Calculus – 11. Series, Solutions

1. Prove that for $a, b > 1$ and $x, y \in \mathbb{R}$
   (a) $(a^x)^y = a^{xy}$;
   (b) $a^x b^x = (ab)^x$;
   (c) $\left(\frac{1}{a}\right)^x = a^{-x}$.

   Proof. We use the new definition of power function, $a^x = \mathrm{e}^{x \log a}$. This implies $\log a^x = \log\left(\mathrm{e}^{x\log a}\right) = x \log a$. (a) Then we have

$$(a^x)^y = \mathrm{e}^{y\log(a^x)} = \mathrm{e}^{yx\log a} = a^{yx}.$$

   (b) Using $\mathrm{e}^{u+v} = \mathrm{e}^u\mathrm{e}^v$ and $\log a + \log b = \log(ab)$ we have

$$a^x a^y = \mathrm{e}^{x\log a}\mathrm{e}^{x\log b} = \mathrm{e}^{x(\log a + \log b)} = \mathrm{e}^{x\log(ab)} = (ab)^x.$$

   (c) $\log b + \log(a/b) = \log(b \cdot a/b) = \log a$ implies $\log\frac{a}{b} = \log a - \log b$. Since $\log 1 = 0$ we have

$$\left(\frac{1}{a}\right)^x = \mathrm{e}^{x\log(1/a)} = \mathrm{e}^{x(\log 1 - \log a)} = \mathrm{e}^{-xa} = a^{-x}.$$

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2. Prove that for all $z, w \in \mathbb{C}$

   (a) $\cosh^2 z - \sinh^2 z = 1$

   (b) $\cos(z+w) = \cos z \cos w - \sin z \sin w$

   (c) $\cos z = \cosh(\mathrm{i}z)$

   (d) $\tanh(z) = -\mathrm{i}\tan(\mathrm{i}z)$

   Prove that for $y \in \mathbb{R}$

   (e) $\operatorname{arcosh}(y) = \log(y + \sqrt{y^2-1}), \quad y \geq 1$

   (f) $\operatorname{arcoth}(y) = \frac{1}{2}\log\frac{y+1}{y-1}, \quad |\,y\,| > 1$

   Proof. (a) Using the binomial formula we have

$$\begin{aligned}\cosh^2 z - \sinh^2 z &= \frac{1}{4}\left(\mathrm{e}^z + \mathrm{e}^{-z}\right)^2 - \frac{1}{4}\left(\mathrm{e}^z - \mathrm{e}^{-z}\right)^2\\ &= \frac{1}{4}\left(\mathrm{e}^{2z} + 2 + \mathrm{e}^{-2z} - \left(\mathrm{e}^{2z} - 2 + \mathrm{e}^{-2z}\right)\right) = 1.\end{aligned}$$

   (b) We use the definition of $\cos z$ and $\sin z$.

$$\cos z\cos w - \sin z \sin w = \frac{1}{4}\left(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}\right)\left(\mathrm{e}^{\mathrm{i}w} + \mathrm{e}^{\mathrm{i}w}\right) - \frac{1}{(2\mathrm{i})^2}\left(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}\right)\left(\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{-\mathrm{i}w}\right)$$

$$= \frac{1}{4}\left(\mathrm{e}^{\mathrm{i}(z+w)} + \mathrm{e}^{-\mathrm{i}(z+w)} + \mathrm{e}^{\mathrm{i}(z-w)} + \mathrm{e}^{\mathrm{i}(w-z)} + \mathrm{e}^{\mathrm{i}(z+w)} + \mathrm{e}^{-\mathrm{i}(z+w)} - \mathrm{e}^{\mathrm{i}(z-w)} - \mathrm{e}^{\mathrm{i}(w-z)}\right)$$

$$= \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}(z+w)} + \mathrm{e}^{-\mathrm{i}(z+w)}\right) = \cos(z+w).$$

(c)

$$\cosh(\mathrm{i}z) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}\right) = \cos z.$$

Similarly,

$$\sinh(\mathrm{i}z) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}\right) = \mathrm{i}\sin z.$$

(d) By (c),

$$-\mathrm{i}\tanh(\mathrm{i}z) = -\mathrm{i}\frac{\sinh(\mathrm{i}z)}{\cosh(\mathrm{i}z)} = -\mathrm{i}\frac{\mathrm{i}\sin z}{\cos z} = \frac{\sin z}{\cos z} = \tan z.$$

(e) Since $\mathrm{e}^{-x} = \dfrac{1}{\mathrm{e}^x}$ and for every $y > 0$ we have $y + 1/y \geq 2$, $\cosh x = \frac{1}{2}(y + 1/y) \geq 1$ where $y = \mathrm{e}^x$. Since $\lim_{x\to+\infty} \mathrm{e}^x = +\infty$ continuity of $\cosh x$ and the intermediate value theorem show, that the image of $\cosh x$ is $[1, +\infty)$. It is easy to show that $\cosh x$ is strictly increasing on $\mathbb{R}_+$. Hence, $\cosh x$ has a strictly increasing continuous inverse function

$$\operatorname{arcosh} : [1, +\infty) \to \mathbb{R}_+$$

given by

$$\operatorname{arcosh}(\cosh x) = x, \quad x \geq 0$$

One gets the inverse function by solving for x

$$y = \cosh x = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right).$$

Put $z = \mathrm{e}^x$,

$$\begin{aligned} y &= \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2z}\left(z^2 + 1\right) \\ 2yz &= z^2 + 1 \\ 0 &= z^2 - 2yz + 1. \end{aligned}$$

This is a quadratic equation in z; its two solutions are

$$z_{1,2} = y \pm \sqrt{y^2 - 1}.$$

Since $x \geq 0$, $y < \mathrm{e}^x = z$. Therefore, the only solution is $z_1 = y + \sqrt{y^2 - 1}$ since $z_2 = y - \sqrt{y^2 - 1} < y$. We conclude

$$\mathrm{e}^x = y + \sqrt{y^2 - 1} \Longrightarrow x = \log(y + \sqrt{y^2 - 1}).$$

(f) Since $|\sinh x| < \cosh x$ for all x, $|\coth x| = \left|\frac{\cosh x}{\sinh x}\right| > 1$. It is easy to see that $\coth x$ is strictly decreasing both on $\mathbb{R}_+ \setminus 0$ and on the negative axes; the image of $\coth x$ is $(-\infty, -1) \cup (1, +\infty)$. Hence $\coth x$ has a strictly decreasing continuous inverse function

$$\operatorname{arcoth} : (-\infty, -1) \cup (1, +\infty) \to \mathbb{R}$$

given by $\operatorname{arcoth}(\tanh x) = x$, $x \neq 0$. We obtain the inverse function by solving for x

$$y = \coth x = \frac{\cosh x}{\sinh x} = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}} = \frac{\mathrm{e}^{2x} + 1}{\mathrm{e}^{2x} - 1}$$

$$y\mathrm{e}^{2x} - y = \mathrm{e}^{2x} + 1 \Longrightarrow \mathrm{e}^{2x}(y-1) = y + 1 \Longrightarrow \mathrm{e}^{2x} = \frac{y+1}{y-1}.$$

$$2x = \log \frac{y+1}{y-1} \Longrightarrow x = \operatorname{arcoth} y = \frac{1}{2} \log \frac{y+1}{y-1}, \quad |\, y \,| > 1.$$

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3. Prove that for all $z \in \mathbb{C}$

$$\mathrm{e}^{\overline{z}} = \overline{\mathrm{e}^z}$$

Hint. Use Proposition 2.31.

Proof. Let (z_n) be convergent complex sequence with $\lim_{n\to\infty} z_n = z$. Proposition 2.31 implies $\lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} z$ and $\lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} z$; hence

$$\lim_{n\to\infty} \overline{z_n} = \lim_{n\to\infty} (\operatorname{Re} z_n - \mathrm{i} \operatorname{Im} z_n) = \operatorname{Re} z - \mathrm{i} \operatorname{Im} z = \overline{z}.$$

That is, the complex conjugation is a continuous operation on $\mathbb{C}$, we have $\overline{\lim z_n} = \lim \overline{z_n}$. In particular, the partial sums of the exponential series

$$s_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$$

converge to e^z as n tends to infinity. Hence, $\overline{s_n(z)}$ converges to $\overline{\mathrm{e}^z}$. But the algebraic properties of conjugation, $\overline{uv} = \overline{u}\,\overline{v}$, $\overline{u+v} = \overline{u} + \overline{v}$, show

$$\overline{s_n(z)} = \overline{\sum_{k=0}^{n} \frac{z^k}{k!}} = \sum_{k=0}^{n} \frac{\overline{z^k}}{k!} = \sum_{k=0}^{n} \frac{\overline{z}^k}{k!} = s_n(\overline{z}).$$

Now, this sequence of partial sums converges to $\mathrm{e}^{\overline{z}}$ (by definition) which completes the proof. ■

4. Compute the following limits

$$\text{(a)} \quad \lim_{x\to 0+0} x \log x$$

$$\text{(b)} \quad \lim_{x\to 0} \frac{\tan x}{x}$$

$$\text{(c)} \quad \lim_{z\to 0} \frac{\mathrm{e}^z - 1}{z}$$

Hint. For (a) substitute $x = 1/\mathrm{e}^z$; for (c) use Proposition 17 with $n = 2$.

Solution. (a) Let (x_n) be a sequence converging to 0 and $x_n > 0$. Then $z_n = -\log(x_n)$ tends to $+\infty$ by Proposition 19 (d). Since $x_n = \mathrm{e}^{-z_n}$ we obtain

$$\lim_{n\to\infty} x_n \log x_n = \lim_{n\to\infty} \frac{1}{\mathrm{e}^{z_n}}(-z_n) = \lim_{z\to+\infty} \frac{-z}{\mathrm{e}^z} = 0,$$

where the last equation is by Proposition 18 (e). This shows $\lim_{x\to 0+0} x\log x = 0$.
(b) Since $\tan x = \sin x/\cos x$, $\lim_{x\to 0}\cos x = \cos 0 = 1$, and $\lim_{x\to 0}\sin x/x = 1$ by Corollary 25 we have

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\cos x = 1.$$

(c) Inserting $n = 2$ into Proposition 17 gives

$$|\, r_2(z)\,| = |\,\mathrm{e}^z - 1 - z\,| \le \frac{2\,|\,z\,|^2}{2!} = |\,z\,|^2 \quad \text{if} \quad |\,z\,| \le \frac{3}{2}.$$

Dividing this inequality by $|\,z\,|$ yields

$$\left| \frac{\mathrm{e}^z - 1}{z} - 1 \right| \le |\,z\,| \quad \text{if} \quad |\,z\,| \le 1.5$$

This shows

$$\lim_{z\to 0}\frac{\mathrm{e}^z - 1}{z} = 1.$$