# Calculus – 10. Series

(turn in: January 9, 2004)

1. Determine all values of $c \in \mathbb{R}$ such that
$$f(x) = \begin{cases} (x-2)^2 & \text{if} \quad x \leq 4, \\ cx^2 - 8 & \text{if} \quad x > 4 \end{cases}$$
is continuous on $\mathbb{R}$.

2. (a) Prove the following fixed point theorem. Let $D = [a, b]$ be a finite closed interval. Every continuous function $f\colon D \to D$ has a fixed point, i.e. there exists $c \in [a, b]$ such that $f(c) = c$.
Give examples of functions $f\colon D \to D$ such that the fixed point theorem fails if
(b) $D$ is a closed infinite interval.
(c) $D = [a, b)$.
(d) $D = [0, 1] \cup [2, 3]$.
(e) $f$ is not continuous.
*Hint.* Use the intermediate value theorem for (a).

3. Let $a$ and $b$ be real numbers with $1 < a < b$. Prove that the equation
$$\frac{x^7 + 1}{x - a} + \frac{x^3 - 1}{x - b} = 0$$
has a solution $x \in (a, b)$.
*Hint.* Define an appropriate function $f$ and apply the intermediate value theorem.

4. Prove. If $f\colon [a, b] \to \mathbb{R}$ is continuous at $x_0 \in (a, b)$ and $f(x_0) = A > 0$ then there exists a real number $\delta > 0$ such that for every $x \in [a, b]$ the inequality $|\, x - x_0 \,| < \delta$ implies $f(x) > A/2$.
(In other words: If a continuous function is nonzero at a point $x_0$, then $f$ is nonzero on a whole neighborhood $U_\delta(x_0)$)

5. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $\mathbb{R}_+$, whereas $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}_+$.

Merry Christmas and a Happy New Year!