Chapter 5

Integration

In the first section of this chapter derivatives will not apper! Roughly speaking, integration generalizes "addition". The formula distance = velocity × time is only valid for constant velocity. The right formula is $s = \int_{t_0}^{t_1} v(t)\,\mathrm{d}t$. We need integrals to compute length of curves, areas of surfaces, and volumes.

The study of integrals requires a long preparation, but once this preliminary work has been completed, integrals will be an invaluable tool for creating new functions, and the derivative will reappear more powerful than ever. The relation between the integral and derivatives is given in the Fundamental Theorem of Calculus.

The integral formalizes a simple intuitive concept—that of area. It is not a surprise that to learn the definition of an intuitive concept can present great difficulties—"area" is certainly not an exception.

5.1 The Riemann–Stieltjes Integral

In this section we will only define the area of some very special regions—those which are bounded by the horizontal axis, the vertical lines through $(a, 0)$ and $(b, 0)$ and the graph of a function f such that $f(x) \geq 0$ for all x in $[a, b]$. If f is negative on a subinterval of $[a, b]$, the integral will represent the difference of the areas above and below the x-axis. All intervals $[a, b]$ are finite intervals.

Definition 5.1 Let $[a, b]$ be an interval. By a *partition* of $[a, b]$ we mean a finite set of points $x_0, x_1, \dots, x_n$, where

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b.$$

We write

$$\Delta x_i = x_i - x_{i-1}, \quad i = 1, \dots, n.$$

Now suppose f is a bounded real function defined on $[a, b]$. Corresponding to each partition P of $[a,b]$ we put

$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} \tag{5.1}$$

$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} \tag{5.2}$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i, \quad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i, \tag{5.3}$$

and finally

$$\overline{\int_a^b} f \,\mathrm{d}x = \inf U(P,f), \tag{5.4}$$

$$\underline{\int_a^b} f \,\mathrm{d}x = \sup L(P,f), \tag{5.5}$$

where the infimum and supremum are taken over all partitions P of $[a,b]$. The left members of (5.4) and (5.5) are called the *upper* and *lower Riemann integrals* of f over $[a,b]$, respectively.
If the upper and lower integrals are equal, we say that f Riemann-integrable on $[a,b]$ and we write $f \in \mathcal{R}$ (that is $\mathcal{R}$ denotes the Riemann-integrable functions), and we denote the common value of (5.4) and (5.5) by

$$\int_a^b f \,\mathrm{d}x \quad \text{or by} \quad \int_a^b f(x)\,\mathrm{d}x. \tag{5.6}$$

This is the *Riemann integral* of f over $[a,b]$.

Since f is bounded, there exist two numbers m and M such that $m \le f(x) \le M$ for all $x \in [a,b]$. Hence for every partition P

$$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a),$$

so that the numbers $L(P,f)$ and $U(P,f)$ form a bounded set. This shows that the upper and the lower integrals are defined for *every* bounded function f. The question of their equality, and hence the question of the integrability of f, is a more delicate one. Instead of investigating it separately for the Riemann integral, we shall immediately consider a more general situation.

Definition 5.2 Let α be a monotonically increasing function on $[a,b]$ (since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bounded on $[a,b]$). Corresponding to each partition P of $[a,b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$

It is clear that $\Delta\alpha_i \ge 0$. For any real function f which is bounded on $[a,b]$ we put

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i, \tag{5.7}$$

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i, \tag{5.8}$$

where M_i and m_i have the same meaning as in Definition 5.1, and we define

$$\overline{\int_a^b} f \,\mathrm{d}\alpha = \inf U(P, f, \alpha), \tag{5.9}$$

$$\underline{\int_a^b} f \,\mathrm{d}\alpha = \sup U(P, f, \alpha), \tag{5.10}$$

where the infimum and the supremum are taken over all partitions P.
If the left members of (5.9) and (5.10) are equal, we denote their common value by

$$\int_a^b f \,\mathrm{d}\alpha \quad \text{or sometimes by} \quad \int_a^b f(x) \,\mathrm{d}\alpha(x). \tag{5.11}$$

This is the *Riemann–Stieltjes integral* (or simply the *Stieltjes integral*) of f with respect to α, over $[a, b]$. If (5.11) exists, we say that f is integrable with respect to α in the Riemann sence, and write $f \in \mathcal{R}(\alpha)$.

By taking $\alpha(x) = x$, the Riemann integral is seen to be a special case of the Riemann–Stieltjes integral. Let us mention explicitely, that in the general case, α need not even be continuous.
We shall now investigate the existence of the integral (5.11). Without saying so every time, f will be assumed real and bounded, and α increasing on $[a, b]$; and we shall write $\int$ in place of $\int_a^b$.

Definition 5.3 We say that a partition P^* is a *refinement* of the partition P if $P^* \supset P$ (that is, every point of P is a point of P^*). Given two partitions, P_1 and P_2, we say that P^* is their *common refinement* if $P^* = P_1 \cup P_2$.

Lemma 5.1 *If P^* is a refinement of P, then*

$$L(P, f, \alpha) \le L(P^*, f, \alpha) \quad \textit{and} \quad U(P, f, \alpha) \ge U(P^*, f, \alpha). \tag{5.12}$$

Proof. We only prove the first inequality of (5.12); the proof of the second one is analogous. Suppose first that P^* contains just one point more than P. Let this extra point be x^*, and suppose $x_{i-1} \le x^* < x_i$, where x_{i-1} and x_i are two consecutive points of P. Put

$$w_1 = \inf\{f(x) \mid x \in [x_{i-1}, x^*]\}, \quad w_2 = \inf\{f(x) \mid x \in [x^*, x_i]\}.$$

Clearly, $w_1 \ge m_i$ and $w_2 \ge m_i$, where, as before, $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$. Hence

$$\begin{aligned} L(P^*, f, \alpha) - L(P, f, \alpha) &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*)) - m_i(\alpha(x_i) - \alpha(x_{i-1})) \\ &= (w_1 - m_i)(\alpha(x^*) - \alpha(x_{i-1})) + (w_2 - m_i)(\alpha(x_i) - \alpha(x^*)) \ge 0. \end{aligned}$$

If P^* contains k points more than P, we repeat this reasoning k times, and arrive at (5.12). ■

Proposition 5.2

$$\underline{\int_a^b} f \, \mathrm{d}\alpha \leq \overline{\int_a^b} f \, \mathrm{d}\alpha.$$

Proof. Let P^* be the common refinement of two partitions P_1 and P_2. By Lemma 5.1

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha).$$

Hence

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha). \tag{5.13}$$

If P_2 is fixed and the supremum is taken over all P_1, (5.13) gives

$$\underline{\int} f \, \mathrm{d}\alpha \leq U(P_2, f, \alpha). \tag{5.14}$$

The proposition follows by taking the infimum over all P_2 in (5.14). ∎

Proposition 5.3 (Riemann criterion) *$f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that*

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon. \tag{5.15}$$

Proof. For every P we have

$$L(P, f, \alpha) \leq \underline{\int} f \, \mathrm{d}\alpha \leq \overline{\int} f \, \mathrm{d}\alpha \leq U(P, f, \alpha).$$

Thus (5.15) implies

$$0 \leq \overline{\int} f \, \mathrm{d}\alpha - \underline{\int} f \, \mathrm{d}\alpha < \varepsilon.$$

since the above inequality can be satisfied for every $\varepsilon > 0$, we have

$$\overline{\int} f \, \mathrm{d}\alpha = \underline{\int} f \, \mathrm{d}\alpha,$$

that is $f \in \mathcal{R}(\alpha)$.
Conversely, suppose $f \in \mathcal{R}(\alpha)$, and let $\varepsilon > 0$ be given. Then there exist partitions P_1 and P_2 such that

$$U(P_2, f, \alpha) - \int f \, \mathrm{d}\alpha < \frac{\varepsilon}{2}, \quad \int f \, \mathrm{d}\alpha - L(P_1, f, \alpha) < \frac{\varepsilon}{2}. \tag{5.16}$$

We choose P to be the common refinement of P_1 and P_2. Then Lemma 5.1, together with (5.16), shows that

$$U(P, f, \alpha) \leq U(P_2, f, \alpha) < \int f \, \mathrm{d}\alpha + \frac{\varepsilon}{2} < L(P_1, f, \alpha) + \varepsilon \leq L(P, f, \alpha) + \varepsilon,$$

so that (5.15) holds for this partition P. ■

Proposition 5.3 furnishes a convenient criterion for integrability. Before we apply it, we state some closely related facts.

Lemma 5.4 (a) *If* (5.15) *holds for P and some ε, then* (5.15) *holds with the same ε for every refinement of P.*
(b) *If* (5.15) *holds for $P = \{x_0, \dots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then*

$$\sum_{i=1}^{n} |\, f(s_i) - f(t_i)\, |\, \Delta\alpha_i < \varepsilon.$$

(c) *If $f \in \mathcal{R}(\alpha)$ and the hypotheses of* (b) *hold, then*

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\, \mathrm{d}\alpha \right| < \varepsilon.$$

Proof. Lemma 5.1 implies (a). Under the assumptions made in (b), both $f(s_i)$ and $f(t_i)$ lie in $[m_i, M_i]$, so that $|\, f(s_i) - f(t_i)\, | \le M_i - m_i$. Thus

$$\sum_{i=1}^{n} |\, f(t_i) - f(s_i)\, |\, \Delta\alpha_i \le U(P, f, \alpha) - L(P, f, \alpha),$$

which proves (b). The obvious inequalities

$$L(P, f, \alpha) \le \sum_i f(t_i)\Delta\alpha_i \le U(P, f, \alpha)$$

and

$$L(P, f, \alpha) \le \int_a^b f\, \mathrm{d}\alpha \le U(P, f, \alpha)$$

prove (c). ■

Theorem 5.5 *If f is continuous on $[a, b]$ then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.*

Proof. Let $\varepsilon > 0$ be given. Choose $\eta > 0$ so that

$$(\alpha(b) - \alpha(a))\eta < \varepsilon.$$

Since f is uniformly continuous on $[a, b]$ (Proposition 3.12), there exists a $\delta > 0$ such that

$$|\, f(x) - f(t)\, | < \eta \tag{5.17}$$

if $x, t \in [a, b]$ and $|\, x - t\, | < \delta$. If P is any partition of $[a, b]$ such that $\Delta x_i < \delta$ for all i, then (5.17) implies that

$$M_i - m_i \le \eta, \quad i = 1, \dots, n \tag{5.18}$$

and therefore

$$U(P,f,\alpha)-L(P,f,\alpha)=\sum_{i=1}^{n}(M_i-m_i)\Delta\alpha_i\le\eta\sum_{i=1}^{n}\Delta\alpha_i=\eta(\alpha(b)-\alpha(a))<\varepsilon.$$

By Proposition 5.3, $f\in\mathcal{R}(\alpha)$. ■

Example 5.1 (a) The proof of Theorem 5.5 together with Lemma 5.4 shows that

$$\left|\sum_{i=1}^{n}f(t_i)\Delta\alpha_i-\int_a^b f\,\mathrm{d}\alpha\right|<\varepsilon$$

if $\Delta x_i<\delta$.
We compute $I=\int_a^b\sin x\,\mathrm{d}x$. Let $\varepsilon>0$. Since $\sin x$ is continuous, $f\in\mathcal{R}$. There exists $\delta>0$ such that $|\,x-t\,|<\delta$ implies

$$|\sin x-\sin t\,|<\frac{\varepsilon}{b-a}. \tag{5.19}$$

In this case (5.15) is satisfied and consequently

$$\left|\sum_{i=1}^{n}\sin(t_i)\Delta x_i-\int_a^b\sin x\,\mathrm{d}x\right|<\varepsilon$$

for every partition P with $\Delta x_i<\delta$, $i=1,\dots,n$.
For we choose an equidistant partition of $[a,b]$, $x_i=a+(b-a)i/n$, $i=0,\dots,n$. Then $h=\Delta x_i=(b-a)/n$ and the condition (5.19) is satisfied provided $n>\dfrac{(b-a)^2}{\varepsilon}$. We have (cf. Homework 4.4)

$$\begin{aligned}\sum_{i=1}^{n}\sin x_i\,\Delta x_i&=\sum_{i=1}^{n}\sin(a+ih)h=\frac{h}{2\sin h/2}\sum_{i=1}^{n}2\,\sin h/2\,\sin(a+ih)\\&=\frac{h}{2\sin h/2}\sum_{i=1}^{n}(\cos(a+(i-1/2)h)-\cos(a+(i+1/2)h))\\&=\frac{h}{2\sin h/2}(\cos(a+h/2)-\cos(a+(n+1/2)h))\\&=\frac{h/2}{\sin h/2}(\cos(a+h/2)-\cos(b+h/2)))\end{aligned}$$

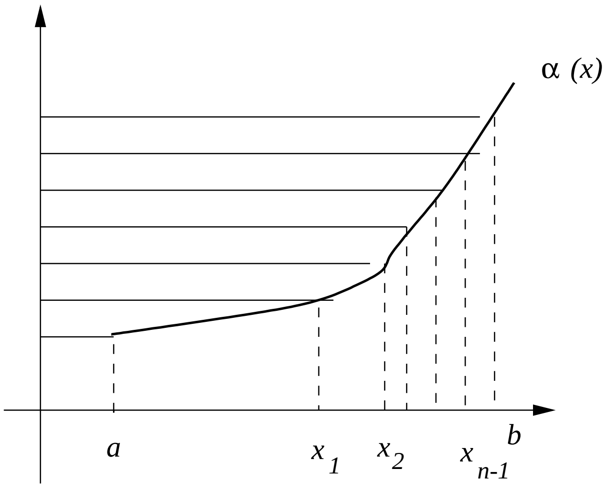
Since $\lim_{h\to0}\sin h/h=1$ and $\cos x$ is continuous, we find that the above expression tends to $\cos a-\cos b$. Hence $\int_a^b\sin x\,\mathrm{d}x=\cos a-\cos b$.
(b) For $x\in[a,b]$ define

$$f(x)=\begin{cases}1, & x\in\mathbb{Q},\\0, & x\notin\mathbb{Q}.\end{cases}$$

We will show $f\notin\mathcal{R}$. Let P be any partition of $[a,b]$. Since any interval contains rational as well as irrational points, $m_i=0$ and $M_i=1$ for all i. Hence $L(P,f)=0$ whereas $U(P,f)=\sum_{i=1}^{n}\Delta x_i=b-a$. We conclude that the upper and lower Riemann integrals don't coincide; $f\notin\mathcal{R}$.

Proposition 5.6 *If f is monotonic on $[a,b]$, and α is continuous on $[a,b]$, then $f \in \mathcal{R}(\alpha)$.*

Proof.



Let $\varepsilon > 0$ be given. For any positive integer n, choose a partition such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n}, \quad i = 1, \dots, n.$$

This is possible by the intermediate value theorem (Theorem 3.5) since α is continuous.

We suppose that f is monotonically increasing (the proof is analogous in the other case). Then

$$M_i = f(x_i), \quad m_i = f(x_{i-1}), \quad i = 1, \dots, n,$$

so that

$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &= \frac{\alpha(b) - \alpha(a)}{n} (f(b) - f(a)) < \varepsilon \end{aligned}$$

if n is taken large enough. By Proposition 5.3, $f \in \mathcal{R}(\alpha)$. ∎

Without proofs which can be found in [8, pp. 126 –128] we note the following facts.

Proposition 5.7 *If f is bounded on $[a,b]$, f has finitely many points of discontinuity on $[a,b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathcal{R}(\alpha)$.*

Proposition 5.8 *If $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \le f(x) \le M$, φ is continuous on $[m,M]$, and $h(x) = \varphi(f(x))$ on $[a,b]$. Then $h \in \mathcal{R}(\alpha)$ on $[a,b]$.*

Proof. Let $\varepsilon > 0$. Since φ is uniformly continuous on $[m,M]$, there exists $\delta > 0$ such that $\delta < \varepsilon$ and $|\,\varphi(s) - \varphi(t)\,| < \varepsilon$ if $|\,s - t\,| < \delta$ and $[s, t \in [m,M]$.
Since $f \in \mathcal{R}(\alpha)$, there exists a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \delta^2. \tag{5.20}$$

Let M_i and m_i have the same meaning as in Definition 5.1, and let M_i^* and m_i^* the analogous numbers for h. Divide the numbers $1, 2, \dots, n$ into two classes: $i \in A$ if $M_i - m_i < \delta$ and $i \in B$ if $M_i - m_i > \delta$. For $i \in A$ our choice of δ shows that $M_i^* - m_i^* \le \varepsilon$. For $i \in B$, $M_i^* - m_i^* \le 2K$ where $K = \sup\{|\,\varphi(t)\,| \mid m \le t \le M\}$. By (5.20), we have

$$\delta \sum_{i\in B} \Delta\alpha_i \le \sum_{i\in B} (M_i - m_i)\Delta\alpha_i < \delta^2 \tag{5.21}$$

so that $\sum_{i\in B} \Delta\alpha_i < \delta$. It follows that

$$U(P,h,\alpha) - L(P,h,\alpha) = \sum_{i\in A}(M_i^* - m_i^*)\Delta\alpha_i + \sum_{i\in B}(M_i^* - m_i^*)\Delta\alpha_i \leq$$
$$\varepsilon(\alpha(b) - \alpha(a)) + 2K\delta < \varepsilon(\alpha(b) - \alpha(a) + 2K).$$

Since ε was arbitrary, Proposition 5.3 implies that $h \in \mathcal{R}(\alpha)$. ∎

Remark 5.1 A bounded function f is Riemann-integrable on $[a,b]$ if and only if f is continuous almost everywhere on $[a,b]$. (The proof of this fact can be found in [8, Theorem 11.33]).
"Almost everywhere" means that the discontinuities form a set of (Lebesgue) measure 0. A set $M \subset \mathbb{R}$ has measure 0 if for given $\varepsilon > 0$ there exist intervals I_n, $n \in \mathbb{N}$ such that $M \subset \bigcup_{n\in\mathbb{N}} I_n$ and $\sum_{n\in\mathbb{N}} |I_n| < \varepsilon$. Here, $|I|$ denotes the length of the interval. Examples of sets of measure 0 are finite sets, countable sets, and the Cantor set (which is uncountable).

5.1.1 Properties of the Integral

Proposition 5.9 (a) *If $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ then $f_1 + f_2 \in \mathcal{R}(\alpha)$, $cf \in \mathcal{R}(\alpha)$ for every constant c and*

$$\int_a^b (f_1 + f_2)\,\mathrm{d}\alpha = \int_a^b f_1\,\mathrm{d}\alpha + \int_a^b f_2\,\mathrm{d}\alpha, \qquad \int_a^b cf\,\mathrm{d}\alpha = c\int_a^b f\,\mathrm{d}\alpha.$$

(b) *If $f_1, f_2 \in \mathcal{R}(\alpha)$ and $f_1(x) \leq f_2(x)$ on $[a,b]$, then*

$$\int_a^b f_1\,\mathrm{d}\alpha \leq \int_a^b f_2\,\mathrm{d}\alpha.$$

(c) *If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$, and*

$$\int_a^b f\,\mathrm{d}\alpha = \int_a^c f\,\mathrm{d}\alpha + \int_c^b f\,\mathrm{d}\alpha.$$

(d) *If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $|f(x)| \leq M$ on $[a,b]$, then*

$$\left|\int_a^b f\,\mathrm{d}\alpha\right| \leq M(\alpha(b) - \alpha(a)).$$

(e) *If $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$, than $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ and*

$$\int_a^b f\mathrm{d}(\alpha_1 + \alpha_2) = \int_a^b f\mathrm{d}\alpha_1 + \int_a^b f\mathrm{d}\alpha_2;$$

if $f \in \mathcal{R}(\alpha)$ and c is a positive constant, then $f \in \mathcal{R}(c\alpha)$ and

$$\int_a^b f\mathrm{d}(c\alpha) = c\int_a^b f\,\mathrm{d}\alpha.$$

Proof. If $f = f_1 + f_2$ and P is any partition of $[a, b]$, we have

$$L(P, f_1, \alpha) + L(P, f_2, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha) \quad (5.22)$$

since $\inf\limits_{I_i} f_1 + \inf\limits_{I_i} f_2 \leq \inf\limits_{I_i}(f_1 + f_2)$ and $\sup\limits_{I_i} f_1 + \sup\limits_{I_i} f_2 \geq \sup\limits_{I_i}(f_1 + f_2)$.
If $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in \mathcal{R}(\alpha)$, let $\varepsilon > 0$ be given. There are partitons P_j, $j = 1, 2$, such that

$$U(P_j, f_j, \alpha) - L(P_j, f_j, \alpha) < \varepsilon.$$

These inequalities persist if P_1 and P_2 are replaced by their common refinement P. Then (5.22) implies

$$U(P, f, \alpha) - L(P, f, \alpha) < 2\varepsilon$$

which proves that $f \in \mathcal{R}(\alpha)$. With the same P we have

$$U(P, f_j, \alpha) < \int f_j \, \mathrm{d}\alpha + \varepsilon, \quad j = 1, 2;$$

hence (5.22) implies

$$\int f \, \mathrm{d}\alpha \leq U(P, f, \alpha) < \int f_1 \, \mathrm{d}\alpha + \int f_2 \, \mathrm{d}\alpha + 2\varepsilon.$$

Since ε was arbitrary, we conclude that

$$\int f \, \mathrm{d}\alpha \leq \int f_1 \, \mathrm{d}\alpha + \int f_2 \, \mathrm{d}\alpha. \quad (5.23)$$

If we replace f_1 and f_2 in (5.23) by $-f_1$ and $-f_2$, respectively, the inequality is reversed, and the equality is proved.
(b) Put $f = f_1 - f_2$. It suffices to prove that $\int f \, \mathrm{d}\alpha \geq 0$. For every partition P we have $m_i \geq 0$ since $f \geq 0$. Hence

$$\int f \, \mathrm{d}\alpha \geq L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \geq 0$$

since in addition $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$ (α is increasing).
The proofs of the other assertions are so similar that we omit the details. In part (c) the point is that (by passing to refinements) we may restrict ourselves to partitions which contain the point c, in approximating $\int f \, \mathrm{d}\alpha$, cf. Homework 14.5. ∎

Proposition 5.10 *If $f, g \in \mathcal{R}(\alpha)$ on $[a, b]$, then*
(a) *$fg \in \mathcal{R}(\alpha)$;*
(b) *$|\, f \,| \in \mathcal{R}(\alpha)$ and $\left| \int_a^b f \, \mathrm{d}\alpha \right| \leq \int_a^b |\, f \,| \, \mathrm{d}\alpha$.*

Proof. If we take $\varphi(t) = t^2$, Proposition 5.8 shows that $f^2 \in \mathcal{R}(\alpha)$ if $f \in \mathcal{R}(\alpha)$. The identity

$$4fg = (f+g)^2 - (f-g)^2$$

completes the proof of (a).
If we take $\varphi(t) = |\,t\,|$, Proposition 5.8 shows that $|\,f\,| \in \mathcal{R}(\alpha)$. Choose $c = \pm 1$ so that $c\int f\,\mathrm{d}\alpha \geq 0$. Then

$$\left|\int f\,\mathrm{d}\alpha\right| = c\int f\,\mathrm{d}\alpha = \int cf\,\mathrm{d}\alpha \leq \int |\,f\,|\,\mathrm{d}\alpha,$$

since $\pm f \leq |\,f\,|$. ∎

The *unit step function* or *Heaviside function* $\theta(x)$ is defined by $\theta(x) = 0$ if $x < 0$ and $\theta(x) = 1$ if $x \geq 0$.

Example 5.2 If $a < s < b$, f is bounded on $[a,b]$, f is continuous at s, and $\alpha(x) = \theta(x-s)$, then

$$\int_a^b f\,\mathrm{d}\alpha = f(s).$$

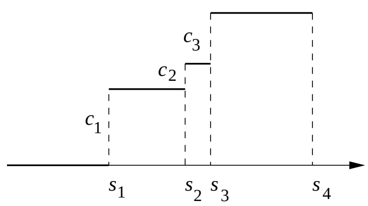
For the proof, consider the partition P with $n = 3$; $a = x_0 < x_1 < s = x_2 < x_3 = b$. Then $\Delta\alpha_1 = \Delta\alpha_3 = 0$, $\Delta\alpha_2 = 1$, and

$$U(P,f,\alpha) = M_2, \quad L(P,f,\alpha) = m_2.$$

Since f is continuous at s, we see that M_2 and m_2 converge to $f(s)$ as $x \to s$.

Proposition 5.11 *Suppose $c_n \geq 0$ for all positive integers $n \in \mathbb{N}$, $\sum c_n$ converges, (s_n) is a sequence of distinct points in (a,b), and*

$$\alpha(x) = \sum_{n=1}^{\infty} c_n\theta(x-s_n). \tag{5.24}$$



Let f be continuous on $[a,b]$. Then

$$\int_a^b f\,\mathrm{d}\alpha = \sum_{n=1}^{\infty} c_n f(s_n). \tag{5.25}$$

Proof. The comparison test shows that the series (5.24) converges for every x. Its sum α is evidently an increasing function with $\alpha(a) = 0$ and $\alpha(b) = \sum c_n$. Let $\varepsilon > 0$ be given, choose N so that

$$\sum_{n=N+1}^{\infty} c_n < \varepsilon.$$

Put

$$\alpha_1(x) = \sum_{n=1}^{N} c_n\theta(x-s_n), \quad \alpha_2(x) = \sum_{n=N+1}^{\infty} c_n\theta(x-s_n).$$

By Proposition 5.9 and Example 5.2

$$\int_a^b f \,\mathrm{d}\alpha_1 = \sum_{n=1}^{N} c_n f(s_n).$$

Since $\alpha_2(b) - \alpha_2(a) < \varepsilon$, by Proposition 5.9 (d),

$$\left| \int_a^b f \,\mathrm{d}\alpha_2 \right| \leq M\varepsilon,$$

where $M = \sup |\, f(x)\,|$. Since $\alpha = \alpha_1 + \alpha_2$ it follows that

$$\left| \int_a^b f \,\mathrm{d}\alpha - \sum_{n=1}^{N} c_n f(s_n) \right| \leq M\varepsilon.$$

If we let $N \to \infty$ we obtain (5.25). ■

Proposition 5.12 *Assume that α is increasing and $\alpha' \in \mathcal{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$.*
Then $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$. In that case

$$\int_a^b f \,\mathrm{d}\alpha = \int_a^b f(x)\alpha'(x) \,\mathrm{d}x. \tag{5.26}$$

The statement remains true if α is continuous on $[a, b]$ and differentiable up to finitely many points $c_1, c_2, \dots c_n$.

Proof. Let $\varepsilon > 0$ be given and apply the Riemann criterion Proposition 5.3 to α': There is a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ such that

$$U(P, \alpha') - L(P, \alpha') < \varepsilon. \tag{5.27}$$

The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$\Delta\alpha_i = \alpha'(t_i)\Delta x_i, \quad \text{for} \quad i = 1, \dots, n.$$

If $s_i \in [x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |\, \alpha'(s_i) - \alpha'(t_i)\,|\, \Delta x_i < \varepsilon \tag{5.28}$$

by (5.27) and Lemma 5.4 (b). Put $M = \sup |\, f(x)\,|$. Since

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i$$

it follows from (5.28) that

$$\left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right| \le M\varepsilon. \tag{5.29}$$

In particular,

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \le U(P, f\alpha') + M\varepsilon,$$

for all choices of $s_i \in [x_{i-1}, x_i]$, so that

$$U(P, f, \alpha) \le U(P, f\alpha') + M\varepsilon.$$

The same argument leads from (5.29) to

$$U(P, f\alpha') \le U(P, f, \alpha) + M\varepsilon.$$

Thus

$$|\, U(P, f, \alpha) - U(P, f\alpha) \,| \le M\varepsilon. \tag{5.30}$$

Now (5.28) remains true if P is replaced by any refinement. Hence (5.29) also remains true. We conclude that

$$\left| \overline{\int_a^b} f \,\mathrm{d}\alpha - \overline{\int_a^b} f(x)\alpha'(x)\,\mathrm{d}x \right| \le M\varepsilon.$$

But ε is arbitrary. Hence

$$\overline{\int_a^b} f \,\mathrm{d}\alpha = \overline{\int_a^b} f(x)\alpha'(x)\,\mathrm{d}x,$$

for *any* bounded f. The equality for the lower integrals follows from (5.29) in exactly the same way. The proposition follows. ∎

We now summarize the two cases.

Proposition 5.13 *Let f be continuous on $[a, b]$. Except for finitely many points $c_0, c_1, \ldots, c_n$ with $c_0 = a$ and $c_n = b$ there exists $\alpha'(x)$ which is continuous and bounded on $[a, b] \setminus \{c_0, \ldots, c_n\}$.*
Then $f \in \mathcal{R}(\alpha)$ and

$$\int_a^b f \,\mathrm{d}\alpha = \int_a^b f(x)\alpha'(x)\,\mathrm{d}x + \sum_{i=1}^{n-1} f(c_i)(\alpha(c_i + 0) - \alpha(c_i - 0)) +$$
$$f(a)(\alpha(a + 0) - \alpha(a)) + f(b)(\alpha(b) - \alpha(b - 0)).$$

Proof (Sketch of proof). (a) Note that $A_i^+ = \alpha(c_i + 0) - \alpha(c_i)$ and $A_i^- = \alpha(c_i)\alpha(c_i - 0)$ exist by Theorem 3.13. Define

$$\alpha_1(x) = \sum_{i=0}^{n-1} A_i^+ \theta(x - c_i) + \sum_{i=1}^{k} -A_i^- \theta(c_i - x).$$

(b) Then $\alpha_2 = \alpha - \alpha_1$ is continuous.
(c) Since α_1 is piecewise constant, $\alpha_1'(x) = 0$ for $x \neq c_k$. Hence $\alpha_2'(x) = \alpha'(x)$. for $x \neq c_i$. Applying Proposition 5.12 gives

$$\int f \mathrm{d}\alpha_2 = \int f\alpha_2' \,\mathrm{d}x = \int f\alpha' \,\mathrm{d}x.$$

Further,

$$\int f \,\mathrm{d}\alpha = \int f \mathrm{d}(\alpha_1 + \alpha_2) = \int f\alpha' \,\mathrm{d}x + \int f \mathrm{d}\alpha_1.$$

By Proposition 5.11

$$\int f \mathrm{d}\alpha_1 = \sum_{i=1}^{n} A_i^+ f(c_i) - \sum_{i=1}^{n-1} A_i^-(-f(c_i)).$$

■

Example 5.3 (a)

$$\int_0^2 x \,\mathrm{d}x^3 = \int_0^2 x \cdot 3x^2 \,\mathrm{d}x = 3\frac{x^4}{4}\bigg|_0^2 = 12.$$

(b) $f(x) = x^2$.

$$\alpha(x) = \begin{cases} x, & 0 \leq x < 1, \\ 7, & x = 1, \\ x^2 + 10, & 1 < x < 2, \\ 64, & x = 2. \end{cases}$$

$$\begin{aligned} \int_0^2 f \,\mathrm{d}\alpha &= \int_0^2 f\alpha' \,\mathrm{d}x + f(1)(\alpha(1+0) - \alpha(1-0)) + f(2)(\alpha(2) - \alpha(2-0)) \\ &= \int_0^1 x^2 \cdot 1 \,\mathrm{d}x + \int_1^2 x^2 \cdot 2x \,\mathrm{d}x + 1(11 - 1) + 4(64 - 14) \\ &= \frac{x^3}{3}\bigg|_0^1 + \frac{x^4}{2}\bigg|_1^2 + 10 + 200 = \frac{1}{3} + 8 - \frac{1}{2} + 210 = 217\frac{5}{6}. \end{aligned}$$

Remark 5.2 The three preceding proposition show the flexibility of the Stieltjes process of integration. If α is a pure step function, the integral reduces to an infinite series. If α has an initegrable derivative, the integral reduces to the ordinary Riemann integral. This makes it possible to study series and integral simultaneously, rather than separately.

5.2 Integration and Differentiation

We shall see that integration and differentiation are, in a certain sense, inverse operations.

Theorem 5.14 *Let $f \in \mathcal{R}$ on $[a,b]$. For $a \le x \le b$ put*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Then F is continuous on $[a,b]$; furthermore, if f is continuous at $x_0 \in [a,b]$ then F is differentiable at x_0 and

$$F'(x_0) = f(x_0).$$

Proof. Since $f \in \mathcal{R}$, f is bounded. Suppose $|\,f(t)\,| \le M$ on $[a,b]$. If $a \le x < y \le b$, then

$$|\,F(y) - F(x)\,| = \left|\int_x^y f(t)\,\mathrm{d}t\right| \le M(y-x),$$

by Proposition 5.9 (c) and (d). Given $\varepsilon > 0$, we see that

$$|\,F(y) - F(x)\,| < \varepsilon,$$

provided that $|\,y - x\,| < \varepsilon/M$. This proves continuity (and, in fact, uniform continuity) of F.

Now suppose that f is continuous at x_0. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$|\,f(t) - f(x_0)\,| < \varepsilon$$

if $|\,t - x_0\,| < \delta$, $t \in [a,b]$. Hence, if

$$x_0 - \delta < s \le x_0 \le t < x_0 + \delta, \quad \text{and} \quad a \le s < t \le b,$$

we have by Proposition 5.9 (d)

$$\left|\frac{F(t)-F(s)}{t-s} - f(x_0)\right| = \frac{1}{t-s}\left|\int_s^t (f(u) - f(x_0))\,\mathrm{d}u\right| < \varepsilon.$$

It follows that $F'(x_0) = f(x_0)$. ∎

Definition 5.4 A function $F\colon D \to \mathbb{R}$ is called an *antiderivative* or a *primitive* of a function $f\colon D \to \mathbb{R}$ if F is differentiable and $F' = f$.

Remarks 5.3 (a) There exist functions f not having an antiderivative, for example $f(x) = 1$ if $x \ge 0$ and $f(x) = 0$ if $x < 0$. f has a simple discontinuity at 0; but by Corollary 4.18 derivatives cannot have simple discontinuities.

(b) The antiderivative F of a function f (if it exists) is unique up to an additive constant. More precisely, if F is a antiderivative, then $F_1(x) = F(x) + c$ is also a antiderivative of f. If F and G are antiderivatives of f, then there is a constant c so that $F(x) - G(x) = c$. The first part is obvious since $F_1'(x) = F'(x) + c' = f(x)$. Suppose F and G are antiderivatives of f. Put $H(x) = F(x) - G(x)$; then $H'(x) = 0$ and $H(x)$ is constant by Corollary 4.11.

Notation for the antiderivative:

$$F(x) = \int f(x)\,\mathrm{d}x = \int f\,\mathrm{d}x.$$

The function f is called the *integrand*. Integration and differentiation are inverse to each other:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int f(x)\,\mathrm{d}x = f(x), \qquad \int f'(x)\,\mathrm{d}x = f(x).$$

Theorem 5.15 (Fundamental Theorem of Calculus) *If $f \in \mathcal{R}$ on $[a, b]$ and if F is an antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Proof. By Theorem 5.14 $G(x) = \int_a^x f(x)\,\mathrm{d}x$ is differentiable with $G' = F' = f$. By the above remark, the antiderivative is unique up to a constant, hence $F(x) - G(x) = C$. Since $G(a) = \int_a^a f(x)\,\mathrm{d}x = 0$ we obtain

$$F(b) - F(a) = (G(b) + C) - (G(a) + C) = G(b) - G(a) = \int_a^b f(x)\,\mathrm{d}x.$$

■

Proposition 5.16 *Let $f, \varphi\colon [a, b] \to \mathbb{R}$ be continuous functions and $\varphi \geq 0$. Then there exists $\xi \in [a, b]$ such that*

$$\int_a^b f(x)\varphi(x)\,\mathrm{d}x = f(\xi)\int_a^b \varphi(x)\,\mathrm{d}x. \tag{5.31}$$

In particular, in case $\varphi = 1$ we have

$$\int_a^b f(x)\,\mathrm{d}x = f(\xi)(b - a)$$

for some $\xi \in [a, b]$.

Proof. Put $m = \inf\{f(x) \mid x \in [a, b]\}$ and $M = \sup\{f(x) \mid x \in [a, b]\}$. Since $\varphi \geq 0$ we obtain $mf \leq vp \leq Mf$. By Proposition 5.9 (a) and (b) we have

$$m\int_a^b \varphi(x)\,\mathrm{d}x \leq \int_a^b f(x)\varphi(x)\,\mathrm{d}x \leq M\int_a^b \varphi(x)\,\mathrm{d}x.$$

Hence there is a $\mu \in [m, M]$ such that

$$\int_a^b f(x)\varphi(x)\,\mathrm{d}x = \mu\int_a^b \varphi(x)\,\mathrm{d}x.$$

Since f is continuous on $[a, b]$ the intermediate value theorem Theorem 3.5 ensures that there is a ξ with $\mu = f(\xi)$. The claim follows. ■

5.3 Antiderivatives

5.3.1 Table of Antiderivatives

By differentiating the right hand side one gets the left hand side of the table.

function	domain	antiderivative
x^α	$\alpha \in \mathbb{R} \setminus \{-1\},\ x > 0$	$\dfrac{1}{\alpha+1} x^{\alpha+1}$
$\dfrac{1}{x}$	$x < 0 \quad \text{or} \quad x > 0$	$\log \lvert x \rvert$
e^x	$x \in \mathbb{C}$	e^x
a^x	$a > 0,\ a \neq 1,\ x \in \mathbb{C}$	$\dfrac{a^x}{\log a}$
$\sin x$		$-\cos x$
$\cos x$		$\sin x$
$\dfrac{1}{\sin^2 x}$	$x \in \mathbb{C} \setminus \{k\pi \mid k \in \mathbb{Z}\}$	$\cot x$
$\dfrac{1}{\cos^2 x}$	$x \in \mathbb{C} \setminus \left\{\dfrac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\right\}$	$\tan x$
$\dfrac{1}{1+x^2}$	$x \in \mathbb{R}$	$\arctan x$
$\dfrac{1}{\sqrt{1+x^2}}$	$x \in \mathbb{R}$	$\operatorname{arsinh} x = \log(x + \sqrt{x^2+1})$
$\dfrac{1}{\sqrt{1-x^2}}$	$x \in \mathbb{R},\ \lvert x \rvert < 1$	$\arcsin x$
$\dfrac{1}{\sqrt{x^2-1}}$	$x < -1 \quad \text{or} \quad x > 1$	$\log(x + \sqrt{x^2-1})$

5.3.2 Integration Rules

The aim of this subsection is to calculate antiderivatives of composed functions using antiderivatives of (already known) simpler functions.
Notation:

$$f(x)\big|_a^b := f(b) - f(a).$$

Proposition 5.17 (a) *Let f and g be functions with antiderivatives F and G, respectively. Then $af(x) + bg(x)$, $a, b \in \mathbb{R}$, has the antiderivative $aF(x) + bG(x)$.*

$$\int (af + bg)\,\mathrm{d}x = a \int f\,\mathrm{d}x + b \int g\,\mathrm{d}x \quad \text{(Linearity.)}$$

(b) *If f and g are differentiable, and $f(x)g'(x)$ has a antiderivative then $f'(x)g(x)$ has a* *antiderivative, too:*

$$\int f'g \,\mathrm{d}x = fg - \int fg' \,\mathrm{d}x, \quad \text{(Integration by parts.)} \tag{5.32}$$

If f and g are continuously differentiable on $[a, b]$ then

$$\int_a^b f'g \,\mathrm{d}x = f(x)g(x)\big|_a^b - \int_a^b fg' \,\mathrm{d}x. \tag{5.33}$$

(c) *If $\varphi\colon D \to \mathbb{R}$ is continuously differentiable with $\varphi(D) \subset I$, and $f\colon I \to \mathbb{R}$ has a antiderivative F, then*

$$\int f(\varphi(x))\varphi'(x) \;\mathrm{d}x = F(\varphi(x)), \quad \text{(Change of variable.)} \tag{5.34}$$

If $\varphi\colon [a, b] \to \mathbb{R}$ is continuously differentiable with $\varphi([a, b]) \subset I$ and $f\colon I \to \mathbb{R}$ is continuous, then

$$\int_a^b f(\varphi(t))\varphi'(t) \,\mathrm{d}t = \int_{\varphi(a)}^{\varphi(b)} f(x) \,\mathrm{d}x.$$

Proof. Since differentiation is linear, (a) follows.
(b) Differentiating the right hand side, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg - \int fg' \,\mathrm{d}x) = f'g + fg' - fg' = f'g$$

which proves the statement.
(c) By the chain rule $F(g(x))$ is differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}x}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x),$$

and (c) follows.
The statements about the Riemann integrals follow from the statements about antiderivatives using the fundamental theorem of calculus. ∎

Corollary 5.18 *Suppose F is the antiderivative of f.*

$$\int f(ax + b) \,\mathrm{d}x = \frac{1}{a}F(ax + b), \quad a \neq 0; \tag{5.35}$$

$$\int \frac{g'(x)}{g(x)} \,\mathrm{d}x = \log|\, g(x)\,|, \quad \textit{(g differentiable and } g(x) \neq 0\textit{).} \tag{5.36}$$

Example 5.4 (a) The antiderivative of a polymnomial. If $p(x) = \sum_{k=0}^n a_k x^k$, then $\int p(x) \,\mathrm{d}x = \sum_{k=0}^n \frac{a_k}{k+1} x^{k+1}$.
(b) Put $f'(x) = \mathrm{e}^x$ and $g(x) = x$, then $f(x) = \mathrm{e}^x$ and $g'(x) = 1$ and we obtain

$$\int x\mathrm{e}^x \,\mathrm{d}x = x\mathrm{e}^x - \int 1 \cdot \mathrm{e}^x \,\mathrm{d}x = \mathrm{e}^x(x - 1).$$

(c) $I=(0,\infty)$. $\int \log x \,\mathrm{d}x = \int 1\cdot \log x\,\mathrm{d}x = x\log x - \int x\frac{1}{x}\,\mathrm{d}x = x\log x - x$.
(d)

$$\begin{aligned}\int \arctan x\,\mathrm{d}x &= \int 1\cdot \arctan x\,\mathrm{d}x = x\arctan x - \int x\frac{1}{1+x^2}\,\mathrm{d}x\\ &= x\arctan x - \frac{1}{2}\int \frac{(1+x^2)'}{1+x^2}\,\mathrm{d}x = x\arctan x - \frac{1}{2}\log(1+x^2).\end{aligned}$$

In the last equation we made use of (5.36).
(e) Recurrent computation of integrals.

$$I_n := \int \frac{\mathrm{d}x}{(1+x^2)^n},\qquad n\in\mathbb{N}.$$

$I_1 = \arctan x$.

$$I_n = \int \frac{(1+x^2)-x^2}{(1+x^2)^n} = I_{n-1} - \int \frac{x^2\,\mathrm{d}x}{(1+x^2)^n}.$$

Put $u=x$, $v' = \dfrac{x}{(1+x^2)^n}$. Then $U'=1$ and

$$v = \int \frac{x\,\mathrm{d}x}{(1+x^2)^n} = \frac{1}{2}\frac{(1+x^2)^{1-n}}{1-n}.$$

Hence,

$$\begin{aligned}I_n &= I_{n-1} - \frac{1}{2}\frac{x(1+x^2)^{1-n}}{1-n} - \frac{1}{2(1-n)}\int (1+x^2)^{1-n}\,\mathrm{d}x\\ I_n &= \frac{x}{(2n-2)(1+x^2)^{n-1}} + \frac{2n-3}{2n-2}\,I_{n-1}.\end{aligned}$$

5.3.3 Integration of Rational Functions

We will give a useful method to compute antiderivatives of an arbitrary rational function. Consider a rational function p/q where p and q are polinomials. We will assume that $\deg p < \deg q$; for otherwise we can express p/q as a polynomial function plus a rational function which *is* of this form, for eample

$$\frac{x^2}{x-1} = x+1+\frac{1}{x-1}.$$

Polynomials

We need some preliminary facts on polynomials which are stated here without proof.

Theorem 5.19 (Fundamental Theorem of Algebra) *Every polynomial p with complex coefficients which is of degree $n\ge 1$ has a complex root, i. e. there exists a complex number z such that $p(z)=0$.*

Lemma 5.20 (Long Division) *Let p and q be polynomials, then there exist unique polynomials r and s such that*
$$p = qs + r, \quad \deg r < \deg q.$$

Lemma 5.21 *Let p be a complex polynomial of degree $n \geq 1$ and leading coefficient a_n. Then there exist n uniquely determined numbers $z_1, \dots, z_n$ (which may be equal) such that*
$$p(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_n).$$

Proof. We use induction over n and the two preceding statements. In case $n = 1$ the linear polynomial $p(z) = az + b$ can be written in the desired form
$$p(z) = a\left(z - \frac{-b}{a}\right) \quad \text{with the unique root} \quad z_1 = -\frac{b}{a}.$$
Suppose the statement is true for all polynomials of degree $n - 1$. We will show it for degree n polynomials. For, let z_n be a complex root of p which exists by Theorem 5.19; $p(z_n) = 0$. Using long division of p by the linear polynomial $q(z) = z - z_n$ we obtain a quotient polynomial $p_1(z)$ and a remainder polynomial $r(z)$ of degree 0 (a constant polynomial) such that
$$p(z) = (z - z_n)p_1(z) + r(z).$$
Inserting $z = z_n$ gives $p(z_n) = 0 = r(z_n)$; hence the constant r vanishes and we have
$$p(z) = (z - z_n)p_1(z)$$
with a polynomial $p_1(z)$ of degree $n - 1$. Applying the induction hypothesis to p_1 the statement follows. ■

A root α of p is said to be a *root of multiplicity* k, $k \in \mathbb{N}$, if α appears exactly k times among the zeros $z_1, z_2, \dots, z_n$.
If p is a real polynomial and α is a root of multiplicity k of p then $\overline{\alpha}$ is also a root of multiplicity k of p. Using this fact, the real version of Lemma 5.21 is as follows.

Lemma 5.22 *Let q be a real polynomial of degree n with leading coefficient a_n. Then there exist real numbers $\alpha_i, \beta_j, \gamma_j$ and multiplicities $r_i, s_j \in \mathbb{N}$, $i = 1, \dots, k$, $j = 1, \dots, l$ such that*
$$q(x) = a_n \prod_{i=1}^{k}(x - \alpha_i)^{r_i} \prod_{j=1}^{l}(x^2 - 2\beta_j x + \gamma_j)^{s_j}.$$
We assume that the quadratic factors cannot be factored further; this means
$$\beta_j^2 - \gamma_j < 0, \quad j = 1, \dots, l.$$
Of course, $\deg q = \sum_i r_i + \sum_j 2s_j = n$.

Example 5.5 (a) $x^2-4=(x^2+2)(x^2-2)=(x-\sqrt{2})(x+\sqrt{2})(x-\mathrm{i}\sqrt{2})(x+\mathrm{i}\sqrt{2})=(x-\sqrt{2})(x+\sqrt{2})(x^2+2)$
(b) x^3+x-2. One can guess the first zero $x_1=1$. Using long division one gets

$$\begin{array}{rrrrl}
x^3 & & +x & -2 & =(x-1)(x^2+x+2)\\
-(x^3 & -x^2) & & & \\
\hline
 & x^2 & +x & -2 & \\
 & -(x^2 & -x &) & \\
\hline
 & & 2x & -2 & \\
 & & -(2x & -2) & \\
\hline
 & & & 0 &
\end{array}$$

There are no further real zeros of x^2+x+2.

5.3.4 Partial Fraction Decomposition

Proposition 5.23 *Let $p(x)$ and $q(x)$ be real polynomials with $\deg p < \deg q$. There exist real numbers A_{ir}, B_{js}, and C_{js} such that*

$$\frac{p(x)}{q(x)}=\sum_{i=1}^{k}\left(\sum_{r=1}^{r_i}\frac{A_{ir}}{(x-\alpha_i)^r}\right)+\sum_{j=1}^{l}\left(\sum_{s=1}^{s_j}\frac{B_{js}x+C_{js}}{(x^2-2\beta_j x+\gamma_j)^s}\right) \tag{5.37}$$

where the α_i, β_j, γ_j, r_i, and s_j have the same meaning as in Lemma 5.22.

Example 5.6

$$I_1=\int\frac{x-1}{x^2+x+1}\,\mathrm{d}x=\frac{1}{2}\int\frac{2x+1}{x^2+x+1}\,\mathrm{d}x+\frac{3}{2}\int\frac{\mathrm{d}x}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$\underset{u=x+\frac{1}{2}}{=}\frac{1}{2}\log(x^2+x+1)-\frac{3}{2}\int\frac{\mathrm{d}u}{u^2+\frac{3}{4}}$$

$$=\frac{1}{2}\log(x^2+x+1)-\frac{3}{2}\frac{2}{\sqrt{3}}\arctan\frac{x+\frac{1}{2}}{\sqrt{3}/2},$$

where we used

$$\int\frac{\mathrm{d}x}{x^2+a^2}=\frac{1}{a}\arctan\frac{x}{a}.$$

Summarizing the results,

$$I=\frac{7}{2}x^2+\frac{2}{3}\log|\,x-1\,|+\frac{1}{6}\log(x^2+x+1)-\frac{1}{\sqrt{3}}\arctan\frac{1}{\sqrt{3}}(2x+1).$$

5.3.5 Other Classes of Elementary Integrable Functions

An *elementary function* is the compositions of rational, exponential, trigonometric functions and their inverse functions, for example

$$f(x)=\frac{\mathrm{e}^{\sin(\sqrt{x}-1)}}{x+\log x}.$$

A function is called *elementary integrable* if it has an elementary antiderivative. Rational functions are elementary integrable. "Most" functions are not elementary integrable as

$$\mathrm{e}^{-x^2}, \quad \frac{\mathrm{e}^x}{x}, \quad \frac{1}{\log x}, \quad \frac{\sin x}{x}.$$

They define "new" functions

$$\begin{aligned}
W(x) &:= \int_0^x \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t, && \text{(Gaussian integral)},\\
\mathrm{li}(x) &:= \int_0^x \frac{\mathrm{d}t}{\log t} && \text{(integral logarithm)}\\
\mathrm{F}(\varphi,k) &:= \int_0^\varphi \frac{\mathrm{d}x}{\sqrt{1-k^2\sin^2 x}} && \text{(elliptic integral of the first kind)},\\
\mathrm{E}(\varphi,k) &:= \int_0^\varphi \sqrt{1-k^2\sin^2 x}\,\mathrm{d}x && \text{(elliptic integral of the second kind)}.
\end{aligned}$$

$\int R(\cos x, \sin x)\,\mathrm{d}x$

Let $R(u,v)$ be a rational function in two variables u and v. We substitute $u = \tan\frac{x}{2}$. Then

$$\sin x = \frac{2u}{1+u^2}, \qquad \cos x = \frac{1-u^2}{1+u^2}, \qquad \mathrm{d}x = \frac{2\mathrm{d}u}{1+u^2}.$$

Hence

$$\int R(\cos x, \sin x)\,\mathrm{d}x = \int R\left(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}\right)\frac{2\mathrm{d}u}{1+u^2} = \int R_1(u)\mathrm{d}u$$

with another rational function $R_1(u)$.

$\int R(x, \sqrt[n]{ax+b})\,\mathrm{d}x$

The substitution

$$t = \sqrt[n]{ax+b}$$

yields $x = (t^n - b)/a,\ \mathrm{d}x = nt^{n-1}\,\mathrm{d}t/a$, and therefore

$$\int R(x, \sqrt[n]{ax+b})\,\mathrm{d}x = \frac{n}{a}\int R\left(\frac{t^n-b}{a}, t\right) t^{n-1}\,\mathrm{d}t.$$

$\int R(x, \sqrt{ax^2+2bx+c})\,\mathrm{d}x$

Using the method of complete squares the above integral can be written in one of the three basic forms

$$\int R(t, \sqrt{t^2+1})\,\mathrm{d}t, \quad \int R(t, \sqrt{t^2-1})\,\mathrm{d}t, \quad \int R(t, \sqrt{1-t^2})\,\mathrm{d}t.$$

Further substitutions

$$\begin{aligned} t &= \sinh u, & \sqrt{t^2+1} &= \cosh u, & \mathrm{d}t &= \cosh u\,\mathrm{d}u,\\ t &= \pm\cosh u, & \sqrt{t^2-1} &= \sinh u, & \mathrm{d}t &= \pm\sinh u\,\mathrm{d}u,\\ t &= \pm\cos u, & \sqrt{1-t^2} &= \sin u, & \mathrm{d}t &= \mp\sin u\,\mathrm{d}u \end{aligned}$$

reduce the integral to already known integrals.

Example 5.7 The trapezoid rule. Let $f\colon [0,1]\to\mathbb{R}$ be twice continuously differentiable. Then there exists $\xi\in[0,1]$ such that

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{1}{2}\,(f(0)+f(1)) - \frac{1}{12}f''(\xi). \tag{5.38}$$

Proof. Let $\varphi(x) = \frac{1}{2}x(1-x)$ such that $\varphi(x)\geq 0$ for $x\in[0,1]$, $\varphi'(x) = \frac{1}{2}-x$, and $\varphi''(x) = -1$. Using integration by parts twice as well as Theorem 5.16 we find

$$\begin{aligned} \int_0^1 f(x)\,\mathrm{d}x &= -\int_0^1 \varphi''(x)f(x)\,\mathrm{d}x = -\varphi'(x)f(x)\big|_0^1 + \int_0^1 \varphi'(x)f'(x)\,\mathrm{d}x\\ &= \frac{1}{2}\,(f(0)+f(1)) + \varphi(x)f'(x)\big|_0^1 - \int_0^1 \varphi(x)f''(x)\,\mathrm{d}x\\ &= \frac{1}{2}\,(f(0)+f(1)) - f''(\xi)\int_0^1 \varphi(x)\,\mathrm{d}x\\ &= \frac{1}{2}\,(f(0)+f(1)) - \frac{1}{12}f''(\xi). \end{aligned}$$

■

5.3.6 Inequalities

Besides the triangle inequality $\left|\int_a^b f\,\mathrm{d}\alpha\right| \leq \int_a^b |\,f\,|\,\mathrm{d}\alpha$ which was shown in Proposition 5.10 we can formulate Hölder's, Minkowski's, and the Cauchy–Schwarz inequalities for Riemann–Stieltjes integrals. For, let $p>0$ be a fixed positive real number and α an increasing function on $[a,b]$. For $f\in\mathcal{R}(\alpha)$ define the L^p-norm

$$\|f\|_p = \left(\int_a^b |\,f\,|^p\,\mathrm{d}\alpha\right)^{\frac{1}{p}}. \tag{5.39}$$

Proposition 5.24 (a) Cauchy–Schwarz inequality. *Suppose $f,g\in\mathcal{R}(\alpha)$, then*

$$\left|\int_a^b fg\,\mathrm{d}\alpha\right| \leq \int_a^b |\,fg\,|\,\mathrm{d}\alpha \leq \sqrt{\int_a^b |\,f\,|^2\,\mathrm{d}\alpha}\,\sqrt{\int_a^b |\,g\,|^2\,\mathrm{d}\alpha} \quad \textit{or} \tag{5.40}$$

$$\int_a^b |\,fg\,|\,\mathrm{d}\alpha \leq \|f\|_2\,\|g\|_2\,. \tag{5.41}$$

(b) Hölder's inequality. *Let p and q be positive real numbers such that $\dfrac{1}{p}+\dfrac{1}{q}=1$. If $f,g\in\mathcal{R}(\alpha)$, then*

$$\left|\int_a^b fg\,\mathrm{d}\alpha\right| \le \int_a^b |\,fg\,|\,\mathrm{d}\alpha \le \|f\|_p\,\|g\|_q\,. \tag{5.42}$$

(c) Minkowski's inequality. *Let $p\ge 1$ and $f,g\in\mathcal{R}(\alpha)$, then*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p\,. \tag{5.43}$$

Proof. We prove (b). The other two statements are consequences, their proofs are along the lines in Section 1.3. The main idea is to approximate the integral on the left by Riemann sums and use Hölder's inequality (1.19). Let $\varepsilon>0$; without loss of generality, let $f,g\ge 0$. By Proposition 5.10 fg, f^p, $g^q\in\mathcal{R}(\alpha)$ and by Proposition 5.3 there exist partitions P_1, P_2, and P_3 of $[a,b]$ such that $U(fg,P_1,\alpha)-L(fg,P_1,\alpha)<\varepsilon$, $U(f^p,P_2,\alpha)-L(f^p,P_2,\alpha)<\varepsilon$, and $U(g^q,P_3,\alpha)-L(g^q,P_3,\alpha)<\varepsilon$. Let $P=\{x_0,x_1,\dots,x_n\}$ be the common refinement of P_1, P_2, and P_3. By Lemma 5.4 (a) and (c)

$$\int_a^b fg\,\mathrm{d}\alpha < \sum_{i=1}^n (fg)(t_i)\Delta\alpha_i + \varepsilon, \tag{5.44}$$

$$\sum_{i=1}^n f(t_i)^p\Delta\alpha_i < \int_a^b f^p\,\mathrm{d}\alpha + \varepsilon, \tag{5.45}$$

$$\sum_{i=1}^n g(t_i)^q\Delta\alpha_i < \int_a^b g^q\,\mathrm{d}\alpha + \varepsilon, \tag{5.46}$$

for any $t_i\in[x_{i-1},x_i]$. Using the two preceding inequalities and Hölder's inequality (1.19) we have

$$\begin{aligned}\sum_{i=1}^n f(t_i)\Delta\alpha_i^{\frac1p} g(t_i)\Delta\alpha_i^{\frac1q} &\le \left(\sum_{i=1}^n f(t_i)^p\Delta\alpha_i\right)^{\frac1p}\left(\sum_{i=1}^n g(t_i)^q\Delta\alpha_i\right)^{\frac1q}\\ &< \left(\int_a^b f^p\,\mathrm{d}\alpha+\varepsilon\right)^{\frac1p}\left(\int_a^b g^q\,\mathrm{d}\alpha+\varepsilon\right)^{\frac1q}.\end{aligned}$$

By (5.44),

$$\int_a^b fg\,\mathrm{d}\alpha < \sum_{i=1}^n (fg)(t_i)\Delta\alpha_i+\varepsilon < \left(\int_a^b f^p\,\mathrm{d}\alpha+\varepsilon\right)^{\frac1p}\left(\int_a^b g^q\,\mathrm{d}\alpha+\varepsilon\right)^{\frac1q}+\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, the claim follows. ∎

5.4 Improper Integrals

The notion of the Riemann integral defined so far is apparently too tight for some applications: we can integrate only over finite intervals and the functions are necessarily bounded.

If the integration interval is unbounded or the function to integrate is unbounded we speak about *improper* integrals. We consider three cases: one limit of the integral is infinite; the function is not defined at one of the end points a or b of the interval; both a and b are critical points (either infinity or the function is not defined there).

5.4.1 Integrals on unbounded intervals

Definition 5.5 Suppose $f \in \mathcal{R}$ on $[a,b]$ for all $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to+\infty} \int_a^b f(x)\,\mathrm{d}x \tag{5.47}$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges if f has been replaced by $|f|$, it is said to *converge absolutely*.

Obviously, if an integral converges absolutely, then it converges where

$$\left| \int_a^\infty f\,\mathrm{d}x \right| \le \int_a^\infty |f|\,\mathrm{d}x.$$

Similarly, one defines $\displaystyle\int_{-\infty}^b f(x)\,\mathrm{d}x$. Moreover,

$$\int_{-\infty}^\infty f\,\mathrm{d}x := \int_{-\infty}^0 f\,\mathrm{d}x + \int_0^\infty f\,\mathrm{d}x$$

if both integrals on the right side converge.

Example 5.8 (a) The integral $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x^s}$ converges for $s > 1$ and diverges for $0 < s \le 1$. Indeed,

$$\int_1^R \frac{\mathrm{d}x}{x^s} = \frac{1}{1-s} \cdot \left.\frac{1}{x^{s-1}}\right|_1^R = \frac{1}{s-1}\left(1 - \frac{1}{R^{s-1}}\right).$$

Since

$$\lim_{R\to+\infty} \frac{1}{R^{s-1}} = \begin{cases} 0, & \text{if} \quad s > 1, \\ +\infty, & \text{if} \quad 0 < s < 1, \end{cases}$$

it follows that

$$\int_0^\infty \frac{\mathrm{d}x}{x^s} = \frac{1}{s-1}, \quad \text{if} \quad s > 1.$$

(b)

$$\int_0^R \mathrm{e}^{-x}\,\mathrm{d}x = -\mathrm{e}^{-x}\Big|_0^R = 1 - \frac{1}{\mathrm{e}^R}.$$

Hence $\displaystyle\int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x = 1.$

Proposition 5.25 (Cauchy criterion) *The improper integral* $\int_a^\infty f\,\mathrm{d}x$ *converges if and only if for every* $\varepsilon > 0$ *there exists some* $b > a$ *such that for all* $c, d > b$

$$\left| \int_c^d f\,\mathrm{d}x \right| < \varepsilon.$$

Proof. The following Cauchy criterion for limits of functions is easily proved using sequences: The limit $\lim_{x\to\infty} F(x)$ exists if and only if

$$\forall \varepsilon > 0\ \exists R > 0\ \forall x, y > R : |\, F(x) - F(y) \,| < \varepsilon.$$

Apply this criterion to the function $F(t) = \int_a^t f\,\mathrm{d}x$; this proves the assertion. ∎

Example 5.9 $\int_1^\infty \frac{\sin x}{x}\,\mathrm{d}x$. Partial integration with $u = \frac{1}{x}$ and $v' = \sin x$ yields $u' = -\frac{1}{x^2}$, $v = -\cos x$ and

$$\int_c^d \frac{\sin x}{x}\,\mathrm{d}x = -\frac{1}{x}\cos x\Big|_c^d - \int_c^d \frac{\cos x}{x^2}\,\mathrm{d}x$$

$$\left|\int_c^d \frac{\sin x}{x}\,\mathrm{d}x\right| \le \left| -\frac{1}{d}\cos d + \frac{1}{c}\cos c \right| + \left| \int_c^d \frac{\mathrm{d}x}{x^2} \right|$$

$$\le \frac{1}{c} + \frac{1}{d} + \left| \frac{1}{d} - \frac{1}{c} \right| \le 2\left(\frac{1}{c} + \frac{1}{d}\right) < \varepsilon$$

if c and d are sufficiently large. Hence, $\int_1^\infty \frac{\sin x}{x}\,\mathrm{d}x$ converges.
The integral does not converge absolutely. For non-negative integers $n \in \mathbb{Z}_+$ we have

$$\int_{n\pi}^{(n+1)\pi} \left| \frac{\sin x}{x} \right| \mathrm{d}x \ge \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin x|\,\mathrm{d}x = \frac{2}{(n+1)\pi};$$

hence

$$\int_0^{(n+1)\pi} \left| \frac{\sin x}{x} \right| \mathrm{d}x \ge \frac{2}{\pi} \sum_{k=0}^{n} \frac{1}{k+1}.$$

Since the harmonic series diverges, so does the integral $\int_0^\infty \left| \frac{\sin x}{x} \right| \mathrm{d}x$.

Proposition 5.26 *Suppose* $f \in \mathcal{R}$ *is nonnegative,* $f \ge 0$. *Then* $\int_a^\infty f\,\mathrm{d}x$ *converges if there exists* $C > 0$ *such that*

$$\int_a^b f\,\mathrm{d}x < C, \quad \textit{for all} \quad b > a.$$

The proof is similar to the proof of Lemma 2.17 (c); we omit it. Analogous propositions are true for integrals $\int_{-\infty}^a f\,\mathrm{d}x$.

Proposition 5.27 (Integral criterion for series) *Assume that* $f \in \mathcal{R}$ *is nonnegative* $f \ge 0$ *and decreasing on* $[1, +\infty)$. *Then* $\int_1^\infty f\,\mathrm{d}x$ *converges if and only if the series* $\sum_{n=1}^\infty f(n)$ *converges.*

Proof. Since $f(n) \le f(x) \le f(n-1)$ for $n-1 \le x \le n$,

$$f(n) \le \int_{n-1}^{n} f\,\mathrm{d}x \le f(n-1).$$

Summation over $n = 2, 3, \dots, N$ yields

$$\sum_{n=2}^{N} f(n) \le \int_{1}^{N} f\,\mathrm{d}x \le \sum_{n=1}^{N-1} f(n).$$

If $\int_1^\infty f\,\mathrm{d}x$ converges the series $\sum_{n=1}^\infty f(n)$ is bounded and therefore convergent. Conversely, if $\sum_{n=1}^\infty f(n)$ converges, the integral $\int_1^R f\,\mathrm{d}x \le \sum_{n=1}^\infty f(n)$ is bounded as $R \to \infty$, hence convergent by Proposition 5.26. ■

Example 5.10 $\sum_{n=2}^\infty \frac{1}{n(\log n)^\alpha}$ converges if and only if $\int_2^\infty \frac{\mathrm{d}x}{x(\log x)^\alpha}$ converges. The substitution $y = \log x$, $\mathrm{d}y = \frac{\mathrm{d}x}{x}$ gives

$$\int_2^\infty \frac{\mathrm{d}x}{x(\log x)^\alpha} = \int_{\log 2}^\infty \frac{\mathrm{d}y}{y^\alpha}$$

which converges if and only if $\alpha > 1$ (see Example 5.8).

5.4.2 Integrals of Unbounded Functions

Definition 5.6 Suppose f is a real function on $[a, b)$ and $f \in \mathcal{R}$ on $[a, t]$ for every t, $a < t < b$. Define

$$\int_a^b f\,\mathrm{d}x = \lim_{t\to b-0} \int_a^t f\,\mathrm{d}x$$

if the limit on the right exists. Similarly, one defines

$$\int_a^b f\,\mathrm{d}x = \lim_{t\to a+0} \int_t^b f\,\mathrm{d}x$$

if f is unbounded at a and integrable on $[t, b]$ for all t with $a < t < b$.

In both cases we say that $\int_a^b f\,\mathrm{d}x$ *converges.*

Example 5.11 (a)

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \lim_{t\to 1-0} \int_0^t \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \lim_{t\to 1-0} \arcsin x\big|_0^t = \lim_{t\to 1-0} \arcsin t = \arcsin 1 = \frac{\pi}{2}.$$

(b)

$$\int_0^1 \frac{\mathrm{d}x}{x^\alpha} = \lim_{t\to 0+0} \int_t^1 \frac{\mathrm{d}x}{x^\alpha} = \lim_{t\to 0+0} \begin{cases} \frac{1}{1-\alpha} x^{1-\alpha}\big|_t^1, & \alpha \ne 1 \\ \log x\big|_t^1, & \alpha = 1 \end{cases} = \begin{cases} \frac{1}{1-\alpha}, & \alpha < 1, \\ +\infty, & \alpha \ge 1. \end{cases}$$

Remarks 5.4 (a) The analogous statements to Proposition 5.25 and Proposition 5.26 are true for improper integrals $\int_a^b f\,\mathrm{d}x$.
(b) If f is unbounded both at a and at b we define the improper integral

$$\int_a^b f\ dx = \int_a^c f\,\mathrm{d}x + \int_c^b f\,\mathrm{d}x$$

if c is between a and b and both improper integrals on the right side exist.
(c) Also, if f is unbounded at a define

$$\int_a^\infty f\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x + \int_b^\infty f\,\mathrm{d}x$$

if the two improper integrals on the right side exist.
(d) If f is unbounded in the interior of the interval $[a, b]$, say at c, we define the improper integral

$$\int_a^b f\,\mathrm{d}x = \int_a^c f\,\mathrm{d}x + \int_c^b f\,\mathrm{d}x$$

if the two improper integrals on the right side exist. For example,

$$\begin{aligned}\int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{|\,x\,|}} &= \int_{-1}^0 \frac{\mathrm{d}x}{\sqrt{|\,x\,|}} + \int_0^1 \frac{\mathrm{d}x}{\sqrt{|\,x\,|}} = \lim_{t\to 0-0} \int_{-1}^t \frac{\mathrm{d}x}{\sqrt{|\,x\,|}} + \lim_{t\to 0+0} \int_t^1 \frac{\mathrm{d}x}{\sqrt{|\,x\,|}} \\ &= \lim_{t\to 0-0} -2\sqrt{-x}\Big|_{-1}^t + \lim_{t\to 0+0} 2\sqrt{x}\Big|_t^1 = 4.\end{aligned}$$

5.4.3 The Gamma function

For $x > 0$ set

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t. \tag{5.48}$$

By Example 5.11, $\Gamma_1(x) = \int_0^1 t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$ converges since for every $t > 0$

$$t^{x-1}\mathrm{e}^{-t} \le \frac{1}{t^{1-x}}.$$

By Example 5.8, $\Gamma_2(x) = \int_1^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$ converges since for every $t \ge t_0$

$$t^{x-1}\mathrm{e}^{-t} \le \frac{1}{t^2}.$$

Note that $\lim_{t\to\infty} t^{x+1}\mathrm{e}^{-t} = 0$ by Proposition 3.18. Hence, $\Gamma(x)$ is defined for every $x > 0$.

Proposition 5.28 *For $n \in \mathbb{N}$ we have $\Gamma(n+1) = n!$ and for every positive x,*

$$x\Gamma(x) = \Gamma(x+1). \tag{5.49}$$

Proof. Using integration by parts,

$$\int_{\varepsilon}^{R} t^{x}\mathrm{e}^{-t}\,\mathrm{d}t = -t^{x}\mathrm{e}^{-t}\Big|_{\varepsilon}^{R} + x\int_{\varepsilon}^{R} t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$

Taking the limits $\varepsilon \to 0{+}0$ and $R \to +\infty$ one has $\Gamma(x{+}1) = x\Gamma(x)$. Since by Example,5.8

$$\Gamma(1) = \int_{0}^{\infty} \mathrm{e}^{-t}\,\mathrm{d}t = 1,$$

it follows from (5.49) that

$$\Gamma(n+1) = n\Gamma(n) = \cdots = n(n-1)(n-2)\cdots\Gamma(1) = n!$$

■

The Γ function interpolates the factorial function $n!$ which is defined only for positive integers n. However, this property alone is not sufficient for a complete characterization of the Gamma function. We need another property.
Let $I \subset \mathbb{R}$ be an interval. A positive function $F\colon I \to \mathbb{R}$ is called *logarithmic convex* if $\log F\colon I \to \mathbb{R}$ is convex, i.e. for every $x, y \in I$ and every λ, $0 \le \lambda \le 1$ we have

$$F(\lambda x + (1-\lambda)y) \le F(x)^{\lambda}\, F(y)^{1-\lambda}.$$

Proposition 5.29 *The Gamma function is logarithmic convex.*

Proof. Let $x, y > 0$ and $0 < \lambda < 1$ be given. Set $p = 1/\lambda$ and $q = 1/(1-\lambda)$. Then $1/p + 1/q = 1$ and we apply Hölder's inequality to the functions

$$f(t) = t^{\frac{x-1}{p}}\mathrm{e}^{-\frac{t}{p}}, \quad g(t) = t^{\frac{y-1}{q}}\mathrm{e}^{-\frac{t}{q}}$$

and obtain

$$\int_{\varepsilon}^{R} f(t)g(t)\,\mathrm{d}t \le \left(\int_{\varepsilon}^{R} f(t)^{p}\,\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_{\varepsilon}^{R} g(t)^{q}\,\mathrm{d}t\right)^{\frac{1}{q}}.$$

Note that

$$f(t)g(t) = t^{\frac{x}{p}+\frac{y}{q}-1}\mathrm{e}^{-t}, \quad f(t)^{p} = t^{x-1}\mathrm{e}^{-t}, \quad g(t)^{q} = t^{y-1}\mathrm{e}^{-t}.$$

Taking the limts $\varepsilon \to 0+0$ and $R \to +\infty$ we obtain

$$\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) \le \Gamma(x)^{\frac{1}{p}}\Gamma(y)^{\frac{1}{q}}.$$

■

Remark 5.5 One can prove that a convex function (see Definition 4.4) is continuous, see Proposition 5.35 in the appendix. Also, an increasing convex function of a convex function f is convex, for example e^{f} is convex if f is. We conclude that $\Gamma(x)$ is continuous for $x > 0$.

Theorem 5.30 *Let* $F\colon (0,+\infty) \to (0,+\infty)$ *be a function with*

(a) $F(1) = 1$,
(b) $F(x+1) = xF(x)$,
(c) F *is logarithmic convex.*

Then $F(x) = \Gamma(x)$ *for all* $x > 0$.

The proof is in the appendix to this chapter.

Stirling's Formula

We give an asymptotic formula for $n!$ as $n \to \infty$. We call two sequences (a_n) and (b_n) to be *asymptotically equal* if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, and we write $a_n \sim b_n$.

Proposition 5.31 (Stirling's Formula) *The asymptotical behavior of* $n!$ *is*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n .$$

Proof. Using the trapezoid rule (5.38) with $f(x) = \log x$, $f''(x) = -1/x^2$ we have

$$\int_k^{k+1} \log x \,\mathrm{d}x = \frac{1}{2}\left(\log k + \log(k+1)\right) + \frac{1}{12\xi_k^2}$$

with $k \le \xi_k \le k+1$. Summation over $k = 1, \dots, n-1$ gives

$$\int_1^n \log x \,\mathrm{d}x = \sum_{k=1}^n \log k - \frac{1}{2}\log n + \frac{1}{12}\sum_{k=1}^{n-1}\frac{1}{\xi_k^2}.$$

Since $\int \log x \,\mathrm{d}x = x\log x - x$ (integration by parts), we have

$$n\log n - n + 1 = \sum_{k=1}^n \log k - \frac{1}{2}\log n + \frac{1}{12}\sum_{k=1}^{n-1}\frac{1}{\xi_k^2}$$

$$\sum_{k=1}^n \log k = \left(n + \frac{1}{2}\right)\log n - n + \gamma_n,$$

where $\gamma_n = 1 - \frac{1}{12}\sum_{k=1}^{n-1}\frac{1}{\xi_k^2}$. Exponentiating both sides of the equation we find with $c_n = \mathrm{e}^{\gamma_n}$

$$n! = n^{n+\frac{1}{2}}\,\mathrm{e}^{-n}\,c_n. \tag{5.50}$$

Since $0 < 1/\xi_k^2 \le 1/k^2$, the limit

$$\gamma = \lim_{n\to\infty}\gamma_n = 1 - \sum_{k=1}^{\infty}\frac{1}{\xi_k^2}$$

exists, and so the limit $c = \lim_{n\to\infty} c_n = \mathrm{e}^\gamma$.

The proof that $c_n \xrightarrow[n\to\infty]{} \sqrt{2\pi}$ uses Wallis's product formula for π and is found in the appendix. ∎

5.5 Integration of Vector-Valued Functions

A mapping $\gamma\colon [a,b] \to \mathbb{R}^k$, $\gamma(t) = (\gamma_1(t), \dots, \gamma_k(t))$ is said to be continuous if all the mappings γ_i, $i = 1, \dots, k$, are continuous. Moreover, if all the γ_i are differentiable, we write $\gamma'(t) = (\gamma_1'(t), \dots, \gamma_k'(t))$.

Definition 5.7 Let $f_1, \dots, f_k$ be real functions on $[a,b]$ and let $f = (f_1, \dots, f_k)$ be the correponding mapping from $[a,b]$ into $\mathbb{R}^k$. If α increases on $[a,b]$, to say that $f \in \mathcal{R}(\alpha)$ means that $f_j \in \mathcal{R}(\alpha)$ for $j = 1, \dots, k$. In this case we define

$$\int_a^b f \,\mathrm{d}\alpha = \left(\int_a^b f_1 \,\mathrm{d}\alpha, \dots, \int_a^b f_k \,\mathrm{d}\alpha \right).$$

In other words $\int_a^b f \,\mathrm{d}\alpha$ is the point in $\mathbb{R}^k$ whose jth coordinate is $\int_a^b f_j \,\mathrm{d}\alpha$. It is clear that parts (a), (c), and (e) of Proposition 5.9 are valid for these vector valued integrals; we simply apply the earlier results to each coordinate. The same is true for Proposition 5.12, Theorem 5.14, and Theorem 5.15. To illustrate this, we state the analog of the fundamental theorem of calculus.

Theorem 5.32 *If $f = (f_1, \dots, f_k) \in \mathcal{R}$ on $[a,b]$ and if $F = (F_1, \dots, F_k)$ is an antiderivative of f on $[a,b]$, then*

$$\int_a^b f(x) \,\mathrm{d}x = F(b) - F(a).$$

The analog of Proposition 5.10 (b) offers some new features.

Proposition 5.33 *If $f = (f_1, \dots, f_k) \in \mathcal{R}(\alpha)$ on $[a,b]$ then $\|f\| \in \mathcal{R}(\alpha)$ and*

$$\left\| \int_a^b f \,\mathrm{d}\alpha \right\| \le \int_a^b \|f\| \,\mathrm{d}\alpha. \tag{5.51}$$

Proof. By (5.39)

$$\|f\| = \left(f_1^2 + f_2^2 + \dots + f_k^2\right)^{\frac{1}{2}}.$$

By Proposition 5.10 (a) each of the functions f_i^2 belong to $\mathcal{R}(\alpha)$; hence so does their sum. Since the square-root is a continuous function on the positive half line. If we apply Proposition 5.8 we see $\|f\| \in \mathcal{R}(\alpha)$.
To prove (5.51), put $y = (y_1, \dots, y_k)$ with $y_j = \int f_j \,\mathrm{d}\alpha$. Then we have $y = \int f \,\mathrm{d}\alpha$, and

$$\|y\|^2 = \sum_{j=1}^k y_j^2 = \sum_{j=1}^k y_j \int f_j \,\mathrm{d}\alpha = \int \sum_{j=1}^k (y_j f_j) \,\mathrm{d}\alpha.$$

By the Cauchy–Schwarz inequality,

$$\sum_{j=1}^k y_j f_j(t) \le \|y\| \, \|f(t)\|, \quad t \in [a,b].$$

Inserting this into the preceding equation, the monotony of the integral gives

$$\|y\|^2 \leq \|y\| \int \|f\| \, d\alpha.$$

If $y = 0$, (5.51) is trivial. If $y \neq 0$, division by $\|y\|$ gives (5.51). ∎

Integration of Complex Valued Functions

This is a special case of the above arguments with $k = 2$. Let $u, v \colon [a, b] \to \mathbb{R}$ real functions. The function $\varphi = u + \mathrm{i}v \colon [a, b] \to \mathbb{C}$ is said to be *integrable* if $u, v \in \mathcal{R}$ on $[a, b]$ and we set

$$\int_a^b \varphi \, dx = \int_a^b u \, dx + \mathrm{i} \int_a^b v \, dx.$$

The fundamental theorem of calculus holds: If the complex function φ is Riemann integrable, $\varphi \in \mathcal{R}$ on $[a, b]$ and $F(x)$ is an antiderivative of φ, then

$$\int_a^b \varphi(x) \, dx = F(b) - F(a).$$

Proof. Let $F = U + \mathrm{i}V$ be the antiderivative of φ where $U' = u$ and $V' = v$. By the fundamental theorem of calculus

$$\int_a^b \varphi \, dx = \int_a^b u \, dx + \mathrm{i} \int_a^b v \, dx = U(b) - U(a) + \mathrm{i}\,(V(b) - V(a)) = F(b) - F(a).$$

∎

Example:

$$\int_a^b e^{\alpha t} \, dt = \frac{1}{\alpha} e^{\alpha t} \Big|_a^b, \quad \alpha \in \mathbb{C}.$$

5.6 Applications

5.6.1 Curves in $\mathbb{R}^k$

We consider concrete geometric objects, curves in $\mathbb{R}^k$. We define the tangent vector, the angle between intersecting curves, and the arc length.

Definition 5.8 A continuous mapping $\gamma \colon [a, b] \to \mathbb{R}^k$ is called a *curve* in $\mathbb{R}^k$. If $\gamma(a) = \gamma(b)$, γ is said to be a *closed curve.*

Of course, a curve is given by a k-tupel $\gamma = (\gamma_1, \dots, \gamma_k)$ where $\gamma_i \colon [a, b] \to \mathbb{R}$ are continuous functions. It should be noted that we define a curve to be *a mapping*, not a point set. Of course, with each curve γ in $\mathbb{R}^k$ there is associated a subset of $\mathbb{R}^k$, namely the range of γ, but different curves may have the same range.

Example 5.12 (a) A *circle* of radius $r > 0$ with midpoint at the origin is described by the curve

$$\gamma\colon [0, 2\pi] \to \mathbb{R}^2, \quad \gamma(t) = (r\cos t, r\sin t).$$

Note that $\tilde{\gamma}\colon [0, 4\pi] \to \mathbb{R}^2$ with $\tilde{\gamma}(t) = \gamma(t)$ has the same range but is different from γ.
(b) Let a, $v \in \mathbb{R}^k$, $v \neq 0$, be given. The map

$$\gamma\colon \mathbb{R} \to \mathbb{R}^k, \quad \gamma(t) = tv + a$$

describes the line in $\mathbb{R}^k$ through a with direction v.
(c) If $f\colon [a, b] \to \mathbb{R}$ is a continuous function, the graph of f can be thought as a curve in $\mathbb{R}^2$:

$$\gamma\colon [a, b] \to \mathbb{R}^2, \quad \gamma(t) = (t, f(t)).$$

Definition 5.9 Let $\gamma\colon [a, b] \to \mathbb{R}^k$ be a differentiable curve. For $t \in [a, b]$ we call

$$\gamma'(t) = (\gamma_1'(t), \dots, \gamma_k'(t))$$

the *tangent vector* of the curve γ at t.
A differentiable curve is said to be *regular* if $\gamma'(t) \neq 0$ for all $t \in [a, b]$. If $\gamma'(t_0) = 0$, γ is *singular* at t_0.

If γ is regular, we also consider the *unit tangent vector* $\gamma'(t)/\,\|\gamma'(t)\|$.

Remark 5.6 Physical interpretation. Let t the time variable and $s(t)$ the coordinates of a point moving in $\mathbb{R}^k$. In this picture $v(t) = s'(t)$ is the velocity vector of the moving point. The instantaneous velocity is the euclidean norm of $v(t)$

$$\|v(t)\| = \sqrt{s_1'(t)^2 + \cdots + s_k'(t)^2}.$$

The *acceleration vector* is the second derivative of $s(t)$, $a(t) = v'(t) = s''(t)$.

Example 5.13 (a) Newton's knot. The curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (t^2 - 1, t^3 - t)$ is not injective since $\gamma(-1) = \gamma(1) = (0, 0) = x_0$. The point x_0 is a *double point* of the curve. In general γ has two different tangent lines at a double point. Since $\gamma'(t) = (2t, 3t^2 - 1)$ we have $\gamma'(-1) = (-2, 2)$ and $\gamma'(1) = (2, 2)$. The curve is regular since $\gamma'(t) \neq 0$ for all t.
(b) Neil's parabola. Let $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ be given by $\gamma(t) = (t^2, t^3)$. Since $\gamma'(t) = (2t, 3t^2)$, the origin is the only singular point.

Definition 5.10 Let $\gamma_i\colon I_i \to \mathbb{R}^k$, $i = 1, 2$, be two regular curves with $\gamma_1(t_1) = \gamma_2(t_2)$. The *angle of intersection* φ between the two curves γ_i at t_i is defined to be the angle between the two tangent lines $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$. Hence,

$$\cos\varphi = \frac{\langle \gamma_1'(t_1), \gamma_2'(t_2)\rangle}{\|\gamma_1'(t_1)\|\,\|\gamma_2'(t_2)\|}, \quad \varphi \in [0, \pi].$$

Example 5.14 Newton's knot. $\gamma_1(t) = \gamma_2(t) = \gamma(t) = (t^2 - 1, t^3 - t)$. Since $\gamma(-1) = \gamma(1) = (0, 0)$, the self-intersection angle φ satisfies

$$\cos\varphi = \frac{\langle(-2, 2), (2, 2)\rangle}{8} = 0,$$

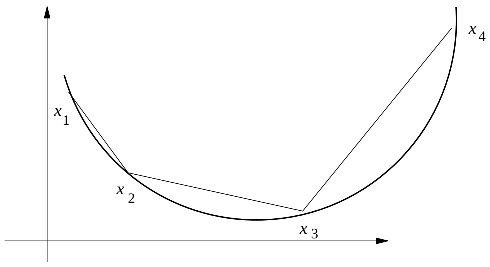
hence $\varphi = 90°$, the intersection is orthogonal.

Rectifiable Curves

We associate to each partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ and to each curve γ the number

$$\ell(P, \gamma) = \sum_{i=1}^{n} \|\gamma(x_i) - \gamma(x_{i-1})\| \,. \tag{5.52}$$

The ith term in this sum is the distance (in $\mathbb{R}^k$) of the points $\gamma(x_{i-1})$ and $\gamma(x_i)$.



Hence $\ell(P, \gamma)$ is the length of the polygonal path with vertices $\gamma(x_0), \ldots, \gamma(x_n)$. As our partition becomes finer and finer, this polygon approaches the range of γ more and more closely.

This makes it seem reasonable to define the *length* of γ as

$$\ell(\gamma) = \sup \ell(P, \gamma)$$

where the supremum is taken over all partitions P of $[a, b]$.
If $\ell(\gamma) < \infty$ we say that γ is *rectifiable*. In certain cases, $\ell(\gamma)$ is given by a Riemann integral. We shall prove this for *continuously differentiable* curves, i. e. for curves γ whose derivative γ' is continuous.

Proposition 5.34 *If γ' is continuous on $[a, b]$, then γ is rectifiable, and*

$$\ell(\gamma) = \int_a^b \|\gamma'(t)\| \, \mathrm{d}t.$$

Proof. If $a \le x_{i-1} < x_i \le b$, by Theorem 5.32, $\gamma(x_i) - \gamma(x_{i-1}) = \int_{x_{i-1}}^{x_i} \gamma'(t)\,\mathrm{d}t$. Applying Proposition 5.33 we have

$$\|\gamma(x_i) - \gamma(x_{i-1})\| = \left\| \int_{x_{i-1}}^{x_i} \gamma'(t)\,\mathrm{d}t \right\| \le \int_{x_{i-1}}^{x_i} \|\gamma'(t)\| \, \mathrm{d}t.$$

Hence

$$\ell(P, \gamma) \le \int_a^b \|\gamma'(t)\| \, \mathrm{d}t$$

for every partition P of $[a, b]$. Consequently,

$$\ell(\gamma) \le \int_a^b \|\gamma'(t)\| \, \mathrm{d}t.$$

To prove the opposite inequality, let $\varepsilon > 0$ be given. Since γ' is uniformly continuous on $[a, b]$, there exists $\delta > 0$ such that

$$\|\gamma'(s) - \gamma'(t)\| < \varepsilon \quad \text{if} \quad |\, s - t \,| < \delta.$$

Let P be a partition with $\Delta x_i \le \delta$ for all i. If $x_{i-1} \le t \le x_i$ it follows that

$$\|\gamma'(t)\| \le \|\gamma'(x_i)\| + \varepsilon.$$

Hence

$$\begin{aligned}\int_{x_{i-1}}^{x_i} \|\gamma'(t)\| \, \mathrm{d}t &\le \|\gamma'(x_i)\| \, \Delta x_i + \varepsilon \Delta x_i \\ &= \left\| \int_{x_{i-1}}^{x_i} (\gamma'(t) - \gamma'(x_i) - \gamma'(t)) \, \mathrm{d}t \right\| + \varepsilon \Delta x_i \\ &\le \left\| \int_{x_{i-1}}^{x_i} \gamma'(t) \, \mathrm{d}t \right\| + \left\| \int_{x_{i-1}}^{x_i} (\gamma'(x_i) - \gamma'(t)) \, \mathrm{d}t \right\| + \varepsilon \Delta x_i \\ &\le \|\gamma'(x_i) - \gamma'(x_{i-1})\| + 2\varepsilon \Delta x_i.\end{aligned}$$

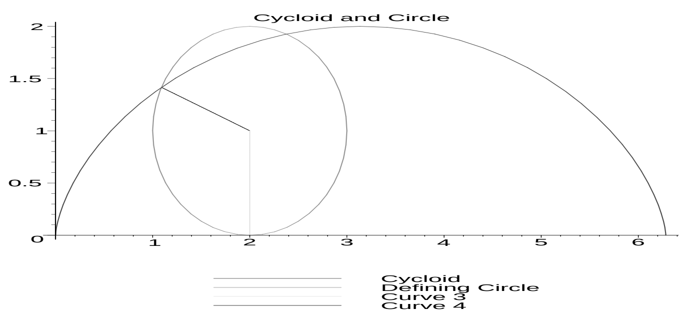
If we add these inequalities, we obtain

$$\int_a^b \|\gamma'(t)\| \, \mathrm{d}t \le \ell(P, \gamma) + 2\varepsilon(b - a) \le \ell(\gamma) + 2\varepsilon(b - a).$$

Since ε was arbitrary,

$$\int_a^b \|\gamma'(t) \, \mathrm{d}t\| \le \ell(\gamma).$$

This completes the proof. ∎



Example 5.15 (a) The position of a bulge in a bicycle tire as it rolls down the street can be parametrized by an angle θ as shown in the figure.

Let the radius of the tire be a. It can be verified by plane trigonometry that

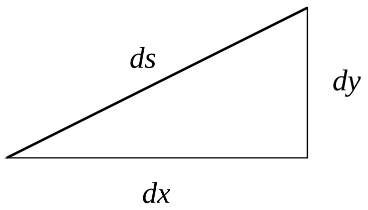
$$\gamma(\theta) = \begin{pmatrix} a(\theta - \sin\theta) \\ a(1 - \cos\theta) \end{pmatrix}.$$

This curve is called a *cycloid.*
Find the distance travelled by the bulge for $0 \le \theta \le 2\pi$.
Using $1 - \cos\theta = 2\sin^2\dfrac{\theta}{2}$ we have

$$\begin{aligned}\gamma'(\theta) &= a(1 - \cos\theta, \sin\theta) \\ \|\gamma'(\theta)\| &= a\sqrt{(1 - \cos\theta)^2 + \sin^2\theta} = a\sqrt{2 - 2\cos\theta} \\ &= a\sqrt{2}\sqrt{1 - \cos\theta} = 2a\sin\frac{\theta}{2}.\end{aligned}$$

Therefore,

$$\ell(\gamma) = 2a \int_0^{2\pi} \sin\frac{\theta}{2}\,\mathrm{d}\theta = 4a\left(\cos\frac{\theta}{2}\right)\Big|_0^{2\pi} = 4a(-\cos\pi + \cos 0) = 8a.$$



(b) *The arc element* ds. Formally the arc element of a plane differentiable curve can be computed using the pythagorean theorem

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 \Longrightarrow \mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

$$\mathrm{d}s = \mathrm{d}x\sqrt{1 + \frac{\mathrm{d}y^2}{\mathrm{d}x^2}}$$

$$\mathrm{d}s = \sqrt{1 + (f'(x))^2}\,\mathrm{d}x.$$

Using Example 5.12 (c) for graphs of functions this can be made rigorous. Since

$$\|\gamma'(t)\| = \|(1, f'(t))\| = \sqrt{1 + (f'(t))^2}$$

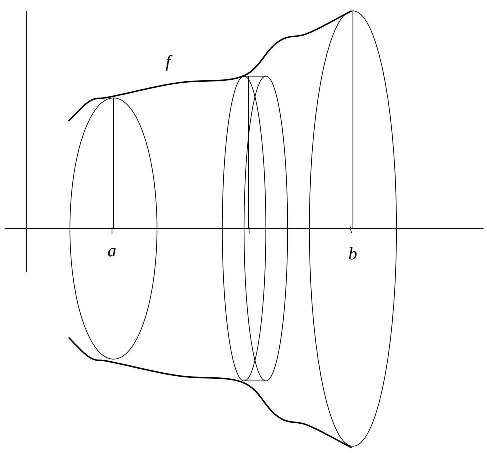
we have by Proposition,5.34

$$\ell(\gamma) = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x.$$

5.6.2 The Cosmopolitan Integral

Originally we introduced the Riemann integral in order to find the area under the graph of a function, but the integral is more versatile than that. In the previous subsection we expressed the length of a curve using integrals.

Rotation around the x-axis



There are some very special solids whose volumes can be expressed by integrals. The simplest such solid V is a "volume of revolution" obtained by revolving the region under the graph of $f \geq 0$ on $[a, b]$ around the horizontal axis.

Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$ and let m_i and M_i have the same meaning as in Definition 5.1. Then

$$\pi\, m_i^2\,(x_i - x_{i-1})$$

is the volume of a disc that lies inside the colid V. Similarly, $\pi\, M_i^2\,(x_i - x_{i-1})$ is the volume of a disc that contains the part of V between x_{i-1} and x_i. Consequently

$$\pi \sum_{i=1}^{n} m_i^2 \Delta x_i \leq \operatorname{vol}(V) \leq \pi \sum_{i=1}^{n} M_i^2 \Delta x_i.$$

But the sums are just the lower and upper sums for f^2 on $[a,b]$:

$$\pi L(P, f^2) \leq \operatorname{vol}(V) \leq \pi U(P, f^2).$$

Consequently, if $f^2 \in \mathcal{R}$, the volume of V must be given by

$$\operatorname{vol}(V) = \pi \int_a^b f(x)^2 \,\mathrm{d}x. \tag{5.53}$$

This method is referred to as the "disc method."

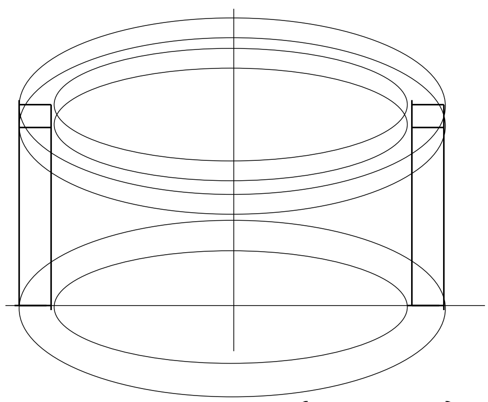
Example 5.16 We compute the volume of the ellipsoid obtained by revolving the graph of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

around the x-axis. We have $y^2 = f(x)^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$; hence

$$\operatorname{vol}(V) = \pi b^2 \int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right) \mathrm{d}x = \pi b^2 \left(x - \frac{x^3}{3a^2}\right) \Big|_{-a}^{a} = \pi b^2 \left(2a - \frac{2a^3}{3a^2}\right) = \frac{4\pi}{3} b^2 a.$$

Rotation around the y-axis



Suppose the solid V is obtained by rotating the region under the graph f around the *vertical* axis. V is the solid left over when we start with a big cylinder of radius b and take away both a small cylinder of radius a and and a solid V_1 sitting in the top of V. In this case we assume $a \geq 0$ as well as $f \geq 0$.

For a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ we consider "shells" obtained by rotating the rectangle with base $[x_{i-1}, x_i]$ and height m_i or M_i . Adding the volumes of these shells we obtain

$$\sum_{i=1}^{n} m_i\, \pi (x_i - x_{i-1})^2 \leq \operatorname{vol}(V) \leq \sum_{i=1}^{n} M_i\, \pi (x_i - x_{i-1})^2,$$

which can be written as

$$\pi \sum_{i=1}^{n} m_i (x_i + x_{i-1})(x_i - x_{i-1}) \leq \operatorname{vol}(V) \leq \pi \sum_{i=1}^{n} M_i (x_i + x_{i-1})(x_i - x_{i-1}).$$

By Lemma 5.4 each of the sum

$$\sum_{i=1}^{n} m_i\, x_i \Delta x_i \quad \text{and} \quad \sum_{i=1}^{n} m_i\, x_{i-1} \Delta x_i$$

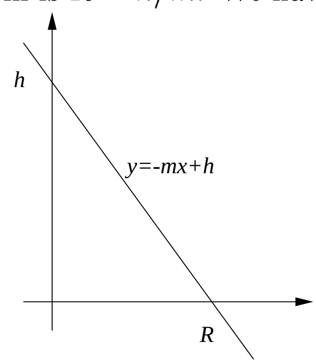
can be made closed to $\int_a^b x f(x)\,\mathrm{d}x$ by choosing Δx_i small enough. The same is true for the sums with M_i. So we find that

$$\mathrm{vol}(V) = 2\pi \int_a^b x f(x)\,\mathrm{d}x; \tag{5.54}$$

this is the so called "shell method" of finding volumes.

Example 5.17 Find the volume of a cone of height h obtained by rotating the graph of $y = -mx + h$, $m > 0$, around the y-axis. The radius of the bottom is $R = h/m$. We have

$$\begin{aligned} V &= 2\pi \int_0^R x(-mx+h)\,\mathrm{d}x = 2\pi \left(-\frac{1}{3}mx^3 + \frac{1}{2}hx^2\right)\Big|_0^R \\ &= 2\pi \left(-\frac{1}{3}mR^3 + \frac{1}{2}hR^2\right) \\ &= \frac{1}{3}\pi R^2 h. \end{aligned}$$



Surface Areas

The surface areas of certain curved regions can also be expressed in terms of integrals. We review some elementary geometry. Recall that the surface area of the frustum of a cone is $A = \pi(r_1 + r_2)s$ where r_1, r_2 are the radii of the top and bottom circles and s is the slant height. If the graph of a positive function f revolves around the x-axis, we obtain the area element of the band to be equal to the circumference $2\pi f(x)$ times the width $\mathrm{d}s$.

$$\mathrm{d}A = 2\pi f(x)\ \mathrm{d}s = 2\pi f(x)\sqrt{1 + (f'(x))^2}\,\mathrm{d}x.$$

The area of a surface formed by revolving the graph of f around the horizontal axis is

$$A = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,\mathrm{d}x \tag{5.55}$$

Example 5.18 Bands of equal width on a sphere have equal area. The shere of radius r is obtained by revolving the graph of $y = f(x) = \sqrt{r^2 - x^2}$ on $[-r, r]$ around the x-axis. Fix a and b with $-r \le a \le b \le r$; we have

$$1 + (f'(x))^2 = 1 + \left(\frac{2x}{2\sqrt{r^2 - x^2}}\right)^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}.$$

Hence,

$$A = 2\pi \int_a^b \sqrt{r^2 - x^2}\frac{r\,\mathrm{d}x}{\sqrt{r^2 - x^2}} = 2\pi r(b - a).$$

In particular, the area of the band depends on $b-a$ only. The full sphere with $b-a=2r$ has the area $4\pi r^2$.

5.7 Appendix D

Proposition 5.35 *Every convex function $f\colon (a,b)\to\mathbb{R}$, $-\infty\le a<b\le+\infty$, is continuous.*

Proof. Let $x\in(a,b)$; choose a finite subinterval (x_1,x_2) with $a<x_1<x<x_2<b$. Since $f(x)\le\lambda f(x_1)+(1-\lambda)f(x_2)$, $\lambda\in[0,1]$, f is bounded above on $[x_1,x_2]$. Chosing x_3 with $x_1<x_3<x$ the convexity of f implies

$$\frac{f(x_3)-f(x_1)}{x_3-x_1}\le\frac{f(x)-f(x_1)}{x-x_1}\Longrightarrow f(x)\ge\frac{f(x_3)-f(x_1)}{x_3-x_1}(x-x_1).$$

This means that f is bounded below on $[x_3,x_2]$ by a linear function; hence f is bounded on $[x_3,x_2]$, say $|\,f(x)\,|\le C$ on $[x_3,x_2]$.
The convexity implies

$$\begin{aligned}f\left(\frac{1}{2}(x+h)+\frac{1}{2}(x-h)\right)&\le\frac{1}{2}\left(f(x+h)+f(x-h)\right)\\ \Longrightarrow f(x)-f(x-h)&\le f(x+h)-f(x).\end{aligned}$$

Iteration yields

$$f(x-(\nu-1)h)-f(x-\nu h)\le f(x+h)-f(x)\le f(x+\nu h)-f(x+(\nu-1)h).$$

Summing up over $\nu=1,\dots,n$ we have

$$\begin{aligned}f(x)-f(x-nh)&\le n\left(f(x+h)-f(x)\right)\le f(x+nh)-f(x)\\ \Longrightarrow\frac{1}{n}\left(f(x)-f(x-nh)\right)&\le f(x+h)-f(x)\le\frac{1}{n}\left(f(x+nh)-f(x)\right).\end{aligned}$$

Let $\varepsilon>0$ be given; choose $n\in\mathbb{N}$ such that $2C/n<\varepsilon$ and choose h such that $x_3<x-nh<x<x+nh<x_2$. The above inequality then implies

$$|\,f(x+h)-f(x)\,|\le\frac{2C}{n}<\varepsilon.$$

This shows continuity of f at x. ∎

If g is an increasing convex function and f is a convex function, then $g\circ f$ is convex since $f(\lambda x+\mu y)\le\lambda f(x)+\mu f(y)$, $\lambda+\mu=1$, $\lambda,\mu\ge0$, implies

$$g(f(\lambda x+\mu y))\le g(\lambda f(x)+\mu g(x))\le\lambda g(f(x))+\mu g(f(y)).$$

Proof of Theorem 5.30. Since $\Gamma(x)$ has the properties (a), (b), and (c) it suffices to prove that F is uniquely determined by (a), (b), and (c). By (b),

$$F(x+n)=F(x)x(x+1)\cdots(x+n)$$

for every positive x and every positive integer n. In particular $F(n+1) = n!$ and it suffices to show that $F(x)$ is uniquely determined for every x with $x \in (0,1)$. Since $n+x = (1-x)n + x(n+1)$ from (c) it follows

$$F(n+x) \le F(n)^{1-x} F(n+1)^x = F(n)^{1-x} F(n)^x \, n^x = (n-1)! n^x.$$

Similarly, from $n+1 = x(n+x) + (1-x)((n+1+x)$ it follows

$$n! = F(n+1) \le F(n+x)^x F(n+1+x)^{1-x} = F(n+x)(n+x)^{1-x}.$$

Combining both inequalities,

$$n!(n+x)^{x-1} \le F(n+x) \le (n-1)! n^x$$

and moreover

$$a_n(x) := \frac{n!(n+x)^{x-1}}{x(x+1)\cdots(x+n-1)} \le F(x) \le \frac{(n-1)! n^x}{x(x+1)\cdots(x+n-1)} =: b_n(x).$$

Since $\frac{b_n(x)}{a_n(x)} = \frac{(n+x)n^x}{n(n+x)^x}$ converges to 1 as $n \to \infty$,

$$F(x) = \lim_{n\to\infty} \frac{(n-1)! n^x}{x(x+1)\cdots(x+n)}.$$

Hence F is uniquely determined. ∎

Proof of $c_n \to \sqrt{2\pi}$. We defined the sequence c_n via

$$n! = n^{n+\frac{1}{2}} \, \mathrm{e}^{-n} \, c_n. \tag{5.56}$$

Using (5.50) we have

$$\frac{c_n^2}{c_{2n}} = \frac{(n!)^2 \sqrt{2n} (2n)^{2n}}{n^{2n+1}(2n)!} = \sqrt{2} \, \frac{2^{2n}(n!)^2}{\sqrt{n}(2n)!}$$

and $\lim_{n\to\infty} \frac{c_n^2}{c_{2n}} = \frac{c^2}{c} = c$. Using Wallis's product formula for π

$$\pi = 2\prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1} = \lim_{n\to\infty} 2 \frac{2{\cdot}2{\cdot}4{\cdot}4 \cdots{\cdot}2n{\cdot}2n}{1{\cdot}3{\cdot}3{\cdot}5 \cdots{\cdot}(2n-1)(2n+1)} \tag{5.57}$$

we have

$$\left(2\prod_{k=1}^{n} \frac{4k^2}{4k^2-1}\right)^{\frac{1}{2}} = \sqrt{2} \frac{2{\cdot}4\cdots 2n}{3{\cdot}5\cdots(2n-1)\sqrt{2n+1}} = \frac{1}{\sqrt{n+\frac{1}{2}}} \cdot \frac{2^2{\cdot}4^2\cdots(2n)^2}{2{\cdot}3{\cdot}4\cdots(2n-1)(2n)}$$

$$= \frac{1}{\sqrt{n+\frac{1}{2}}} \cdot \frac{2^{2n}(n!)^2}{(2n)!},$$

such that

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2^{2n}(n!)^2}{\sqrt{n}(2n)!}.$$

Consequently, $c = \sqrt{2\pi}$ which completes the proof. ∎