Chapter 9

Integration of Functions of Several Variables

References to this chapter are [10, Section 4] which is quite elemantary and good accessible. Another elementary approach is [9, Chapter 17] (part III). A more advanced but still good accessible treatment is [13, Chapter 3]. This will be our main reference here. In particular the main difference between content zero and measure zero is explained in full detail. Rudin's book [12] is not recommendable for an introduction to integration.

9.1 Basic Definition

The definition of the Riemann integral of a function $f\colon A \to \mathbb{R}$, where $A \subset \mathbb{R}^n$ is a closed rectangle, is so similar to that of the ordinary integral that a rapid treatment will be given, see Section 5.1.
If nothing is specified otherwise, A denotes a rectangle. A *rectangle* A is the cartesian product of n intervals,

$$A = [a_1, b_1] \times \cdots \times [a_n, b_n] = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_k \leq x_k \leq b_k,\ k = 1, \dots, n\}.$$

Recall that a *partition* of a closed interval $[a, b]$ is a sequence $t_0, \dots, t_k$ where $a = t_0 \leq t_1 \leq \cdots \leq t_k = b$. The partition divides the interval $[a, b]$ in to k subintervals $[t_{i-1}, t_i]$.
A *partition* of a rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is a collection $P = (P_1, \dots, P_n)$ where each P_i is a partition of the interval $[a_i, b_i]$. Suppose for example that $P_1 = (t_0, \dots, t_k)$ is a partition of $[a_1, b_1]$ and $P_2 = (s_0, \dots, s_l)$ is a partition of $[a_2, b_2]$. Then the partition $P = (P_1, P_2)$ of $[a_1, b_1] \times [a_2, b_2]$ divides the closed rectangle $[a_1, b_1] \times [a_2, b_2]$ into kl subrectangles, a typical one being $[t_{i-1}, t_i] \times [s_{j-1}, s_j]$. In general, if P_i divides $[a_i, b_i]$ into N_i subintervals, then $P = (P_1, \dots, P_n)$ divides $[a_1, b_1] \times \cdots \times [a_n, b_n]$ into $N_1 \cdots N_n$ subrectangles. These subrectangles will be called *subrectangles of the partition* P.
Suppose now A is a rectangle, $f\colon A \to \mathbb{R}$ is a bounded function, and P is a partition of A. For each subrectangle S of the partition let

$$m_S = \inf\{f(x) \mid x \in S\}, \quad M_S = \sup\{f(x) \mid x \in S\},$$

and let $v(S)$ be the volume of the rectangle S. Note that *volume* of the rectangle $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is

$$v(A) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

The *lower* and the *upper sums* of f for P are defined by

$$L(P, f) = \sum_S m_S \, v(S) \quad \text{and} \quad U(P, f) = \sum_S M_S \, v(S),$$

where the sum is taken over all subrectangles S of the partition P. Clearly, if f is bounded with $m \leq f(x) \leq M$ on the rectangle $x \in R$,

$$m \, v(R) \leq L(P, f) \leq U(P, f) \leq M \, v(R),$$

so that the numbers $L(P, f)$ and $U(P, f)$ form bounded sets. Lemma 5.1 remains true; the proof is completely the same.

Lemma 9.1 *Suppose the partition P^* is a refinement of P (that is, each subrectangle of P^* is contained in a subrectangle of P). Then*

$$L(P, f) \leq L(P^*, f) \quad \textit{and} \quad U(P^*, f) \leq U(P, f).$$

Corollary 9.2 *If P and P' are any two partitions, then $L(P, f) \leq U(P', f)$.*

It follows from the above corollary that all lower sums are bounded above by any upper sum and vice versa.

Definition 9.1 Let $f\colon A \to \mathbb{R}$ be a bounded function. The function f is called *Riemann integrable* on the rectangle A if

$$\underline{\int_A} f \, \mathrm{d}x := \sup_P \{L(P, f)\} = \inf_P \{U(P, f)\} =: \overline{\int_A} f \, \mathrm{d}x,$$

where the supremum and the infimum are taken over all partitions P of A. This common number is the *Riemann integral* of f on A and is denoted by

$$\int_A f \, \mathrm{d}x \quad \text{or} \quad \int_A f(x_1, \dots, x_n) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

$\underline{\int_A} f \, \mathrm{d}x$ and $\overline{\int_A} f \, \mathrm{d}x$ are called the *lower* and the *upper* integral of f on A, respectively. They always exist. The set of integrable function on A is denoted by $\mathcal{R}(A)$.

As in the one dimensional case we have the following criterion.

Proposition 9.3 (Riemann Criterion) *A bounded function $f\colon A \to \mathbb{R}$ is integrable if and only if for every $\varepsilon > 0$ there exists a partition P of A such that $U(P, f) - L(P, f) < \varepsilon$.*

Example 9.1 (a) Let $f\colon A \to \mathbb{R}$ be a constant function $f(x) = c$. Then for any Partition P and any subrectangle S we have $m_S = M_S = c$, so that

$$L(P,f) = U(P,f) = \sum_S c\,v(S) = c\sum_S v(S)c = v(A).$$

Hence, $\int_A c\,\mathrm{d}x = cv(A)$.
(b) Let $f\colon [0,1]\times[0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0, & \text{if } x \text{ is rational,}\\ 1, & \text{if } x \text{ is irrational.}\end{cases}$$

If P is a partition, then every subrectangle S will contain points (x,y) with x rational, and also points (x,y) with x irrational. Hence $m_S = 0$ and $M_S = 1$, so

$$L(P,f) = \sum_S 0v(S) = 0,$$

and

$$U(P,f) = \sum_S 1v(S) = v(A) = v([0,1]\times[0,1]) = 1.$$

Therefore, $\overline{\int}_A f\,\mathrm{d}x = 1 \neq 0 = \underline{\int}_A f\,\mathrm{d}x$ and f is not integrable.

9.1.1 Properties of the Riemann Integral

We briefly write $\mathcal{R}$ for $\mathcal{R}(A)$.

Remark 9.1 (a) $\mathcal{R}$ is a linear space and $\int_A (\cdot)\,\mathrm{d}x$ is a linear functional, i. e. $f,g \in \mathcal{R}$ imply $\lambda f + \mu g \in \mathcal{R}$ for all $\lambda,\mu \in \mathbb{R}$ and

$$\int_A (\lambda f + \mu g)\,\mathrm{d}x = \lambda\int_A f\,\mathrm{d}x + \mu\int_A g\,\mathrm{d}x.$$

(b) $\mathcal{R}$ is a *lattice*, i. e., $f \in \mathcal{R}$ implies $|\,f\,| \in \mathcal{R}$. If $f,g \in \mathcal{R}$, then $\max\{f,g\} \in \mathcal{R}$ and $\min\{f,g\} \in \mathcal{R}$.

(c) $\mathcal{R}$ is an algebra, i. e., $f,g \in \mathcal{R}$ imply $fg \in \mathcal{R}$.

(d) The triangle inequality holds:

$$\left|\int_A f\,\mathrm{d}x\right| \leq \int_A |\,f\,|\,\mathrm{d}x.$$

(e) $\mathrm{C}(A) \subset \mathcal{R}(A)$.

(f) $f \in \mathcal{R}(A)$ and $f(A) \subset [a,b]$, $g \in \mathrm{C}[a,b]$. Then $g\circ f \in \mathcal{R}(A)$.

(g) If $f \in \mathcal{R}$ and $f = g$ except at finitely many points, then $g \in \mathcal{R}$ and $\int_A f\,\mathrm{d}x = \int_A g\,\mathrm{d}x$.

(h) Let $f\colon A \to \mathbb{R}$ and let P be a partition of A. Then $f \in \mathcal{R}(A)$ if and only if $f{\restriction}S$ is integrable for each subrectangle S. In this case

$$\int_A f\,\mathrm{d}x = \sum_S \int_S f{\restriction}S\,\mathrm{d}x.$$

9.2 Integrable Functions

We are going to characterize integrable functions. For, we need the notion of a set of *content zero* and of *measure zero.*

Definition 9.2 Let A be a subset of $\mathbb{R}^n$.
(a) A has (n-dimensional) *measure zero* if for every $\varepsilon > 0$ there exists a sequence $(U_i)_{i\in\mathbb{N}}$ of closed rectangles U_i which cover A such that $\sum_{i=1}^{\infty} v(U_i) < \varepsilon$.
(b) A has (n-dimensional) *content zero* if for every $\varepsilon > 0$ there exists a finite cover $\{U_1, \dots, U_k\}$ of A by closed rectangles such that $\sum_{i=1}^{k} v(U_i) < \varepsilon$.

Open rectangles can also be used in the definition.

Remark 9.2 (a) Any countable set has measure 0. If each $(A_i)_{i\in\mathbb{N}}$ has measure 0 then $A = A_1 \cup A_2 \cup \cdots$ has measure 0.
(b) If A has content 0, then A clearly has measure 0.
(c) If A is compact and has measure 0, then A has content 0. For, let $\varepsilon > 0$. Since A has measure 0 there is a countable open covering $(U_1, U_2, \dots)$ of A such that $\sum_i v(U_i) < \varepsilon$. Since A is compact, there is a finite subcover, say $U_1, \dots, U_n$ of A. Clearly $\sum_{i=1}^{n} v(U_i) < \varepsilon$.
(d) If $a < b$ all intervals from a to b (open, closed, half-open, ...) don't have measure 0 (in fact, their measure and content are $b - a$).
(e) The conclusion (c) is wrong if A is not compact. For example, let $A = \mathbb{Q} \cup [0, 1]$. Then A has measure 0 since A is countable. Suppose, however $A \subset [a_1, b_1] \cup \cdots \cup [a_n, b_n] =: A_n$, then A_n is a *closed* set. Hence, $\overline{A} = [0, 1] \subset A_n$. Since $[0, 1]$ has nonzero content, so has A.

Theorem 9.4 *Let A be a closed rectangle and $f\colon A \to \mathbb{R}$ a bounded function. Let $B = \{x \in A \mid f \text{ is discontinuous at } x\}$.*
Then f is integrable if and only if B is a set of measure 0.

For the proof see [13, 3-8 Theorem] or [12, Theorem 11.33].
We have so far dealt only with integrals of functions over rectangles. Integrals over other sets are easily reduced to this type.
If $C \subset \mathbb{R}^n$, the *characteristic function* χ_C of C is defined by

$$\chi_C(x) = \begin{cases} 1, & x \in C, \\ 0, & x \notin C. \end{cases}$$

Definition 9.3 Let $f\colon C \to \mathbb{R}$ be bounded and A a rectangle, $C \subset A$. We call f *Riemann integrable* on C if the product function $f{\cdot}\chi_C\colon A \to \mathbb{R}$ is Riemann integrable on A. In this case we define

$$\int_C f \,\mathrm{d}x = \int_A f\chi_C \,\mathrm{d}x.$$

This certainly occurs if both f and χ_C are integrable on A.
For every $x \in A$ exactly one of the following three cases occurs:

(a) x has a neighborhood which is completely contained in C (x is an inner point of C),

(b) x has a neighborhood which is completely contained in C^c (x is an inner point of C^c),

(c) every neighborhood of x intersects both C and C^c. In this case we say, x is the *boundary* ∂C of C. By definition $\partial C = \overline{C} \cap \overline{C^c}$.

By the above discussion, A is the disjoint union of two open and a closed set:

$$A = C^\circ \cup \partial C \cup (C^c)^\circ.$$

Theorem 9.5 *The characteristic function $\chi_C \colon A \to \mathbb{R}$ is integrable if and only if the boundary of C has measure* 0.

Proof. Since the boundary ∂C is closed and inside the bounded set, ∂C is compact. Suppose first x is an inner point of C. Then there is an open set $U \subset C$ containing x. Thus $\chi_C(x) = 1$ on $x \in U$; clearly χ_C is continuous at x (since it is locally constant). Similarly, if x is an inner point of C^c, $\chi_C(x)$ is locally constant, namely $\chi_C = 0$ in a neighborhood of x. Hence χ_C is continuous at x. Finally, if x is in the boundary of C for every open neighborhood U of x there is $y_1 \in U \cap C$ and $y_2 \in U \cap C^c$, so that $\chi_C(y_1) = 1$ whereas $\chi_C(y_2) = 0$. Hence, χ_C is not continuous at x. Thus, the set of discontinuity of χ_C is exactly the boundary ∂C. The rest follows from Theorem 9.4. ∎

Definition 9.4 A bounded set C whose boundary has measure 0 is called *Jordan measuralbe* or simply a *Jordan set*. The integral $\int_C 1\,\mathrm{d}x$ is called the n-dimensional *content* of C or the n-dimensional *volume* of C. Naturally, the one-dimensional volume in the *length*, and the two-dimensional volume is the *area*.

A typical example of a Jordan measurable subset D of $\mathbb{R}^{n+1}$ is

$$D = \{(x, y) \mid x \in K,\ 0 \le y \le f(x)\},$$

where $K \subset \mathbb{R}^n$ is a compact set and $f \colon K \to \mathbb{R}$ is continuous. In particular, the graph of f has measure 0 in $\mathbb{R}^{n+1}$.

9.2.1 Fubini's Theorem and Iterated Integrals

So far there was no method to compute multiple integrals.

Theorem 9.6 (Fubini's Theorem) *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be closed rectangles, and let $f \colon A \times B \to \mathbb{R}$ be integrable. For $x \in A$ let $g_x \colon B \to \mathbb{R}$ be defined by $g_x(y) = f(x, y)$ and let*

$$\mathcal{L}(x) = \underline{\int_B} g_x\,\mathrm{d}y = \underline{\int_B} f(x, y)\,\mathrm{d}y,$$

$$\mathcal{U}(x) = \overline{\int_B} g_x\,\mathrm{d}y = \overline{\int_B} f(x, y)\,\mathrm{d}y.$$

Then $\mathfrak{L}(x)$ *and* $\mathfrak{U}(x)$ *are integrable on* A *and*

$$\int_{A\times B} f \, \mathrm{d}x\mathrm{d}y = \int_A \mathfrak{L}(x)\,\mathrm{d}x = \int_A \left(\underline{\int}_B f(x,y)\,\mathrm{d}y \right) \mathrm{d}x,$$
$$\int_{A\times B} f \, \mathrm{d}x\mathrm{d}y = \int_A \mathfrak{U}(x)\,\mathrm{d}x = \int_A \left(\overline{\int}_B f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

The integrals on the right are called iterated integrals.

Proof. Let P_A be a partition of A and P_B a partition of B. Together they give a partition P of $A \times B$ for which any subrectangle S is of the form $S_A \times S_B$, where S_A is a subrectangle of the partition P_A, and S_B is a subrectangle of the partition P_B. Thus

$$L(P,f) = \sum_S m_S\, v(S) = \sum_{S_A, S_B} m_{S_A\times S_B}\, v(S_A \times S_B)$$
$$= \sum_{S_A} \left(\sum_{S_B} m_{S_A\times S_B}\, v(S_B) \right) v(S_A).$$

Now, if $x \in S_A$, then clearly $m_{S_A\times S_B}(f) \le m_{S_B}(g_x)$ since the reference set $S_A \times S_B$ on the left is bigger than the reference set $\{x\} \times S_B$ on the right. Consequently, for $x \in S_A$ we have

$$\sum_{S_B} m_{S_A\times S_B}\, v(S_B) \le \sum_{S_B} m_{S_B}(g_x)\, v(S_B) \le \underline{\int}_B g_x \,\mathrm{d}y = \mathfrak{L}(x)$$
$$\sum_{S_B} m_{S_A\times S_B}\, v(S_B) \le m_{S_A}(\mathfrak{L}(x)).$$

Therefore,

$$\sum_{S_A} \left(\sum_{S_B} m_{S_A\times S_B}\, v(S_B) \right) v(S_A) \le \sum_{S_A} m_{S_A}(\mathfrak{L}(x)) v(S_A) = L(P_A, \mathfrak{L}).$$

We thus obtain

$$L(P,f) \le L(P_A,\mathfrak{L}) \le U(P_A,\mathfrak{L}) \le U(P_A,\mathfrak{U}) \le U(P,f),$$

where the proof of the last inequality is entirely analogous to the proof of the first. Since f is integrable, $\sup\{L(P,f)\} = \inf\{U(P,f)\} = \int_{A\times B} f \,\mathrm{d}x\mathrm{d}y$. Hence,

$$\sup\{L(P_A,\mathfrak{L})\} = \inf\{U(P_A,\mathfrak{L})\} = \int_{A\times B} f \,\mathrm{d}x\mathrm{d}y.$$

In other words, $\mathfrak{L}(x)$ is integrable on A and $\int_{A\times B} f \,\mathrm{d}x\mathrm{d}y = \int_A \mathfrak{L}(x)\,\mathrm{d}x$.
The assertion for $\mathfrak{U}(x)$ follows similarly from the inequalities

$$L(P,f) \le L(P_A,\mathfrak{L}) \le L(P_A,\mathfrak{U}) \le U(P_A,\mathfrak{U}) \le U(P,f).$$

■

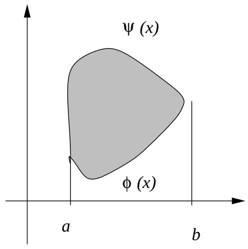
Remarks 9.3 (a) A similar proof shows that

$$\int_{A\times B} f \,\mathrm{d}x\mathrm{d}y = \int_B \left(\underline{\int}_A f(x,y)\,\mathrm{d}x \right) \mathrm{d}y = \int_B \left(\overline{\int}_A f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

These integrals are called *iterated integrals* for f in the reverse order from those of the theorem. The possibility of interchanging the orders of iterated integrals has many consequences.
(b) In practice it is often the case that each g_x is integrable so that $\int_{A\times B} f\,\mathrm{d}x\mathrm{d}y = \int_A \left(\int_B f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x$. This certainly occurs if f is continuous.
(c) If $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $f\colon A \to \mathbb{R}$ is sufficiently nice, we can apply Fubini's theorem repeatedly to obtain

$$\int_A f\,\mathrm{d}x = \int_{a_n}^{b_n} \left(\cdots \left(\int_{a_1}^{b_1} f(x_1, \dots, x_n)\,\mathrm{d}x_1 \right) \cdots \right) \mathrm{d}x_n.$$

(d) If $C \subset A \times B$, Fubini's theorem can be used to compute $\int_C f\,\mathrm{d}x$ since this is by definition $\int_{A\times B} f\chi_C\,\mathrm{d}x$. Here are two examples in case $n = 2$ and $n = 3$.
Let $a < b$ and $\varphi(x)$ and $\psi(x)$ continuous real valued functions on $[a, b]$ with $\varphi(x) < \psi(x)$ on $[a, b]$. Put



$$C = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, \quad \varphi(x) \le y \le \psi(x)\}.$$

Let $f(x,y)$ be continuous on C. Then f is integrable on C and

$$\iint\limits_C f\,\mathrm{d}x\mathrm{d}y = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

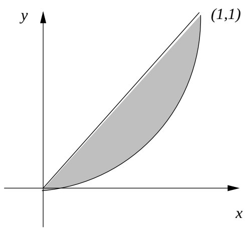
Let

$$G = \{(x,y,z) \in \mathbb{R}^3 \mid a \le x \le b,\ \varphi(x) \le y \le \psi(x),\ \alpha(x,y) \le z \le \beta(x,y)\},$$

where all functions are sufficiently nice. Then

$$\iiint\limits_G f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_a^b \left(\int_{\varphi(x)}^{\psi(y)} \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x.$$

(e) Cavalieri's Principle. Let A and B be Jordan sets in $\mathbb{R}^3$ and let $A_c = \{(x,y) \mid (x,y,c) \in A\}$; B_c is defined similar. Suppose each A_c and B_c is Jordan measurable (in $\mathbb{R}^2$) and they have the same area.
Then A and B have the same volume.



Example 9.2 (a) Let $f(x,y) = xy$ and

$$\begin{aligned} C &= \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x^2 \le y \le x\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1,\ y \le x \le \sqrt{y}\}. \end{aligned}$$

Then

$$\iint_C xy\,\mathrm{d}x\mathrm{d}y = \int_0^1 \int_{x^2}^x xy\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \frac{xy^2}{2}\bigg|_{y=x^2}^x \mathrm{d}x = \frac{1}{2}\int_0^1 (x^3 - x^5)\,\mathrm{d}x$$
$$= \frac{x^4}{8} - \frac{x^6}{12}\bigg|_0^1 = \frac{1}{8} - \frac{1}{12} = \frac{1}{24}.$$

Interchanging the order of integration we obtain

$$\iint_C xy\,\mathrm{d}x\mathrm{d}y = \int_0^1 \int_y^{\sqrt{y}} xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \frac{x^2y}{2}\bigg|_y^{\sqrt{y}} = \frac{1}{2}\int_0^1 (y^2 - y^3)\,\mathrm{d}y$$
$$= \frac{y^3}{6} - \frac{y^4}{8}\bigg|_0^1 = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.$$

(b) Let $G = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0,\ x+y+z \leq 1\}$ and $f(x, y, z) = 1/(x+y+z+1)^3$. The set G can be parametrized as follows

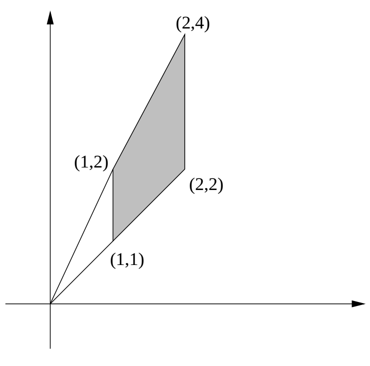
$$\iiint_G f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} \frac{\mathrm{d}z}{(1+x+y+z)^3}\right) \mathrm{d}y\right) \mathrm{d}x$$
$$= \int_0^1 \left(\int_0^{1-x} \frac{1}{2}\frac{-1}{(1+x+y+z)^2}\bigg|_0^{1-x-y} \mathrm{d}y\right) \mathrm{d}x$$
$$= \int_0^1 \left(\int_0^{1-x} \left(\frac{1}{2}\frac{1}{(1+x+y)^2} - \frac{1}{8}\right) \mathrm{d}y\right) \mathrm{d}x$$
$$= \frac{1}{2}\int_0^1 \left(\frac{1}{x+1} + \frac{x-3}{4}\right) \mathrm{d}x = \frac{1}{2}\left(\log 2 - \frac{5}{8}\right).$$

(c) Let $C = \{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 1,\ x^2 \leq y \leq 2 - x\}$. We want to express $\iint_D f(x, y)\,\mathrm{d}x\mathrm{d}y$ as iterated integrals. Clearly,

$$\iint_D f\,\mathrm{d}x\mathrm{d}y = \int_{-2}^1 \mathrm{d}x \int \mathrm{d}y f(x, y).$$

Interchanging the order of integration we have

$$\iint_D f\,\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}y \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\,\mathrm{d}x + \int_1^4 \mathrm{d}y \int_{-\sqrt{y}}^{2-y} f(x, y)\,\mathrm{d}x.$$



(d) Let $f(x, y) = \mathrm{e}^{y/x}$ and D the above region. Compute the integral of f on D.
D can be parametrized as follows $D = \{(x, y) \mid 1 \leq x \leq 2,\ x \leq y \leq 2x\}$ Hence,

$$\iint_D f\,\mathrm{d}x\mathrm{d}y = \int_1^2 \mathrm{d}x \int_x^{2x} \mathrm{e}^{\frac{y}{x}}\,\mathrm{d}y$$
$$= \int_1^2 \mathrm{d}x\ x\mathrm{e}^{\frac{y}{x}}\bigg|_x^{2x} = \int_1^2 (\mathrm{e}^2 x - \mathrm{e}x)\,\mathrm{d}x = \frac{3}{2}(\mathrm{e}^2 - \mathrm{e}).$$

But trying to reverse the order of integration we encounter two problems. First, we must break D in several regions:

$$\iint\limits_D f\,\mathrm{d}x\mathrm{d}y = \int_1^2 \mathrm{d}y \int_1^y \mathrm{e}^{y/x}\,\mathrm{d}x + \int_2^4 \mathrm{d}y \int_{y/2}^2 \mathrm{e}^{y/x}\,\mathrm{d}x.$$

This is not a serious problem. A greater problem is that $\mathrm{e}^{1/x}$ has no elementary antiderivative, so $\int_1^y \mathrm{e}^{y/x}\,\mathrm{d}x$ and $\int_{y/2}^2 \mathrm{e}^{y/x}\,\mathrm{d}x$ are very difficult to evaluate. In this example, there is a considerable advantage in one order of integration over the other.

9.3 Change of Variable

We want to generalize change of variables formula $\int_{g(a)}^{g(b)} f(x)\,\mathrm{d}x = \int_a^b f(g(y))g'(y)\,\mathrm{d}y$.
If B_R is the ball in $\mathbb{R}^3$ with radius R around the origin we have in cartesian coordinates

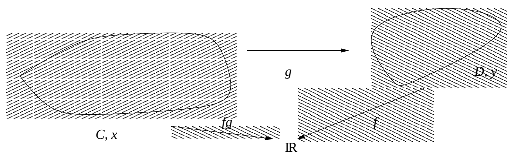
$$\iiint\limits_{B_R} f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{-R}^R \mathrm{d}x \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \mathrm{d}y \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} \mathrm{d}z f(x,y,z).$$

Usually, the complicated limits yield hard computations. Here spherical coordinates are appropriate.
To motivate the formula consider $f = 1$ and a *linear* transformation of variables $g\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\det g \neq 0$, $C = [0,1] \times [0,1]$, and $D = g(C)$. We want to compare the area of C, $\iint_C \mathrm{d}x = 1$ with the area of the transformed set D, $\iint_D \mathrm{d}x$. Let $g(1,0) = (a_1, a_2)$ and $g(0,1) = (b_1, b_2)$ it is easy to see that D is the parallelogram spanned by the two vectors (a_1, a_2) and (b_1, b_2). Its area is

$$v(D) = \left| \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \right| = |\det g|\,.$$

This is true for any $\mathbb{R}^n$ and any regular map, $v(g(C)) = |\det g|\, v(C)$.

Theorem 9.7 (Change of variable) *Let C and D be compact Jordan set in $\mathbb{R}^n$; let $M \subset C$ a set of measure 0. Let $g\colon C \to D$ be continuously differentiable with the following properties*
(i) *g is injective on $C \setminus M$.*
(ii) *$g'(x)$ is regular on $C \setminus M$.*



Let $f\colon D \to \mathbb{R}$ be continuous.
Then

$$\int_D f(y)\,\mathrm{d}y = \int_C f(g(x)) \left| \frac{\partial(g_1,\dots,g_n)}{\partial(x_1,\dots,x_n)}(x) \right| \mathrm{d}x. \qquad (9.1)$$

Remark 9.4 Why the *absolute value* of the Jacobian? In $\mathbb{R}^1$ we don't have the absolute value. But in contrast to $\mathbb{R}^n$, $n \geq 1$, we have an orientation of the integration set $\int_a^b f\,\mathrm{d}x = -\int_b^a f\,\mathrm{d}x$.

For the proof see [12, 10.9 Theorem]. The main steps of the proof are: 1) In a small open set g can be written as the composition of n "flips" and n "primitive mappings". A flip changes two variables x_i and x_k, wheras a primitive mapping H is equal to the identity except for one variable, $H(x) = x + (h(x) - x)e_m$ where $h\colon U \to \mathbb{R}$.
2) If the statement is true for transformations S and T, then it is true for the composition $S\circ T$ which follows from $\det(AB) = \det A \det B$.
3) Use a partition of unity.

Example 9.3 (a) Polar coordinates. Let $A = \{(r, \varphi) \mid 0 \le r \le R,\ 0 \le \varphi < 2\pi\}$ be a rectangle in polar coordinates. The mapping $g(r, \varphi) = (x, y)$, $x = r\cos\varphi$, $y = r\sin\varphi$ maps this rectangle continuously differentiable onto the disc D with radius R. Let $M = \{(r, \varphi) \mid r = 0\}$. Since $\frac{\partial(x,y)}{\partial(r,\varphi)} = r$, the map g is bijective and regular on $R \setminus M$. The assumptions of the theorem are satisfied and we have

$$\begin{aligned}\iint\limits_D f(x, y)\,\mathrm{d}x\mathrm{d}y &= \iint\limits_A f(r\cos\varphi, r\sin\varphi) r\mathrm{d}r\mathrm{d}\varphi \\ &\underset{\text{Fubini}}{=} \int_0^R \int_0^{2\pi} f(r\cos\varphi, r\sin\varphi) r\,\mathrm{d}r\mathrm{d}\varphi.\end{aligned}$$

(b) Spherical coordinates. Recall from the exercise class the spherical coordinates $r \in [0, \infty)$, $\varphi \in [0, 2\pi]$, and $\vartheta \in [0, \pi]$

$$\begin{aligned}x &= r\sin\vartheta\cos\varphi, \\ y &= r\sin\vartheta\sin\varphi, \\ z &= r\cos\vartheta.\end{aligned}$$

The Jacobian reads

$$\frac{\partial(x, y, z)}{\partial(r, \vartheta, \varphi)} = \begin{vmatrix} x_r & x_\vartheta & x_\varphi \\ y_r & y_\vartheta & y_\varphi \\ z_r & z_\vartheta & z_\varphi \end{vmatrix} = \begin{vmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{vmatrix} = r^2\sin\vartheta$$

Sometimes one uses

$$\frac{\partial(x, y, z)}{\partial(r, \varphi, \vartheta)} = -r^2\sin\vartheta.$$

Hence

$$\iiint\limits_{B_1} f(x, y, z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \int_0^{2\pi} \int_0^\pi f(x, y, z)\sin^2\vartheta\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\vartheta.$$

Compute the volume of the ellipsoid E given by $u^2/a^2 + v^2/b^2 + w^2/c^2 = 1$. We use scaled spherical coordinates:

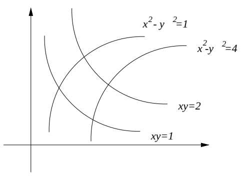
$$\begin{aligned}u &= ar\sin\vartheta\cos\varphi, \\ v &= br\sin\vartheta\sin\varphi, \\ w &= cr\cos\vartheta,\end{aligned}$$

where $r \in [0,1]$, $\vartheta \in [0,\pi]$, $\varphi \in 0, 2\pi]$. Since the rows of the spherical Jacobian matrix $\frac{\partial(x,y,z)}{\partial(r,\vartheta,\varphi)}$ are simply multiplied by a, b, and c, respectively, we have

$$\frac{\partial(u,v,w)}{\partial(r,\vartheta,\varphi)} = abcr^2 \sin\vartheta.$$

Hence, if B_1 is the unit ball around 0 we have using iterated integrals

$$\begin{aligned} v(E) &= \iiint\limits_E \mathrm{d}u\mathrm{d}v\mathrm{d}w = abc \iiint\limits_{B_1} r^2 \sin\vartheta \, \mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\varphi \\ &= abc \int_0^1 \mathrm{d}r r^2 \int_0^{2\pi} \mathrm{d}\varphi \int_0^\pi \sin\vartheta \mathrm{d}\vartheta \\ &= \frac{1}{3} abc \, 2\pi \, \left(-\cos\vartheta\right)\big|_0^\pi = \frac{4\pi}{3} abc. \end{aligned}$$



(c) $\iint\limits_C (x^2+y^2)\,\mathrm{d}x\mathrm{d}y$ where C is bounded by the four hyperbolas $xy = 1$, $xy = 2$, $x^2 - y^2 = 1$, $x^2 - y^2 = 4$.
We change coordinates $g(x,y) = (u,v)$

$$u = xy, \quad v = x^2 - y^2.$$

The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = -2(x^2+y^2).$$

The Jacobian of the inverse transform is

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2(x^2+y^2)}.$$

In the (u,v)-plane, the region is a rectangle $D = \{(u,v) \in \mathbb{R}^2 \mid 1 \le u \le 2,\ 1 \le v \le 4\}$. Hence,

$$\iint\limits_C (x^2+y^2)\,\mathrm{d}x\mathrm{d}y = \iint\limits_D (x^2+y^2) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\mathrm{d}v = \iint\limits_D \frac{x^2+y^2}{2(x^2+y^2)} \mathrm{d}u\mathrm{d}v = \frac{1}{2} v(D) = \frac{3}{2}.$$

Physical Applications

If $\varrho(x) = \rho(x_1, x_2, x_3)$ is a mass density of a solid $C \subset \mathbb{R}^3$, then

$$m = \int_C \rho \,\mathrm{d}x \quad \text{is the mass of } C \text{ and}$$

$$\overline{x_i} = \frac{1}{m} \int_C x_i\, \rho(x)\, \mathrm{d}x,\ i = 1, \dots, 3 \quad \text{are the coordinates of the mass center } \overline{x} \text{ of } C.$$

The *moments of inertia* of C are defined as follows

$$I_{xx} = \iiint\limits_C (y^2+z^2)\rho\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, I_{yy} = \iiint\limits_C (x^2+z^2)\rho\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, I_{zz} = \iiint\limits_C (x^2+y^2)\rho\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

$$I_{xy} = \iiint\limits_C xy\rho\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, \qquad I_{xz} = \iiint\limits_C xz\rho\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, \qquad I_{yz} = \iiint\limits_C yz\rho\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Here I_{xx}, I_{yy}, and I_{zz} are the moments of inertia of the solid with respect to the x-axis, y-axis, and z-axis, respectively.

Example 9.4 Compute the mass center of a homogeneous half-plate of radius R, $C = \{(x, y) \mid x^2 + y^2 \le R^2,\ y \ge 0\}$.
Solution. By the symmetry of C with respect to the y-axis, $\overline{x} = 0$. Using polar coordinates we find

$$\overline{y} = \frac{1}{m} \iint\limits_C y \,\mathrm{d}x\mathrm{d}y = \frac{1}{m} \int_0^R \int_0^\pi r \sin\varphi r \mathrm{d}\varphi \,\mathrm{d}r = \frac{1}{m} \int_0^R r^2 \,\mathrm{d}r \,(-\cos\varphi)\big|_0^\pi = \frac{1}{m}\frac{2R^3}{3}.$$

Since the mass is proportional to the area, $m = \pi\frac{R^2}{2}$ and we find $(0, \frac{4R}{3\pi})$ is the mass center of the half-plate.

Q 25. Compute the volume of the region bounded by the two surfaces $z = x^2 + y^2$ and $x^2 + y^2 + z^2 = 2$.
Q 26. Compute the volume of the region which is bounded by the 5 surfaces $\mid x \mid + \mid y \mid = a$ and $x^2 + z^2 = a^2$, $a > 0$.

9.4 Surface integrals

9.4.1 Surfaces in $\mathbb{R}^3$

A *domain* G is an open and *connected* subset in $\mathbb{R}^n$; connected means that for any two points x and y in G, there exist points $x_0, x_1, \ldots, x_k$ with $x_0 = x$ and $x_k = y$ such that every segment $\overline{x_{i-1}x_i}$, $i = 1, \ldots, k$, is completely contained in G.

Definition 9.5 Let $G \subset \mathbb{R}^2$ be a domain and $F\colon G \to \mathbb{R}^3$ continuously differentiable. The mapping F as well as the set $\mathcal{F} = F(G) = \{F(s,t) \mid (s,t) \in G\}$ is called an *open regular surface* if the Jacobian matrix $F'(s,t)$ has rank 2 for all $(s,t) \in G$.

If

$$F(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix},$$

the Jacobian matrix of F is

$$F'(s,t) = \begin{pmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{pmatrix}.$$

The two column vectors of $F'(s,t)$ span the tangent plane to F at (s,t):

$$D_1F(s,t) = \left(\frac{\partial x}{\partial s}(s,t), \frac{\partial y}{\partial s}(s,t), \frac{\partial z}{\partial s}(s,t)\right),$$
$$D_2F(s,t) = \left(\frac{\partial x}{\partial t}(s,t), \frac{\partial y}{\partial t}(s,t), \frac{\partial z}{\partial t}(s,t)\right).$$

Justification: Suppose $(s, t_0) \in G$ where t_0 is fixed. Then $\gamma(s) = F(s, t_0)$ defines a curve in $\mathfrak{F}$ with tangent vector $\gamma'(s) = D_1F(s, t_0)$. Similarly, for fixed s_0 we obtain another curve $\tilde{\gamma}(t) = F(s_0, t)$ with tangent vector $\tilde{\gamma}'(t) = D_2F(s_0, t)$. Since $F'(s, t)$ has rank 2 at every point of G, the vectors D_1F and D_2F are linearly independent; hence they span a plane.

Definition 9.6 Let $F\colon G \to \mathbb{R}^3$ be an open regular surface, and $(s_0, t_0) \in G$. Then

$$\vec{x} = F(s_0, t_0) + \alpha D_1F(s_0, t_0) + \beta D_2F(s_0, t_0), \quad \alpha, \beta \in \mathbb{R}$$

is called the *tangent plane* E to F at $F(s_0, t_0)$. The line through $F(s_0, t_0)$ which is orthogonal to E is called the *normal line* to F at $F(s_0, t_0)$.

Recall that the vector product $\vec{x} \times \vec{y}$ of vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ from $\mathbb{R}^3$ is the vector

$$\vec{x} \times \vec{y} = \begin{vmatrix} \mathrm{e}_1 & \mathrm{e}_2 & \mathrm{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

It is orthogonal to the plane spanned by the parallelogram P with edges $\vec{x}$ and $\vec{y}$. Its length is the area of the parallelogram P.
A vector which points in the direction of the normal line is

$$D_1F(s_0, t_0) \times D_2F(s_0, t_0) = \begin{vmatrix} \mathrm{e}_1 & \mathrm{e}_2 & \mathrm{e}_3 \\ x_s & y_s & z_s \\ x_t & y_t & z_t \end{vmatrix} \tag{9.2}$$

$$\vec{n} = \pm \frac{D_1F \times D_2F}{\|D_1F \times D_2F\|}, \tag{9.3}$$

where $\vec{n}$ is a unit vector in the direction of the normal line.

Example 9.5 Let F be given by the graph of a function $f\colon G \to \mathbb{R}$, namely $F(x, y) = (x, y, f(x, y))$. By definition

$$D_1F = (1, 0, f_x), \quad D_2F = (0, 1, f_y),$$

hence

$$D_1f \times D_2f = \begin{vmatrix} \mathrm{e}_1 & \mathrm{e}_2 & \mathrm{e}_3 \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1).$$

Therefore, the tangent plane has the equation

$$-f_x(x - x_0) - f_y(y - y_0) + 1(z - z_0) = 0.$$

Further, the unit normal vector to the tangent plane is

$$\vec{n} = \pm \frac{(f_x, f_y, -1)}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

Recall that the tangent plane to the graph of f at $(x_0, y_0, f(x_0, y_0))$ is

$$(x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) = z - z_0.$$

9.4.2 The Area of a Surface

Let F and $\mathfrak{F}$ be as above. We assume that the continuous vector fields D_1F and D_2F on G can be extended to continuous functions on the closure $\overline{G}$.

Definition 9.7 The number

$$|\,\mathfrak{F}\,| = |\,\overline{\mathfrak{F}}\,| := \iint\limits_{G} \|D_1F \times D_2F\| \,\mathrm{d}s\mathrm{d}t \tag{9.4}$$

is called the *area* of $\mathfrak{F}$ and of $\mathfrak{F}$.
We call

$$\mathrm{d}S = \|D_1F \times D_2F\| \,\mathrm{d}s\mathrm{d}t$$

the scalar surface element of F.

9.4.3 Scalar Surface Integrals

Let $\mathfrak{F}$ and $\overline{\mathfrak{F}}$ be as above, and $f\colon \overline{\mathfrak{F}} \to \mathbb{R}$ a continuous function on the compact subset $\overline{\mathfrak{F}} \subset \mathbb{R}^3$.

Definition 9.8 The number

$$\iint\limits_{\overline{\mathfrak{F}}} f(\vec{x})\,\mathrm{d}S := \iint\limits_{\overline{G}} f(F(s,t))\,\|D_1F(s,t) \times D_2(s,t)\| \,\mathrm{d}s\mathrm{d}t$$

is called the *scalar surface integral of f on $\overline{\mathfrak{F}}$*.

Physical Application

(a) If $\rho(x,y,z)$ is the mass density on a surface $\overline{\mathfrak{F}}$, $\iint\limits_{\overline{\mathfrak{F}}} \rho\,\mathrm{d}S$ is the total mass of $\mathfrak{F}$.

(b) If $\sigma(\vec{x})$ is a charge density on a surface $\mathfrak{F}$. Then

$$U(\vec{y}) = \iint\limits_{\mathfrak{F}} \frac{\sigma(\vec{x})}{\|\vec{y}-\vec{x}\|}\,\mathrm{d}S(\vec{x}), \quad \vec{y} \notin \mathfrak{F}$$

is the potential generated by $\mathfrak{F}$.

Proposition 9.8 *Let $M = \{(x,y,z) \in \mathbb{R}^3 \mid \rho \le \|(x,y,z)\| \le R\}$ where $R > \rho \ge 0$. Let $f\colon M \to \mathbb{R}$ be continuous. Put $S_r = \{(x,y,z) \in \mathbb{R}^3 \mid \|(x,y,z)\| = r\}$. Then*

$$\iiint\limits_{M} f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_\rho^R \mathrm{d}r \left(\iint\limits_{S_r} f(\vec{x})\,\mathrm{d}S\right) = \int_\rho^R r^2 \left(\iint\limits_{S_1} f(r\vec{x})\,\mathrm{d}S(\vec{x})\right) \mathrm{d}r.$$

Other Forms for dS

If $\mathcal{F}$ is given by $F(s,t)$ we have

$$\mathrm{d}S = \sqrt{EG - F^2}\,\mathrm{d}s\mathrm{d}t,$$

where

$$E = x_s^2 + y_s^2 + z_s^2, \quad G = x_t^2 + y_t^2 + z_t^2, \quad F = x_s x_t + y_s y_t + z_s z_t.$$

Let the surface be given implicitly as $F(x,y,z) = 0$. Suppose F is locally solvable for z in a neighborhood of some point (x_0, y_0, z_0). Then the surface element (up to the sign) is given by

$$\mathrm{d}S = \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|\,F_z\,|}\,\mathrm{d}x\mathrm{d}y = \frac{\|\,\mathrm{grad}\,F\|}{|\,F_z\,|}\,\mathrm{d}x\mathrm{d}y.$$

One checks that $DF_1 \times DF_2 = (F_x, F_y, F_z)/F_z$.
Let the surface be given as the graph of a function $(x, y, f(x,y))$. Then

$$\mathrm{d}S = \sqrt{1 + f_x^2 + f_y^2}\,\mathrm{d}x\mathrm{d}y.$$

Example 9.6 (a) We give two different form for the scalar surface element of a sphere. By the previous example, the sphere $x^2 + y^2 + z^2 = R^2$ has surface element

$$\mathrm{d}S = \frac{\|(2x, 2y, 2z)\|}{2z}\,\mathrm{d}x\mathrm{d}y = \frac{R}{z}\,\mathrm{d}x\mathrm{d}y.$$

If

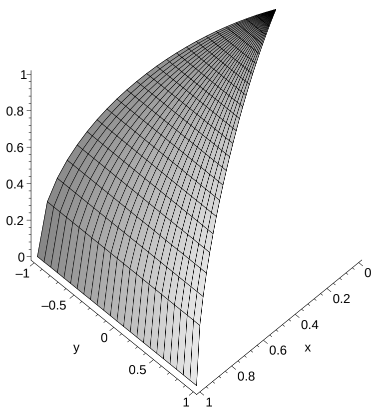
$$x = R\cos\varphi\sin\vartheta, \quad y = R\sin\varphi\sin\vartheta, \quad z = R\cos\vartheta,$$

we obtain

$$\begin{aligned}
D_1 = F_\vartheta &= R(\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, -\sin\vartheta),\\
D_2 = F_\varphi &= R(-\sin\varphi\sin\vartheta, \cos\varphi\sin\vartheta, 0),\\
D_1 \times D_2 &= R^2(\cos\varphi\sin^2\vartheta, \sin\varphi\sin^2\vartheta, \sin\vartheta\cos\vartheta).
\end{aligned}$$

Hence,

$$\mathrm{d}S = \|D_1 \times D_2\|\,\mathrm{d}\vartheta\mathrm{d}\varphi = R^2\sin\vartheta\mathrm{d}\vartheta\mathrm{d}\varphi.$$



(b) Compute the area of the solid which is bounded by the two cylinders $x^2 + z^2 = a^2$ and $y^2 + z^2 = a^2$. The main problem is to get an idea of the geometry of the solid with this boundary. In the figure there is 1/8 of the surface with $a = 1$. This $\mathcal{F}$ can be parametrized as follows

Part of $x^2 + z^2 = a^2$.

$$\overline{\mathcal{F}} = \{(x, y, f(x,y)) \mid 0 \le x \le a,\ -x \le y \le x,\ z = f(x,y) = \sqrt{a^2 - x^2}\}.$$

Since $\mathcal{F}$ is the graph of the function $f(x,y)=\sqrt{a^2-x^2}$,

$$\mathrm{d}S=\sqrt{1+f_x^2+f_y^2}\,\mathrm{d}x\mathrm{d}y=\sqrt{1+\frac{x^2}{a^2-x^2}+0}\,\mathrm{d}x\mathrm{d}y=\frac{a}{\sqrt{a^2-x^2}}\,\mathrm{d}x\mathrm{d}y.$$

Thus

$$\begin{aligned}\operatorname{area}(\overline{\mathcal{F}})&=\iint_{\overline{\mathcal{F}}}\mathrm{d}S=\int_0^a\mathrm{d}x\int_{-x}^{x}\frac{a}{\sqrt{a^2-x^2}}\,\mathrm{d}y=\int_0^a\frac{2ax\,\mathrm{d}x}{\sqrt{a^2-x^2}}\\&\underset{t=x^2,\,\mathrm{d}t=2x\,\mathrm{d}x}{=}a\int_0^{a^2}\frac{\mathrm{d}t}{\sqrt{a^2-t}}=-2a\sqrt{a^2-t^2}\Big|_0^{a^2}=2a^2.\end{aligned}$$

The area of the surface is $16a^2$.

Guldin's Rule—the Area of a Surfaces

(Paul Guldin, 1577–1643, Swiss Mathematician) Let f be a continuously differentiable function on $[a,b]$ with $f(x)\geq 0$ for all $x\in[a,b]$. Let the graph of f revolve around the x-axis and let $\overline{\mathcal{F}}$ be the corresponding surface. We have

$$v(\overline{\mathcal{F}})=2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,\mathrm{d}x.$$

Proof. Using polar coordinates in the y-z-plane, we obtain a parametrization of $\mathcal{F}$

$$\overline{\mathcal{F}}=\{(x,f(x)\cos\varphi,f(x)\sin\varphi)\mid x\in[a,b],\varphi\in[0,2\pi]\}.$$

We have

$$\begin{aligned}D_1F&=(1,f'(x)\cos\varphi,f'(x)\sin\varphi),\quad D_1F=(0,-f\sin\varphi,f\cos\varphi),\\D_1F\times D_2F&=(ff',-f\cos\varphi,-f\sin\varphi);\end{aligned}$$

so that $\mathrm{d}S=f(x)\sqrt{1+f'(x)^2}\,\mathrm{d}x\mathrm{d}\varphi$. Hence

$$v(\overline{\mathcal{F}})=\int_a^b\int_0^{2\pi}f(x)\sqrt{1+f'(x)^2}\mathrm{d}\varphi\,\mathrm{d}x=2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,\mathrm{d}x.$$

∎

We have recovered the formula from Subsection 5.6.2.

9.4.4 Surface Integrals

Orientation

We want to define the notion of *orientation* for a regular surface. Let $\mathcal{F}$ be a regular (injective) surface with or without boundary. Then for every point $x_0\in\mathcal{F}$ there exists the tangent plane E_{x_0}; the normal line to $\mathcal{F}$ at x_0 is uniquely defined.
However, a unit vector on the normal line can have two different directions.

Definition 9.9 (a) Let $\mathcal{F}$ be a surface as above. A *unit normal field* to $\mathcal{F}$ is a continuous function $\vec{n}\colon \mathcal{F} \to \mathbb{R}^n$ with the following two properties for every $x_0 \in \mathcal{F}$

(i) $\vec{n}(x_0)$ is orthogonal to the tangent plane to $\mathcal{F}$ at x_0.
(ii) $\|\vec{n}(x_0)\| = 1$.

(b) A regular surface $\mathcal{F}$ is called *orientable*, if there exists a unit normal field on $\mathcal{F}$.

It turns out that for a regular surface $\mathcal{F}$ there either exists exactly two unit normal fields or there is no normal field. Examples of non-orientable surfaces are the *Möbius band* and the *real projective plane*. Analytically the Möbius band is given by

$$F(s,t) = \begin{pmatrix} \left(1 + t\cos\frac{s}{2}\right)\sin s \\ \left(1 + t\cos\frac{s}{2}\right)\cos s \\ t\sin\frac{s}{2} \end{pmatrix}, \quad (s,t) \in [0, 2\pi] \times \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Suppose $\mathcal{F}$ is an oriented, open, regular surface with piecewise smooth boundary $\partial\mathcal{F}$. Let $F(s,t)$ be a parametrization of $\mathcal{F}$. We assume that the vector functions F, DF_1, and DF_2 can be extended to continuous functions on $\overline{\mathcal{F}}$. The unit normal vector is given by

$$\vec{n} = \varepsilon \frac{D_1F \times D_2F}{\|D_1F \times D_2F\|},$$

where $\varepsilon = +1$ or $\varepsilon = -1$ fixes the *orientation* of $\mathcal{F}$.

Definition 9.10 Let $\vec{f}\colon \overline{\mathcal{F}} \to \mathbb{R}^3$ be a continuous vector field on $\overline{\mathcal{F}}$. The number

$$\iint\limits_{\mathcal{F}} \vec{f}(\vec{x}) \cdot \vec{n}\, \mathrm{d}S \tag{9.5}$$

is called the *surface integral of the vector field $\vec{f}$ on $\overline{\mathcal{F}}$*. We call

$$\mathrm{d}\vec{S} = \vec{n}\,\mathrm{d}S = \varepsilon\, D_1F \times D_2F\, \mathrm{d}s\mathrm{d}t$$

the surface element of $\mathcal{F}$.

Remark 9.5 (a) The surface integral is independently of the parametrization of $\mathcal{F}$ but depends on the orientation. For, let $(s,t) = (s(\xi,\eta), t(\xi,\eta))$ be a new parametrization with $F(s(\xi,\eta), t(\xi,\eta)) = G(\xi,\eta)$. Then the Jacobian is

$$\mathrm{d}s\mathrm{d}t = \frac{\partial(s,t)}{\partial(\xi,\eta)} = (s_\xi t_\eta - s_\eta t_\xi)\,\mathrm{d}\xi\mathrm{d}\eta.$$

Further

$$D_1G = D_1F\, s_\xi + D_2F\, t_\xi, \quad D_2G = D_1F\, s_\eta + D_2F t_\eta,$$

so that using $\vec{x}\times\vec{x}=0$, $\vec{x}\times\vec{y}=-\vec{y}\times\vec{x}$

$$\begin{aligned} D_1G\times D_2G\mathrm{d}\xi\mathrm{d}\eta &= (D_1F\,s_\xi + D_2F\,t_\xi)\times(D_1F\,s_\eta + D_2Ft_\eta)\mathrm{d}\xi\mathrm{d}\eta,\\ &= (s_\xi t_\eta - s_\eta t_\xi)D_1F\times D_2F\mathrm{d}\xi\mathrm{d}\eta\\ &= D_1F\times D_2F\,\mathrm{d}s\mathrm{d}t. \end{aligned}$$

(b) The scalar surface integral $\iint f\,\mathrm{d}S$ is a special case of the surface integral, namely $\iint f\vec{n}\cdot\vec{n}\,\mathrm{d}S$.
(c) Special cases. Let $\mathfrak{F}$ be the graph of a function f, $\mathfrak{F}=\{(x,y,f(x,y))\mid (x,y)\in C\}$, then

$$\mathrm{d}\vec{S}=\pm(f_x,f_y,1)\,\mathrm{d}x\mathrm{d}y.$$

If the surface is given implicitly by $F(x,y,z)=0$ and it is locally solvable for z, then

$$\mathrm{d}\vec{S}=\pm\frac{\operatorname{grad}F}{F_z}.$$

(d) Still another form of $\mathrm{d}\vec{S}$.

$$\iint\limits_{\mathfrak{F}}\vec{f}\,\mathrm{d}\vec{S}=\varepsilon\iint\limits_{G} f(F(s,t))\cdot(D_1F\times D_2F)\,\mathrm{d}s\mathrm{d}t$$

$$\vec{f}(F(s,t))\cdot(D_1F\times D_2F)=\begin{vmatrix} f_1(F(s,t)) & f_2(F(s,t)) & f_3(F(s,t))\\ x_s(s,t) & y_s(s,t) & z_s(s,t)\\ x_t(s,t) & y_t(s,t) & z_t(s,t)\end{vmatrix}. \tag{9.6}$$

(e) Again another notation. Computing the previous determinant or the determinant (9.2) explicitely we have

$$\vec{f}\cdot(D_1F\times D_2F)=f_1\begin{vmatrix} y_s & z_s\\ y_t & z_t\end{vmatrix}+f_2\begin{vmatrix} z_s & x_s\\ z_t & x_t\end{vmatrix}+f_3\begin{vmatrix} x_s & y_s\\ x_t & y_t\end{vmatrix}=f_1\frac{\partial(y,z)}{\partial(s,t)}+f_2\frac{\partial(z,x)}{\partial(s,t)}+f_3\frac{\partial(x,y)}{\partial(s,t)}.$$

Hence,

$$\begin{aligned} \mathrm{d}\vec{S} &= D_1F\times D_2F\,\mathrm{d}s\mathrm{d}t=\left(\frac{\partial(y,z)}{\partial(s,t)}\,\mathrm{d}s\mathrm{d}t,\ \frac{\partial(z,x)}{\partial(s,t)}\,\mathrm{d}s\mathrm{d}t,\ \frac{\partial(x,y)}{\partial(s,t)}\,\mathrm{d}s\mathrm{d}t\right)\\ \mathrm{d}\vec{S} &= (\,\mathrm{d}y\mathrm{d}z,\ \mathrm{d}z\mathrm{d}x,\ \mathrm{d}x\mathrm{d}y\,). \end{aligned}$$

Therefore we can write

$$\iint\limits_{\mathfrak{F}}\vec{f}\cdot\,\mathrm{d}\vec{S}=\iint\limits_{\mathfrak{F}}(f_1\,\mathrm{d}y\mathrm{d}z+f_2\,\mathrm{d}z\mathrm{d}x+f_3\,\mathrm{d}x\mathrm{d}y).$$

In this setting

$$\iint\limits_{\mathfrak{F}} f_1\,\mathrm{d}y\mathrm{d}z=\iint\limits_{\mathfrak{F}}(f_1,0,0)\cdot\,\mathrm{d}\vec{S}=\pm\iint\limits_{G} f_1(F(s,t))\frac{\partial(y,z)}{\partial(s,t)}\,\mathrm{d}s\mathrm{d}t.$$

Sometimes one uses

$$\mathrm{d}\vec{S} = (\cos(\vec{n}, \mathrm{e}_1), \cos(\vec{n}, \mathrm{e}_2), \cos(\vec{n}, \mathrm{e}_3))\,\mathrm{d}S,$$

since $\cos(\vec{n}, \mathrm{e}_i) = \langle \vec{n}, \mathrm{e}_i \rangle = n_i$ and $\mathrm{d}\vec{S} = \vec{n}\,\mathrm{d}S$.
Note that we have surface integrals in the last two lines, not ordinary double integrals since $\mathcal{F}$ is a surface in $\mathbb{R}^3$ and $f_1 = f_1(x, y, z)$ can also depend on x.
The physical meaning of $\iint_{\mathcal{F}} \vec{f} \cdot \mathrm{d}\vec{S}$ is the flow of the vector field $\vec{f}$ through the surface $\mathcal{F}$. The flow is (locally) positive if $\vec{n}$ and $\vec{f}$ are on the same side of the tangent plane to $\mathcal{F}$ and negative in the other case.

Example 9.7 (a) Compute the surface integral

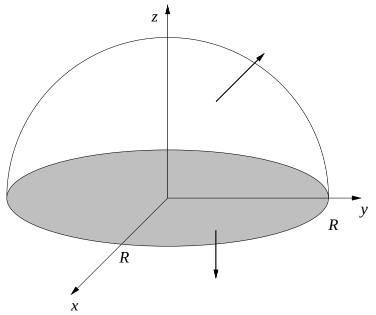
$$\iint\limits_{\mathcal{F}} f\,\mathrm{d}z\mathrm{d}x$$

of $f(x, y, z) = x^2yz$ where $\mathcal{F}$ is the graph of $F(x, y) = x^2 + y$ over the square $Q = [0, 1] \times [0, 1]$ with the downward directed unit normal field.
By Remark 9.5 (c)

$$\mathrm{d}\vec{S} = (F_x, F_y, -1)\,\mathrm{d}x\mathrm{d}y = (2x, 1, -1)\,\mathrm{d}x\mathrm{d}y.$$

Hence

$$\begin{aligned}\iint\limits_{\mathcal{F}} f\,\mathrm{d}z\mathrm{d}x &= \iint\limits_{\mathcal{F}} (0, f, 0) \cdot \mathrm{d}\vec{S} \\ &= (R)\iint\limits_{Q} x^2y(x^2 + y)\,\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^1 (x^4y + x^2y^2)\,\mathrm{d}y = \frac{19}{90}.\end{aligned}$$



(b) Let G denote the upper half ball of radius R in $\mathbb{R}^3$:

$$G = \{(x, y, z) \mid x^2 + y^2 + z^2 \le R^2, \quad z \ge 0\},$$

and let $\mathcal{F}$ be the boundary of G with the orientation of the outer normal. Then $\mathcal{F}$ consists of the upper half sphere $\mathcal{F}_1$

$$\mathcal{F}_1 = \{(x, y, \sqrt{R^2 - x^2 - z^2}) \mid x^2 + y^2 \le R^2\}$$

with the upper orientation of the unit normal field and of the disc $\mathcal{F}_2$ in the x-y-plane

$$\mathcal{F}_2 = \{(x, y, 0) \mid x^2 + y^2 \le R^2\}$$

with the downward directed normal. Let $\vec{f}(x, y, z) = (ax, by, cz)$. We want to compute

$$\iint\limits_{\mathcal{F}} \vec{f} \cdot \mathrm{d}\vec{S}.$$

By Remark 9.5 (c), the surface element of the half-sphere $\mathcal{F}_1$ is $\mathrm{d}\vec{S} = \frac{1}{z}(x, y, z)\,\mathrm{d}x\mathrm{d}y$. Hence

$$I_1 = \iint\limits_{\mathcal{F}_1} \vec{f} \cdot \mathrm{d}\vec{S} = \iint\limits_{B_R} (ax, by, cz) \cdot \frac{1}{z}(x, y, z)\,\mathrm{d}x\mathrm{d}y = \iint\limits_{B_R} \frac{1}{z}(ax^2 + by^2 + cz^2)\,\mathrm{d}x\mathrm{d}y.$$

Using polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, $r \in [0, R]$, and $z = \sqrt{R^2 - x^2 - y^2} = \sqrt{R^2 - r^2}$ we get

$$I_1 = \int_0^{2\pi} \mathrm{d}\varphi \int_0^R \frac{ar^2\cos^2\varphi + br^2\sin^2\varphi + c(R^2 - r^2)}{\sqrt{R^2 - r^2}} r\mathrm{d}r.$$

Noting $\int_0^{2\pi} \sin^2\varphi \mathrm{d}\varphi = \int_0^{2\pi} \cos^2\varphi \mathrm{d}\varphi = \pi$ we continue

$$I_1 = \pi \int_0^r \left(\frac{ar^3}{\sqrt{R^2 - r^2}} + \frac{br^3}{\sqrt{R^2 - r^2}} + 2cr\sqrt{R^2 - r^2} \right) \mathrm{d}r.$$

Using $r = R\sin t$, $\mathrm{d}r = R\cos t\,\mathrm{d}t$ we have

$$\int_0^R \frac{r^3}{\sqrt{R^2 - r^2}} \mathrm{d}r = \int_0^{\frac{\pi}{2}} \frac{R^3 \sin^3 t R\cos t\,\mathrm{d}t}{R\sqrt{1 - \sin^2 t}} = R^3 \int_0^{\frac{\pi}{2}} \sin^3 t\,\mathrm{d}t = \frac{2}{3}R^3.$$

Hence,

$$\begin{aligned} I_1 &= \frac{2\pi}{3}R^3(a + b) + \pi c \int_0^R (R^2 - r^2)^{\frac{1}{2}} \mathrm{d}(r^2) \\ &= \frac{2\pi}{3}R^3(a + b) + \pi c - \frac{2}{3}(R^2 - r^2)^{\frac{3}{2}} \Big|_0^R \\ &= \frac{2\pi}{3}R^3(a + b + c). \end{aligned}$$

In case of the disc $\mathcal{F}_2$ we have $z = f(x, y) = 0$, such that $f_x = f_y = 0$ and

$$\mathrm{d}\vec{S} = (0, 0, -1)\,\mathrm{d}x\mathrm{d}y$$

by Remark 9.5 (c). Hence

$$\iint\limits_{\mathcal{F}_2} \vec{f} \cdot \mathrm{d}\vec{S} = \iint\limits_{B_R} (ax, by, cz) \cdot (0, 0, -1)\,\mathrm{d}x\mathrm{d}y = -c \iint\limits_{B_R} z\,\mathrm{d}x\mathrm{d}y \underset{z=0}{=} 0.$$

Hence,

$$\iint\limits_{\mathcal{F}} (ax, by, cz) \cdot \mathrm{d}\vec{S} = \frac{2\pi}{3}R^3(a + b + c).$$

9.4.5 Gauß' Divergence Theorem

The aim is to generalize the fundamental theorem of calculus to higher dimensions:

$$\int_a^b g'(x)\,\mathrm{d}x = g(b) - g(a).$$

Note that a and b form the boundary of the segment $[a, b]$. There are three possibilities to do this

$$\begin{array}{lcll} \iiint\limits_G f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z & \Longrightarrow & \iint\limits_{\partial G} \vec{g}\cdot \mathrm{d}\vec{S} & \text{Gauß' theorem in } \mathbb{R}^3, \\ \iint\limits_G f\,\mathrm{d}x\mathrm{d}y & \Longrightarrow & \int\limits_{\partial G} \vec{g}\cdot \mathrm{d}\vec{x} & \text{Gauß' theorem in } \mathbb{R}^2, \\ \iint\limits_{\mathcal{F}} \vec{f}\cdot \mathrm{d}\vec{S} & \Longrightarrow & \int\limits_{\partial G} \vec{g}\cdot \mathrm{d}\vec{x} & \text{Stokes' theorem .} \end{array}$$

Let $G \subset \mathbb{R}^3$ be a bounded domain (open, connected) such that its boundary $\mathcal{F} = \partial G$ satisfies the following assumptions:

1. $\mathcal{F}$ is a union of regular, orientable surfaces $\mathcal{F}_i$. The parametrization $F_i(s,t)$, $(s,t) \in \overline{C_i}$, of $\mathcal{F}_i$ as well as D_1F_i and D_2F_i are continuous vector functions on $\overline{C_i}$; G is a domain in $\mathbb{R}^2$.
2. Let $\mathcal{F}_i$ be oriented by the **outer normal** (with respect to G).
3. There is given a continuously differentiable vector field $\vec{f}\colon \overline{G} \to \mathbb{R}^3$ on G (More precisely, there exist an open set $U \supset G$ and a continuously differentiable function $\tilde{f}\colon U \to \mathbb{R}^3$ such that $\tilde{f}\restriction\overline{G} = \vec{f}$.)

Theorem 9.9 (Gauß' Divergence Theorem) *Under the above assumptions we have*

$$\iiint\limits_{\overline{G}} \operatorname{div} \vec{f}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint\limits_{\partial G} \vec{f}\,\mathrm{d}\vec{S} \tag{9.7}$$

Sometimes the theorem is called Gauß–Ostrogadski theorem or simply Ostrogadski theorem in the russian literature.
Other writings:

$$\iiint\limits_{\overline{G}} \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint\limits_{\partial G} (f_1\,\mathrm{d}y\mathrm{d}z + f_2\,\mathrm{d}z\mathrm{d}x + f_3\,\mathrm{d}x\mathrm{d}y) \tag{9.8}$$

The theorem holds for more general regions $G \subset \mathbb{R}^3$.
Proof. We give a proof for

$$G = \{(x,y,z) \mid (x,y) \in C,\ \alpha(x,y) \le z \le \beta(x,y)\},$$

where $C \subset \mathbb{R}^2$ is a domain and $\alpha, \beta \in \mathrm{C}^1(C)$ define regular top and bottom surfaces $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$, respectively. We prove only one part of (9.8)

$$\iiint\limits_{\overline{G}} \frac{\partial f_3}{\partial z}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint\limits_{\partial G} f_3\,\mathrm{d}x\mathrm{d}y. \tag{9.9}$$

By Fubini's theorem, the left side reads

$$\iiint\limits_{\overline{G}} \frac{\partial f_3}{\partial x}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint\limits_{C} \left(\int_{\alpha(x,y)}^{\beta(x,y)} \frac{\partial f_3}{\partial z}\,\mathrm{d}z \right) \mathrm{d}x\mathrm{d}y$$
$$= \iint\limits_{C} \left(f_3(x,y,\beta(x,y)) - f_3(x,y,\alpha(x,y)) \right) \mathrm{d}x\mathrm{d}y, \qquad (9.10)$$

where the last equality is by the fundamental theorem of calculus.
Now we are going to compute the surface integral. The outer normal for the top surface is $(-\beta_x(x,y), -\beta_y(x,y), 1)$ such that

$$I_1 = \iint\limits_{\mathcal{F}_1} f_3\,\mathrm{d}x\mathrm{d}y = \iint\limits_{C} (0,0,f_3)\cdot(-\beta_x(x,y), -\beta_y(x,y), 1)\,\mathrm{d}x\mathrm{d}y$$
$$= \iint\limits_{C} f_3(x,y,\beta(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Since the bottom surface $\mathcal{F}_2$ is oriented downward, the outer normal is $(\alpha_x(x,y), \alpha_y(x,y), -1)$ such that

$$I_2 = \iint\limits_{\mathcal{F}_2} f_3\,\mathrm{d}x\mathrm{d}y = -\iint\limits_{C} f_3(x,y,\alpha(x,y))\,\mathrm{d}x\mathrm{d}y.$$

Finally, the shell $\mathcal{F}_3$ is parametrized by an angle φ and z:

$$\mathcal{F}_3 = \{(r(\varphi)\cos\varphi, r(\varphi)\sin\varphi, z) \mid \alpha(x,y) \le z \le \beta(x,y)\}.$$

Since $D_2F = (0,0,1)$, the normal vector is orthogonal to the z-axis, $\vec{n} = (n_1, n_2, 0)$. Therefore,

$$I_3 = \iint\limits_{\mathcal{F}_3} f_3\,\mathrm{d}x\mathrm{d}y = \iint\limits_{\mathcal{F}_3} (0,0,f_3)\cdot(n_1,n_2,0)\,\mathrm{d}S = 0.$$

Comparing $I_1 + I_2 + I_3$ with (9.10) proves the theorem in this special case. ∎

Remarks 9.6 (a) Gauß' divergence theorem can be used to compute the volume of the domain $G \subset \mathbb{R}^3$. Suppose the boundary ∂G of G has the orientation of the outer normal. Then

$$v(G) = \iint\limits_{\partial G} x\,\mathrm{d}y\mathrm{d}z = \iint\limits_{\partial G} y\,\mathrm{d}z\mathrm{d}x = \iint\limits_{\partial G} z\,\mathrm{d}x\mathrm{d}y.$$

(b) Applying the mean value theorem to the left-hand side of Gauß' formula we have for any bounded region G containing x_0

$$\operatorname{div}\vec{f}(x_0+h) \iiint\limits_{G} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \operatorname{div}\vec{f}(x_0+h)v(G) = \iint\limits_{\partial G} \vec{f}\,\mathrm{d}\vec{S},$$

where h is a small vector. The integral on the left is the volume $v(G)$. Hence

$$\operatorname{div} \vec{f}(x_0) = \lim_{G \to x_0} \frac{1}{v(G)} \iint_{\partial G} \vec{f} \, \mathrm{d}\vec{S},$$

where the region G tends to x_0. The right hand side can be thought as to be the *source density* of the field $\vec{f}$. In particular, the right side gives a basis independent description of $\operatorname{div} \vec{f}$.

9.5 Line Integrals

A lot of physical applications are to be found in [9, Chapter 18]. Integration of vector fields along curves is of fundamental importance in both mathematics and physics. We use the concept of *work* to motivate the material in this section.
The motion of an object is described by a parametric curve $\vec{x} = \vec{x}(t) = (x(t), y(t), z(t))$. By differentiating this function, we obtain the velocity $\vec{v}(t) = \vec{x}'(t)$ and the acceleration $\vec{a}(t) = \vec{x}''(t)$. We use the physicist notation $\dot{\vec{x}}(t)$ and $\ddot{\vec{x}}(t)$ to denote derivatives with respect to the time t.
According to Newton's law, the total force $\tilde{F}$ acting on an object of mass m is

$$\tilde{F} = m\vec{a}.$$

Since the kinetic energy K is defined by $K = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\vec{v} \cdot \vec{v}$ we have

$$\dot{K}(t) = \frac{1}{2}m(\dot{\vec{v}} \cdot \vec{v} + \vec{v} \cdot \dot{\vec{v}}) = m\vec{a} \cdot \vec{v} = \tilde{F} \cdot \vec{v}.$$

The total change of the kinetic energy from time t_1 to t_2, denoted W, is called the *work done by the force $\tilde{F}$ along the path $\vec{x}(t)$*:

$$W = \int_{t_1}^{t_2} \dot{K}(t) \, \mathrm{d}t = \int_{t_1}^{t_2} \tilde{F} \cdot \vec{v} \, \mathrm{d}t = \int_{t_1}^{t_2} \tilde{F}(t) \cdot \dot{\vec{x}}(t) \, \mathrm{d}t.$$

Let us now suppose that the force $\tilde{F}$ at time t depends only on the position $\vec{x}(t)$. That is, we assume that there is a vector field $\vec{F}(\vec{x})$ such that $\tilde{F}(t) = \vec{F}(\vec{x}(t))$ (gravitational and electrostatic attraction are position-dependent while magnetic forces are velocity-dependent). Then we may rewrite the above integral as

$$W = \int_{t_1}^{t_2} \vec{F}(\vec{x}(t)) \cdot \dot{\vec{x}}(t) \, \mathrm{d}t.$$

In the one-dimensional case, by a change of variables, this can be simplified to

$$W = \int_a^b F(x) \, \mathrm{d}x,$$

where a and b are the starting and ending positions.

Definition 9.11 Let $\Gamma = \{\vec{x}(t) \mid t \in [r,s]\}$, be a continuously differentiable curve $\vec{x}(t) \in \mathrm{C}^1([r,s])$ in $\mathbb{R}^n$ and $\vec{f}\colon \Gamma \to \mathbb{R}^n$ a continuous vector field on Γ. The integral

$$\int_\Gamma \vec{f}(\vec{x}) \cdot \mathrm{d}\vec{x} = \int_r^s \vec{f}(\vec{x}(t)) \cdot \dot{\vec{x}}(t)\,\mathrm{d}t$$

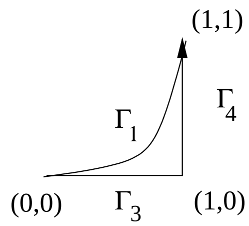
is called the *line integral* of the vector field $\vec{f}$ along the curve Γ.

Remark 9.7 (a) The definition of the line integral does not depend on the parametrization of Γ.
(b) If we take different curves between the same endpoints, the line integral may be different.
(c) If the vector field $\vec{f}$ is orthogonal to the tangent vector, then $\int_\Gamma \vec{f} \cdot \mathrm{d}\vec{x} = 0$.
(d) Other notations. If $\vec{f} = (P, Q)$ is a vector field in $\mathbb{R}^2$,

$$\int_\Gamma \vec{f} \cdot \mathrm{d}\vec{x} = \int_\Gamma P\,\mathrm{d}x + Q\,\mathrm{d}y,$$

where the right side is either a symbol or $\int_\Gamma P\,\mathrm{d}x = \int_\Gamma (P, 0) \cdot \mathrm{d}\vec{x}$.

Example 9.8 (a) Find the line integral $\int_{\Gamma_i} y\,\mathrm{d}x + (x-y)\,\mathrm{d}y$, $i = 1, 2$, where



$$\Gamma_1 = \{\vec{x}(t) = (t, t^2) \mid t \in [0,1]\} \quad \text{and} \quad \Gamma_2 = \Gamma_3 \cup \Gamma_4,$$

with $\Gamma_3 = \{(t,0) \mid t \in [0,1]\}$, $\Gamma_4 = \{(1,t) \mid t \in [0,1]\}$.
In the first case $\dot{\vec{x}}(t) = (1, 2t)$; hence

$$\int_\Gamma y\,\mathrm{d}x + (x-y)\,\mathrm{d}y = \int_0^1 (t^2 \cdot 1 + (t - t^2)2t)\,\mathrm{d}t = \int_0^1 (3t^2 - 2t^3)\,\mathrm{d}t = \frac{1}{2}.$$

In the second case $\int_\Gamma f\,\mathrm{d}\vec{x} = \int_{\Gamma_1} f\,\mathrm{d}\vec{x} + \int_{\Gamma_2} f\,\mathrm{d}\vec{x}$. For the first part $(\mathrm{d}x,\,\mathrm{d}y) = (\mathrm{d}t, 0)$, for the second part $(\mathrm{d}x,\,\mathrm{d}y) = (0,\,\mathrm{d}t)$ such that

$$\int_\Gamma f\,\mathrm{d}\vec{x} = \int_\Gamma y\,\mathrm{d}x + (x-y)\,\mathrm{d}y = \int_0^1 0\,\mathrm{d}t + (t-0)\cdot 0 + \int_0^1 t \cdot 0 + (1-t)\,\mathrm{d}t = t - \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}.$$

(b) Find the work done by the force field $\vec{F}(x,y,z) = (y, -x, 1)$ as a particle moves from $(1,0,0)$ to $(1,0,1)$ along the following paths $\varepsilon = \pm 1$:

$\vec{x}(t)_\varepsilon = (\cos t, \varepsilon \sin t, \frac{t}{2\pi})$, $t \in [0, 2\pi]$,

We find

$$\begin{aligned}\int_{\Gamma_\varepsilon} \vec{F} \cdot \mathrm{d}\vec{x} &= \int_0^{2\pi} (\varepsilon \sin t, -\cos t, 1) \cdot (-\sin t, \varepsilon \cos t, 1/(2\pi))\,\mathrm{d}t \\ &= \int_0^{2\pi} \left(-\varepsilon \sin^2 t - \varepsilon \cos^2 t + \frac{1}{2\pi}\right)\,\mathrm{d}t \\ &= -2\pi\varepsilon + 1.\end{aligned}$$

In case $\varepsilon = 1$, the motion is "with the force", so the work is positive; for the path $\varepsilon = -1$, the motion is against the force and the work is negative.

Properties of Line Integrals

Remark 9.8 (a) Linearity.

$$\int_{\Gamma} (\vec{f} + \vec{g}) \, \mathrm{d}\vec{x} = \int_{\Gamma} \vec{f} \, \mathrm{d}\vec{x} + \int_{\Gamma} \vec{g} \, \mathrm{d}\vec{x}, \qquad \int_{\Gamma} \lambda \vec{f} \, \mathrm{d}\vec{x} = \lambda \int_{\Gamma} \vec{f} \, \mathrm{d}\vec{x}.$$

(b) Change of direction. If $\vec{x}(t)$, $t \in [r, s]$ defines a curve Γ which goes from $a = \vec{x}(r)$ to $b = \vec{x}(s)$, then $\vec{y}(t) = \vec{x}(r + s - t)$, $t \in [r, s]$, defines the curve $-\Gamma$ which goes in the opposite direction from b to a. It is easy to see that

$$\int_{-\Gamma} \vec{f} \, \mathrm{d}\vec{x} = -\int_{\Gamma} \vec{f} \, \mathrm{d}\vec{x}.$$

(c) Triangle inequality.

$$\left| \int_{\Gamma} \vec{f} \, \mathrm{d}\vec{x} \right| \leq \ell(\Gamma) \sup_{x \in \Gamma} \left\| \vec{f}(x) \right\| .$$

(d) Splitting. If Γ_1 and Γ_2 are two curves such that the ending point of Γ_1 equals the starting point of Γ_2 then

$$\int_{\Gamma_1 \cup \Gamma_2} \vec{f} \, \mathrm{d}\vec{x} = \int_{\Gamma_1} \vec{f} \, \mathrm{d}\vec{x} + \int_{\Gamma_2} \vec{f} \, \mathrm{d}\vec{x}.$$

9.5.1 Path Independence

Problem: For which vector fields $\vec{f}$ the line integral from a to b does not depend upon the path (see Example 9.8 (a) Example 8.2)?

Definition 9.12 A vector field $\vec{f}\colon G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, is called *conservative* if for any points a and b in G and any curves Γ_1 and Γ_2 from a to b we have

$$\int_{\Gamma_1} \vec{f} \, \mathrm{d}\vec{x} = \int_{\Gamma_2} \vec{f} \, \mathrm{d}\vec{x}.$$

In this case we say that the line integral $\int_{\Gamma} \vec{f} \, \mathrm{d}\vec{x}$ is *path independent* and we use the notation $\int_a^b \vec{f} \, \mathrm{d}\vec{x}$.

Definition 9.13 A vector field $\vec{f}\colon G \to \mathbb{R}^n$ is called *potential field* or *gradient vector field* if there exists a continuously differentiable function $U\colon G \to \mathbb{R}$ such that $\vec{f}(x) = \operatorname{grad} U(x)$ for $x \in G$. We call U the *potential* or *antiderivative* of $\vec{f}$.

Example 9.9 The gravitational force

$$\vec{F}(x) = -\gamma m M \frac{x}{\|x\|^3}$$

is a potential field with potential

$$U(x) = \gamma m M \frac{1}{\|x\|}$$

This follows from Example 8.2 (a), $\operatorname{grad} f(\|x\|) = f'(\|x\|) \frac{x}{\|x\|}$ with $f(y) = 1/y$ and $f'(y) = -1/y^2$.

Remark 9.9 (a) $\vec{f}$ is conservative if and only if the line integral over any *closed* curve in G is 0.
(b) Let G be connected. If it exists, $U(x)$ is uniquely determined up to a constant. Indeed, if $\operatorname{grad} U_1(x) = \operatorname{grad} U_2(x) = \vec{f}$ we have $\operatorname{grad}(U_1 - U_2) = 0$. Since G is connected, $U_1 - U_2$ is constant on G (all partial derivatives of $U_1 - U_2$ are identically 0).

Theorem 9.10 *Let $G \subset \mathbb{R}^n$ be a domain.*
(i) *If $U\colon G \to \mathbb{R}$ is continuously differentiable and $\vec{f} = \operatorname{grad} U$. Then for every (piecewise continuously differentiable) curve Γ from a to b, $a, b \in G$, we have*

$$\int_\Gamma \vec{f}\,\mathrm{d}\vec{x} = U(b) - U(a).$$

(ii) *Let $\vec{f}\colon G \to \mathbb{R}^n$ be a continuous conservative vector field and $a \in G$. Put*

$$U(x) = \int_a^x \vec{f}\,\mathrm{d}\vec{y}, \quad x \in G.$$

Then $U(x)$ is an antiderivative for $\vec{f}$, that is $\operatorname{grad} U = \vec{f}$.
(iii) *A continuous vector field $\vec{f}$ is conservative in G if and only if it is a potential field.*

Proof. (i) Let $\Gamma = \{\vec{x}(t) \mid t \in [r, s]\}$, be a continuously differentiable curve from $a = \vec{x}(r)$ to $b = \vec{x}(s)$. We define $\varphi(t) = U(\vec{x}(t))$ and compute the derivative using the chain rule

$$\dot{\varphi}(t) = \operatorname{grad} U(\vec{x}(t)) \cdot \dot{\vec{x}}(t) = \vec{f}(\vec{x}(t)) \cdot \dot{\vec{x}}(t).$$

By definition of the line integral we have

$$\int_\Gamma \vec{f}\,\mathrm{d}\vec{x} = \int_r^s \vec{f}(\vec{x}(t))\,\dot{\vec{x}}(t)\,\mathrm{d}t.$$

Inserting the above expression and applying the fundamental theorem of calculus, we find

$$\int_\Gamma \vec{f}\,\mathrm{d}\vec{x} = \int_r^s \dot{\varphi}(t)\,\mathrm{d}t = \varphi(s) - \varphi(r) = U(\vec{x}(s)) - U(\vec{x}(r)) = U(b) - U(a).$$

(ii) Choose $h \in \mathbb{R}^n$ small such that $x + th \in G$ for all $t \in [0, 1]$. By the path independence of the line integral

$$U(x+h) - U(x) = \int_a^x \vec{f} \cdot \mathrm{d}\vec{y} - \int_a^{x+h} \vec{f} \cdot \mathrm{d}\vec{y} = \int_x^{x+h} \vec{f} \cdot \mathrm{d}\vec{y}$$

Consider the curve $\vec{x}(t) = x + th$, $t \in [0, 1]$ from x to $x + h$. Then $\dot{\vec{x}}(t) = h$. By the mean value theorem of integration (Theorem 5.16 with $\varphi = 1$, $a = 0$ and $b = 1$) we have

$$\int_x^{x+h} \vec{f} \cdot \mathrm{d}\vec{y} = \int_0^1 \vec{f}(\vec{x}(t)) \cdot h\,\mathrm{d}t = \vec{f}(x + \theta h) \cdot h,$$

where $\theta \in [0,1]$. We check $\operatorname{grad} U(x) = \vec{f}(x)$ using the definition of the derivative:

$$\frac{\left| U(x+h) - U(x) - \vec{f}(x)\cdot h \right|}{\|h\|} = \frac{\left| (\vec{f}(x+\theta h) - \vec{f}(x))\cdot h \right|}{\|h\|} \underset{\text{CSI}}{\leq} \frac{\left\| \vec{f}(x+\theta h) - \vec{f}(x) \right\| \|h\|}{\|h\|}$$
$$= \left\| \vec{f}(x+\theta h) - \vec{f}(x) \right\| \xrightarrow[h\to 0]{} 0,$$

since $\vec{f}(x)$ is continuous.
(iii) follows immediately from (i) and (ii). ∎

Remark 9.10 (a) In case $n = 2$, a simple path to compute the line integral (and so the potential U) in (ii) consists of 2 segments: from $(0,0)$ via $(x,0)$ to (x,y). The line integral of $P\,\mathrm{d}x + Q\,\mathrm{d}y$ then reads as ordinary Riemann integrals

$$U(x,y) = \int_0^x P(t,0)\,\mathrm{d}t + \int_0^y Q(x,t)\,\mathrm{d}t.$$

(b) Case $n = 3$. You can also use just one single segment from the origin to the endpoint (x,y,z). This path is parametrized by the curve

$$\vec{x}(t) = (tx, ty, tz), \quad t \in [0,1], \quad \dot{\vec{x}}(t) = (x,y,z).$$

We obtain

$$U(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} f_1\,\mathrm{d}x + f_2\,\mathrm{d}y + f_3\,\mathrm{d}z \tag{9.11}$$
$$= x\int_0^1 f_1(tx,ty,tz)\,\mathrm{d}t + y\int_0^1 f_2(tx,ty,tz)\,\mathrm{d}t + z\int_0^1 f_3(tx,ty,tz)\,\mathrm{d}t. \tag{9.12}$$

(c) Although Theorem 9.10 gives a necessary and sufficient condition for a vector field to be conservative, we are missing an easy criterion. The next proposition fills this gap.

Definition 9.14 A connected open subset G (a region) of $\mathbb{R}^n$ is called *simply connected* if every closed polygonal path inside G can be shrunk inside G to a single point.

Roughly speaking, simply connected sets do not have holes. Every convex subset of $\mathbb{R}^n$ is simply connected while the torus $\mathrm{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and any ring $R = \{(x,y) \in \mathbb{R}^2 \mid r^2 < x^2 + y^2 \leq R^2\}$, $0 \leq r < R \leq \infty$, are not.
The precise mathematical term for a curve γ to be "shrinkable to a point" is to be *null-homotopic*: for the closed curve $\gamma\colon [1,0] \to X$ there exists a continuous mapping $h\colon [0,1]\times[0,1] \to X$ such that $h(t,0) = \gamma(t)$ for all t and $h(t,1) = \{\mathrm{pt}\}$.

Proposition 9.11 *Let $\vec{f} = (f_1, f_2, f_3)$ a continuously differentiable vector field on a simply connected region $G \subset \mathbb{R}^3$. Then $\vec{f}$ is conservative if and only if* $\operatorname{curl} \vec{f} = 0$, *i. e.*

$$\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} = 0, \quad \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} = 0, \quad \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = 0.$$

In case $n = 2$, $\vec{f} = (P, Q)$, $G \subset \mathbb{R}^2$ *is simply connected, the condition is*

$$\operatorname{curl} \vec{f} = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0.$$

Proof. (a) Let $\vec{f}$ be conservative; by Theorem 9.10 there exists a potential U, $\operatorname{grad} U = \vec{f}$. However, $\operatorname{curl} \operatorname{grad} U = 0$ since

$$\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} = \frac{\partial^2 U}{\partial x_2 \partial x_3} - \frac{\partial^2 U}{\partial x_3 \partial x_2} = 0$$

by Schwarz's Lemma.
(b) This will be an application of Stokes' theorem, see below. ■

Remark 9.11 (a) $\operatorname{curl} \vec{f} = 0$ is called the *integrability condition* for the vector field $\vec{f}$.
(b) The statement is false if the condition " G is simply connected" is dropped. For example, let $G = \mathbb{R}^2 \setminus \{(0, 0)\}$ and

$$\vec{f} = (P, Q) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

The vector field is not conservative (Prove!), however, the condition $P_y = Q_x$ is nevertheless satisfied, see homework 30.1

Example 9.10 Let on $\mathbb{R}^3$, $\vec{f} = (P, Q, R) = (6xy^2 + \mathrm{e}^x, 6x^2 y, 1)$. Then

$$\operatorname{curl} \vec{f} = (R_y - Q_z, P_z - R_x, Q_x - P_y) = (0, 0, 12xy - 12xy) = 0;$$

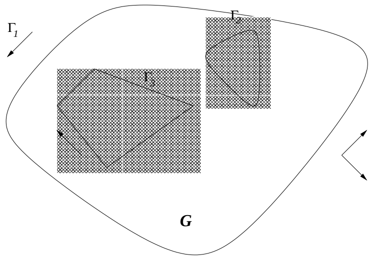
hence, $\vec{f}$ is conservative with the potential (see Remark 9.10 (b))

$$\begin{aligned} U(x, y, z) &= x \int_0^1 f_1(tx, ty, tz)\, \mathrm{d}t + y \int_0^1 f_2(tx, ty, tz)\, \mathrm{d}t + z \int_0^1 f_3(tx, ty, tz)\, \mathrm{d}t \\ &= x \int_0^1 (6t^3 xy^2 + \mathrm{e}^{tx})\, \mathrm{d}t + y \int_0^1 6t^3 x^2 y\, \mathrm{d}t + z \int_0^1 \mathrm{d}t \\ &= 3x^2 y^2 + \mathrm{e}^x + z. \end{aligned}$$

9.6 Stokes' Theorem

Roughly speaking, Stokes' theorem relates a surface integral over a surface $\mathcal{F}$ with a line integral over the boundary $\partial\mathcal{F}$. In case of a plane surface in $\mathbb{R}^2$, it is called Green's theorem.

9.6.1 Green's Theorem



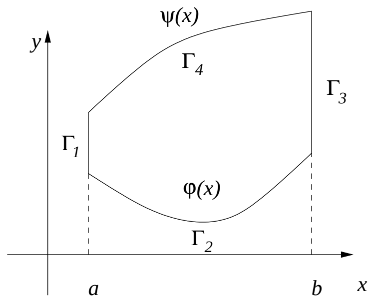
Let G be a domain in $\mathbb{R}^2$ with picewise smooth (differentiable) boundaries $\Gamma_1, \Gamma_2, \dots, \Gamma_k$. We give an orientation to the boundary: the outer curve is oriented counter clockwise (mathematical positive), the inner boundaries are oriented in the opposite direction.

Theorem 9.12 (Green's Theorem) *Let (P, Q) be a continuously differentiable vector field on $\overline{G}$ and let the boundary $\Gamma = \partial G$ be oriented as above. Then*

$$\iint\limits_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \int_\Gamma P\,\mathrm{d}x + Q\,\mathrm{d}y. \tag{9.13}$$

Proof. (a) First, we consider a region G of type 1 in the plane, as shown in the figure and we will prove that

$$-\iint\limits_G \frac{\partial P}{\partial y}\,\mathrm{d}x\mathrm{d}y = \int_\Gamma P\,\mathrm{d}x. \tag{9.14}$$



The double integral on the left may be evaluated as an iterated integral (Fubini's theorem), we have

$$\begin{aligned}\iint\limits_G \frac{\partial P}{\partial y}\,\mathrm{d}x\mathrm{d}y &= \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} P_y(x,y)\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_a^b (P(x,\psi(x)) - P(x,\varphi(x)))\,\mathrm{d}x.\end{aligned}$$

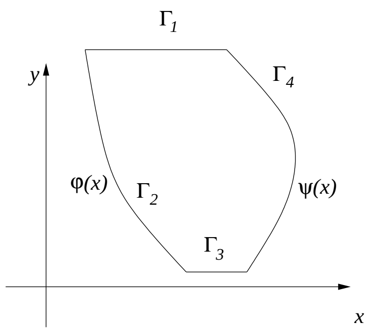
The latter equality is due to the fundamental theorem of calculus. To compute the line integral, we parametrize the four parts of Γ in a natural way:

$$\begin{array}{llll}\vec{x}_1(t) = (a, -t), & t \in [-\psi(a), -\varphi(a)], & \mathrm{d}x = 0, & \mathrm{d}y = -\,\mathrm{d}t, \\ \vec{x}_2(t) = (t, \varphi(t)), & t \in [a,b], & \mathrm{d}x = \mathrm{d}t, & \mathrm{d}y = \varphi'(t)\,\mathrm{d}t, \\ \vec{x}_3(t) = (b, t), & t \in [\varphi(b), \psi(b)], & \mathrm{d}x = 0, & \mathrm{d}y = \mathrm{d}t, \\ \vec{x}_4(t) = (-t, \psi(-t)), & t \in [-b, -a], & \mathrm{d}x = -\,\mathrm{d}t, & \mathrm{d}y = -\psi'(t)\,\mathrm{d}t.\end{array}$$

Since $\mathrm{d}x = 0$ on Γ_1 and Γ_3 we are left with the line integrals over Γ_2 and Γ_4:

$$\begin{aligned}\int_\Gamma P\,\mathrm{d}x &= \int_a^b P(t,\varphi(t))\,\mathrm{d}t + \int_{-b}^{-a} P(-t,\psi(-t))(-\,\mathrm{d}t) \\ &= \int_a^b P(t,\varphi(t))\,\mathrm{d}t - \int_a^b P(t,\psi(t))\,\mathrm{d}t.\end{aligned}$$

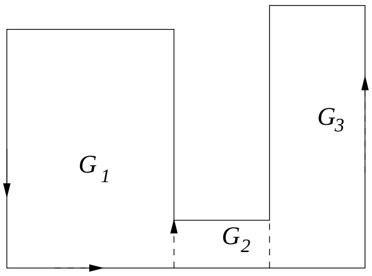
This completes the proof for type 1 regions.



Exactly in the same way, we can prove that if G is a type 2 region then for $Q = Q(x, y)$ with continuous partial derivatives,

$$\iint\limits_G \frac{\partial Q}{\partial x}\,\mathrm{d}x\mathrm{d}y = \int_\Gamma Q\,\mathrm{d}y. \tag{9.15}$$

If G is both a region of type 1 and 2, then (9.14) and (9.15) are both valid and we can add them and obtain (9.13).



(b) Breaking a region G up into smaller regions, each of which is both of type 1 and 2, Green's theorem is valid for G. The line integrals along the inner boundary cancel leaving the line integral around the boundary of G.

(c) If the region has a hole, one can split it into two simply connected regions, for which Green's theorem is valid by the arguments of (b). ∎

Application

If Γ is a curve which bounds a region G, then the area of G is

$$A = \frac{1}{2}\int_\Gamma x\,\mathrm{d}y - y\,\mathrm{d}x = \int_\Gamma x\,\mathrm{d}y = -\int_\Gamma y\,\mathrm{d}x. \tag{9.16}$$

Proof. Choosing $P = -\alpha y$, $Q = (1-\alpha)x$ one has

$$\begin{aligned} A &= \iint\limits_G \mathrm{d}x\mathrm{d}y = \iint\limits_G ((1-\alpha) - (-\alpha))\,\mathrm{d}x\mathrm{d}y = \iint\limits_G (Q_x - P_y)\,\mathrm{d}x\mathrm{d}y = \int_\Gamma P\,\mathrm{d}x + Q\,\mathrm{d}y \\ &= -\alpha\int_\Gamma y\,\mathrm{d}x + (1-\alpha)\int_\Gamma x\,\mathrm{d}y. \end{aligned}$$

Inserting $\alpha = 0$, $\alpha = 1$, and $\alpha = \frac{1}{2}$ yields the assertion. ∎

Example 9.11 Find the area bounded by the ellipse Γ: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. We parametrize Γ by $\vec{x}(t) = (a\cos t, b\sin t)$, $t \in [0, 2\pi]$, $\dot{\vec{x}}(t) = (-a\sin t, b\cos t)$. Then (9.16) gives

$$A = \frac{1}{2}\int_0^{2\pi} a\cos t\, b\sin t\,\mathrm{d}t - b\sin t(-a\sin t)\,\mathrm{d}t = \frac{1}{2}\int_0^{2\pi} ab\,\mathrm{d}t = \pi ab.$$

9.6.2 Stokes' Theorem

Conventions: Let $\mathcal{F}$ be a regular, oriented surface. Let $\Gamma = \partial\mathcal{F}$ be the boundary of $\mathcal{F}$ with the induced orientation: the orientation of the surface together with the orientation of the boundary form a right-oriented screw.

Theorem 9.13 (Stokes' theorem) *Let $\mathcal{F}$ be a smooth regular oriented surface with a parametrization $F \in \mathrm{C}^2(G)$ and G is a plane region to which Green's theorem applies. Let $\Gamma = \partial\mathcal{F}$ be the boundary with the above orientation. Further, let $\vec{f}$ be a continuously differentiable vector field on $\overline{\mathcal{F}}$.*
Then we have

$$\iint\limits_{\mathcal{F}} \operatorname{curl} \vec{f}\, \mathrm{d}\vec{S} = \int_\Gamma \vec{f}\, \mathrm{d}\vec{x}. \tag{9.17}$$

This can also be written as

$$\iint\limits_{\mathcal{F}} \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) \mathrm{d}y\mathrm{d}z + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) \mathrm{d}z\mathrm{d}x + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \mathrm{d}x\mathrm{d}y.$$

Remark 9.12 (a) Green's theorem is a special case with $\mathcal{F} = G \times \{0\}$ and $\vec{f} = (P, Q, 0)$.
(b) We complete the proof of Proposition 9.11 and show that for a *simply* connected region $G \subset \mathbb{R}^3$ $\operatorname{curl} \vec{f} = 0$ implies $\vec{f}$ to be conservative. Indeed, any closed, regular, piecewise differentiable curve $\gamma \subset G$ is the boundary of a suitable smooth regular oriented surface $\mathcal{F}$, $\gamma = \partial\mathcal{F}$. Inserting $\operatorname{curl} \vec{f} = 0$ into Stokes' theorem gives $\int_\Gamma \vec{f}\, \mathrm{d}\vec{x} = 0$; the line integral is path independent and hence, $\vec{f}$ is conservative. The region must be simply connected; otherwise its boundary has more than one component and the line integral is not path independent, in general.

Proof. Main idea: Reduction to Green's theorem. Since both sides of the equation are additive with respect to the vector field $\vec{f}$, it suffices to proof the statement for the vector fields $(f_1, 0, 0)$, $(0, f_2, 0)$, and $(0, 0, f_3)$. We show the theorem for $\vec{f} = (f, 0, 0)$, the other cases are quite analogous:

$$\iint\limits_{\mathcal{F}} \left(\frac{\partial f}{\partial z} \mathrm{d}z\mathrm{d}x - \frac{\partial f}{\partial y} \mathrm{d}x\mathrm{d}y\right) = \int_{\partial\mathcal{F}} f\, \mathrm{d}x.$$

Let $F(u, v)$, $u, v \in G$ be the parametrization of the surface $\mathcal{F}$. Then

$$\mathrm{d}x = \frac{\partial x}{\partial u} \mathrm{d}u + \frac{\partial x}{\partial v} \mathrm{d}v,$$

such that the line integral on the right reads

$$\begin{aligned}
\int_{\partial\mathcal{F}} f\,\mathrm{d}x &= \int_{\partial G} f\,x_u\,\mathrm{d}u + f\,x_v\,\mathrm{d}v = \int_{\partial G} P\,\mathrm{d}u + Q\,\mathrm{d}v \underset{\text{Green's th.}}{=} \iint\limits_G \left(-\frac{\partial P}{\partial v} + \frac{\partial Q}{\partial u}\right)\,\mathrm{d}u\,\mathrm{d}v\\
&= \iint\limits_G -(f_v x_u + f\,x_{vu}) + (f_u x_v + f\,x_{uv})\,\mathrm{d}u\,\mathrm{d}v = \iint\limits_G -f_v x_u + f_u x_v\,\mathrm{d}u\,\mathrm{d}v\\
&= \iint\limits_G -(f_x\,x_v + f_y\,y_v + f_z\,z_v)x_u + (f_x\,x_u + f_y\,y_u + f_z\,z_u)x_v\,\mathrm{d}u\,\mathrm{d}v\\
&= \iint\limits_G \left(-f_y(x_u y_v - x_v y_u) + f_z(z_u x_v - z_v x_u)\right)\,\mathrm{d}u\,\mathrm{d}v\\
&= \iint\limits_G \left(-f_y\frac{\partial(x,y)}{\partial(u,v)} + f_z\frac{\partial(z,x)}{\partial(u,v)}\right)\,\mathrm{d}u\,\mathrm{d}v = \iint\limits_{\mathcal{F}} -f_y\,\mathrm{d}x\mathrm{d}y + f_z\,\mathrm{d}z\mathrm{d}x.
\end{aligned}$$

This completes the proof. ∎

Remark 9.13 (a) The right side of (9.17) is called *the circulation of the vector field $\vec{f}$ over the closed curve Γ*. Now let $\vec{x}_0 \in \mathcal{F}$ be fixed and consider smaller and smaller neighborhoods $\mathcal{F}_0$ of $\vec{x}_0$ with boundaries Γ_0. By Stokes' theorem

$$\int_{\Gamma_0} \vec{f}\,\mathrm{d}\vec{x} = \iint\limits_{\mathcal{F}_0} \operatorname{curl}\vec{f}\cdot\vec{n}(y)\,\mathrm{d}\vec{S}(y) = \operatorname{curl}\vec{f}(\vec{x}_0)\vec{n}(\vec{x}_0)\operatorname{area}(\mathcal{F}_0).$$

Hence,

$$\operatorname{curl}\vec{f}(\vec{x}_0)\cdot\vec{n}(\vec{x}_0) = \lim_{\mathcal{F}_0\to\vec{x}_0} \frac{\int_{\partial\mathcal{F}_0}\vec{f}\,\mathrm{d}\vec{x}}{\operatorname{area}(\mathcal{F}_0)}.$$

We call $\operatorname{curl}\vec{f}(\vec{x}_0)\cdot\vec{n}(\vec{x}_0)$ *the infinitesimal circulation of the vector field $\vec{f}$ at $\vec{x}$ corresponding to the unit normal vector $\vec{n}$.*
(b) Stokes' theorem then says that the integral over the infinitesimal circulation of a vector field $\vec{f}$ corresponding to the unit normal vector $\vec{n}$ over $\mathcal{F}$ equals the circulation of the vector field along the boundary of $\mathcal{F}$.

Green's Identities in $\mathbb{R}^3$

We formulate consequences from Gauß' divergence theorem which play an important role in partial differential equations.
Recall (Proposition 8.9 (Prop. 8.9)) that the *directional derivative* of a function $v\colon U\to\mathbb{R}$, $U\subset\mathbb{R}^n$, at x_0 in the direction of the unit vector $\vec{n}$ is given by $D_{\vec{n}}f(x_0) = \langle\operatorname{grad} f(x_0),\vec{n}\rangle$.

Definition 9.15 Let $\mathcal{F} \subset U \subset \mathbb{R}^3$ be an oriented, regular surface with the unit normal vector $\vec{n}(x_0)$ at $x_0 \in \mathcal{F}$. Let $g\colon U \to \mathbb{R}$ be differentiable.
Then

$$\frac{\partial g}{\partial \vec{n}}(x_0) = \langle \operatorname{grad} g(x_0), \vec{n}\rangle \tag{9.18}$$

is called the *normal derivative* of g on $\mathcal{F}$ at x_0.

Proposition 9.14 *Let G be a region as in Gauß' theorem, the boundary ∂G is oriented with the outer normal, u, v are twice continuously differentiable on an open set U with $\overline{G} \subset U$. Then we have Green's identities:*

$$\iiint\limits_G \nabla(u)\cdot\nabla(v)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint\limits_{\partial G} u\frac{\partial v}{\partial \vec{n}}\,\mathrm{d}S - \iiint\limits_G u\Delta(v)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, \tag{9.19}$$

$$\iiint\limits_G (u\Delta(v) - v\Delta(u))\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint\limits_{\partial G}\left(u\frac{\partial v}{\partial \vec{n}} - v\frac{\partial u}{\partial \vec{n}}\right)\mathrm{d}S, \tag{9.20}$$

$$\iiint\limits_G \Delta(u)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint\limits_{\partial G}\frac{\partial u}{\partial \vec{n}}\,\mathrm{d}S. \tag{9.21}$$

Proof. Put $f = u\nabla(v)$. Then

$$\begin{aligned}\operatorname{div} f &= \nabla(u\nabla v) = \nabla(u)\cdot\nabla(v) + u\nabla(\nabla v)\\ &= \operatorname{grad} u \cdot \operatorname{grad} v + u\Delta(v).\end{aligned}$$

Applying Gauß' theorem, we obtain

$$\begin{aligned}\iiint\limits_G \operatorname{div} f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iiint\limits_G (\operatorname{grad} u\cdot\operatorname{grad} v)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint\limits_G u\Delta(v)\,\mathrm{d}x\mathrm{d}y\mathrm{d}zp\\ &= \iint\limits_{\partial G} u\operatorname{grad} v\cdot\vec{n}\,\mathrm{d}S = \iint\limits_{\partial G} u\frac{\partial v}{\partial \vec{n}}\,\mathrm{d}S.\end{aligned}$$

This proves Green's first identity. Changing the role of u and v and taking the difference, we obtain the second formula.
Inserting $v = -1$ into (9.20) we get (9.21). ∎

Application: Let u_i, $i = 1, 2$, be harmonic functions on G, i.e. $\Delta u_1 = \Delta u_2 = 0$ and $u_1(x) = u_2(x)$ for all $x \in \partial G$. Then $u_1 \equiv u_2$ in G (cf. Homework 30.2).

9.6.3 Vector Potential and the Inverse Problem of Vector Analysis

Let $\vec{f}$ be a continuously differentiable vector field on the *simply connected* region $G \subset \mathbb{R}^3$.

Definition 9.16 The vector field $\vec{f}$ on G is called a *source-free* field (solenoidal field, divergence-zero field) if there exists a vector field $\vec{g}$ on G with $\vec{f} = \operatorname{curl}\vec{g}$. Then $\vec{g}$ is called the *vector potential* to $\vec{f}$.

Theorem 9.15 *$\vec{f}$ is source-free if and only if* $\operatorname{div}\vec{f} = 0$.

Proof. (a) If $\vec{f} = \operatorname{curl}\vec{g}$ then $\operatorname{div}\vec{f} = \operatorname{div}(\operatorname{curl}\vec{g}) = 0$.
(b) To simplify notations, we skip the arrows. We explicitly construct a vector potential g to f with $g = (g_1, g_2, 0)$ and $\operatorname{curl} g = f$. This means

$$\begin{aligned}
f_1 &= -\frac{\partial g_2}{\partial z},\\
f_2 &= \frac{\partial g_1}{\partial z},\\
f_3 &= \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}.
\end{aligned}$$

The first two equations are satisfied setting

$$\begin{aligned}
g_2 &= -\int_{z_0}^{z} f_1(x,y,t)\,\mathrm{d}t + h(x,y),\\
g_1 &= \int_{z_0}^{z} f_2(x,y,t)\,\mathrm{d}t.
\end{aligned}$$

Inserting this into the third equation, we obtain

$$\begin{aligned}
\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} &= -\int_{z_0}^{z} \frac{\partial f_1}{\partial x}(x,y,t)\,\mathrm{d}t + h_x(x,y) - \int_{z_0}^{z} \frac{\partial f_2}{\partial y}(x,y,t)\,\mathrm{d}t\\
&= -\int_{z_0}^{z} \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right)\,\mathrm{d}t + h_x\\
&\underset{\operatorname{div} f = 0}{=} \int_{z_0}^{z} \frac{\partial f_3}{\partial z}(x,y,t)\,\mathrm{d}t + h_x\\
&= f_3(x,y,z) - f_3(x,y,z_0) + h_x(x,y).
\end{aligned}$$

Choosing $h_x(x,y) = f_3(x,y,z_0)$, the third equation is satisfied and $\operatorname{curl} g = f$. ∎

Remarks 9.14 (a) The proof of second direction is a constructive one; you can use this method to calculate a vector potential explicitly, see homework 30.3. You can also try another ansatz, say $g = (0, g_2, g_3)$ or $g = (g_1, 0, g_3)$.
(b) If g is a vector potential for f and $U \in \mathrm{C}^2(G)$, then $\tilde{g} = g + \operatorname{grad} U$ is also a vector potential for f. Indeed

$$\operatorname{curl}\tilde{g} = \operatorname{curl} g + \operatorname{curl}\operatorname{grad} U = f.$$

The Inverse Problem of Vector Analysis

Let h be a function and $\vec{a}$ be a vector field on G; both continuously differentiable. Problem: Does there exist a vector field $\vec{f}$ such that

$$\operatorname{div} \vec{f} = h \quad \text{and} \quad \operatorname{curl} \vec{f} = \vec{a}.$$

Proposition 9.16 *The above problem has a solution if and only if* $\operatorname{div} \vec{a} = 0$.

Proof. The condition is necessary since $\operatorname{div} \vec{a} = \operatorname{div} \operatorname{curl} \vec{f} = 0$. We skip the vector arrows. For the other direction we use the ansatz $f = r + s$ with

$$\operatorname{curl} r = 0, \qquad \operatorname{div} r = h, \tag{9.22}$$

$$\operatorname{curl} s = a, \qquad \operatorname{div} s = 0. \tag{9.23}$$

Since $\operatorname{curl} r = 0$, by Proposition 9.11 there exists a potential U with $r = \operatorname{grad} U$. Then $\operatorname{curl} r = 0$ and $\operatorname{div} r = \operatorname{div} \operatorname{grad} U = \Delta(U)$. Hence (9.22) is satisfied if and only if $r = \operatorname{grad} U$ and $\Delta(U) = h$.
Since $\operatorname{div} a = \operatorname{div} \operatorname{curl} s = 0$, there exists a vector potential g such that $\operatorname{curl} a = s$. Let φ be twice continuously differentiable on G and set $s = g + \operatorname{grad} \varphi$. Then $\operatorname{curl} s = \operatorname{curl} g = a$ and $\operatorname{div} s = \operatorname{div} g + \operatorname{div} \operatorname{grad} \varphi = \operatorname{div} g + \Delta(\varphi)$. Hence, $\operatorname{div} s = 0$ if and only if $\Delta(\varphi) = -\operatorname{div} g$.
Both equations $\Delta(U) = h$ and $\Delta(\varphi) = -\operatorname{div} g$ are so called Poisson equations which can be solved within the theory of partial differential equations (PDE). ∎

The inverse problem has *not* a unique solution. Choose a harmonic function ψ, $\Delta(\psi) = 0$ and put $f_1 = f + \operatorname{grad} \psi$. Then

$$\begin{aligned}
\operatorname{div} f_1 &= \operatorname{div} f + \operatorname{div} \operatorname{grad} \psi = \operatorname{div} f + \Delta(\psi) = \operatorname{div} f = h, \\
\operatorname{curl} f_1 &= \operatorname{curl} f + \operatorname{curl} \operatorname{grad} \psi = \operatorname{curl} f = a.
\end{aligned}$$

9.7 Differential Forms on $\mathbb{R}^n$

We show that Gauß', Green's and Stokes' theorems are three cases of a "general" theorem which is also named after Stokes. The appearance of the Jacobian in the change of variable theorem will become clear. We formulate the Poincaré lemma.

9.7.1 The Exterior Algebra $\Lambda(\mathbb{R}^n)$

Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$; for $h \in \mathbb{R}^n$ we write $h = (h_1, \dots, h_n)$ with respect to the standard basis, $h = \sum_i h_i e_i$.

The Dual Vector Space V^*

The interplay between a normed space E and its dual space E' forms the basis of *functional analysis*. We start with the definition of the *algebraic* dual.

Definition 9.17 Let V be a $\mathbb{K}$-linear space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). The *(algebraic) dual* vector space to V is the set of all $\mathbb{K}$-linear functionals $f\colon V \to \mathbb{K}$,

$$V^* = \{f\colon V \to \mathbb{K} \mid f \text{ is linear}\}.$$

We denote it by V^*.

The *evaluation* of $f \in V^*$ on $v \in V$ is denoted by

$$f(v) = \langle f, v\rangle \in \mathbb{K}.$$

In this case, the brackets denote the *dual pairing* between V^* and V. Obviously, V^* is again a $\mathbb{K}$-linear space if we define the linear structure as follows

$$\langle \lambda f + \mu g, v\rangle := \lambda \langle f, v\rangle + \mu \langle g, v\rangle, \quad \lambda,\, \mu \in \mathbb{K}, \quad f, g \in V^*, \quad v \in V.$$

Example 9.12 (a) If $V = \mathbb{R}^n$ with the above standard basis $\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$, V^* has also dimension n and we can define the *dual basis* $\{\,\mathrm{d}x_1, \dots, \mathrm{d}x_n\}$ of V^* to $\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$ by

$$\mathrm{d}x_i(\mathrm{e}_j) = \delta_{ij}, \quad i, j = 1, \dots, n.$$

The functional $\mathrm{d}x_i$ associates to each vector $h \in V$ its ith coordinate h_i: $\mathrm{d}x_i(h) = h_i$. We call $\mathrm{d}x_i$ the *ith coordinate functional.*
(b) If $V = \mathrm{C}([0,1])$, the continuous functions on $[0,1]$ and α is an increasing on $[0,1]$ function, then

$$\varphi_\alpha(f) = \int_0^1 f\,\mathrm{d}\alpha, \quad f \in V$$

defines a linear functional φ_α on V. If $a \in [0,1]$,

$$\psi_a(f) = f(a), \quad f \in V$$

defines another linear functional on V.
(c) Let $a \in \mathbb{R}^n$. Then $\langle a, x\rangle = \sum_{i=1}^n a_i x_i$, $x \in \mathbb{R}^n$ defines a linear functional on $\mathbb{R}^n$. In Functional Analysis we will learn that this is already the most general form of a continuous linear functional, cf. Theorem of Riesz.

Definition 9.18 Let $k \in \mathbb{N}$. An *alternating multilinear form of degree k* on $\mathbb{R}^n$, a *k-form* for short, is a mapping $\omega\colon \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$, k factors $\mathbb{R}^n$, which is *multilinear* and *antisymmetric*, i. e.

$$\text{MULT} \quad \omega(\cdots, \alpha x_i + \beta y_i, \cdots) = \alpha\omega(\cdots, x_i, \cdots) + \beta\omega(\cdots, y_i, \cdots), \tag{9.24}$$

$$\text{ANT} \quad \omega(\cdots, x_i, \cdots, x_j, \cdots) = -\omega(\cdots, x_j, \cdots, x_i, \cdots), \quad i, j = 1, \dots, k, i \neq j. \tag{9.25}$$

We denote the linear space of all k-forms on $\mathbb{R}^n$ by $\Lambda^k(\mathbb{R}^n)$ with the convention $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$ and $\Lambda^1(\mathbb{R}^n) = (\mathbb{R}^n)^*$.

Let $f_1, \dots, f_k \in (\mathbb{R}^n)^*$ be linear functionals on $\mathbb{R}^n$. Then we define the k-form $f_1 \wedge \dots \wedge f_k \in \Lambda^k(\mathbb{R}^n)$ (read: "f_1 wedge f_2 ... wedge f_k") as follows

$$f_1 \wedge \dots \wedge f_k(h_1, \dots, h_k) = \begin{vmatrix} f_1(h_1) & \cdots & f_1(h_k) \\ \vdots & & \vdots \\ f_k(h_1) & \cdots & f_k(h_k) \end{vmatrix} \tag{9.26}$$

In particular, let $i_1, \dots, i_k \in \{1, \dots, n\}$ be fixed and choose $f_j = \mathrm{d}x_{i_j}$, $j = 1, \dots, k$. Then

$$\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_k}(h_1, \dots, h_k) = \begin{vmatrix} h_{1i_1} & \cdots & h_{ki_1} \\ \vdots & & \vdots \\ h_{1i_k} & \cdots & h_{ki_k} \end{vmatrix}$$

Remark 9.15 (a) $f_1 \wedge \dots \wedge f_k$ is indeed a k-form since the f_i are linear, the determinant is multilinear

$$\begin{vmatrix} \lambda a + \mu a' & b & c \\ \lambda d + \mu d' & e & f \\ \lambda g + \mu g' & h & i \end{vmatrix} = \lambda \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \mu \begin{vmatrix} a' & b & c \\ d' & e & f \\ g' & h & i \end{vmatrix},$$

and antisymmetric

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = - \begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix}.$$

(b) For example, let $y = (y_1, \dots, y_n)$, $z = (z_1, \dots, z_n) \in \mathbb{R}^n$,

$$\mathrm{d}x_3 \wedge \mathrm{d}x_1(y, z) = \begin{vmatrix} y_3 & z_3 \\ y_1 & z_1 \end{vmatrix} = y_3 z_1 - y_1 z_3.$$

If $f_r = f_s$ for some $r \neq s$, we have $f_1 \wedge \dots \wedge f_k = 0$ since determinants with identical rows vanish.

Proposition 9.17 *For $k \leq n$ the k-forms $\{\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$ form a basis of the vector space $\Lambda^k(\mathbb{R}^n)$. A k-form with $k > n$ is identically zero. We have*

$$\dim \Lambda^k(\mathbb{R}^n) = \binom{n}{k}.$$

Proof. Any k-form ω is uniquely determined by its values on the k-tuple of vectors $(\mathrm{e}_{i_1}, \dots, \mathrm{e}_{i_k})$ with $1 \leq i_1 < i_2 < \dots < i_k \leq n$. Indeed, using antisymmetry of ω, we know ω on all k-tuples of basis vectors; using linearity in each component, we get ω on all k-tuples of vectors. This shows that the $\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_k}$ with $1 \leq i_1 < i_2 < \dots < i_k \leq n$ generate the linear space $\Lambda^k(\mathbb{R}^n)$. We make this precise in case $k = 2$. With $y = \sum_i y_i \mathrm{e}_i$, $z = \sum_j z_j \mathrm{e}_j$ we have by linearity and antisymmetry of ω

$$\begin{aligned} \omega(y, z) &= \sum_{i,j=1}^{n} y_i z_j \omega(\mathrm{e}_i, \mathrm{e}_j) \underset{\mathrm{ANT}}{=} \sum_{1 \leq i < j \leq n} (y_i z_j - y_j z_i) \omega(\mathrm{e}_i, \mathrm{e}_j) \\ &= \sum_{i<j} \omega(\mathrm{e}_i, \mathrm{e}_j) \begin{vmatrix} y_i & z_i \\ y_j & z_j \end{vmatrix} = \sum_{i<j} \omega(\mathrm{e}_i, \mathrm{e}_j)\, \mathrm{d}x_i \wedge \mathrm{d}x_j(y, z). \end{aligned}$$

Hence,

$$\omega = \sum_{i<j} \omega(\mathrm{e}_i, \mathrm{e}_j)\, \mathrm{d}x_i \wedge \mathrm{d}x_j.$$

This shows that $\{\, \mathrm{d}x_i \wedge \mathrm{d}x_j \mid i < j\}$ generates $\Lambda^2(\mathbb{R}^n)$. These are $\binom{n}{2}$ elements. We show its linear independence. Suppose that $\sum_{i<j} \alpha_{ij}\, \mathrm{d}x_i \wedge \mathrm{d}x_j = 0$ for some α_{ij}. Evaluating this on $(\mathrm{e}_r, \mathrm{e}_s)$, $r < s$, gives

$$\begin{aligned} 0 &= \sum_{i<j} \alpha_{ij}\, \mathrm{d}x_i \wedge \mathrm{d}x_j(\mathrm{e}_r, \mathrm{e}_s) = \sum_{i<j} \alpha_{ij} \begin{vmatrix} \delta_{ri} & \delta_{si} \\ \delta_{rj} & \delta_{sj} \end{vmatrix} \\ &= \sum_{i<j} \alpha_{ij}(\delta_{ri}\delta_{sj} - \delta_{rj}\delta_{si}) = \alpha_{rs}; \end{aligned}$$

hence, the above 2-forms are linearly independent. The arguments for general k are similar. ∎

For $\omega \in \Lambda^k(\mathbb{R}^n)$ there exist unique numbers $a_{i_1 \cdots i_k} \in \mathbb{R}$ such that

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1 \cdots i_k}\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

Example 9.13 Let $n = 3$.
$k = 1$ $\{\, \mathrm{d}x_1,\, \mathrm{d}x_2,\, \mathrm{d}x_3\}$ is a basis of $\Lambda^1(\mathbb{R}^3)$.
$k = 2$ $\{\, \mathrm{d}x_1 \wedge \mathrm{d}x_2,\ \mathrm{d}x_1 \wedge \mathrm{d}x_3,\ \mathrm{d}x_2 \wedge \mathrm{d}x_3\}$ is a basis of $\Lambda^2(\mathbb{R}^3)$.
$k = 3$ $\{\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3\}$ is a basis of $\Lambda^3(\mathbb{R}^3)$.
$\Lambda^k(\mathbb{R}^3) = \{0\}$ for $k \ge 4$.

Definition 9.19 An *algebra* A over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) is a $\mathbb{K}$-linear space together with a product map $(a, b) \to ab$, $A \times A \to A$, such that the following holds for all $a, b, c \in A$ an d $\alpha \in \mathbb{K}$

(i) $a(bc) = (ab)c$ (associative),
(ii) $(a + b)c = ac + bc, \quad a(b + c) = ab + ac$,
(iii) $\alpha(ab) = (\alpha a)b = a(\alpha b)$.

Standard examples are $\mathrm{C}(X)$, the continuous functions on a metric space X or $\mathrm{Mat}(n \times n, \mathbb{K})$, the full $n \times n$-matrix algebra over $\mathbb{K}$ or the polynomial algebra.
Let $\Lambda(\mathbb{R}^n) = \bigoplus_{k=0}^{n} \Lambda^k(\mathbb{R}^n)$ be the direct sum of linear spaces.

Proposition 9.18 (i) *$\Lambda(\mathbb{R}^n)$ is an $\mathbb{R}$-algebra with unity 1 and product $\wedge$ defined by*

$$(\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) \wedge (\,\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l}) = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l}$$

(ii) *If $\omega_k \in \Lambda^k(\mathbb{R}^n)$ and $\omega_l \in \Lambda^l(\mathbb{R}^n)$ then $\omega_k \wedge \omega_l \in \Lambda^{k+l}(\mathbb{R}^n)$ and*

$$\omega_k \wedge \omega_l = (-1)^{kl} \omega_l \wedge \omega_k.$$

Proof. We show (ii) for $\omega_k = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ and $\omega_l = \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l}$. We already know $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i$. There are kl transpositions $\mathrm{d}x_{i_r} \leftrightarrow \mathrm{d}x_{j_s}$; hence the sign is $(-1)^{kl}$. ∎

In particular, $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$. The formula $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i$ determines the product in $\Lambda(\mathbb{R}^n)$ uniquely.

Definition 9.20 $\Lambda(\mathbb{R}^n)$ is called the *exterior algebra* of the vector space $\mathbb{R}^n$.

The Pull-Back of k-forms

Definition 9.21 Let $A \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ a linear mapping and $k \in \mathbb{N}$. For $\omega \in \Lambda^k(\mathbb{R}^m)$ we define a k-form $A^*(\omega) \in \Lambda^k(\mathbb{R}^n)$ by

$$(A^*\omega)(h_1, \dots, h_k) = \omega(Ah_1, Ah_2, \dots, Ah_k), \quad h_1, \dots, h_k \in \mathbb{R}^n$$

We call $A^*(\omega)$ the *pull-back* of ω under A.

Note that $A^* \in \mathrm{L}(\Lambda^k(\mathbb{R}^m), \Lambda^k(\mathbb{R}^n))$ is a linear mapping. In case $k = 1$ we call A^* the *dual mapping* to A. In case $k = 0$, $\omega \in \mathbb{R}$ we simply set $A^*\omega = \omega$. We have $A^*(\omega \wedge \eta) = A^*\omega \wedge A^*\eta$.

Orientation of $\mathbb{R}^n$

If $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ are two bases of $\mathbb{R}^n$ there exists a unique regular matrix $A = (a_{ij})$ ($\det A \neq 0$) such that $e_i = \sum_j a_{ij} f_j$. We say that $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ are *equivalent* if and only if $\det A > 0$. Since $\det A \neq 0$, there are exactly two equivalence classes.

Definition 9.22 An *orientation* of $\mathbb{R}^n$ is given by fixing one of the two equivalence classes.

Example 9.14 (a) In $\mathbb{R}^2$ the bases $\{\mathrm{e}_1, \mathrm{e}_2\}$ and $\{\mathrm{e}_2, \mathrm{e}_1\}$ have different orientations since $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\det A = -1$.
(b) In $\mathbb{R}^3$ the bases $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$, $\{\mathrm{e}_3, \mathrm{e}_1, \mathrm{e}_2\}$ and $\{\mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_1\}$ have the same orientation whereas $\{\mathrm{e}_1, \mathrm{e}_3, \mathrm{e}_2\}$, $\{\mathrm{e}_2, \mathrm{e}_1, \mathrm{e}_3\}$, and $\{\mathrm{e}_3, \mathrm{e}_2, \mathrm{e}_1\}$ have opposite orientation.

9.7.2 Differential Forms

Let $U \subset \mathbb{R}^n$ be an open set.

Definition 9.23 A *k-form on U* is a mapping $\omega \colon U \to \Lambda^k(\mathbb{R}^n)$, i. e. to every point $x \in U$ we associate a k-form $\omega(x) \in \Lambda^k(\mathbb{R}^n)$.

Let ω be a k-form on U. Since $\{\,\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_k}\mid 1\le i_1<i_2<\cdots<i_k\le n\}$ forms a basis of $\Lambda^k(\mathbb{R}^n)$ there exist uniquely determined functions $a_{i_1\cdots i_k}$ on U such that

$$\omega(x)=\sum_{1\le i_1<\cdots<i_k\le n} a_{i_1\cdots i_k}(x)\ \mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_k}. \tag{9.27}$$

If all functions $a_{i_1\cdots i_k}$ are in $\mathrm{C}^\infty(U)$ we say ω is a *differential k-form* on U. The linear space of differential k-forms on U is denoted by $\Omega^k(U)$. We define $\Omega^0(U)=\mathrm{C}^\infty(U)$ and $\Omega(U)=\bigoplus_{k=0}^{n}\Omega^k(U)$. The product in $\Lambda(\mathbb{R}^n)$ defines a product in $\Omega(U)$:

$$(\omega\wedge\eta)(x)=\omega(x)\wedge\eta(x),\quad x\in U,$$

hence $\Omega(U)$ is an algebra.

(a) Differentiation of Differential Forms

Let $U\subset\mathbb{R}^n$ be open.

Definition 9.24 Let $f\in\Omega^0(U)=\mathrm{C}^\infty(U)$ and $x\in U$. We define

$$\mathrm{d}f(x)=Df(x);$$

then $\mathrm{d}f$ is a differential 1-form on U.
If $\omega(x)=\sum_{1\le i_1<\cdots<i_k\le n} a_{i_1\cdots i_k}(x)\ \mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_k}$ is a differential k-form, we define

$$\mathrm{d}\omega(x)=\sum_{1\le i_1<\cdots<i_k\le n}\mathrm{d}a_{i_1\cdots i_k}(x)\wedge\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_k}. \tag{9.28}$$

Then $\mathrm{d}\omega$ is a differential $(k+1)$-form. The linear operator $\mathrm{d}\colon\Omega^k(U)\to\Omega^{k+1}(U)$ is called the *exterior differential*.

Remarks 9.16 (a) Note, that for a function $f\colon U\to\mathbb{R}$, $Df\in\mathrm{L}(\mathbb{R}^n,\mathbb{R})=\Lambda^1(\mathbb{R}^n)$. By Example 8.7 (a)

$$Df(x)(h)=\operatorname{grad}f(x)\cdot h=\sum_{i=1}^n\frac{\partial f}{\partial x_i}(x)h_i=\sum_{i=1}^n\frac{\partial f}{\partial x_i}(x)\,\mathrm{d}x_i(h),$$

hence

$$\mathrm{d}f(x)=\sum_{i=1}^n\frac{\partial f}{\partial x_i}(x)\ \mathrm{d}x_i. \tag{9.29}$$

Viewing $x_i\colon U\to\mathbb{R}$ as a C^∞-function, by the above formula

$$\mathrm{d}x_i(x)=\mathrm{d}x_i.$$

This justifies the notation $\mathrm{d}x_i$. If $f\in\mathrm{C}^\infty(\mathbb{R})$ we have $\mathrm{d}f(x)=f'(x)\ \mathrm{d}x$.
(b) One can show that the definition of $\mathrm{d}\omega$ does not depend on the choice of the basis $\{\,\mathrm{d}x_1,\dots,\mathrm{d}x_n\}$ of $\Lambda^1(\mathbb{R}^n)$.

Example 9.15 (a) $G = \mathbb{R}^2$, $\omega = \mathrm{e}^{xy}\,\mathrm{d}x + xy^3\,\mathrm{d}y$. Then

$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}(\mathrm{e}^{xy}) \wedge \mathrm{d}x + \mathrm{d}(xy^3) \wedge \mathrm{d}y\\ &= (y\mathrm{e}^{xy}\,\mathrm{d}x + x\mathrm{e}^{xy}\,\mathrm{d}y) \wedge \mathrm{d}x + (y^3\,\mathrm{d}x + 3xy^2\,\mathrm{d}y) \wedge \mathrm{d}y\\ &= (-x\mathrm{e}^{xy} + y^3)\,\mathrm{d}x \wedge \mathrm{d}y.\end{aligned}$$

(b) Let f be continuously differentiable. Then

$$\mathrm{d}f = f_x\,\mathrm{d}x + f_y\,\mathrm{d}y + f_z\,\mathrm{d}z = \operatorname{grad} f \cdot (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z).$$

(c) Let $v = (v_1, v_2, v_3)$ be a C^1-vector field. Put $\omega = v_1\,\mathrm{d}x + v_2\,\mathrm{d}y + v_3\,\mathrm{d}z$. Then we have

$$\begin{aligned}\mathrm{d}\omega &= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y\\ &= \operatorname{curl}(v) \cdot (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y).\end{aligned}$$

(d) Let v be as above. Put $\omega = v_1\,\mathrm{d}y \wedge \mathrm{d}z + v_2\,\mathrm{d}z \wedge \mathrm{d}x + v_3\,\mathrm{d}x \wedge \mathrm{d}y$. Then we have

$$\mathrm{d}\omega = \operatorname{div}(v)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Proposition 9.19 *The exterior differential* d *is a linear mapping which satisfies*

(i) $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta, \quad \omega \in \Omega^k(U),\ \eta \in \Omega(U).$
(ii) $\mathrm{d}(\mathrm{d}\omega) = 0, \quad \omega \in \Omega(U).$

Proof. (i) For $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_r)$ we abbreviate $\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_k}$ and $\mathrm{d}x_J = \mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_r}$. Let $\omega = \sum_I a_I\,\mathrm{d}x_I$ and $\eta = \sum_J b_J\,\mathrm{d}x_J$. By definition

$$\begin{aligned}\mathrm{d}(\omega \wedge \eta) &= \mathrm{d}\left(\sum_{I,J} a_I b_J\,\mathrm{d}x_I \wedge \mathrm{d}x_J\right)\\ &= \sum_{I,J} (\mathrm{d}a_I\, b_J + a_I\,\mathrm{d}b_J) \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J\\ &= \sum_{I,J} \mathrm{d}a_I \wedge \mathrm{d}x_I \wedge b_J\,\mathrm{d}x_J + \sum_{I,J} a_I\,\mathrm{d}x_I \wedge \mathrm{d}b_J \wedge \mathrm{d}x_J(-1)^k\\ &= \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta,\end{aligned}$$

where in the third line we used $\mathrm{d}b_J \wedge \mathrm{d}x_I = (-1)^k\,\mathrm{d}x_I \wedge \mathrm{d}b_J$.
(ii) Again by the definition of d:

$$\begin{aligned}\mathrm{d}(\mathrm{d}\omega) &= \sum_I \mathrm{d}\,(\mathrm{d}a_I \wedge \mathrm{d}x_I)\\ &= \sum_{I,j} \mathrm{d}\left(\frac{\partial a_I}{\partial x_j}\,\mathrm{d}x_j \wedge \mathrm{d}x_I\right)\\ &= \sum_{I,i,j} \frac{\partial^2 a_I}{\partial x_i \partial x_j}\,\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_I\\ &\underset{\text{Schwarz' lemma}}{=} \sum_{I,i,j} \frac{\partial^2 a_I}{\partial x_j \partial x_i}\,(-\,\mathrm{d}x_j \wedge \mathrm{d}x_i \wedge \mathrm{d}x_I) = -\mathrm{d}(\mathrm{d}\omega).\end{aligned}$$

It follows that $\mathrm{d}(\mathrm{d}\omega) = \mathrm{d}^2\,\omega = 0$. ∎

(b) The Pull-Back of Differential Forms

Definition 9.25 Let $f\colon U \to V$ be a differentiable function with open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$. Let $\omega \in \Omega^k(V)$ be a differential k-form. We define a differential k-form $f^*(\omega) \in \Omega^k(U)$ by

$$\begin{aligned}(f^*\omega)(x) &= (Df(x)^*)\omega(f(x)),\\ (f^*\omega)(x;h_1,\dots,h_k) &= \omega(f(x);Df(x)(h_1),\dots,Df(x)(h_k)), \quad x \in U,\ h_1,\dots,h_k \in \mathbb{R}^n.\end{aligned}$$

In case $k = 0$ and $\omega \in \Omega^0(V) = \mathrm{C}^\infty(V)$ we simply set

$$(f^*\omega)(x) = \omega(f(x)), \quad f^*\omega = \omega{\circ}f.$$

We call $f^*(\omega)$ the *pull-back* of the differential k-form ω with respect to f.

Note that by definition the pull-back f^* is a linear mapping from the space of differential k-forms on V to the space of differential k-forms on U, $f^*\colon \Omega^k(V) \to \Omega^k(U)$.

Proposition 9.20 *Let f be as above and $\omega, \eta \in \Omega(V)$. Let $\{\,\mathrm{d}y_1,\dots,\,\mathrm{d}y_m\}$ be the dual basis to the standard basis in $\mathbb{R}^m$. Then we have the following properties*

$$f^*(\,\mathrm{d}y_i) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}\,\mathrm{d}x_j, \quad i = 1,\dots,m. \tag{9.30}$$

$$f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega), \tag{9.31}$$

$$f^*(a\omega) = (a{\circ}f)f^*(\omega), \quad a \in \mathrm{C}^\infty(V), \tag{9.32}$$

$$f^*(\omega\wedge\eta) = f^*(\omega)\wedge f^*(\eta). \tag{9.33}$$

If $n = m$ and $\{y_1,\dots,y_n\}$ is the standard basis in the image space $\mathbb{R}^m$, then

$$f^*(\,\mathrm{d}y_1\wedge\cdots\wedge\,\mathrm{d}y_n) = \frac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_n)}\,\mathrm{d}x_1\wedge\cdots\wedge\,\mathrm{d}x_n. \tag{9.34}$$

Proof. We show (9.30). Let $h \in \mathbb{R}^n$; by Definition 9.24 and the definition of the derivative we have

$$\begin{aligned}f^*(\,\mathrm{d}y_i)(h) = \,\mathrm{d}y_i(Df(x)(h)) &= \left\langle \,\mathrm{d}y_i, \left(\sum_{j=1}^n \left(\frac{\partial f_k(x)}{\partial x_j}\right) h_j\right)_{k=1,\dots,m}\right\rangle\\ &= \sum_{j=1}^n \left(\frac{\partial f_i(x)}{\partial x_j}\right) h_j = \sum_{j=1}^n \left(\frac{\partial f_i(x)}{\partial x_j}\right)\,\mathrm{d}x_j(h).\end{aligned}$$

This shows (9.30). Equation (9.32) is a special case of (9.33). We skip the proof.
To show (9.31) in view of (9.28) and (9.33), it suffices to show $f^*(\mathrm{d}g) = \mathrm{d}(f^* g)$ for *functions* $g\colon U \to \mathbb{R}$. By (9.29) and (9.32) we have

$$\begin{aligned}
f^*(\mathrm{d}g)(x) &= f^*\left(\sum_{i=1}^m \frac{\partial g}{\partial y_i}\,\mathrm{d}y_i\right)(x) = \sum_{i=1}^m \frac{\partial g(f(x))}{\partial y_i} f^*(\,\mathrm{d}y_i)\\
&= \sum_{i=1}^m \frac{\partial g(f(x))}{\partial y_i} \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(x)\,\mathrm{d}x_j\\
&= \sum_{j=1}^n \left(\sum_{i=1}^m \frac{\partial g(f(x))}{\partial y_i}\frac{\partial f_i(x)}{\partial x_j}\right)\mathrm{d}x_j\\
&\underset{\text{chain rule}}{=} \sum_{j=1}^n \frac{\partial}{\partial x_j}(g\circ f)(x)\,\mathrm{d}x_j\\
&= \mathrm{d}(g\circ f)(x) = \mathrm{d}(f^* g)(x).
\end{aligned}$$

We finally prove (9.34). By (9.33) and (9.31) we have

$$\begin{aligned}
f^*(\,\mathrm{d}y_1\wedge\cdots\wedge\,\mathrm{d}y_n) &= f^*(\,\mathrm{d}y_1)\wedge\cdots\wedge f^*(\,\mathrm{d}y_n)\\
&= \sum_{i_1=1}^n \frac{\partial f_1}{\partial x_{i_1}}\,\mathrm{d}x_{i_1}\wedge\cdots\wedge\sum_{i_n=1}^n \frac{\partial f_n}{\partial x_{i_n}}\,\mathrm{d}x_{i_n}\\
&= \sum_{i_1,\dots,i_n=1}^n \frac{\partial f_1}{\partial x_{i_1}}\cdots\frac{\partial f_n}{\partial x_{i_n}}\,\mathrm{d}x_{i_1}\wedge\cdots\wedge\,\mathrm{d}x_{i_n}.
\end{aligned}$$

Since the square of a 1-form vanishes, the only non-vanishing terms in the above sum are the permutations $(i_1,\dots,i_n)$ of $(1,\dots,n)$. Using antisymmetry to write $\mathrm{d}x_{i_1}\wedge\cdots\wedge\,\mathrm{d}x_{i_n}$ as a multiple of $\mathrm{d}x_1\wedge\cdots\wedge\,\mathrm{d}x_n$, we obtain the sign of the permutation $(i_1,\dots,i_n)$:

$$\begin{aligned}
f^*(\,\mathrm{d}y_1\wedge\cdots\wedge\,\mathrm{d}y_n) &= \sum_{I=(i_1,\dots,i_n)\in \mathrm{S}_n} \operatorname{sign}(I)\frac{\partial f_1}{\partial x_{i_1}}\cdots\frac{\partial f_n}{\partial x_{i_n}}\,\mathrm{d}x_1\wedge\cdots\wedge\,\mathrm{d}x_n\\
&= \frac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_n)}\,\mathrm{d}x_1\wedge\cdots\wedge\,\mathrm{d}x_n.
\end{aligned}$$

■

Example 9.16 (a) Let $f(r,\varphi) = (r\cos\varphi, r\sin\varphi)$ be given on $\mathbb{R}^2\setminus(0\times\mathbb{R})$ and let $\{\,\mathrm{d}r,\mathrm{d}\varphi\}$ and $\{\,\mathrm{d}x,\,\mathrm{d}y\}$ be the dual bases to $\{\mathrm{e}_r,\mathrm{e}_\varphi\}$ and $\{\mathrm{e}_1,\mathrm{e}_2\}$. We have

$$\begin{aligned}
f^*(x) &= r\cos\varphi, & f^*(y) &= r\sin\varphi,\\
f^*(\,\mathrm{d}x) &= \cos\varphi\,\mathrm{d}r - r\sin\varphi\mathrm{d}\varphi, & f^*(\,\mathrm{d}y) &= \sin\varphi\,\mathrm{d}r + r\cos\varphi\mathrm{d}\varphi,\\
f^*(\,\mathrm{d}x\wedge\,\mathrm{d}y) &= r\,\mathrm{d}r\wedge\mathrm{d}\varphi,
\end{aligned}$$

$$f^*\left(\frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y\right) = \mathrm{d}\varphi.$$

(b) Let $k \in \mathbb{N}$, $r \in \{1, \dots, k\}$, and $\alpha \in \mathbb{R}$. Define a mapping a mapping from $I\colon \mathbb{R}^k \to \mathbb{R}^{k+1}$ and $\omega \in \Omega^k(\mathbb{R}^{k+1})$ by

$$I(x_1, \dots, x_k) = (x_1, \dots, x_{r-1}, \alpha, x_r, \dots x_k),$$
$$\omega(y_1, \dots, y_{k+1}) = \sum_{i=1}^{k+1} f_i(y)\, \mathrm{d}y_1 \wedge \cdots \widehat{\mathrm{d}y_i} \cdots \wedge \mathrm{d}y_{k+1},$$

where $f_i \in \mathrm{C}^\infty(\mathbb{R}^{k+1})$ for all i; the hat means omission of the factor $\mathrm{d}y_i$. Then

$$I^*(\omega)(x) = f_r(x_1, \dots, x_{r-1}, \alpha, x_r, \dots, x_k)\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k.$$

This easily follows from

$$\begin{aligned} I^*(\mathrm{d}y_i) &= \mathrm{d}x_i, \quad i = 1, \dots, r-1, \\ I^*(\mathrm{d}y_r) &= 0, \\ I^*(\mathrm{d}y_{i+1}) &= \mathrm{d}x_i, \quad i = r, \dots, k. \end{aligned}$$

Roughly speaking: $f^*\omega$ is obtained by substituting the new variables at all places.

(c) Closed and Exact Differential Forms

Motivation: Let $f(x)$ be a continuous function on $\mathbb{R}$. Then $\omega = f(x)\,\mathrm{d}x$ is a 1-form. By the fundamental theorem of calculus, there exists an antiderivative $F(x)$ to $f(x)$ such that $\mathrm{d}\,F(x) = f(x)\,\mathrm{d}x = \omega$.
Problem: Given $\omega \in \Omega^k(U)$. Does there exist $\eta \in \Omega^{k-1}(U)$ with $\mathrm{d}\eta = \omega$?

Definition 9.26 $\omega \in \Omega^k(U)$ is called *closed* if $\mathrm{d}\,\omega = 0$.
$\omega \in \Omega^k(U)$ is called *exact* if there exists $\eta \in \Omega^{k-1}(U)$ such that $\mathrm{d}\,\eta = \omega$.

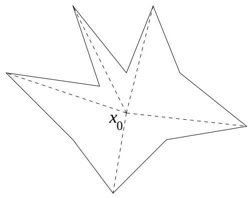
Remarks 9.17 (a) An exact form ω is closed; indeed, $\mathrm{d}\omega = \mathrm{d}(\mathrm{d}\eta) = 0$.
(b) A 1-form $\omega = \sum_i f_i\,\mathrm{d}x_i$ is closed if and only if $\operatorname{curl} \vec{f} = 0$ for the corresponding vector field $\vec{f} = (f_1, \dots, f_n)$. Here the general curl can be defined as a vector with $n(n-1)/2$ components

$$(\operatorname{curl} \vec{f})_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}.$$

The form ω is exact if and only if $\vec{f}$ is conservative, that is, $\vec{f}$ is a gradient vector field with $\vec{f} = \operatorname{grad}(U)$. Then $\omega = \mathrm{d}\,U$.
(c) There are closed forms that are not exact; for example, the winding form

$$\omega = \frac{-y}{x^2 + y^2}\,\mathrm{d}x + \frac{x}{x^2 + y^2}\,\mathrm{d}y$$

on $\mathbb{R}^2 \setminus \{(0,0)\}$ is not exact, cf. homework 30.1.
(d) If $\mathrm{d}\,\eta = \omega$ then $\mathrm{d}(\eta + \mathrm{d}\xi) = \omega$, too, for all $\xi \in \Omega^{k-2}(U)$.



Definition 9.27 An open set U is called *star-shaped* if there exists an $x_0 \in U$ such that for all $x \in U$ the segment from x_0 to x is in U, i.e. $(1-t)x_0 + tx \in U$ for all $t \in [0,1]$.

Convex sets U are star-shaped (take any $x_0 \in U$); any star-shaped set is connected and simply connected.

Lemma 9.21 *Let $U \subset \mathbb{R}^n$ be star-shaped with respect to the origin. Let $\omega = \sum\limits_{i_1<\cdots<i_k} a_{i_1\cdots i_k}\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \in \Omega^k(U)$. Define*

$$I(\omega)(x) = \sum_{i_1<\cdots<i_k} \sum_{r=1}^{k} (-1)^{r-1} \left(\int_0^1 t^{k-1} a_{i_1\cdots i_k}(tx)\, \mathrm{d}t \right) x_{i_r}\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_{i_r}} \wedge \cdots \wedge \mathrm{d}x_{i_k}, \tag{9.35}$$

where the hat means omission of the factor $\mathrm{d}x_{i_r}$. Then we have

$$I(\mathrm{d}\,\omega) + \mathrm{d}(I\,\omega) = \omega. \tag{9.36}$$

(Without proof.)

Example 9.17 (a) Let $k=1$ and $\omega = a_1\,\mathrm{d}x_1 + a_2\,\mathrm{d}x_2 + a_3\,\mathrm{d}x_3$. Then

$$I(\omega) = x_1 \int_0^1 a_1(tx)\,\mathrm{d}t + x_2 \int_0^1 a_2(tx)\,\mathrm{d}t + x_3 \int_0^1 a_3(tx)\,\mathrm{d}t.$$

Note that this is exactly the formula for the potential $U(x_1,x_2,x_3)$ from Remark 9.10 (b). Let (a_1,a_2,a_3) be a vector field on U with $\mathrm{d}\omega = 0$. This is equivalent to $\operatorname{curl} a = 0$ by Example 9.15 (c). The above lemma shows $\mathrm{d}U = \omega$ for $U = I(\omega)$; this means $\operatorname{grad} U = (a_1,a_2,a_3)$, U is the potential to the vector field (a_1,a_2,a_3).
(b) Let $k=2$ and $\omega = a_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + a_2\,\mathrm{d}x_3 \wedge \mathrm{d}x_1 + a_3\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$ where a is a C^1-vector field on U. Then

$$\begin{aligned} I(\omega) = \left(x_3 \int_0^1 t a_2(tx)\,\mathrm{d}t - x_2 \int_0^1 t a_3(tx)\,\mathrm{d}t \right) \mathrm{d}x_1 + {}\\ + \left(x_1 \int_0^1 t a_3(tx)\,\mathrm{d}t - x_3 \int_0^1 t a_1(tx)\,\mathrm{d}t \right) \mathrm{d}x_2 + {}\\ + \left(x_2 \int_0^1 t a_1(tx)\,\mathrm{d}t - x_1 \int_0^1 t a_2(tx)\,\mathrm{d}t \right) \mathrm{d}x_3. \end{aligned}$$

By Example 9.15 (d), ω is closed if and only if $\operatorname{div}(a) = 0$ on U. Let $\eta = b_1\,\mathrm{d}x_1 + b_2\,\mathrm{d}x_2 + b_3\,\mathrm{d}x_3$ such that $\mathrm{d}\eta = \omega$. This means $\operatorname{curl} b = a$. The Poincaré lemma shows that b with $\operatorname{curl} b = a$ exists if and only if $\operatorname{div}(a) = 0$. Then b is the vector potential to a. In case $\mathrm{d}\,\omega = 0$ we can choose $\vec{b}\,\mathrm{d}\vec{x} = I(\omega)$.

Theorem 9.22 (Poincaré Lemma) *Let U be star-shaped. Then every closed differential form is exact.*

Proof. This is an easy consequence of Lemma 9.36. Without loss of generality let U be star-shaped with respect to the origin and $d\omega = 0$. By Lemma 9.21, $d(I\omega) = \omega$. ∎

Remarks 9.18 (a) Let U be star-shaped, $\omega \in \Omega^k(U)$. Suppose $d\,\eta_0 = \omega$ for some $\eta \in \Omega^{k-1}(U)$. Then the general solution of $d\,\eta = \omega$ is given by $\eta_0 + d\xi$ with $\xi \in \Omega^{k-2}(U)$. Indeed, let η be a second solution of $d\,\eta = \omega$. Then $d(\eta - \eta_0) = 0$. By the Poincaré lemma, there exists $\xi \in \Omega^{k-2}(U)$ with $\eta - \eta_0 = d\xi$, hence $\eta = \eta_0 + d\xi$.
(b) Let V be a linear space and W a linear subspace of V. We define an equivalence relation on V by $v_1 \sim v_2$ if $v_1 - v_2 \in W$. The equivalence class of v is denoted by $v + W$. One easily sees that the set of equivalence classes, denoted by V/W, is again a linear space: $\alpha(v + W) + \beta(u + W) := \alpha v + \beta u + W$.
Let U be an arbitrary open subset of $\mathbb{R}^n$. We define

$C^k(U) = \{\omega \in \Omega^k(U) \mid d\,\omega = 0\}$, the *cocycles* on U,
$B^k(U) = \{\omega \in \Omega^k(U) \mid \omega$ is exact$\}$, the *coboundaries* on U.

Since exact forms are closed, $B^k(U)$ is a linear subspace of $C^k(U)$. The factor space

$$H^k_{\text{deR}}(U) = C^k(U)/B^k(U)$$

is called the *de Rham cohomology* of U. If U is star-shaped, $H^k_{\text{deR}}(U) = 0$ for $k \geq 1$, by Poincaré's lemma. The first de Rham cohomology H^1_{deR} of $\mathbb{R}^2 \setminus \{(0,0)\}$ is non-zero. The winding form is a non-zero element. We have

$$H^0_{\text{deR}}(U) \cong \mathbb{R}^p,$$

if and only if U has exactly p components which are not connected $U = U_1 \cup \cdots \cup U_p$ (disjoint union). Then, the characteristic functions χ_{U_i}, $i = 1, \dots, p$, form a basis of the 0-cycles $C^0(U)$ ($B^0(U) = 0$).

9.7.3 Stokes' Theorem

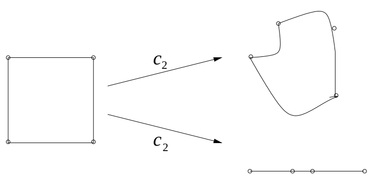
(a) Singular Cubes, Singular Chains, and the Boundary Operator

A very nice treatment of the topics to this Subsection is [13, Chapter 4]. The set $[0,1]^k = [0,1] \times \cdots \times [0,1] = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, i = 1, \dots, n\}$ is called the *n-dimensional unit cube*. Let $U \subset \mathbb{R}^n$ be open.

Definition 9.28 (a) A *singular k-cube* in $U \subset \mathbb{R}^n$ is a continuously differentiable mapping $c_k \colon [0,1]^k \to U$.
(b) A *singular k-chain* in U is a formal sum

$$s_k = n_1 c_{k,1} + \cdots + n_r c_{k,r}$$

with singular k-cubes $c_{k,i}$ and integers $n_i \in \mathbb{Z}$.



A singular 0-cube is a point, a singular 1-cube is a curve, in general, a singular 2-cube (in $\mathbb{R}^3$) is a surface with a boundary of 4 pieces which are differentiable curves. Note that a singular 2-cube can also be a single point—that is where the name "singular" comes from.

Let $I_k\colon [0,1]^k \to \mathbb{R}^k$ be the identity map, i. e. $I_k(x) = x$, $x \in [0,1]^k$. It is called the *standard k-cube in* $\mathbb{R}^k$. We are going to define the *boundary* ∂s_k of a singular k-chain s_k. For $i = 1, \dots, k$ define

$$I^k_{(i,0)}(x_1, \dots, x_{k-1}) = (x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{k-1}),$$
$$I^k_{(i,1)}(x_1, \dots, x_{k-1}) = (x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{k-1}).$$

Insert a 0 and a 1 at the ith component, respectively.
The boundary of the standard k-cube I_k is now defined by $\partial I_k\colon [0,1]^{k-1} \to [0,1]^k$

$$\partial I_k = \sum_{i=1}^{k} (-1)^i \left(I^k_{(i,0)} - I^k_{(i,1)} \right). \tag{9.37}$$

It is the formal sum of $2k$ singular $(k-1)$-cubes, the faces of the k-cube.
The boundary of an arbitrary singular k-cube $c_k\colon [0,1]^k \to U \subset \mathbb{R}^n$ is defined by the composition of the above mapping $\partial I_k\colon [0,1]^{k-1} \to [0,1]^k$ and the k-cube c_k:

$$\partial c_k = c_k{\circ}\partial I_k = \sum_{i=1}^{k} (-1)^i \left(c_k{\circ} I^k_{(i,0)} - c_k{\circ} I^k_{(i,1)} \right), \tag{9.38}$$

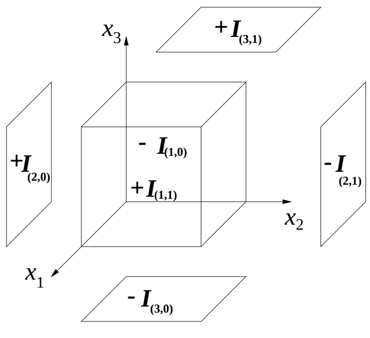
and for a singular k-chain $s_k = n_1 c_{k,1} + \cdots + n_r c_{k,r}$ we set

$$\partial s_k = n_1 \partial c_{k,1} + \cdots + n_r \partial c_{k,r}.$$

The boundary operator ∂c_k associates to each singular k-chain a singular $(k-1)$-chain (since both $I^k_{(i,0)}$ and $I^k_{(i,1)}$ depend on $k-1$ variables, all from the segment $[0,1]$).
One can show that

$$\partial(\partial s_k) = 0$$

for any singular k-chain s_k.



Example 9.18 (a) In case $n = k = 3$ have

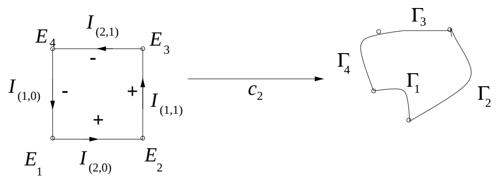
$$\partial I_3 = -I^3_{(1,0)} + I^3_{(1,1)} + I^3_{(2,0)} - I^3_{(2,1)} - I^3_{(3,0)} + I^3_{(3,1)},$$

where

$$-I^3_{(1,0)}(x_1, x_2) = -(0, x_1, x_2), \quad +I^3_{(1,1)}(x_1, x_2) = +(1, x_1, x_2),$$
$$+I^3_{(2,0)}(x_1, x_2) = +(x_1, 0, x_2), \quad -I^3_{(2,1)}(x_1, x_2) = -(x_1, 1, x_2),$$
$$-I^3_{(3,0)}(x_1, x_2) = -(x_1, x_2, 0), \quad +I^3_{(3,1)}(x_1, x_2) = +(x_1, x_2, 1).$$

Note, if we take care of the signs in (9.37) all 6 unit normal vectors $D_1 I^k_{(i,j)} \times D_2 I^k_{(i,j)}$ to the faces have the orientation of the outer normal with respect to the unit 3-cube $[0,1]^3$. The above sum ∂I_3 is a *formal* sum of singular 2-cubes. You are not allowed to add componentwise: $-(0, x_1, x_2) + (1, x_1, x_2) \neq (1, 0, 0)$.

(b) In case $k = 2$ we have



$$\begin{aligned} \partial I_2(x) &= I^2_{(1,1)} - I^2_{(1,0)} + I^2_{(2,0)} - I^2_{(2,1)} \\ &= (1, x) - (0, x) + (x, 0) - (x, 1). \\ \partial\partial I_2 &= (E_3 - E_2) - (E_4 - E_1) + \\ &\quad + (E_2 - E_1) - (E_3 - E_4) = 0. \end{aligned}$$

Here we have $\partial c_2 = \Gamma_1 + \Gamma_2 - \Gamma_3 - \Gamma_4$.

(c) Let $c_2 \colon [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3 \setminus 0$ be the singular 2-cube

$$c(s, t) = (\cos s \sin t, \sin s \sin t, \cos t).$$

By (b)

$$\begin{aligned} \partial c_2(x) &= c_2 \circ \partial I_2 = \\ &= c_2(2\pi, x) - c_2(0, x) + c_2(x, 0) - c_2(x, \pi) \\ &= (\cos 2\pi \sin x, \sin 2\pi \sin x, \cos 2\pi) - (\cos 0 \sin x, \sin 0 \sin x, \cos x) + \\ &\quad + (\cos x \sin 0, \sin x \sin 0, \cos 0) - (\cos x \sin 2\pi, \sin x \sin 2\pi, \cos \pi) \\ &= (\sin x, 0, \cos x) - (\sin x, 0, \cos x) + (0, 0, 1) - (0, 0, -1) \\ &= (0, 0, 1) - (0, 0, -1). \end{aligned}$$

Hence, the boundary ∂c_2 of the singular 2-cube c_2 is a degenerate singular 1-chain. We come back to this example.

(b) Integration of Differential Forms

Definition 9.29 Let $c_k \colon [0,1]^k \to U \subset \mathbb{R}^n$, $\vec{x} = c_k(t_1, \dots, t_k)$, be a singular k-cube and ω a k-form on U. Then $(c_k)^*(\omega)$ is a k-form on the unit cube $[0,1]^k$. Thus there exists a unique function $f(t)$, $t \in [0,1]^k$, such that

$$(c_k)^*(\omega) = f(t)\, \mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_k.$$

Then

$$\int_{c_k} \omega = \int_{I_k} (c_k^*)\, \omega = \int_{[0,1]^k} f(t)\, \mathrm{d}t_1 \cdots \mathrm{d}t_k$$

is called the *integral of* ω *over the singular cube* c_k; on the right there is the k-dimensional Riemann integral.

If $s_k = \sum_{i=1}^{r} n_i c_{k,i}$ is a k-chain, set

$$\int_{s_k} \omega = \sum_{i=1}^{r} n_i \int_{c_{k,i}} \omega.$$

If $k=0$, a 0-cube is a single point $c_0(0)=x_0$ and a 0-form is a function $\omega\in C^\infty(G)$. We set $\int_{c_0}\omega = c_0^*(\omega)|_{t=0}=\omega(c_0(0))=\omega(x_0)$. We discuss two special cases $k=1$ and $k=n$.

Example 9.19 (a) $k=1$. Let $c\colon [0,1]\to\mathbb{R}^n$ be an oriented, smooth curve Γ. Let $\omega=f_1(x)\,\mathrm{d}x_1+\cdots+f_n(x)\,\mathrm{d}x_n$ be a 1-form on $\mathbb{R}^n$, then

$$c^*(\omega)=(f_1(c(t))c_1'(t)+\cdots f_n(c(t))c_n'(t))\ \mathrm{d}t$$

is a 1-form on $[0,1]$ such that

$$\int_c\omega=\int_{[0,1]}c^*\,\omega=\int_0^1 f_1(c(t))c_1'(t)+\cdots+f_n(c(t))c_n'(t)\,\mathrm{d}t=\int_\Gamma \vec{f}\cdot\,\mathrm{d}\vec{x}.$$

Obviously, $\int_c\omega$ is the line integral of $\vec{f}$ over Γ.
(b) $k=n$. Let $c\colon[0,1]^k\to\mathbb{R}^k$ be continuously differentiable and let $x=c(t)$. Let $\omega=f(x)\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_k$ be a differential k-form on $\mathbb{R}^k$. Then

$$c^*(\omega)=f(c(t))\,\frac{\partial(c_1,\dots,c_k)}{\partial(t_1,\dots,t_k)}\,\mathrm{d}t_1\wedge\ldots\wedge\,\mathrm{d}t_k.$$

Therefore,

$$\int\limits_c\omega=\int\limits_{[0,1]^k}f(c(t))\,\frac{\partial(c_1,\dots,c_k)}{\partial(t_1,\dots,t_k)}\,\mathrm{d}t_1\ldots\mathrm{d}t_k \tag{9.39}$$

Let $c=I_k$ be the standard k-cube in $[0,1]^k$. Then

$$\int\limits_{I_k}\omega=\int\limits_{[0,1]^k}f(x)\,\mathrm{d}x_1\cdots\mathrm{d}x_k$$

is the k-dimensional Riemann integral of f over $[0,1]^k$.
Let $\tilde{I}_k(x_1,\dots,x_k)=(x_2,x_1,x_3,\dots,x_k)$. Then $I_k([0,1]^k)=\tilde{I}_k([0,1]^k)=[0,1]^k$, however

$$\int\limits_{\tilde{I}_k}\omega=\int\limits_{[0,1]^k}f(x_2,x_1,x_3,\dots,x_k)(-1)\,\mathrm{d}x_1\cdots\mathrm{d}x_k$$
$$=-\int\limits_{[0,1]^k}f(x_1,x_2,x_3,\dots,x_k)\,\mathrm{d}x_1\cdots\mathrm{d}x_k=-\int\limits_{I_k}\omega.$$

We see that $\int\limits_{I_k}\omega$ is an *oriented* Riemann integral. Note that in the above formula (9.39) we do *not* have the absolute value of the Jacobian.

(c) Stokes' Theorem

Theorem 9.23 *Let U be an open subset of $\mathbb{R}^n$, $k\geq 0$ a non-negative integer and $s_{k+1}\colon[0,1]^{k+1}\to U$ a singular $(k+1)$-chain. Let ω be a differential k-form on U. Then we have*

$$\int\limits_{\partial s_{k+1}}\omega=\int\limits_{s_{k+1}}\mathrm{d}\,\omega.$$

Proof. (a) Let $s_{k+1} = I_{k+1}$ be the standard $(k+1)$-cube, in particular $n = k+1$. Let $\omega = \sum_{i=1}^{k+1} f_i(x)\, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_{k+1}$. Then

$$\mathrm{d}\,\omega = \sum_{i=1}^{k+1} (-1)^{i+1} \frac{\partial f_i(x)}{\partial x_i}\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{k+1},$$

hence by Example 9.16 (b), Fubini's theorem and the fundamental theorem of calculus

$$\begin{aligned}
\int_{I_{k+1}} \mathrm{d}\,\omega &= \sum_{i=1}^{k+1} (-1)^{i+1} \int_{[0,1]^{k+1}} \frac{\partial f_i}{\partial x_i}\, \mathrm{d}x_1 \cdots \mathrm{d}x_{k+1} \\
&= \sum_{i=1}^{k+1} (-1)^{i+1} \int_{[0,1]^{k}} \left(\int_0^1 \frac{\partial f_i}{\partial x_i}(x_1, \dots, \underset{\widehat{i}}{t}, \dots x_{k+1})\, \mathrm{d}t \right) \mathrm{d}x_1 \cdots \mathrm{d}x_{i-1}\, \mathrm{d}x_{i+1} \cdots \mathrm{d}x_{k+1} \\
&= \sum_{i=1}^{k+1} (-1)^{i+1} \int_{[0,1]^{k}} \left(f_i(x_1, \dots, \underset{\widehat{i}}{1}, \dots, x_{k+1}) - f_i(x_1, \dots, \underset{\widehat{i}}{0}, \dots, x_{k+1}) \right) \mathrm{d}x_1 \overset{\widehat{i}}{\cdots} \mathrm{d}x_{k+1} \\
&\underset{\text{Example 9.16 (b)}}{=} \sum_{i=1}^{k+1} (-1)^{i+1} \left(\int_{[0,1]^{k}} \left(I_{(i,1)}^{k+1} \right)^* \omega - \int_{[0,1]^{k}} \left(I_{(i,0)}^{k+1} \right)^* \omega \right) \\
&= \int_{\partial I_{k+1}} \omega,
\end{aligned}$$

by definition of ∂I_{k+1}. The assertion is shown in case of identity map.
(b) The general case. Let I_{k+1} be the standard $(k+1)$-cube. Since the pull-back and the differential commute (Proposition 9.20) we have

$$\begin{aligned}
\int_{c_{k+1}} \mathrm{d}\,\omega &= \int_{I_{k+1}} (c_{k+1})^* (\mathrm{d}\,\omega) = \int_{I_{k+1}} \mathrm{d}\left((c_{k+1})^* \omega \right) = \int_{\partial I_{k+1}} (c_{k+1})^* \omega \\
&= \sum_{i=1}^{k+1} (-1)^i \left(\int_{I_{(i,0)}^{k+1}} (c_{k+1})^* \omega - \int_{I_{(i,0)}^{k+1}} (c_{k+1})^* \omega \right) \\
&= \sum_{i=1}^{k+1} (-1)^i \int_{c_{k+1} \circ I_{(i,0)}^{k+1} - c_{k+1} \circ I_{(i,1)}^{k+1}} \omega = \int_{\partial c_{k+1}} \omega.
\end{aligned}$$

■

Remark 9.19 Stokes' theorem is valid for arbitrary oriented compact differentiable k-dimensional manifolds $\mathcal{F}$ and continuously differentiable $(k-1)$-forms ω on $\mathcal{F}$.

Example 9.20 We come back to Example 9.18 (c). Let $\omega = (x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y)/r^3$ be a 2-form on $\mathbb{R}^3 \setminus 0$. It is easy to show that ω is closed, $\mathrm{d}\,\omega = 0$. We compute

$$\int_{c_2} \omega$$

We have $c_2^*(r^3) = 1$ and

$$c_2^*(\omega) = c_2^*(x\,\mathrm{d}y\wedge \mathrm{d}z+y\,\mathrm{d}z\wedge \mathrm{d}x+z\,\mathrm{d}x\wedge \mathrm{d}y) = (x,y,z)\cdot \mathrm{d}\vec{S} = (x,y,z)\cdot D_1c_2\times D_2c_2\,\mathrm{d}s\wedge \mathrm{d}t.$$

This gives

$$c_2^*(\omega) = -\sin t\,\mathrm{d}s \wedge \mathrm{d}t,$$

such that

$$\int_{c_2} \omega = \int_{[0,2\pi]\times[0,\pi]} c_2^*(\omega) = \int_0^{2\pi}\int_0^{\pi} (-\sin t)\,\mathrm{d}s\mathrm{d}t = -4\pi.$$

Stokes' theorem shows that ω is not exact on $\mathbb{R}^3 \setminus 0$. Suppose to the contrary that $\omega = \mathrm{d}\,\eta$ for some $\eta \in \Omega^1(\mathbb{R}^3 \setminus 0)$. Since by Example 9.18 (c), ∂c_2 is a degenerate 1-chain (it consists of two points), the pull-back $(\partial c_2)^*(\eta)$ is 0 and so is the integral

$$0 = \int_{I_1} (\partial c_2)^*(\eta) = \int_{\partial c_2} \eta = \int_{c_2} \mathrm{d}\eta = \int_{c_2} \omega = -4\pi,$$

a contradiction; hence, ω is not exact.

Q 27. Let $G \subseteq \mathbb{R}^n$ be an open set, $U\colon G \to \mathbb{R}$ in $\mathrm{C}^\infty(G)$, and let $\omega(x) = U_{x_1}(x)\,\mathrm{d}x_1 + \cdots + U_{x_n}(x)\,\mathrm{d}x_n$ be a 1-form on G.
(a) Is ω closed?
(b) Is ω exact? If so, find η with $\omega = \mathrm{d}\eta$.
(c) Let $c\colon [0,1] \to G$ be a singular 1-cube in G. Compute

$$\int_c \omega$$

Compare this result with the assertion of Stokes' theorem in case $k = 0$.
We come back to the two special cases $k = 1$, $n = 2$ and $k = 1$, $n = 3$.

(d) Special Cases

$k = 1, n = 3$. Let $c\colon [0,1]^2 \to U \subseteq \mathbb{R}^3$ be a singular 2-cube, $\mathcal{F} = c([0,1]^2)$ is a regular smooth surface in $\mathbb{R}^3$. Then $\partial\mathcal{F}$ is a closed path consisting of 4 parts with the counter-clockwise orientation. Let $\omega = f_1\,\mathrm{d}x_1 + f_2\,\mathrm{d}x_2 + f_3\,\mathrm{d}x_3$ be a differential 1-form on U. By Example 9.19 (a)

$$\int_{\partial c_2} \omega = \int_{\partial\mathcal{F}} f_1\,\mathrm{d}x_1 + f_2\,\mathrm{d}x_2 + f_3\,\mathrm{d}x_3$$

On the other hand by Example 9.15 (c)

$$\mathrm{d}\,\omega = \operatorname{curl} f \cdot (\mathrm{d}x_2 \wedge \mathrm{d}x_2,\, \mathrm{d}x_3 \wedge \mathrm{d}x_2,\, \mathrm{d}x_1 \wedge \mathrm{d}x_2).$$

In this case Stokes' theorem gives

$$\int\limits_{\partial c_2} \omega = \int\limits_{c_2} \mathrm{d}\omega$$

$$\int\limits_{\partial \mathcal{F}} f_1\,\mathrm{d}x_1 + f_2\,\mathrm{d}x_2 + f_3\,\mathrm{d}x_3 = \int\limits_{\mathcal{F}} \operatorname{curl} f \cdot (\,\mathrm{d}x_2 \wedge \mathrm{d}x_3,\ \mathrm{d}x_3 \wedge \mathrm{d}x_1,\ \mathrm{d}x_1 \wedge \mathrm{d}x_2)$$

If $\mathcal{F}$ is in the x_1–x_2 plane, we get Green's theorem.
$k = 2$, $n = 3$. Let c_3 be a singular 3-cube in $\mathbb{R}^3$ and $G = c_3([0,1])$. Further let

$$\omega = v_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + v_2\,\mathrm{d}x_3 \wedge \mathrm{d}x_1 + v_3\,\mathrm{d}x_1 \wedge \mathrm{d}x_2,$$

with a continuously differentiable vector field $v \in \mathrm{C}^1(G)$. By Example 9.15 (d), $\mathrm{d}\,\omega = \operatorname{div}(v)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$. The boundary of G consists of the 6 faces $\partial c_3([0,1]^3)$. They are oriented with the outer unit normal vector. Stokes' theorem then gives

$$\int_{c_3} \mathrm{d}\omega = \int_{\partial c_3} v_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + v_2\,\mathrm{d}x_3 \wedge \mathrm{d}x_1 + v_3\,\mathrm{d}x_1 \wedge \mathrm{d}x_2,$$

$$\int_G \operatorname{div} v\ \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{\partial G} \vec{v} \cdot \,\mathrm{d}\vec{S}.$$

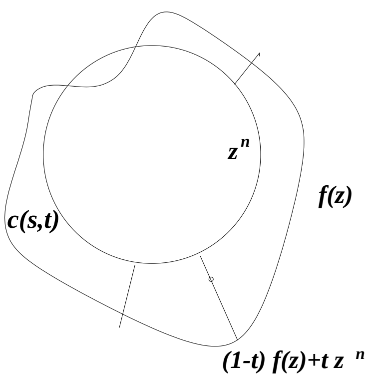
This is Gauß' divergence theorem.

Application—The Fundamental Theorem of Algebra

We give a first proof of the fundamental theorem of algebra, Theorem 5.19:

> Every polynomial $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ with complex coefficients $a_i \in \mathbb{C}$ has a root in $\mathbb{C}$.

We use two facts, the winding form ω on $\mathbb{R}^2 \setminus 0$ is closed but not exact and z^n and $f(z)$ are "close together" for sufficiently large $|\,z\,|$.



We view $\mathbb{C}$ as $\mathbb{R}^2$ with $(a,b) = a + b\mathrm{i}$. Define the following singular 1-cubes on $\mathbb{R}^2$

$$c_{R,n}(s) = (R^n \cos(2\pi n s), R^n \sin(2\pi n s)) = z^n, \quad (9.40)$$

$$c_{R,f}(s) = f{\circ}c_{R,1}(s) = f(R\cos(2\pi s), R\sin(2\pi s)) = f(z), \quad (9.41)$$

where $z = z(s) = R(\cos 2\pi s + \mathrm{i}\sin 2\pi s)$, $s \in [0,1]$. Note that $|\,z\,| = R$. Further, let

$$c(s,t) = (1-t)c_{R,f}(s) + t c_{R,n} = (1-t)f(z) + t z^n, \quad (s,t) \in [0,1]^2, \quad (9.42)$$

$$b(s,t) = f((1-t)\,R(\cos 2\pi s, \sin 2\pi s)) = f((1-t)z), \quad (s,t) \in [0,1]^2 \quad (9.43)$$

be singular 2-cubes in $\mathbb{R}^2$.

Lemma 9.24 *If $|z| = R$ is sufficiently large, then*

$$|c(s,t)| \geq \frac{R^n}{2}, \quad (s,t) \in [0,1]^2.$$

Proof. Since $f(z) - z^n$ is a polynomial of degree less than n,

$$\left| \frac{f(z) - z^n}{z^n} \right| \underset{z\to\infty}{\longrightarrow} 0,$$

in particular $|f(z) - z^n| \leq R^n/2$ if R is sufficiently large. Then we have

$$\begin{aligned} |c(s,t)| &= |(1-t)f(z) + tz^n| = |z^n + (1-t)(f(z) - z^n)| \\ &\geq |z^n| - (1-t)|f(z) - z^n| \geq R^n - \frac{R^n}{2} = \frac{R^n}{2}. \end{aligned}$$

■

The only fact we need is $c(s,t) \neq 0$ for sufficiently large R; hence, c maps the unit square into $\mathbb{R}^2 \setminus 0$.

Lemma 9.25 *Let $\omega = \omega(x,y) = (-y\,\mathrm{d}x + x\,\mathrm{d}y)/(x^2+y^2)$ be the winding form on $\mathbb{R}^2 \setminus 0$. Then we have*
(a)

$$\begin{aligned} \partial c &= c_{R,f} - c_{R,n}, \\ \partial b &= f(z) - f(0). \end{aligned}$$

(b) *For sufficiently large R, c, $c_{R,n}$, and $c_{R,f}$ are chains in $\mathbb{R}^2 \setminus 0$ and*

$$\int_{c_{R,n}} \omega = \int_{c_{R,f}} \omega = 2\pi n.$$

Proof. (a) Note that $z(0) = z(1) = R$. Since $\partial I_2(x) = (x,0) - (x,1) + (1,x) - (0,x)$ we have

$$\begin{aligned} \partial c(s) &= c(s,0) - c(s,1) + c(1,s) - c(0,s) \\ &= f(z) - z^n - ((1-s)f(R) + sR^n) + ((1-s)f(R) + sR^n) = f(z) - z^n. \end{aligned}$$

This proves (a). Similarly, we have

$$\begin{aligned} \partial b(s) &= b(s,0) - b(s,1) + b(1,s) - b(0,s) \\ &= f(z) - f(0) + f((1-s)R) - f((1-s)R) = f(z) - f(0). \end{aligned}$$

(b) By the Lemma 9.24, c is a singular 2-chain in $\mathbb{R}^2 \setminus 0$ for sufficiently large R. Hence ∂c is a 1-chain in $\mathbb{R}^2 \setminus 0$. In particular, both $c_{R,n}$ and $c_{R,f}$ are defined on $\mathbb{R}^2 \setminus 0$. Hence $(\partial c)^*(\omega)$ is well-defined. We compute $c^*_{R,n}(\omega)$ using the pull-backs of $\mathrm{d}x$ and $\mathrm{d}y$

$$\begin{aligned} {c_{R,n}}^*(x^2+y^2) &= R^{2n}, \\ {c_{R,n}}^*(\,\mathrm{d}x) &= -2\pi n R^n \sin(2\pi n s)\,\mathrm{d}s, \\ {c_{R,n}}^*(\,\mathrm{d}y) &= 2\pi n R^n \cos(2\pi n s)\,\mathrm{d}s, \\ {c_{R,n}}^*(\omega) &= 2\pi n\,\mathrm{d}s. \end{aligned}$$

Hence

$$\int_{c_{R,n}} \omega = \int_0^1 2\pi n \,\mathrm{d}s = 2\pi n.$$

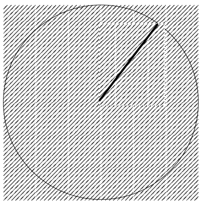
By Stokes' theorem and since ω is closed,

$$\int_{\partial c} \omega = \int_c \mathrm{d}\,\omega = 0,$$

such that by (a), and the above calculation

$$0 = \int_{\partial c} \omega = \int_{c_{R,n}} \omega - \int_{c_{R,f}} \omega, \quad \text{hence} \quad \int_{c_{R,n}} \omega = \int_{c_{R,f}} \omega = 2\pi n.$$

■



$b(s,t)=(1-t)\, z^n$
for $f=1$.

We complete the proof of the fundamental theorem of algebra. Suppose to the contrary that the polynomial $f(z)$ is non-zero in $\mathbb{C}$, then b as well as ∂b are singular chains in $\mathbb{R}^2 \setminus 0$.

By Lemma 9.25 (b) and again by Stokes' theorem we have

$$\int_{c_{R,f}} \omega = \int_{c_{R,f}-f(0)} \omega = \int_{\partial b} \omega = \int_b \mathrm{d}\omega = 0.$$

But this is a contradiction to Lemma 9.25 (b). Hence, b is not a 2-chain in $\mathbb{R}^2 \setminus 0$, that is there exist $s, t \in [0,1]$ such that $b(s,t) = f((1-t)z) = 0$. We have found that $(1-t)R(\cos(2\pi s) + \mathrm{i}\sin(2\pi s))$ is a zero of f. Actually, we have shown a little more. There is a zero of f in the disc $\{z \in \mathbb{C} \mid |\, z\,| \le R\}$ where $R \ge \max\{1, 2\sum_i |\, a_i \,|\}$. Indeed, in this case

$$|\, f(z) - z^n \,| \le \sum_{k=1}^{n-1} |\, a_k \,| \; |\, z^{n-k} \,| \le \sum_{k=1}^{n-1} |\, a_k \,| \; R^{n-1} \le \frac{R^n}{2}$$

and this condition ensures $|\, c(s,t) \,| \ne 0$ as in the proof of Lemma 9.24.