Chapter 7

Sequences and Series of Functions

In the present chapter we confine our attention to complex-valued functions (including the real-valued), although many of the theorems and proofs which follow extend to vector-valued functions without difficulty and even to mappings into general metric spaces. We stay within this simple framework in order to focus attention on the most important aspects of the problem that arise when **limit processes are interchanged.**

7.1 Discussion of the Main Problem

Definition 7.1 Suppose (f_n), $n \in \mathbb{N}$, is a sequence of functions defined on a set E, and suppose that the sequence of numbers $(f_n(x))$ converges for every $x \in E$. We can then define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad x \in E. \tag{7.1}$$

Under these circumstances we say that (f_n) *converges* on E and f is the *limit* (or the *limit function*) of (f_n). Sometimes we say that "(f_n) *converges pointwise* to f on E" if (7.1) holds. Similarly, if $\sum f_n(x)$ converges for every $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad x \in E, \tag{7.2}$$

the function f is called the *sum* of the series $\sum f_n$.

The main problem which arises is to determine whether important properties of the functions f_n are preserved under the limit operations (7.1) and (7.2). For instance, if the functions f_n are continuous, or differentiable, or integrable, is the same true of the limit function? What are the relations between f_n' and f', say, or between the integrals of f_n and that of f? To say that f is continuous at x means

$$\lim_{t\to x} f(t) = f(x).$$

Hence, to ask whether the limit of a sequence of continuous functions is continuous is the same as to ask whether

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t) \tag{7.3}$$

i. e. whether the order in which limit processes are carried out is immaterial. We shall now show by means of several examples that limit processes cannot in general be interchanged without affecting the result. Afterwards, we shall prove that under certain conditions the order in which limit operations are carried out is inessential.

Example 7.1 (a) Our first example, and the simplest one, concerns a "double sequence." For positive integers $m, n \in \mathbb{N}$ let

$$s_{mn} = \frac{m}{m+n}.$$

Then, for fixed n

$$\lim_{m\to\infty} s_{mn} = 1,$$

so that $\lim\limits_{n\to\infty} \lim\limits_{m\to\infty} s_{mn} = 1$. On the other hand, for every fixed m,

$$\lim_{n\to\infty} s_{mn} = 0,$$

so that $\lim\limits_{m\to\infty} \lim\limits_{n\to\infty} s_{mn} = 0$.
(b) Let

$$f_n(x) = \frac{x^2}{(1+x^2)^n}, \quad x \in \mathbb{R},\, n \in \mathbb{N},$$

and consider

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}. \tag{7.4}$$

Since $f_n(0) = 0$, we have $f(0) = 0$. For $x \neq 0$, the last series is a convergent geometric series with sum $1 + x^2$. Hence

$$f(x) = \begin{cases} 0, & x = 0, \\ 1 + x^2, & x \neq 0, \end{cases}$$

so that a convergent sum of continuous functions may have a discontinuous sum.

(c) Let

$$f_n(x) = \frac{\sin(nx)}{\sqrt{n}}, \quad x \in \mathbb{R},\, n \in \mathbb{N}, \tag{7.5}$$

$f(x) = \lim\limits_{n\to\infty} f_n(x) = 0$. Then $f'(x) = 0$, and

$$f_n'(x) = \sqrt{n}\cos(nx),$$

so that (f_n') does not converge to f'. For instance

$$f_n'(0) = \sqrt{n} \xrightarrow[n\to\infty]{} +\infty$$

as $n \to \infty$, whereas $f'(0) = 0$. Note that (f_n) converges uniformly to 0.
(d) Let

$$f_n(x) = n^2\, x(1 - x^2)^n, \quad x \in [0,1],\ n \in \mathbb{N}. \tag{7.6}$$

For $0 < x \le 1$ we have $\lim_{n\to\infty} f_n(x) = 0$. Since $f_n(0) = 0$ we see that

$$\lim_{n\to\infty} f_n(x) = 0 \quad x \in [0,1]. \tag{7.7}$$

A simple calculation shows that

$$\int_0^1 x(1 - x^2)^n \,\mathrm{d}x = \frac{1}{2n+2}.$$

Thus

$$\int_0^1 f_n(x)\,\mathrm{d}x = \frac{n^2}{2n+2} \to +\infty$$

as $n \to \infty$. If, in (7.6), we replace n^2 by n, (7.7) still holds, but we now have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \frac{n}{2n+2} = \frac{1}{2},$$

whereas

$$\int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) \mathrm{d}x = 0.$$

Thus the limit of the integral need not to be equal the integral of the limit, even if both are finite.

After these examples, which show what can go wrong if limit processes are interchanged carelessly, we now define a new notion of convergence, stronger than pointwise convergence as defined in Definition 7.1, which will enable us to arrive at positive results.

7.2 Uniform Convergence

Definition 7.2 We say that a sequence of functions (f_n) converges *uniformly* on E to a function f if for every $\varepsilon > 0$ there is a positive integer n_0 such that $n \ge n_0$ implies

$$|\, f_n(x) - f(x)\, | \le \varepsilon \tag{7.8}$$

for all $x \in E$. We write $f_n \rightrightarrows f$ on E.

It is clear that every uniformly convergent sequence is pointwise convergent. Quite explicitly, the difference between the two concepts is this: If (f_n) converges pointwise on E to a function f, for every $\varepsilon > 0$ and for every $x \in E$, there exists an integer n_0 depending on both ε *and* $x \in E$ such that (7.8) holds if $n \ge n_0$. If (f_n) converges uniformly on E it is possible, for each $\varepsilon > 0$ to find *one* integer n_0 which will do for *all* $x \in E$.

We say that the series $\sum f_n(x)$ converges *uniformly* on E if the sequence $(s_n(x))$ of partial sums defined by

$$s_n(x) = \sum_{i=1}^{n} f_i(x)$$

converges uniformly on E.

Proposition 7.1 (Cauchy criterion) (a) *The sequence of functions (f_n) defined on E converges uniformly on E if and only if for every $\varepsilon > 0$ there is an integer n_0 such that $n, m \geq n_0$ and $x \in E$ imply*

$$|\, f_n(x) - f_m(x) \,| \leq \varepsilon. \tag{7.9}$$

(b) The series of functions $\sum_{k=1}^{\infty} g_k(x)$ defined on E converges uniformly on E if and only if for every $\varepsilon > 0$ there is an integer n_0 such that $n, m \geq n_0$ and $x \in E$ imply

$$\left| \sum_{k=m}^{n} g_k(x) \right| \leq \varepsilon.$$

Proof. Suppose (f_n) converges uniformly on E and let f be the limit function. Then there is an integer n_0 such that $n \geq n_0$, $x \in E$ implies

$$|\, f_n(x) - f(x) \,| \leq \frac{\varepsilon}{2},$$

so that

$$|\, f_n(x) - f_m(x) \,| \leq |\, f_n(x) - f(x) \,| + |\, f_m(x) - f(x) \,| \leq \varepsilon$$

if $m, n \geq n_0$, $x \in E$.

Conversely, suppose the Cauchy condition holds. By Proposition 2.16, the sequence $(f_n(x))$ converges for every x to a limit which may we call $f(x)$. Thus the sequence (f_n) converges pointwise on E to f. We have to prove that the convergence is uniform. Let $\varepsilon > 0$ be given, choose n_0 such that (7.9) holds. Fix n and let $m \to \infty$ in (7.9). Since $f_m(x) \to f(x)$ as $m \to \infty$ this gives

$$|\, f_n(x) - f(x) \,| \leq \varepsilon$$

for every $n \geq n_0$ and $x \in E$.

(b) immediately follows from (a) with $f_n(x) = \sum_{k=1}^{n} g_k(x)$. ■

Example 7.2 Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \tag{7.10}$$

If $f(x)$ exists, so does $f(x+2\pi) = f(x)$, and $f(0) = 0$. We will show that the series converges uniformly on $[\delta, 2\pi - \delta]$ for every $\delta > 0$. For, put

$$s_n(x) = \sum_{k=1}^{n} \sin kx = \operatorname{Im}\left(\sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}kx}\right).$$

If $\delta \le x \le 2\pi - x$ we have

$$|\, s_n(x)\,| \le \left| \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}kx} \right| = \left| \frac{\mathrm{e}^{\mathrm{i}(n+1)x} - \mathrm{e}^{\mathrm{i}x}}{\mathrm{e}^{\mathrm{i}x} - 1} \right| \le \frac{2}{|\,\mathrm{e}^{\mathrm{i}x/2} - \mathrm{e}^{-\mathrm{i}x/2}\,|} = \frac{1}{\sin\frac{x}{2}} \le \frac{1}{\sin\frac{\delta}{2}}.$$

Note that $|\operatorname{Im} z\,| \le |\,z\,|$ and $\left|\mathrm{e}^{\mathrm{i}x}\right| = 1$. Since $\sin\frac{x}{2} \ge \sin\frac{\delta}{2}$ for $\delta/2 \le x/2 \le \pi - \delta/2$ we have for $0 < m < n$

$$\begin{aligned}
\left| \sum_{k=m}^{n} \frac{\sin kx}{k} \right| &= \left| \sum_{k=m}^{n} \frac{s_k(x) - s_{k-1}(x)}{k} \right| \\
&= \left| \sum_{k=m}^{n} s_k(x) \left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{s_n(x)}{n+1} - \frac{s_{m-1}(x)}{m} \right| \\
&\le \frac{1}{\sin\frac{\delta}{2}} \left(\left| \sum_{k=m}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{1}{n+1} \right| + \left| \frac{1}{m} \right| \right) \\
&\le \frac{1}{\sin\frac{\delta}{2}} \left(\frac{1}{m} - \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{m} \right) \le \frac{2}{m\sin\frac{\delta}{2}}
\end{aligned}$$

The right side becomes arbitrary small as $m \to \infty$. Using Proposition 7.1 (b) uniform convergence of (7.10) on $[\delta, 2\pi - \delta]$ follows. Moreover f is continuous on $[\delta, 2\pi - \delta]$ by Theorem 7.4 below.
In the section on Fourier series we will see that

$$f(x) = \begin{cases} 0, & x = 0, \\ \frac{\pi - x}{2}, & x \in (0, 2\pi), \end{cases}$$

and $f(x + 2n\pi) = f(x)$ for all $n \in \mathbb{Z}$.

The following criterion is sometimes useful.

Proposition 7.2 *Suppose*

$$\lim_{n\to\infty} f_n(x) = f(x), \quad x \in E.$$

Put

$$M_n = \sup_{x\in E} |\, f_n(x) - f(x)\,|\,.$$

Then $f_n \rightrightarrows f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Proof. Suppose the criterion of the proposition is satisfied. To $\varepsilon > 0$ choose n_0 such that $n \geq n_0$ implies $M_n < \varepsilon$. Then we have

$$| f_n(x) - f(x) | \leq \varepsilon, \quad n \geq n_0, \quad x \in E,$$

and (f_n) converges uniformly to f. The other direction is also immediate. ∎

Theorem 7.3 (Weierstraß) *Suppose (f_n) is a sequence of functions defined on E, and suppose*

$$| f_n(x) | \leq M_n, \quad x \in E, \, n \in \mathbb{N}. \tag{7.11}$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Proof. If $\sum M_n$ converges, then, for arbitrary $\varepsilon > 0$ there exists n_0 such that $m, n \geq n_0$ implies

$$\left| \sum_{i=m}^{n} f_i(x) \right| \leq \sum_{i=m}^{n} M_i \leq \varepsilon, \quad x \in E.$$

Uniform convergence now follows from Proposition 7.1. ∎

7.2.1 Uniform Convergence and Continuity

Theorem 7.4 *Let $E \subset \mathbb{C}$ be a subset and $f_n \colon E \to \mathbb{C}$, $n \in \mathbb{N}$, be a sequence of continuous functions on E uniformly converging to some function $f \colon E \to \mathbb{C}$.*
Then f is continuous on E.

Proof. Let $a \in E$ and $\varepsilon > 0$ be given. Since $f_n \rightrightarrows f$ there is an $r \in \mathbb{N}$ such that

$$| f_r(x) - f(x) | \leq \varepsilon/3 \quad \text{for all} \quad x \in E.$$

Since f_r is continuous on E, there exists $\delta > 0$ such that $| x - a | < \delta$ implies

$$| f_r(x) - f(a) | \leq \varepsilon/3.$$

Hence $| x - a | < \delta$ implies

$$| f(x) - f(a) | \leq | f(x) - f_r(x) | + | f_r(x) - f_r(a) | + | f_r(a) - f(a) | \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This proves the assertion. ∎

The converse is not true; that is, a sequence of continuous functions may converge to a continuous function; although the convergence is not uniform. Example 7.1 (e) is of this kind. Example 7.1 (b) and (c) show that continuity of f_n alone (without uniform convergence) may give a discontinuous limit function f.
We give a first application to power series.

Proposition 7.5 *Let*

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C} \tag{7.12}$$

be a power series with radius of convergence $R > 0$.
(a) Then (7.12) converges uniformly on the compact disc $\{z \mid |z| \le r\}$ for every r with $0 < r < R$.
(b) The power series

$$\sum_{n=0}^{\infty} n a_n z^{n-1}$$

has the same radius of convergence R as the series (7.12) and hence also converges uniformly on the compact disc $\{z \mid |z| \le r\}$.

Proof. Choose s such that

$$r < s < R \iff \frac{1}{r} > \frac{1}{s} > \frac{1}{R} \tag{7.13}$$

Since $\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|} = 1/R$, Homework 6.3 (a) shows that

$$\sqrt[n]{|a_n|} \le \frac{1}{s} \quad \text{for all but finitely many } n.$$

Hence $|a_n| < 1/s^n$ if $n \ge n_0$, and we can replace

$$|a_n z^n| \le \frac{1}{s^n} |z|^n \le \left(\frac{r}{s}\right)^n = q^n$$

if $n \ge n_0$ where $0 < q < 1$. By Theorem 7.3 and convergence of the geometric series, (7.12) converges uniformly on the disc $|z| \le r < R$.
(b) This simply follows from the fact that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{(n+1)\,|a_{n+1}|} = \lim_{n\to\infty} \sqrt[n]{n+1}\, \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \alpha.$$

∎

Remark 7.1 (a) A power series defines a continuous functions on $|z| < R$.
(b) Note that the power series in general does *not* converge uniformly on the whole open disc of convergence $|z| < R$. As an example, consider the geometric series

$$f(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \quad |z| < 1.$$

To $\varepsilon = 1$ and every $n \in \mathbb{N}$ choose $z_n = \frac{n}{n+1}$ and we obtain, using Bernoulli's inequality,

$$z_n^n = \left(1 - \frac{1}{n+1}\right)^n \ge 1 - n\frac{1}{n+1} = 1 - z_n, \quad \text{hence} \quad \frac{z_n^n}{1-z_n} \ge 1. \tag{7.14}$$

so that

$$| s_{n-1}(z_n) - f(z_n) | = \left| \sum_{k=0}^{n-1} z_n^k - \frac{1}{1-z_n} \right| = \left| \sum_{k=n}^{\infty} z_n^k \right| = \frac{z_n^n}{1-z_n} \underset{(7.14)}{\geq} 1.$$

The geometric series doesn't converge uniformly on the whole open unit disc.

Definition 7.3 If X is a metric space $\mathrm{C}(X)$ will denote the set of all continuous, bounded functions with domain X. We associate with each $f \in \mathrm{C}(X)$ its *supremum norm*

$$\|f\|_\infty = \|f\| = \sup_{x \in X} | f(x) | . \tag{7.15}$$

Since f is assumed to be bounded, $\|f\| < \infty$. Note that boundedness of X is redundant if X is a *compact* metric space (Proposition 6.22). Thus $\mathrm{C}(X)$ contains of all continuous functions in that case.

It is clear that $\mathrm{C}(X)$ is a vector space since the sum of bounded functions is again a bounded function (see the triangle inequality below) and the sum of continuous functions is a continuous function (see Proposition 6.17). We show that $\|f\|_\infty$ is indeed a norm on $\mathrm{C}(X)$.

(i) Obviously, $\|f\|_\infty \geq 0$ since the absolute value $| f(x) |$ is nonnegative. Further $\|0\| = 0$. Suppose now $\|f\| = 0$. This implies $| f(x) | = 0$ for all x; hence $f = 0$.

(ii) Clearly, for every (real or complex) number λ we have

$$\|\lambda f\| = \sup_{x \in X} | \lambda f(x) | = | \lambda | \sup_{x \in X} | f(x) | = | \lambda | \, \|f\| .$$

(iii) If $h = f + g$ then

$$| h(x) | \leq | f(x) | + | g(x) | \leq \|f\| + \|g\| , \quad x \in X;$$

hence

$$\|f + g\| \leq \|f\| + \|g\| .$$

We have thus made $\mathrm{C}(X)$ into a normed vector space. Proposition 7.2 can be rephrased as

> *A sequence (f_n) converges to f with respect to the norm in $\mathrm{C}(X)$ if and only if $f_n \to f$ uniformly on X.*

Accordingly, closed subsets of $\mathrm{C}(X)$ are sometimes called *uniformly closed*, the closure of a set $A \subset \mathrm{C}(X)$ is called the *uniform closure*, and so on.

Theorem 7.6 *The above norm makes $\mathrm{C}(X)$ into a Banach space (a complete normed space).*

Proof. Let (f_n) be a Cauchy sequence of $\mathrm{C}(X)$. This means to every $\varepsilon > 0$ corresponds an $n_0 \in \mathbb{N}$ such that $n, m \geq n_0$ implies $\|f_n - f_m\| < \varepsilon$. It follows by Proposition 7.1 that there is a function f with domain X to which (f_n) converges uniformly. By Theorem 7.4,

f is continuous. Moreover, f is bounded, since there is an n such that $|\, f(x) - f_n(x)\,| < 1$ for all $x \in X$, and f_n is bounded.
Thus $f \in \mathrm{C}(X)$, and since $f_n \to f$ uniformly on X, we have $\|f - f_n\| \to 0$ as $n \to \infty$. ∎

7.2.2 Uniform Convergence and Integration

Theorem 7.7 *Let α be an increasing function on $[a,b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ for all $n \in \mathbb{N}$ and suppose $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and*

$$\int_a^b f \,\mathrm{d}\alpha = \lim_{n\to\infty} \int_a^b f_n \,\mathrm{d}\alpha. \tag{7.16}$$

Proof. Put

$$\varepsilon_n = \sup_{x\in[a,b]} |\, f_n(x) - f(x)\,|\,.$$

Then

$$f_n - \varepsilon_n \le f \le f_n + \varepsilon_n,$$

so that the upper and the lower integrals of f satisfy

$$\int_a^b (f_n - \varepsilon_n)\,\mathrm{d}\alpha \le \underline{\int} f \,\mathrm{d}\alpha \le \overline{\int} f \,\mathrm{d}\alpha \le \int_a^b (f_n + \varepsilon_n)\,\mathrm{d}\alpha. \tag{7.17}$$

Hence,

$$0 \le \overline{\int} f \,\mathrm{d}\alpha - \underline{\int} f \,\mathrm{d}\alpha \le 2\varepsilon_n(\alpha(b) - \alpha(a)).$$

Since $\varepsilon_n \to 0$ as $n \to \infty$ (Proposition 7.2), the upper and the lower integrals of f are equal. Thus $f \in \mathcal{R}(\alpha)$. Another application of (7.17) yields

$$\left| \int_a^b f \,\mathrm{d}\alpha - \int_a^b f_n \,\mathrm{d}\alpha \right| \le \varepsilon_n((\alpha(b) - \alpha(a)).$$

This implies (7.16). ∎

Corollary 7.8 *If $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ and if the series*

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad a \le x \le b$$

converges uniformly on $[a,b]$, then

$$\int_a^b f \,\mathrm{d}\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \,\mathrm{d}\alpha.$$

In other words, the series may be integrated term by term.

Example 7.3 (a) For every real $t \in (-1,1)$ we have

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} \mp \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n. \tag{7.18}$$

Proof. In Homework 13.5 (a) there was computed the Taylor series

$$T(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

of $\log(1+x)$ and it was shown that $T(x) = \log(1+x)$ if $x \in (0,1)$.
By Proposition 7.5 the geometric series $\sum_{n=0}^{\infty} (-1)^n x^n$ converges uniformly to the function $\frac{1}{1+x}$ on $[-r,r]$ for all $0 < r < 1$. By Theorem 7.7 we have for all $t \in [-r,r]$

$$\begin{aligned}\log(1+t) = \log(1+x)\big|_0^t &= \int_0^t \frac{\mathrm{d}x}{1+x} = \int_0^t \sum_{n=0}^{\infty} (-1)^n x^n \,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \int_0^t (-1)^n x^n \,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}\bigg|_0^t = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n\end{aligned}$$

■

(b) For $|\,t\,| < 1$ we have

$$\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} \mp \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} \tag{7.19}$$

As in the previous example we use the uniform convergence of the geometric series on $[-r,r]$ for every $0 < r < 1$ that allows to exchange integration and summation

$$\arctan t = \int_0^t \frac{\mathrm{d}x}{1+x^2} = \int_0^t \sum_{n=0}^{\infty} (-1)^n x^{2n} \,\mathrm{d}x = \sum_{n=0}^{\infty} (-1)^n \int_0^t x^{2n} \,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} t^{2n+1}.$$

Note that you are, in general, not allowed to insert $t = 1$ into the equations (7.18) and (7.19). However, the following proposition (the proof is in the appendix to this chapter) fills this gap.

Proposition 7.9 (Abel's Limit Theorem) *Let* $\sum_{n=0}^{\infty} a_n$ *a convergent series of real numbers.*
Then the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for $x \in [0,1]$ *and is continuous on* $[0,1]$.

As a consequence of the above proposition we have

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n},$$
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{2n+1}.$$

Q 22. Define $f_n(x) = \dfrac{x}{n^2} \mathrm{e}^{-x/n}$, $n \in \mathbb{N}$, $x \in \mathbb{R}_+$.
Prove that $(f_n(x))$ converges uniformly to 0 on $\mathbb{R}_+$. However

$$\lim_{n\to\infty} \int_0^{\infty} f_n \,\mathrm{d}x = 1.$$

Hint. Use $\mathrm{e}^z > 1 + z + z^2/2$ for $z \in \mathbb{R}_+$ and consider when seeking a suitable n the two cases $x < n$ and $x > n$ parallel.
Solution. Since $\mathrm{e}^z > 1 + z + z^2/2$ we have for $x \geq 0$

$$\left| \frac{x}{n^2 \mathrm{e}^{\frac{x}{n}}} \right| \leq \left| \frac{x}{n^2(1 + \frac{x}{n} + \frac{x^2}{2n})} \right| \leq \left| \frac{x}{n^2 + nx + x^2/2} \right|.$$

Given $\varepsilon > 0$ choose $n > 1/\varepsilon$. In case $x \leq n$ use $n^2 + nx + x^2/2 \geq n^2$ and therefore

$$\left| \frac{x}{n^2 + nx + x^2/2} \right| \leq \frac{x}{n^2} \leq \frac{1}{n} < \varepsilon. \tag{7.20}$$

In case $x > n$, $n^2 + nx + x^2/2 \geq x^2/2$, and we have

$$\left| \frac{x}{n^2 + nx + x^2/2} \right| \leq \frac{x}{x^2/2} \leq \frac{2}{x} \leq \frac{2}{n} < 2\varepsilon. \tag{7.21}$$

By (7.20) and (7.21) we have for $n \geq 1/\varepsilon$ and all $x \geq 0$, $|\, f_n(x) \,| < 2\varepsilon$ which shows uniform convergence of (f_n) to 0 on $\mathbb{R}_+$.
Integration by parts with $u = x$, $u' = 1$, $v' = \mathrm{e}^{-x/n}$, and $v = -n\mathrm{e}^{-x/n}$ gives

$$\begin{aligned}\int_0^R \frac{x\mathrm{e}^{-x/n}}{n^2} \,\mathrm{d}x &= \frac{1}{n^2} \left(-nx\mathrm{e}^{-x/n}\Big|_0^R + n \int_0^R \mathrm{e}^{-x/n} \,\mathrm{d}x \right) \\ &= \frac{1}{n^2} \left(-nR\mathrm{e}^{-R/n} + n\left(-n\mathrm{e}^{-R/n} + n\right)\right) \\ &= 1 - \frac{n+R}{n} \mathrm{e}^{-R/n}.\end{aligned}$$

One can see that for every fixed R the term on the right tends to 0 as $n \to \infty$. On the other hand

$$\int_0^{\infty} f_n \,\mathrm{d}x = \lim_{R\to\infty} \left(1 - \frac{n+R}{n} \mathrm{e}^{-R/n} \right) = 1,$$

which proves the second assertion.

Q 23. Compute the sum of the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^3} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\cos nx}{n^4}, \qquad x \in \mathbb{R}.$$

Q 24. For $x \in (-1, 1)$ compute the sum of the series

$$\sum_{n=1}^{\infty} n^2 x^n, \quad \sum_{n=1}^{\infty} n^3 x^n, \quad \sum_{n=0}^{\infty} \frac{x^n}{n}.$$

7.2.3 Uniform Convergence and Differentiation

We have already seen, in Example 7.1 (c) that uniform convergence of (f_n) implies nothing about the sequence (f_n'). Thus stronger hypothesis are required for for the assertion that $f_n \to f$ implies $f_n' \to f'$.

Theorem 7.10 *Suppose (f_n) is a sequence of continuously differentiable functions on $[a, b]$ (pointwise) converging to some function f. Suppose further that (f_n') converges uniformly on $[a, b]$.*
Then f is differentiable and on $[a, b]$, and

$$f'(x) = \lim_{n\to\infty} f_n'(x), \qquad a \le x \le b. \tag{7.22}$$

Proof. Put $g(x) = \lim_{n\to\infty} f_n'(x)$, then g is continuous by Theorem 7.4. By the Theorem 5.14 Thm. 5.14 ,

$$f_n(x) = f_n(a) + \int_a^x f_n'(t)\,\mathrm{d}t.$$

By Theorem 7.7 the sequence

$$\left(\int_a^x f_n'(t)\,\mathrm{d}t \right)$$

converges on $[a, b]$ to $\int_a^x g(t)\,\mathrm{d}t$. Therefore,

$$f(x) = f(a) + \int_a^x g(t)\,\mathrm{d}t.$$

Differentiation yields $f'(x) = g(x)$ (Theorem 5.14 Thm. 5.14), which completes the proof.

∎

Remark 7.2 (a) Note that under the assumption of the theorem it follows that (f_n) uniformly converges to f.
(b) For a more general result (without the additional assumption of continuity of f_n') see [8, 7.17 Theorem].

Corollary 7.11 *Let $f(x) = \sum a_n x^n$ be a power series with radius R of convergence. Then for all x with $x \in (-R, R)$ we have*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}. \tag{7.23}$$

Proof. This is immediate from Theorem 7.10 and Proposition 7.5 (b). ∎

Example 7.4 For $x \in (-1, 1)$ we have

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$

Since the geometric series $f(x) = \sum_{n=0}^{\infty} x^n$ equals $1/(1-x)$ on $(-1, 1)$ by Corollary 7.11 we have

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \sum_{n=1}^{\infty} (x^n)' = \sum_{n=1}^{\infty} n x^{n-1}.$$

Multiplying the preceding equation by x gives the result.

Corollary 7.12 *Let $f(x) = \sum a_n x^n$ be a power series with radius R of convergence. Then f is infinitely often differentiable on the interval $(-R, R)$ and we have*

$$a_n = \frac{1}{n!} f^{(n)}(0), \quad n \in \mathbb{N}_0. \tag{7.24}$$

In particular, f coincides with its Taylor series.

Proof. Iterated application of Corollary 7.11 yields

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k}.$$

In particular,

$$f^{(k)}(0) = k! a_k, \quad \text{that is} \quad a_k = \frac{f^{(k)}(0)}{k!}.$$

These are exactly the Taylor coefficients of f. ∎

Note that these two corollaries are also true for complex power series. However, we haven't defined complex differentiability, yet. We obtain that $f(z) = \sum_n a_n z^n$ is infinitely often differentiable on the disc $D = \{z \mid |z| < R\}$; and f coincides on D with its Taylor series around 0.

7.3 Stone–Weierstraß Theorem

The main part of this section is to prove that a continuous function on a compact interval can be uniformly approximated by polynomials. This theorem has a nice generalization: A characterization of certain dense subalgebras of $\mathrm{C}(X)$ where X is an arbitrary compact set.
A linear subspace A of $\mathrm{C}(X)$ is a called a *subalgebra* if A contains the constant function 1 and $f, g \in A$ implies $f \cdot g \in A$.
We say *A separates the points of X* if given any two different points x and y in X there is a function $f \in A$ such that $f(x) \neq f(y)$.

Example 7.5 (a) If $X = [0, 1]$, the set of polynomials form a subalgebra of $\mathrm{C}(X)$ since the product of two polynomials is again a polynomial. Also, the convergent on $[0, 1]$ power series $f(x) = \sum a_n x^n$ form a subalgebra of $\mathrm{C}(X)$.
The polynomials f in the variables x and $1/x$, i. e. $f(x) = \sum_{n=-m}^{m} a_n x^n$ form a subalgebra of $\mathrm{C}([1, 2])$.
(b) Let $X = [0, 1] \times [0, 1]$ and $A = \{f \in \mathrm{C}(X) \mid f(x, y) = g(x+y), g \in \mathrm{C}([0, 2])\}$. Then A is a subalgebra of $\mathrm{C}(X)$ which does not separate the points of X since $f((0, 1)) = f((1, 0))$ for all $f \in A$. We need more functions in A to separate the points of X.
(c) Let X as above and consider

$$A = \{f \in \mathrm{C}(X) \mid f_1(x, y) = g(x+y) \quad \text{or} \quad f_2(x, y) = g(x-y),\ g \in \mathrm{C}([0, 2])\}_{\mathrm{alg}}.$$

Then A separates the points of X. For, if (x_1, y_1) and (x_2, y_2) satisfy $x_1 + y_1 \neq x_2 + y_2$ choose $g(x) = x$ and we find $f_1(x_1, y_1) \neq f(x_2, y_2)$. If $x_1 - y_1 \neq x_2 - y_2$ then $f_2(x_1, y_1) \neq f_2(x_2, y_2)$. However, if $x_1 + y_1 = x_2 + y_2$ and $x_1 - y_1 = x_2 - y_2$, then the points are equal.

Our first goal is to show that every closed subalgebra A of $\mathrm{C}(X)$ contains square roots, more precisely, $f \in A$ implies $\sqrt{f} \in A$.
For, note that the binomial series

$$\sqrt{1-t} = (1-t)^{\frac{1}{2}} = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} t^n = \sum_{n=0}^{\infty} a_n t^n$$

where

$$a_n = \binom{\frac{1}{2}}{n} = \frac{\frac{1}{2} \cdot \left(\frac{1}{2} - 1\right) \cdots \left(\frac{1}{2} - n + 1\right)}{n!} = (-1)^{n-1} \frac{1}{2^{2n-1}(2n-1)} \binom{2n-1}{n}.$$

converges uniformly on $[0, 1]$ since by Stirling's formula there exists a constant C with

$$a_n \leq C \frac{1}{n^{\frac{3}{2}}}, \quad n \in \mathbb{N},$$

see Homework 21.3. Note that it is easy to see that the binomial series has radius of convergence $R = 1$ (using the quotient test), so that it converges uniformly on any closed subinterval of $(-1, 1)$. But we need more, the uniform convergence on $[0, 1]$.

Proposition 7.13 *Let X be a compact metric space and A a (uniformly) closed subalgebra of* $\mathrm{C}(X)$. *If $f \in A$, $f \geq 0$, then $\sqrt{f} \in A$.*

Proof. Without loss of generality we may assume that $0 \leq f \leq 1$ (replace f by $f/\|f\|_\infty$). Put $g = 1 - f$, then $f = 1 - g$ and $0 \leq g \leq 1$. We have

$$\sqrt{f(x)} = \sqrt{1 - g(x)} = 1 - \sum_{n=1}^{\infty} a_n g(x)^n \tag{7.25}$$

Since $|\, a_n g(x)^n \,| \leq C n^{-3/2}$ and $\sum n^{-3/2}$ converges, Theorem 7.3 ensures the uniform convergence of (7.25). Since all partial sums are in A and A is uniformly closed, $\sqrt{f} \in A$. ∎

Let us denote by $\mathrm{C}(X, \mathbb{R})$ and $\mathrm{C}(X, \mathbb{C})$ the space of bounded continuous real valued and complex valued functions on X, respectively.

Theorem 7.14 (Stone–Weierstraß) *Let X be a compact metric space and A a uniformly closed subalgebra of* $\mathrm{C}(X, \mathbb{R})$ *such that A separates the points of X.*
Then $A = \mathrm{C}(X, \mathbb{R})$.

Proof. By Proposition 7.13 $f \in A$ implies $|\, f \,| = \sqrt{f^2} \in A$. Therefore $f, g \in A$ implies that

$$\min(f, g) = \frac{1}{2}(f + g - |\, f - g \,|), \quad \max(f, g) = \frac{1}{2}(f + g + |\, f - g \,|)$$

are also contained in A.
Let $x, y \in X$ be different points; by assumption there exists $h \in A$ such that $h(x) \neq h(y)$. Since $1 \in A$ for every $\lambda, \mu \in \mathbb{R}$ we see that

$$g(t) = \mu + (\lambda - \mu) \frac{h(t) - h(y)}{h(x) - h(y)}$$

is in A and $g(x) = \lambda$, $g(y) = \mu$.
Now let $\varepsilon > 0$, $f \in \mathrm{C}(X)$, and $x, y \in X$, $x \neq y$, be given. By the above argument there exists a function $f_{x,y} \in A$ such that $f_{x,y}(x) = f(x)$ and $f_{x,y}(y) = f(y)$. Then

$$U_y = \{\xi \in X \mid f_{x,y}(\xi) < f(\xi) + \varepsilon\}$$

is an open neighborhood of y (We can write

$$U_y = \{\xi \mid (f_{x,y} - f)(\xi) < \varepsilon\} = (f_{x,y} - f)^{-1}((-\infty, \varepsilon)).$$

Since $f_{x,y} - f$ is continuous and (∞, ε) is open, the preimage U_y is open. Since $f_{x,y}(y) = f(y) < f(y) + \varepsilon$, $y \in U_y$.) Since X is compact there exist $y_1, \dots, y_n \in X$ such that $X = \bigcup_j U_{y_j}$. Put $h_x = \min\{f_{x,y_j} \mid j = 1, \dots, n\}$ then $h_x \in A$ and we have $h_x(x) = f(x)$ and $h_x(t) < f(t) + \varepsilon$ for all $t \in X$ (Since $t \in X$ there is some i, $1 \leq i \leq n$, such that $t \in U_{y_i}$. Hence $h_x(t) \leq f_{x,y_i}(t) < f(t) + \varepsilon$.) So far we have found a set of lower bounds $h_x \in A$ which are close to f. We are going to find a close upper bound of f in A. The set

$$V_x = \{\xi \in X \mid h_x(\xi) > f(\xi) - \varepsilon\}$$

is an open neighborhood of x. Since X is compact there exist $x_1, \dots, x_m \in X$ such that $X = \bigcup_{i=1}^m V_{x_i}$; and we set $g = \max\{h_{x_k} \mid k = 1, \dots, m\}$. Then $g \in A$ with

$$f - \varepsilon < g < f + \varepsilon.$$

Hence, $\|f - g\|_\infty < \varepsilon$, i. e. every ε-neighborhood of f contains elements of A (namely g). Thus $f \in \overline{A} = A$. ∎

Theorem 7.15 *Let X be a compact metric space and A a uniformly closed subalgebra of* $\mathrm{C}(X, \mathbb{C})$ *such that*

(i) *A separates the points of X;*
(ii) *$f \in A$ implies $\overline{f} \in A$.*

Then $A = \mathrm{C}(X, \mathbb{C})$.

Proof. Set $A_0 = \{\operatorname{Re} f \mid f \in A\}$. Since $\operatorname{Re} f = \frac{1}{2}(f + \overline{f})$, (ii) implies that A_0 is a subalgebra of A. Since

$$\operatorname{Im} f = \operatorname{Re}(-\mathrm{i} f), \tag{7.26}$$

$A_0 = \{\operatorname{Im} f \mid f \in A\}$ and $A = A_0 + \mathrm{i} A_0$. We show that A_0 separates the points of X. Suppose $x \neq y$ and $f(x) \neq f(y)$ for some $f \in A$. Hence $\operatorname{Re} f(x) + \mathrm{i} \operatorname{Im} f(x) \neq \operatorname{Re} f(y) + \mathrm{i} \operatorname{Im} f(y)$. If $\operatorname{Re} f(x) \neq \operatorname{Re} f(y)$ we are done. Otherwise $\operatorname{Im} f(x) \neq \operatorname{Im} f(y)$. By (7.26), $\operatorname{Re}(-\mathrm{i} f) \in A_0$ separates x and$y$. Since A_0 is a closed subalgebra of $\mathrm{C}(X, \mathbb{R})$ separating the points of X, from Theorem 7.14 it follows that $A_0 = \mathrm{C}(X, \mathbb{R})$. Therefore, $A = \mathrm{C}(X, \mathbb{R}) + \mathrm{i}\, \mathrm{C}(X, \mathbb{R}) = \mathrm{C}(X, \mathbb{C})$. ∎

Theorem 7.16 (Approximation Theorem of Weierstraß) *Let X be a nonempty compact subset of $\mathbb{R}^n$. Then any continuous function on X is the uniform limit of polynomials on X.*

Proof. We consider the cases of real-valued and the complex-valued continuous functions at the same time. Let $x = (x_1, \dots, x_n) \in X$. Put

$$P(X, \mathbb{R}) = \{f \in \mathrm{C}(X, \mathbb{R}) \mid f(x) = \sum_{\text{finite}} a_{k_1 \cdots k_n}\, x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}, x \in X,\ k_i \in \mathbb{N}_0, a_{k_1 \cdots k_n} \in \mathbb{R})\}$$

$$P(X, \mathbb{C}) = \{f \in \mathrm{C}(X, \mathbb{C}) \mid f(x) = \sum_{\text{finite}} a_{k_1 \cdots k_n}\, x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}, x \in X,\ k_i \in \mathbb{N}_0,\ a_{k_1 \cdots k_n} \in \mathbb{C}\}$$

and let A be the uniform closure of $P(X)$. Then A is a closed subalgebra of $\mathrm{C}(X)$. Note that both $P(X, \mathbb{R})$ and $P(X, \mathbb{C})$ separate the points of X: If $y \neq z$ are two different points in X. Then $z = (z_1, \dots, z_n)$ and $y = (y_1, \dots, y_n)$ differ at at least one coordinate, say $z_i \neq y_i$. Then the polynomial $p(x) = x_i$ separates z from y. Theorem 7.14 shows that

$A = \mathrm{C}(X, \mathbb{R})$.
Let $f(x) = \sum_{\text{finite}} a_{k_1\cdots k_n}\, x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \in P(X, \mathbb{C})$ with complex coefficients $a_{k_1\cdots k_n}$. Then

$$\overline{f}(x) = \overline{f(x)} = \sum_{\text{finite}} \overline{a}_{1\cdots k_n}\, x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

is also in $P(X, \mathbb{C})$. By Theorem 7.15 $A = \mathrm{C}(X, \mathbb{C})$. ∎

The same statement is true for compact subsets $X \subset \mathbb{C}^n$.

Remarks 7.3 (a) If X is a nonempty compact subset of $\mathbb{C}^1$ a (complex-valued) continuous function on X can be uniformly approximated by complex polynomials in z and $\overline{z}$:

$$Q(X) = \left\{ f \in \mathrm{C}(X) \mid f(z) = \sum_{j,k=0}^{n} a_{jk} z^j\, \overline{z}^k,\ n \in \mathbb{N}_0,\ a_{jk} \in \mathbb{C} \right\}.$$

Note that $z = x + \mathrm{i}y$, $\overline{z} = x - \mathrm{i}y$, $x = \frac{1}{2}(z + \overline{z})$ and $y = \frac{1}{2\mathrm{i}}(z - \overline{z})$ with real x, y gives a bijection between $Q(X)$ and $P(X, \mathbb{C})$, where X is viewed as a subset of $\mathbb{R}^2$.
(b) If $\mathrm{S}^1 = \{z \in \mathbb{C} \mid |\, z \,| = 1\}$ is the unit circle, any continuous function on S^1 can be uniformly approximated by the functions

$$T(\mathrm{S}) = \left\{ f \in \mathrm{C}(\mathrm{S}^1) \mid f(z) = \sum_{k=-n}^{n} a_k z^k \mid n \in \mathbb{N}_0,\ a_k \in \mathbb{C} \right\}.$$

This is immediate from (a) and $\overline{z} = z^{-1}$ on S^1.
(c) A complex-valued function on $\mathbb{R}$ is said to be *periodic* with period L if

$$f(x + L) = f(x), \quad \text{for all} \quad x \in \mathbb{R}.$$

Obviously, $f(x + nL) = f(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Changing the variable we can restrict ourselves to periodic functions with period 2π. Indeed, if f has period L then

$$F(x) = f\left(\frac{L}{2\pi}\, x\right)$$

has period 2π. From F we come back to f using the formula $f(x) = F(2\pi x/L)$. In the remainder of this chapter every periodic function has period 2π.
Any complex-valued periodic continuous function on $\mathbb{R}$ is the uniform limit of trigonometric polynomials

$$T(\mathbb{C}) = \left\{ f \in \mathrm{C}(\mathbb{R}) \mid f(x) = \sum_{k=-n}^{n} a_k \mathrm{e}^{\mathrm{i}kx} \mid n \in \mathbb{N}_0,\ a_k \in \mathbb{C} \right\}$$

This is immediate from (b) since any periodic function $f(x)$ on $\mathbb{R}$ can be identified using $z = \mathrm{e}^{\mathrm{i}x}$ with a continuous function $g(z) = f(x)$ on the unit circle S. For example, $g(z) = z^2$ on the circle S^1 corresponds to the periodic function

$$f(x) = g(z) = z^2 = \left(\mathrm{e}^{\mathrm{i}x}\right)^2 = \mathrm{e}^{2\mathrm{i}x} = \cos 2x + \mathrm{i}\sin 2x = (\cos x + \mathrm{i}\sin x)^2,$$

The function $\cos x$ on $\mathbb{R}$ corresponds to $(z+1/z)/2$.
(d) Any real-valued periodic continuous function on $\mathbb{R}$ is the uniform limit of trigonometric polynomials

$$T(\mathbb{R}) = \{f \in \mathrm{C}(\mathbb{R}) \mid f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx, \quad a_k, b_k \in \mathbb{R}\}.$$

This is immediate from (c) and $T(\mathbb{R}) = \mathrm{Re}\,(T(\mathbb{C})) = \mathrm{Im}\,(T(\mathbb{C}))$.

7.4 Fourier Series

In this section we consider basic notions and results of the theory of Fourier series. The question is to write a periodic function as a series of $\cos kx$ and $\sin kx$, $k \in \mathbb{N}$. In contrast to Taylor expansions the periodic function need not to be infinitely often differentiable. Two Fourier series may have the same behavior in one interval, but may behave in different ways in some other interval. We have here a very striking contrast between Fourier series and power series.
Special periodic functions are the trigonometric polynomials.

Definition 7.4 A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *trigonometric polynomial* if there are real numbers a_k, b_k, $k = 0, \dots, n$ with

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx. \tag{7.27}$$

The coefficients a_k and b_k are uniquely determined by f since

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,\mathrm{d}x, \quad k = 0, 1, \dots, n, \\ b_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,\mathrm{d}x, \quad k = 1, \dots, n. \end{aligned} \tag{7.28}$$

This is immediate from

$$\begin{aligned} &\int_0^{2\pi} \cos kx\, \sin mx\,\mathrm{d}x = 0, \\ &\int_0^{2\pi} \cos kx\, \cos mx\,\mathrm{d}x = \pi\delta_{km}, \quad k, m \in \mathbb{N}, \\ &\int_0^{2\pi} \sin kx\, \sin mx\,\mathrm{d}x = \pi\delta_{km}, \end{aligned} \tag{7.29}$$

where $\delta_{km} = 1$ if $k = m$ and $\delta_{km} = 0$ if $k \neq m$ is the so called *Kronecker symbol*, see

Homework 21.4. For example, if $m \geq 1$ we have

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\,\mathrm{d}x &= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{a_0}{2}+\sum_{k=1}^{n} a_k\cos kx + b_k \sin kx\right)\cos mx\,\mathrm{d}x\\ &= \frac{1}{\pi}\left(\sum_{k=1}^{n}\int_0^{2\pi}\left(a_k\cos kx\cos mx + b_k\sin kx\cos mx\right)\,\mathrm{d}x\right)\\ &= \frac{1}{\pi}\left(\sum_{k=1}^{n} a_k\pi\delta_{km}\right) = a_m.\end{aligned}$$

Sometimes it is useful to consider complex trigonometric polynomials. Using the formulas expressing $\cos x$ and $\sin x$ in terms of $\mathrm{e}^{\mathrm{i}x}$ and $\mathrm{e}^{-\mathrm{i}x}$ we can write the above polynomial (7.27) as

$$f(x) = \sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}kx}, \tag{7.30}$$

where $c_0 = a_0/2$ and

$$c_k = \frac{1}{2}\left(a_k - \mathrm{i}b_k\right), \quad c_{-k} = \frac{1}{2}\left(a_k + \mathrm{i}b_k\right), \quad k \geq 1.$$

To obtain the coefficients c_k using integration we need the notion of an integral of a complex-valued function, see Section 5.5. If $m \neq 0$ we have

$$\int_a^b \mathrm{e}^{\mathrm{i}mx}\,\mathrm{d}x = \frac{1}{\mathrm{i}m}\mathrm{e}^{\mathrm{i}mx}\Big|_a^b.$$

If $a = 0$ and $b = 2\pi$ and $m \in \mathbb{Z}$ we obtain

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}mx}\,\mathrm{d}x = \begin{cases} 0, & m \in \mathbb{Z}\setminus\{0\},\\ 2\pi, & m = 0.\end{cases} \tag{7.31}$$

We conclude,

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, \quad k = 0, \pm 1, \dots, \pm n.$$

Definition 7.5 Let $f\colon \mathbb{R} \to \mathbb{C}$ be a periodic function with $f \in \mathfrak{R}$ on $[0, 2\pi]$. We call

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, \quad k \in \mathbb{Z} \tag{7.32}$$

the *Fourier coefficients* of f, and the series

$$\sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}kx}, \tag{7.33}$$

i. e. the sequence of partial sums

$$s_n = \sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}kx}, \quad n \in \mathbb{N},$$

the *Fourier series* of f.

The Fourier series can also be written as

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx. \tag{7.34}$$

where a_k and b_k are given by (7.28). One can ask whether the Fourier series of a function converges to the function itself. It is easy to see: If the function f is the *uniform* limit of a series of trigonometric polynomials

$$f(x) = \sum_{k=-\infty}^{\infty} \gamma_k \mathrm{e}^{\mathrm{i}kx} \tag{7.35}$$

then f coincides with its Fourier series. Indeed, since the series (7.35) converges uniformly, by Proposition 7.7 we can change the order of summation and integration and obtain

$$\begin{aligned} c_k &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{m=-\infty}^{\infty} \gamma_m \mathrm{e}^{\mathrm{i}mx} \right) \mathrm{e}^{-\mathrm{i}kx} \,\mathrm{d}x \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{2\pi} \gamma_m \mathrm{e}^{\mathrm{i}(m-k)x} \,\mathrm{d}x = \gamma_k. \end{aligned}$$

In general, the Fourier series of f neither converges uniformly nor pointwise to f. For Fourier series convergence with respect to the L^2-norm

$$\| f \|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |\, f \,|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \tag{7.36}$$

is the appropriate notion.

7.4.1 A Scalar Product on the Periodic Functions

Let V be the linear space of periodic functions $f\colon \mathbb{R} \to \mathbb{C}$, $f \in \mathcal{R}$ on $[0, 2\pi]$. We introduce an inner product on V by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \overline{f(x)}\, g(x) \,\mathrm{d}x, \quad f, g \in V.$$

One easily checks the following properties for $f, g, h \in V$, $\lambda, \mu \in \mathbb{C}$.

$$\begin{aligned} \langle f + g, h \rangle &= \langle f, h \rangle + \langle g, h \rangle, \\ \langle f, g + h \rangle &= \langle f, g \rangle + \langle f, h \rangle, \\ \langle \lambda f, \mu g \rangle &= \overline{\lambda} \mu \, \langle f, g \rangle, \\ \langle f, g \rangle &= \overline{\langle g, f \rangle}. \end{aligned}$$

For every $f \in V$ we have $\langle f, f \rangle = 1/(2\pi) \int_0^{2\pi} |\, f \,|^2 \,\mathrm{d}x \geq 0$. However, $\langle f, f \rangle = 0$ does not imply $f = 0$ (you can change f at finitely many points without any impact on $\langle f, f \rangle$). If $f \in V$ is continuous, then $\langle f, f \rangle = 0$ implies $f = 0$, see Homework 14.3. Put

$\|f\|_2 = \sqrt{\langle f, f \rangle}$.
Note that in the mathematical literature the inner product in $L^2(X)$ is often *linear* in the first component and antilinear in the second component. Define for $k \in \mathbb{Z}$ the periodic function $e_k \colon \mathbb{R} \to \mathbb{C}$ by $e_k(x) = e^{ikx}$, the Fourier coefficients of $f \in V$ take the form

$$c_k = \langle e_k, f \rangle , \quad k \in \mathbb{Z}.$$

From (7.31) it follows that the functions e_k, $k \in \mathbb{Z}$, satisfy

$$\langle e_k, e_l \rangle = \delta_{kl}. \tag{7.37}$$

Any such subset $\{e_k \mid k \in \mathbb{N}\}$ of an inner product space V satisfying (7.37) is called an *orthonormal system (ONS)*. Using $e_k(x) = \cos kx + i \sin kx$ the real orthogonality relations (7.29) immediately follow from (7.37).
Note that the notions of convergent and Cauchy sequences as defined in Section 6.2 still makes sense; however, the limit in V is not unique. Let $f_n, f \in V$, we say that (f_n) *converges to f in* L_2 (denoted by $f_n \underset{\|\cdot\|_2}{\longrightarrow} f$) if

$$\lim_{n\to\infty} \|f_n - f\|_2 = 0.$$

Explicitly

$$\frac{1}{2\pi} \int_0^{2\pi} |\, f_n(x) - f(x) \,|^2 \, dx \underset{n\to\infty}{\longrightarrow} 0.$$

Lemma 7.17 *Suppose $f \in V$ has the Fourier coefficients c_k, $k \in \mathbb{Z}$ and let $\gamma_k \in \mathbb{C}$ be arbitrary. Then*

$$\left\| f - \sum_{k=-n}^{n} c_k e_k \right\|_2^2 \le \left\| f - \sum_{k=-n}^{n} \gamma_k e_k \right\|_2^2 , \tag{7.38}$$

and equality holds if and only if $c_k = \gamma_k$ for all k. Further,

$$\left\| f - \sum_{k=-n}^{n} c_k e_k \right\|_2^2 = \|f\|_2^2 - \sum_{k=-n}^{n} |\, c_k \,|^2 . \tag{7.39}$$

Proof. Let $\sum$ always denote $\sum\limits_{k=-n}^{n}$. Put $g_n = \sum \gamma_k e_k$. Then

$$\langle f, g_n \rangle = \left\langle f, \sum \gamma_k \, e_k \right\rangle = \sum \gamma_k \, \langle f, e_k \rangle = \sum \overline{c_k} \gamma_k$$

and $\langle e_k, g_n \rangle = \gamma_k$ such that

$$\langle g_n, g_n \rangle = \sum |\, \gamma_k \,|^2 .$$

Noting that $|\, a - b \,|^2 = (a-b)(\overline{a} - \overline{b}) = |\, a \,|^2 + |\, b \,|^2 - \overline{a} b - a \overline{b}$, it follows that

$$\begin{aligned} \|f - g_n\|_2^2 &= \langle f - g_n, f - g_n \rangle = \langle f, f \rangle - \langle f, g_n \rangle - \langle g_n, f \rangle + \langle g_n, g_n \rangle \\ &= \|f\|_2^2 - \sum \overline{c_k} \gamma_k - \sum c_k \overline{\gamma_k} + \sum |\, \gamma_k \,|^2 \\ &= \|f\|_2^2 - \sum |\, c_k \,|^2 + \sum |\, \gamma_k - c_k \,|^2 \end{aligned} \tag{7.40}$$

which is evidently minimized if and only if $\gamma_k = c_k$. Inserting this into (7.40), equation (7.39) follows. ∎

Corollary 7.18 (Bessel's Inequality) *Under the assumptions of the above lemma we have*

$$\sum_{k=-\infty}^{\infty} |\, c_k\,|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |\, f\,|^2 \,\mathrm{d}x. \tag{7.41}$$

Proof. By the lemma, for every $n \in \mathbb{N}$ we have

$$\sum_{k=-n}^{n} |\, c_k\,|^2 \leq \|f\|_2^2 .$$

Taking the limit $n \to \infty$ shows the assertion. ∎

Note that Lemma 7.17 implies that the Fourier series $\sum_{k=-\infty}^{\infty} c_k \mathrm{e}_k$ of f converges in L^2 to f if and only if

$$\sum_{k=-\infty}^{\infty} |\, c_k\,|^2 = \|f\|_2^2 . \tag{7.42}$$

Our first main result is that (7.42) is indeed true for any $f \in V$. Let use write

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}kx}$$

to express the fact that (c_k) are the (complex) Fourier coefficients of f. Further

$$s_n(f) = s_n(f;x) = \sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}kx} \tag{7.43}$$

denotes the nth partial sum.

Theorem 7.19 (Parseval's Completeness Theorem) *The ONS* $\{\mathrm{e}_k \mid k \in \mathbb{Z}\}$ *is complete. More precisely, if* $f, g \in V$ *with*

$$f \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}_k, \quad g \sim \sum_{k=-\infty}^{\infty} \gamma_k \mathrm{e}_k,$$

then

(i) $$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |\, f - s_n(f)\,|^2 \,\mathrm{d}x = 0, \tag{7.44}$$

(ii) $$\frac{1}{2\pi}\int_0^{2\pi} \overline{f}\, g \,\mathrm{d}x = \sum_{k=-\infty}^{\infty} \overline{c_k}\, \gamma_k, \tag{7.45}$$

(iii) $$\frac{1}{2\pi}\int_0^{2\pi} |\, f\,|^2 \,\mathrm{d}x = \sum_{k=-\infty}^{\infty} |\, c_k\,|^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) \quad \textit{Parseval's formula.} \tag{7.46}$$

Proof. Let $\varepsilon > 0$ be given. Since $f \in \mathcal{R}$ on $[0, 2\pi]$ there exists a *continuous* function $h \in V$ such that

$$\|f - h\|_2 < \varepsilon, \tag{7.47}$$

see Homework 16.5. By Remarks 7.3 (c), there is a trigonometric polynomial p, such that $|\, h(x) - p(x)\,| < \varepsilon$ for all x. Note that for $k \in V$ continuous,

$$\|k\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |\, k\,|^2 \,\mathrm{d}x \leq \frac{1}{2\pi}\int_0^{2\pi} \|k\|_\infty^2 \,\mathrm{d}x = \|k\|_\infty^2 .$$

Hence

$$\|h - p\|_2 < \varepsilon \tag{7.48}$$

If p has degree n_0, Lemma 7.17 shows that

$$\|h - s_n(h)\|_2 \leq \|h - p\|_2 \quad \text{for all} \quad n \geq n_0. \tag{7.49}$$

By Bessel's inequality (Corollary 7.18), $\|s_n(k)\|_2 \leq \|k\|_2$, $k \in V$. This gives with $h - f$ in place of k

$$\|s_n(h - f)\|_2 = \|s_n(h) - s_n(f)\|_2 \leq \|h - f\|_2 < \varepsilon. \tag{7.50}$$

Now the triangle inequality for $\|\cdot\|_2$ combined with (7.48), (7.49), and (7.50) gives

$$\|f - s_n(f)\|_2 \leq \|f - h\|_2 + \|h - s_n(h)\| + \|s_n(h) - s_n(f)\| < 3\varepsilon \quad (n \geq n_0).$$

This proves (7.44).
We have

$$\langle s_n(f), g\rangle = \langle \sum c_k \mathrm{e}_k, g\rangle = \sum \overline{c_k}\, \langle \mathrm{e}_k, g\rangle = \sum \overline{c_k}\gamma_k. \tag{7.51}$$

By the Cauchy–Schwarz inequality we have

$$|\,\langle f, g\rangle - \langle s_n(f), g\rangle\,| = |\,\langle f - s_n(f), g\rangle\,| \leq \|f - s_n(f)\|_2\, \|g\|_2 \tag{7.52}$$

which tends to 0 as $n \to \infty$, by (7.44). Comparing (7.51) and (7.52) gives (7.45). Finally, (7.46) is the special case of (7.45) with $g = f$. ∎

Example 7.6 (a) Consider the periodic function $f \in V$ given by

$$f(x) = \begin{cases} 1, & 0 \leq x < \pi \\ -1, & \pi \leq x < 2\pi. \end{cases}$$

Since f is an odd function the coefficients a_k vanish (see Homework 22.1). We compute the Fourier coefficients b_k. end nein

$$b_k = \frac{2}{\pi}\int_0^\pi \sin kx \,\mathrm{d}x = \frac{2}{k\pi}\left(-\cos kx\,|_0^\pi\right. = \frac{2}{k\pi}\left((-1)^{n+1} + 1\right) = \begin{cases} 0, & \text{if } k \text{ is even,} \\ \frac{4}{k\pi}, & \text{if } k \text{ is odd..} \end{cases}$$

The Fourier series of f reads

$$f \sim \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}.$$

Noting that

$$\sum_{k\in\mathbb{Z}} |\, c_k \,|^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n\in\mathbb{N}}(a_n^2 + b_n^2)$$

Parseval's formula gives

$$\|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x = 1 = \frac{1}{2}\sum_{n\in\mathbb{N}} b_n^2 = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} =: \frac{8}{\pi^2}s_1 \Longrightarrow s_1 = \frac{\pi^2}{8}.$$

Now we can compute $s = \sum_{n=1}^{\infty}\frac{1}{n^2}$. Since this series converges absolutely we are allowed to rearrange the elements in such a way that we first add all the odd terms, which gives s_1 and then all the even terms which gives s_0. Using $s_1 = \pi^2/8$ we find

$$\begin{aligned}
s &= s_1 + s_0 = s_1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots \\
s &= s_0 + \frac{1}{2^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots\right) = s_1 + \frac{s}{4} \\
s &= \frac{4}{3}s_1 = \frac{\pi^2}{6}.
\end{aligned}$$

(b) Fix $a \in [0, 2\pi]$ and consider $f \in V$ with

$$f(x) = \begin{cases} 1, & 0 \le x \le a, \\ 0, & a < x \le 2\pi \end{cases}$$

The Fourier coefficients of f are $c_0 = \dfrac{1}{2\pi}\displaystyle\int_0^a \mathrm{d}x = \dfrac{a}{2\pi}$ and

$$c_k = \langle \mathrm{e}_k, f\rangle = \frac{1}{2\pi}\int_0^a \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{\mathrm{i}}{2\pi k}\left(\mathrm{e}^{-\mathrm{i}ka} - 1\right), \quad k \neq 0.$$

If $k \neq 0$,

$$|\, c_k \,|^2 = \frac{1}{4\pi^2k^2}\left(1 - \mathrm{e}^{\mathrm{i}ka}\right)\left(1 - \mathrm{e}^{-\mathrm{i}ka}\right) = \frac{1 - \cos ka}{2\pi^2k^2},$$

hence Parseval's formula gives

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |\, c_k \,|^2 &= \frac{a^2}{4\pi^2} + \sum_{k=1}^{\infty}\frac{1 - \cos ak}{\pi^2k^2} \\
&= \frac{a^2}{4\pi^2} + \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} - \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos ak}{k^2} \\
&= \frac{a^2}{4\pi^2} + \frac{1}{\pi^2}\left(s - \sum_{k=1}^{\infty}\frac{\cos ak}{k^2}\right),
\end{aligned}$$

where $s = \sum 1/k^2$. On the other hand

$$\|f\|_2^2 = \frac{1}{2\pi}\int_0^a \mathrm{d}x = \frac{a}{2\pi}.$$

Hence, (7.44) reads

$$\frac{a^2}{4\pi^2} + \frac{1}{\pi^2}\left(s - \sum_{k=1}^{\infty}\frac{\cos ka}{k^2}\right) = \frac{a}{2\pi}$$

$$\sum_{k=1}^{\infty}\frac{\cos ka}{k^2} = \frac{a^2}{4} - \frac{a\pi}{2} + \frac{\pi^2}{6} = \frac{(a-\pi)^2}{4} - \frac{\pi^2}{12}. \tag{7.53}$$

Since the series

$$\sum_{k=1}^{\infty}\frac{\cos kx}{k^2} \tag{7.54}$$

converges uniformly on $\mathbb{R}$ (use Theorem 7.3 and $\sum_k 1/k^2$ is an upper bound) (7.54) is the Fourier series of the function

$$\frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}, \quad x \in [0, 2\pi].$$

Since the term by term differentiated series converges uniformly on $[\delta, 2\pi - \delta]$, see Example 7.2, we obtain

$$-\sum_{k=1}^{\infty}\frac{\sin kx}{k} = \sum_{k=1}^{\infty}\left(\frac{\cos kx}{k^2}\right)' = \left(\frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}\right)' = \frac{x-\pi}{2}$$

which is true for $x \in (0, 2\pi)$.

Theorem 7.20 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous periodic function which is piecewise continuously differentiable, i. e. there exists a partition $\{t_0, \dots, t_r\}$ of $[0, 2\pi]$ such that $f|[t_{i-1}, t_i]$ is continuously differentiable.*
Then the Fourier series of f converges uniformly to f.

Proof. Let $\varphi_i\colon [t_{i-1}, t_i] \to \mathbb{R}$ denote the continuous derivative of $f|[t_{i-1}, t_i]$ and $\varphi\colon \mathbb{R} \to \mathbb{R}$ the periodic function that coincides with φ_i on $[t_{i-1}, t_i]$. By Bessel's inequality, the Fourier coefficients γ_k of φ satisfy

$$\sum_{k=-\infty}^{\infty} |\,\gamma_k\,|^2 \le \|\varphi\|_2^2 < \infty.$$

If $k \neq 0$ the Fourier coefficients c_k of f can be found using integration by parts from the Fourier coefficients of γ_k.

$$\int_{t_{i-1}}^{t_i} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{\mathrm{i}}{k}\left(f(x)\mathrm{e}^{-\mathrm{i}kx}\Big|_{t_{i-1}}^{t_i} - \int_{t_{i-1}}^{t_i}\varphi(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\right).$$

Hence summation over $i = 1, \dots, r$ yields,

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{2\pi}\sum_{i=1}^{r}\int_{t_{i-1}}^{t_i} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$$
$$c_k = \frac{-\mathrm{i}}{2\pi k}\int_0^{2\pi} \varphi(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{-\mathrm{i}\gamma_k}{k}.$$

Note that the term

$$\sum_{i=1}^{r} f(x)\mathrm{e}^{-\mathrm{i}kx}\Big|_{t_{i-1}}^{t_i}$$

vanishes since f is continuous and $f(2\pi) = f(0)$. Since for $\alpha, \beta \in \mathbb{C}$ we have $|\,\alpha\beta\,| \le \frac{1}{2}(|\,\alpha\,|^2 + |\,\beta\,|^2)$, we obtain

$$|\,c_k\,| \le \frac{1}{2}\left(\frac{1}{|\,k\,|^2} + |\,\gamma_k\,|^2\right).$$

Since both $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ and $\displaystyle\sum_{k=-\infty}^{\infty}|\,\gamma_k\,|^2$ converge,

$$\sum_{k=-\infty}^{\infty}|\,c_k\,| < \infty.$$

Thus, the Fourier series converges uniformly to a continuous function g (see Theorem 7.4). Since the Fourier series converges both to f and to g in the L^2 norm, $\|f-g\|_2 = 0$. Since both f and g are continuous, they coincide. This completes the proof. ■

Note that for any $f \in V$, the series $\sum_{k\in\mathbb{Z}}|\,c_k\,|^2$ converges while the series $\sum_{k\in\mathbb{Z}}|\,c_k\,|$ converges only if the Fourier series converges uniformly to f.

7.5 Appendix F

Proposition 7.21 *There exists a real continuous function on the real line which is nowhere differentiable.*

Proof. Define

$$\varphi(x) = |\,x\,|, \quad x \in [-1, 1]$$

and extend the definition of φ to all real x by requiring periodicity

$$\varphi(x+2) = \varphi(x).$$

Then for all $s, t \in \mathbb{R}$,

$$|\,\varphi(s) - \varphi(t)\,| \le |\,s - t\,|. \tag{7.55}$$

In particular, φ is continuous on $\mathbb{R}$. Define

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x). \tag{7.56}$$

Since $0 \le \varphi \le 1$, Theorem 7.3 shows that the series (7.56) converges uniformly on $\mathbb{R}$. By Theorem 7.4, f is continuous on $\mathbb{R}$.
Now fix a real number x and a positive integer $m \in \mathbb{N}$. Put

$$\delta_m = \frac{\pm 1}{2 \cdot 4^m}$$

where the sign is chosen that no integer lies between $4^m x$ and $4^m(x + \delta_m)$. This can be done since $4^m \mid \delta_m \mid = \frac{1}{2}$. It follows that $\mid \varphi(4^m x) - \varphi(4^m x + 4^m \delta_m) \mid = \frac{1}{2}$. Define

$$\gamma_n = \frac{\varphi(4^n(x + \delta_m)) - \varphi(4^n x)}{\delta_m}.$$

When $n > m$, then $4^n \delta_m$ is an even integer, so that $\gamma_n = 0$ by peridicity of φ. When $0 \le n \le m$, (7.55) implies $\mid \gamma_n \mid \le 4^m$. Since $\mid \gamma_m \mid = 4^m$, we conclude that

$$\left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| = \left| \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n \right| \ge 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m + 1).$$

As $m \to \infty$, $\delta_m \to 0$. It follows that f is not differentiable at x. ∎

Proof of Proposition 7.9. By Proposition 7.5, the series converges on $(-1, 1)$ and the limit function is continuous there since the radius of convergence is at least 1, by assumption. Hence it suffices to proof continuity at $x = 1$, i. e. that $\lim_{x \to 1-0} f(x) = f(1)$. Put $r_n = \sum_{k=n}^{\infty} a_k$; then $r_0 = f(1)$ and $r_{n+1} - r_n = -c_n$ for all nonnegative integers $n \in \mathbb{Z}_+$ and $\lim_{n\to\infty} r_n = 0$. Hence there is a constant C with $\mid r_n \mid \le C$ and the series $\sum_{n=0}^{\infty} r_{n+1} x^n$ converges for $\mid x \mid < 1$ by the comparison test. We have

$$\begin{aligned}
(1 - x) \sum_{n=0}^{\infty} r_{n+1} x^n &= \sum_{n=0}^{\infty} r_{n+1} x^n + \sum_{n=0}^{\infty} r_{n+1} x^{n+1} \\
&= \sum_{n=0}^{\infty} r_{n+1} x^n - \sum_{n=0}^{\infty} r_n x^n + r_0 = -\sum_{n=0}^{\infty} a_n x^n + f(1),
\end{aligned}$$

hence,

$$f(1) - f(x) = (1 - x) \sum_{n=0}^{\infty} r_{n+1} x^n.$$

Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $n \ge N$ implies $\mid r_n \mid < \varepsilon$. Put $\delta = \varepsilon/(CN)$; then $x \in (1 - \delta, 1)$ implies

$$\begin{aligned}
\mid f(1) - f(x) \mid &\le (1 - x) \sum_{n=0}^{N-1} \mid r_{n+1} \mid x^n + (1 - x) \sum_{n=N}^{\infty} \mid r_{n+1} \mid x^n \\
&\le (1 - x)CN + (1 - x)\varepsilon \sum_{n=0}^{\infty} x^n = 2\varepsilon;
\end{aligned}$$

hence f tends to $f(1)$ as $x \to 1 - 0$. ■