Calculus – 14. Series, Solutions

1. Compute directly (without using the Fundamental Theorem of Calculus)

$$\int_0^1 x \, \mathrm{d}x^2.$$

Hint. Consider equidistant partitions of $[0,1]$ and use $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ and $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

Solution. By Theorem 5 and the remark to Example 1, since $f(x) = x$ is continuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that for all partitions with $\Delta x_i < \delta$,

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f \, \mathrm{d}\alpha \right| < \varepsilon, \tag{1}$$

where $t_i \in [x_{i-1}, x_i]$.
Consider the equidistant partition $P_n = \{x_0, x_1, \dots, x_n\}$, $x_i = i/n$, $i = 1, \dots, n$ of $[0,1]$. Since f is increasing,

$$U(P_n, f, \alpha) = \sum_{i=1}^{n} x_i \, \Delta\alpha_i,$$

where

$$\Delta\,\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \frac{i^2 - (i-1)^2}{n^2} = \frac{2i}{n^2}.$$

Using the hint we have

$$U(P, f, \alpha) = \sum_{i=1}^{n} \frac{i}{n}\frac{2i}{n^2} = \frac{2}{n^3}\sum_{i=1}^{n} i^2 = \frac{2}{n^3}\frac{1}{6}n(n+1)(2n+1) = \frac{n(n+1)(2n+1)}{3n^2}.$$

Taking the limit $n \to \infty$ we assure $\Delta x_i = 1/n < \delta$; now (1) shows that $U(P_n, f, \alpha)$ converges to the integral

$$\int_0^1 x \, \mathrm{d}x^2 = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{3n^2} = \frac{2}{3}.$$

2. Suppose α increases on $[a,b]$, $a \le c \le b$, α is continuous at c, $f(c) = 1$, and $f(x) = 0$ if $x \ne c$.
Prove that $f \in \mathcal{R}(\alpha)$ and that $\int_a^b f \, \mathrm{d}\alpha = 0$.

Proof. Consider the partition $P = \{x_0 = a, x_1, x_2, x_3 = b\}$ of $[a,b]$ with $x_1 < c < x_2$. Since $m_i = 0$, $i = 1,2,3$, and $M_1 = M_3 = 0$, $M_2 = 1$, we have

$$U(P, f, \alpha) = \Delta\alpha_2, \quad \text{and} \quad L(P, f, \alpha) = 0.$$

Since α is continuous at c,

$$\lim_{\substack{x_1 \to c-0 \\ x_2 \to c+0}} (\alpha(x_2) - \alpha(x_1)) = \alpha(c) - \alpha(c) = 0.$$

Therefore both the upper and lower integrals of f with respect to α are 0. This shows $f \in \mathcal{R}(\alpha)$ and $\int f \,\mathrm{d}\alpha = 0$. ■

3. Suppose α strictly increases on $[a, b]$, $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f \,\mathrm{d}\alpha = 0$.
Prove that $f(x) = 0$ for all $x \in [a, b]$. Compare this with homework 14.2.
Hint. Make an indirect proof; use homework 10.4 and Proposition 9 (b) and (c).

Proof. Suppose to the contrary, $f(c) = A > 0$ for some $c \in [a, b]$. By homework 10.4, there exists $\delta > 0$ such that

$$f(x) > \frac{A}{2} \quad \text{if} \quad |\, x - c \,| \leq \delta, \quad x \in [a, b].$$

Put $r = \max\{a, c - \delta\}$ and $s = \min\{b, c + \delta\}$. By Proposition 9 we have

$$\int_a^b f \,\mathrm{d}\alpha = \int_a^r f \,\mathrm{d}\alpha + \int_r^s f \,\mathrm{d}\alpha + \int_s^b f \,\mathrm{d}\alpha \geq 0 + \frac{A}{2}\,(\alpha(s) - \alpha(r)) + 0.$$

Since α is *strictly* increasing and $s > r$ one has $\alpha(s) > \alpha(r)$ and the right hand side of the above inequalities is positive. However this contradicts the assumption $\int f \,\mathrm{d}\alpha = 0$; hence $f(x) = 0$ on $[a, b]$.
Homework 14.2 shows that the continuity of f is a necessary assumption to conclude $f(x) = 0$. ■

4. Define three functions β_j, $j = 1, 2, 3$, as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$, $\beta_1(0) = 0$, $\beta_2(0) = 1$, and $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.
(a) Prove that $f \in \mathcal{R}(\beta_1)$ if and only if $\lim\limits_{x \to 0+0} f(x) = f(0)$ and that

$$\int_{-1}^{1} f \mathrm{d}\beta_1 = f(0).$$

(b) State and prove a similar result for β_2.
(c) Prove that $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.
(d) If f is continuous at 0 prove that

$$\int_{-1}^{1} f \mathrm{d}\beta_1 = \int_{-1}^{1} f \mathrm{d}\beta_2 = \int_{-1}^{1} f \mathrm{d}\beta_3 = f(0).$$

Proof. Consider the partition $P = \{x_0 = -1,\ x_1,\ x_2 = 0,\ x_3,\ x_4 = 1\}$ with $x_1 < 0 < x_3$. Then $\Delta(\beta_j)_1 = \Delta(\beta_j)_4 = 0$, $j = 1, 2, 3$. Since $\beta_j(x_1) = 0$ and

$\beta_j(x_3) = 1$ we have

$$\begin{aligned} L(P, f, \beta_j) &= m_2\Delta(\beta_j)_2 + m_3\Delta(\beta_j)_2 = m_2\beta_j(0) + m_3(1 - \beta_j(0)) \\ U(P, f, \beta_j) &= M_2\beta_j(0) + M_3(1 - \beta_j(0)) \end{aligned} \tag{2}$$

(a) In case $j = 1$, $\beta_1(0) = 0$ and (2) yields

$$L(P, f, \beta_1) = m_3, \quad U(P, f, \beta_1) = M_3. \tag{3}$$

If $f \in \mathcal{R}(\beta_1)$, the Riemann criterion and (3) show that given $\varepsilon > 0$ there exists $x_3 > 0$ such that

$$\varepsilon > M_3 - m_3 \geq |\, f(t) - f(y)\,|, \quad \text{for all } t,\ y \text{ with} \quad 0 \leq t, y \leq x_3.$$

In particular, inserting $y = 0$ and $t > 0$ we have

$$|\, f(t) - f(0)\,| < \varepsilon \quad \text{if} \quad 0 < t < x_3.$$

This means $\lim\limits_{t\to 0+0} f(t) = f(0)$.

Conversely, suppose $\lim\limits_{t\to 0+0} f(t)$ exists and equals $f(0)$. Then for $\varepsilon > 0$ there exist $x_3 > 0$ such that

$$|\, f(t) - f(0)\,| \leq \varepsilon \quad \text{if} \quad 0 \leq t \leq x_3.$$

It follows

$$\begin{aligned} &f(0) - \varepsilon < f(t) < f(0) + \varepsilon, \quad t \in [0, x_3] \\ &f(0) - \varepsilon \leq m_3 < f(0) + \varepsilon \\ &f(0) - \varepsilon < M_3 \leq f(0) + \varepsilon. \end{aligned}$$

Hence $M_3 - m_3 \leq 2\varepsilon$. The Riemann criterion is satisfied, $f \in \mathcal{R}(\beta_1)$. Since the integral is always inbetween the lower and the upper sums we have

$$f(0) - \varepsilon \leq m_3 \leq \int_{-1}^{1} f\,\mathrm{d}\beta_1 \leq M_3 \leq f(0) + \varepsilon$$

Since ε was arbitrary, $\int f\,\mathrm{d}\beta_1 = f(0)$.

(b) In case $j = 2$, $\beta_2(0) = 1$ and (2) yields

$$L(P, f, \beta_2) = m_2, \qquad U(P, f, \beta_2) = M_2 \tag{4}$$

The same arguments as in (a) show that $f \in \mathcal{R}(\beta_2)$ if and only if $\lim\limits_{t\to 0-0} f(t) = f(0)$. In this case

$$\int_{-1}^{1} f\,\mathrm{d}\beta_2 = f(0).$$

(c) If f is continuous at 0, $f \in \mathcal{R}(\beta_1)$ and $f \in \mathcal{R}(\beta_2)$ by (a) and (b). Since $\beta_3 = \frac{1}{2}\beta_1 + \frac{1}{2}\beta_2$, it follows from Proposition 9 (e) and that $f \in \mathcal{R}(\beta_3)$ and

$$\int_{-1}^{1} f\,\mathrm{d}\beta_3 = \frac{1}{2}\int_{-1}^{1} f\,\mathrm{d}\beta_1 + \frac{1}{2}\int_{-1}^{1} f\,\mathrm{d}\beta_2 = f(0).$$

Conversely, suppose $f \in \mathcal{R}(\beta_3)$ on $[-1, 1]$, then $f \in \mathcal{R}(\beta_3)$ on $[-1, 0]$ and on $[0, 1]$ by Proposition 9 (c). Then $f \in \mathcal{R}(\frac{1}{2}\beta_2)$ on $[-1, 0]$ since $\frac{1}{2}\beta_2 = \beta_3$ on $[-1, 0]$ and $f \in \mathcal{R}(\frac{1}{2}\beta_1)$ on $[0, 1]$ since $\beta_3 = \frac{1}{2}\beta_1 + \frac{1}{2}$ on $[0, 1]$.
Hence $f \in \mathcal{R}(\beta_2)$ on $[-1, 0]$ and $f \in \mathcal{R}(\beta_1)$ on $[0, 1]$ by Proposition 9 (e). Since $\beta_2 = 1$ is constant on $[0, 1]$ *every* bounded function is in $\mathcal{R}(\beta_2)$ on $[0, 1]$. Since the converse statement to Proposition 9 (c) is also true (see below, the proof of homework 5) $f \in \mathcal{R}(\beta_2)$ on $[-1, 1]$. Similarly, $f \in \mathcal{R}(\beta_1)$ on $[-1, 1]$. From (a) and (b) it follows

$$\lim_{x \to 0+0} f(x) = f(0) = \lim_{x \to 0-0} f(x).$$

That is, f is continuous at 0.
(d) If f is continuous at 0, the left-hand and the right-hand limits exist at 0 and they coincide with $f(0)$. The statement follows from (a), (b), and (c). ■

5. If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and if $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $[a, c]$ and on $[c, b]$ and

$$\int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha.$$

Hint. Let $\varepsilon > 0$, consider a partition P of $[a, b]$ with $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$; pass to a refinement P^* of P which contains the point c and write this in terms of upper and lower sums on the intervals $[a, c]$ and $[c, b]$.

Proof. Let $\varepsilon > 0$ be given; consider a partition P_0 of $[a, b]$ with

$$U(P_0, f, \alpha) - L(P_0, f, \alpha) < \varepsilon$$

and pass to a refinement P of P_0 which contains the point c. By Lemma 4 (a),

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Since $c \in P$, the partition $P = \{x_0, \dots, x_n\}$ can be viewed as the union of the partitions $P_1 = \{x_0, \dots, c\}$ of $[a, c]$ and $P_2 = \{c, \dots, x_n\}$ of $[c, b]$. By the above inequality

$$U(P_1, f, \alpha) + U(P_2, f, \alpha) - L(P_1, f, \alpha) - L(P_2, f, \alpha) < \varepsilon.$$

This shows $U(P_i, f, \alpha) - L(P_i, f, \alpha) < \varepsilon$ for $i = 1, 2$; hence $f \in \mathcal{R}(\alpha)$ on $[a, c]$ and on $[c, b]$.
With the same P_1 and P_2 we have

$$U(P_1, f, \alpha) - \int_a^c f \, d\alpha < \varepsilon, \quad \text{and} \quad U(P_2, f, \alpha) - \int_c^b f \, d\alpha < \varepsilon.$$

Hence

$$\int_a^b f \, d\alpha \le U(P, f, \alpha) = U(P_1, f, \alpha) + U(P_2, f, \alpha) < \int_a^c f \, d\alpha + \int_c^b f \, d\alpha + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary,

$$\int_a^b f \, \mathrm{d}\alpha \leq \int_a^c f \, \mathrm{d}\alpha + \int_c^b f \, \mathrm{d}\alpha.$$

If we replace f by $-f$, the above inequality is reversed and we obtain equality.

Note that the reverse statement is also true. If $f \in \mathcal{R}(\alpha)$ on $[a, c]$ and $f \in \mathcal{R}(\alpha)$ on $[c, b]$ then $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Start with partitions P_1 and P_2 of $[a, c]$ and $[c, b]$, respectivly, satisfying the Riemann criterion with $\varepsilon > 0$. Then

$$U(P, f, \alpha) - L(P, f, \alpha) < 2\varepsilon$$

for the common refinement P of P_1 and P_2 on the interval $[a, b]$. The rest is completely the same as above ($\int_a^b f \, \mathrm{d}\alpha = \int_a^c f \, \mathrm{d}\alpha + \int_c^b f \, \mathrm{d}\alpha$.) ∎