Calculus – 14. Series

(turn in: April 13, 2004)

1. Compute directly (without using the Fundamental Theorem of Calculus)

$$\int_0^1 x \, \mathrm{d}x^2.$$

Hint. Consider equidistant partitions of $[0,1]$ and use $\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ and $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

2. Suppose α increases on $[a,b]$, $a \le c \le b$, α is continuous at c, $f(c) = 1$, and $f(x) = 0$ if $x \ne c$.
Prove that $f \in \mathcal{R}(\alpha)$ and that $\int_a^b f \, \mathrm{d}\alpha = 0$.

3. Suppose α strictly increases on $[a,b]$, $f \ge 0$, f is continuous on $[a,b]$, and $\int_a^b f \, \mathrm{d}\alpha = 0$.
Prove that $f(x) = 0$ for all $x \in [a,b]$. Compare this with homework 14.2.
Hint. Make an indirect proof; use homework 10.4 and Proposition 9 (b) and (c).

4. Let

$$f(x) := \frac{x(1-x)}{x^2+1}.$$

Find the area of the bounded region enclosed by $f(x)$ and $-f(x)$.

5. If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$ and

$$\int_a^b f \, \mathrm{d}\alpha = \int_a^c f \, \mathrm{d}\alpha + \int_c^b f \, \mathrm{d}\alpha.$$

Hint. Let $\varepsilon > 0$, consider a partition P of $[a,b]$ with $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$; pass to a refinement P^* of P which contains the point c and write this in terms of upper and lower sums over the intervals $[a,c]$ and $[c,b]$.