Calculus – 12. Series, Solutions

1. Compute the derivatives of $f\colon (0,1) \to \mathbb{R}$ where

$$\begin{aligned} &\text{(a)} \quad f(x) = x^{x^x} \\ &\text{(b)} \quad f(x) = (x^x)^x \\ &\text{(c)} \quad f(x) = x^{a^x} \end{aligned}$$

and $a > 0$ is a constant. Note that $a^{b^c} = a^{(b^c)}$ by definition.

Solution. We first compute the derivative of $g(x) = x^x = \mathrm{e}^{x \log x}$. The chain rule gives

$$g'(x) = \mathrm{e}^{x \log x}\,(x' \log x + x(\log x)') = x^x(\log x + 1).$$

(a) We have $f(x) = x^{g(x)}$. Since $x > 0$ $\log f(x) = g(x)\log x$ is well-defined. Differentiating this equation using the chain rule we obtain

$$\begin{aligned} (\log f(x))' &= \frac{1}{f(x)}\, f'(x) = g'(x)\log x + g(x)\frac{1}{x} \\ \frac{f'(x)}{f(x)} &= x^x\,(\log x + 1)\log x + x^x\frac{1}{x} \\ f'(x) &= x^{x^x} x^x \left((\log x)^2 + \log x + \frac{1}{x} \right). \end{aligned}$$

(b) We have $f(x) = (x^x)^x = x^{x\cdot x} = x^{x^2} = \mathrm{e}^{x^2 \log x}$. The chain rule gives

$$f'(x) = x^{x^2}\left((x^2)' \log x + x^2(\log x)'\right) = x^{x^2}\,(2x \log x + x) = x^{x^2+1}\,(2\log x + 1)\,.$$

(c) We have $f(x) = x^{a^x} = \mathrm{e}^{a^x \log x}$. The chain rule gives

$$f'(x) = f(x)\left(a^x \log a \log x + \frac{a^x}{x}\right) = x^{a^x} a^x \left(\log x \log a + \frac{1}{x}\right).$$

2. Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^2 \cos\frac{1}{x}, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Prove that $g(x)$ is differentiable for all $x \in \mathbb{R}$ and compute $g'(x)$. Prove that g' is not continuous at $x = 0$. What kind of discontinuity has g' at $x = 0$?

Proof. If $x \neq 0$, g is differentiable at x as the composition of differentiable functions. The derivative is

$$g'(x) = 2x \cos\frac{1}{x} + x^2\left(-\sin\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = 2x\cos\frac{1}{x} + \sin\frac{1}{x}.$$

In case $x = 0$ we compute the limit explicitly. Since $g(0) = 0$,

$$g'(0) = \lim_{h\to 0} \frac{g(h)}{h} = \lim_{h\to 0} h \cos\frac{1}{h} = 0.$$

Hence g is differentiable on $\mathbb{R}$ with

$$g'(x) = \begin{cases} 2x\cos\frac{1}{x} + \sin\frac{1}{x} & , \quad \text{if} \quad x \neq 0, \\ 0 & , \quad \text{if} \quad x = 0. \end{cases}$$

Since $\lim_{x\to 0} 2x\cos\frac{1}{x} = 0$ but both one-sided limits $\lim_{x\to 0\pm 0} \sin\frac{1}{x}$ do not exist, g' has a discontinuity of the second kind at $x = 0$.
Remark. Derivatives cannot obey discontinuities of the first kind. ∎

3. Compute the derivatives of
(a) $\cosh x$, $\sinh x$, and $\tanh x$,
(b) $\operatorname{arcosh} x$, $\operatorname{arsinh} x$, and $\operatorname{artanh} x$,
(c) $\arccos x$.

Solution. (a)

$$(\cosh x)' = \frac{1}{2}\left((\mathrm{e}^x)' + \left(\mathrm{e}^{-x}\right)'\right) = \frac{1}{2}\left(\mathrm{e}^x - \mathrm{e}^{-x}\right) = \sinh x.$$
$$(\sinh x)' = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right) = \cosh x;$$
$$(\tanh x)' = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{(\sinh x)'\cosh x - \sinh x(\cosh x)'}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x},$$

where the last equation follows from Homework 11.2.
(b) We set $y = \cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$ and $x = \operatorname{arcosh} y$, $x \geq 0$, $y \geq 1$.
First solution using Proposition 5. By the above calculations in (a)

$$(\operatorname{arcosh} y)' = \frac{1}{(\cosh x)'} = \frac{1}{\sinh x}.$$

Put $z = \sinh x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$, then $y + z = \mathrm{e}^x$ and $y - z = \mathrm{e}^{-x}$. Taking the product of these equations, we have $y^2 - z^2 = 1$ and, since $z \geq 0$, $z = \sqrt{y^2 - 1}$. Finally,

$$(\operatorname{arcosh} y)' = \frac{1}{\sqrt{y^2 - 1}}.$$

Second solution. By Homework 11.2 (e), $\operatorname{arcosh} y = \log(y + \sqrt{y^2 - 1})$. Using the chain rule we have

$$(\operatorname{arcosh} y)' = \frac{1}{y + \sqrt{y^2 - 1}}\left(1 + \frac{2y}{2(\sqrt{y^2 - 1})}\right) = \frac{1}{y + \sqrt{y^2 - 1}}\,\frac{\sqrt{y^2 - 1} + y}{\sqrt{y^2 - 1}} = \frac{1}{\sqrt{y^2 - 1}}.$$

Setting $y = \sinh x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$, $x = \operatorname{arsinh} y$ and using Proposition 5

$$(\operatorname{arsinh} y)' = \frac{1}{(\sinh x)'} = \frac{1}{\cosh x}.$$

Put $z = \cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$, then $y + z = \mathrm{e}^x$ whereas $z - y = \mathrm{e}^{-x}$. The product of both equations gives $z^2 - y^2 = 1$ and finally $z = \sqrt{y^2+1}$ since $z \geq 1$. Inserting this gives

$$(\operatorname{arsinh} y)' = \frac{1}{\sqrt{y^2+1}}, \qquad y \in \mathbb{R}.$$

Setting

$$y = \tanh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}},$$

$x = \operatorname{artanh} y$, $|\, y \,| < 1$, $x \in \mathbb{R}$ we have by (a)

$$(\operatorname{artanh} y)' = \frac{1}{(\tanh x)'} = \cosh^2 x =: z$$

Using $\cosh^2 x - \sinh^2 x = 1$ (Homework 11.2)

$$y^2 = \frac{\sinh^2 x}{\cosh^2 x} = \frac{z-1}{z}.$$

This yields

$$z(y^2 - 1) = -1 \Longrightarrow z = \frac{1}{1-y^2}.$$

This gives

$$(\operatorname{artanh} y)' = \frac{1}{1-y^2}.$$

(c) Setting $y = \cos x$, $x = \arccos y$, $y \in [-1, 1]$, $x \in [0, \pi]$, we have

$$(\arccos y)' = \frac{1}{(\cos x)'} = -\frac{1}{\sin x}.$$

Since $x \in [0, \pi]$, $\sin x \geq 0$, namely $\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - y^2}$. This yields

$$(\arccos y)' = -\frac{1}{\sqrt{1-y^2}}, \qquad |\, y \,| < 1.$$

4. Compute $(x^3 \mathrm{e}^x)^{(2003)}$. *Solution.* Since $(x^3)' = 3x^2$, $(x^3)'' = 6x$, $(x^3)''' = 6$, and $(x^3)^{(k)} = 0$ for $k \geq 4$ we obtain by Proposition 6

$$(x^3 \mathrm{e}^x)^{(2003)} = \sum_{k=0}^{3} \binom{2003}{k} (x^3)^{(k)} \mathrm{e}^x = \mathrm{e}^x \left(x^3 + 2003 \cdot 3x^2 + \binom{2003}{2} \cdot 6x + \binom{2003}{3} \cdot 6 \right).$$

5. Let $f\colon (a,b) \to \mathbb{R}$ be a function and $c \in (a,b)$.
(a) Prove: If f is differentiable at c then

$$\lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h} \tag{1}$$

exists and is equal to $f'(c)$.
(b) Suppose the limit (1) exists. Does this imply that f is differentiable at c?

Proof. (a) Since f is differentiable at c,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h} &= \lim_{h\to 0} \frac{1}{2} \frac{f(c+h) - f(c) + f(c) - f(c-h)}{h} \\
&= \frac{1}{2} \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} + \frac{1}{2} \lim_{h\to 0} \frac{f(c) - f(c-h)}{h} \\
&= \frac{1}{2} f'(c) + \frac{1}{2} \lim_{h\to 0} \frac{f(c+(-h)) - f(c)}{-h} = \frac{1}{2} f'(c) + \frac{1}{2} f'(c) \\
&= f'(c).
\end{aligned}$$

The second limit exists and equals $f'(c)$ since $-h$ tends to 0 as h approaches 0.
(b) No. A counterexample is $f(x) = |\,x\,|$ at $c = 0$. We have

$$\lim_{h\to 0} \frac{|\,h\,| - |-h\,|}{2h} = \lim_{h\to 0} \frac{|\,h\,| - |\,h\,|}{2h} = 0.$$

However, f is not differentiable at $c = 0$. Also, the condition is not sufficient for the continuity of f at c: redefine $f(x) = |\,x\,|$ at $c = 0$ by $f(0) = 1$, then the limit still exists but f has a simple discontinuity at $c = 0$. ∎