} \qquad \Big| \cdot \frac{1}{\sqrt{a-b}} \\
1 &= \sqrt{-1}\sqrt{-1} = \mathrm{i}^2 \\
1 &= \mathrm{i}^2.
\end{aligned}$$

Solution. Only roots of *non-negative* real numbers are uniquely defined in the real field. The square root of a negative real number or of an arbitrary complex number w has always two values z and $-z$. In particular, $\sqrt{-1}$ can take the two values i and $-\mathrm{i}$.

The first mistake appears here

$$\sqrt{(-1)(b-a)} = \sqrt{-1}\sqrt{b-a}$$

since both $\sqrt{-1}$ and $\sqrt{b-a}$ are not well-defined.