Calculus – 12. Series

(turn in: January 23, 2004)

1. Compute the derivatives of $f\colon (0,1) \to \mathbb{R}$ where

$$\begin{array}{ll} \text{(a)} & f(x) = x^{x^x} \\ \text{(b)} & f(x) = (x^x)^x \\ \text{(c)} & f(x) = x^{a^x} \end{array}$$

   and $a > 0$ is a constant. Note that $a^{b^c} = a^{(b^c)}$ by definition.

2. Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^2 \cos \frac{1}{x}, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0. \end{cases}$$

   Prove that $g(x)$ is differentiable for all $x \in \mathbb{R}$ and compute $g'(x)$. Prove that g' is not continuous at $x = 0$. What kind of discontinuity has g' at $x = 0$?

3. Compute the derivatives of
   (a) $\cosh x$, $\sinh x$, and $\tanh x$,
   (b) $\operatorname{arcosh} x$, $\operatorname{arsinh} x$, and $\operatorname{artanh} x$,
   (c) $\arccos x$.

4. Compute $(x^3 e^x)^{(2003)}$.

5. Let $f\colon (a,b) \to \mathbb{R}$ be a function and $c \in (a,b)$.
   (a) Prove: If f is differentiable at c then

$$\lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h} \tag{1}$$

   exists and is equal to $f'(c)$.
   (b) Suppose the limit (1) exists. Does this imply that f is differentiable at c?