Chapter 3

Functions and Continuity

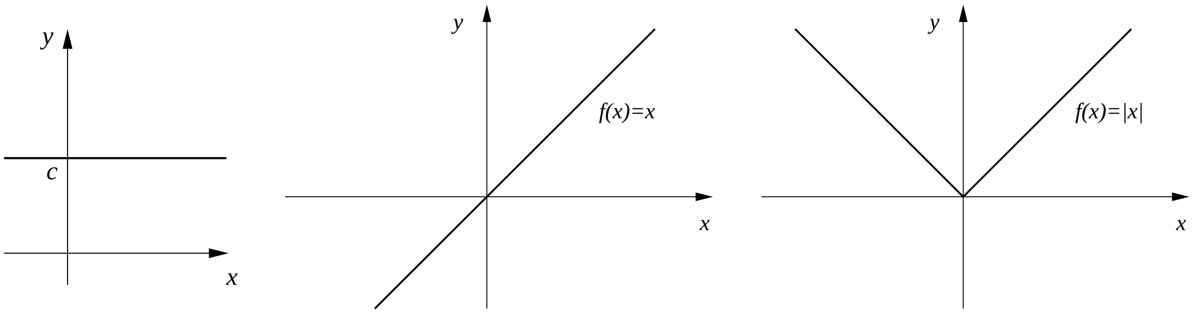
This chapter is devoted to another central notion in analysis—the notion of a continuous function. We will see that sums, product, quotients, and compositions of continuous functions are continuous. If nothing is specified otherwise D will denote a finite union of intervals.

Definition 3.1 Let $D \subset \mathbb{R}$ be a subset of $\mathbb{R}$. A *function* is a map $f\colon D \to \mathbb{R}$.
(a) The set D is called the *domain* of f; we write $D = D(f)$.
(b) If $A \subseteq D$, $f(A) := \{f(x) \mid x \in A\}$ is called the *image* of A under f.
The function $f \restriction A\colon A \to \mathbb{R}$ given by $f \restriction A(a) = f(a)$, $a \in A$, is called the *restriction* of f to A.
(c) If $B \subset \mathbb{R}$, we call $f^{-1}(B) := \{x \in D \mid f(x) \in B\}$ the *preimage* of B under f.
(d) The *graph* of f is the set $\operatorname{graph}(f) := \{(x, f(x)) \mid x \in D\}$.

Later we will consider functions in a wider sense: From the complex numbers into complex numbers and from $\mathbb{F}^n$ into $\mathbb{F}^m$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

We say that a function $f\colon D \to \mathbb{R}$ is *bounded*, if $f(D) \subset \mathbb{R}$ is a bounded set of real numbers, i. e. there is a $C > 0$ such that $|f(x)| \le C$ for all $x \in D$.

Example 3.1 (a) Polynomials and rational functions are the first main examples of functions.
Given a real number $c \in \mathbb{R}$. The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := c$ is called the *constant* function. This is the general form of a polynomial of degree 0.
(b) $\mathrm{id}_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x$ is called the *identity* function (this is a special linear polynomial).
(c) $\mathrm{abs}\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$ is called the *absolute value* function; the image of f is $f(\mathbb{R}) = \mathbb{R}_+$.



The graphs of the constant, the identity, and absolute value functions.

3.1 Limits of a Function

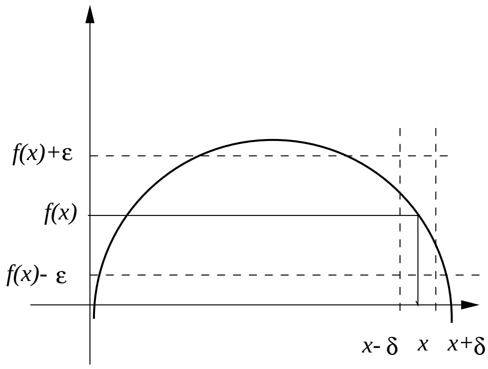
Definition 3.2 (ε-δ definition) Let $f\colon (a,b) \to \mathbb{R}$ be a function of the (possibly infinite) interval (a,b) into $\mathbb{R}$ and x_0 such that $a < x_0 < b$. We call $A \in \mathbb{R}$ the *limit of f in x_0* ("The limit of $f(x)$ is A as x approaches x_0"; "f approaches A near x_0").

> Given $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in (a,b)$ and $0 < |\, x - x_0 \,| < \delta$ imply $|\, f(x) - A \,| < \varepsilon$.

We write

$$\lim_{x \to x_0} f(x) = A.$$

Roughly speaking, if x is close to x_0, $f(x)$ must be closed to A.



Using quantifiers $\lim\limits_{x \to x_0} f(x) = A$ reads as

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \;\; \forall\, x \in D(f) : \quad 0 < |\, x - x_0 \,| < \delta \Longrightarrow |\, f(x) - A \,| < \varepsilon.$$

Note that the formal negation of $\lim\limits_{x \to x_0} f(x) = A$ is

$$\exists\, \varepsilon > 0 \;\; \forall\, \delta > 0 \;\; \exists\, x \in D(f) : \quad 0 < |\, x - x_0 \,| < \delta \quad \text{and} \quad |\, f(x) - A \,| \ge \varepsilon.$$

Proposition 3.1 (limit definition using sequences) *Let $f(x)$ and x_0 be as above. Then $\lim\limits_{x \to x_0} f(x) = A$ if and only if for every sequence (x_n) with $x_n \in (a,b)$, $x_n \neq x_0$ for all n, and $\lim\limits_{n \to \infty} x_n = x_0$ we have $\lim\limits_{n \to \infty} f(x_n) = A$.*

Proof. Suppose $\lim_{x\to x_0} f(x) = A$, and $x_n \to x_0$. Given $\varepsilon > 0$ we find $\delta > 0$ such that $|f(x) - A| < \varepsilon$ if $0 < |x - x_0| < \delta$. Since $x_n \to x_0$, there is a positive integer n_0 such that $n \geq n_0$ implies $|x_n - x_0| < \delta$. Therefore $n \geq n_0$ implies $|f(x_n) - A| < \varepsilon$. That is, $\lim_{n\to\infty} f(x_n) = A$.
Suppose that the condition of the proposition is fulfilled but $\lim_{x\to x_0} f(x) \neq A$. Then there is some $\varepsilon > 0$ such that for all $\delta = 1/n$, $n \in \mathbb{N}$, there is an $x_n \in (a, b)$ such that $0 < |x_n - x_0| < 1/n$, but $0|f(x_n) - A| \geq \varepsilon$. We have constructed a sequence (x_n), $x_n \neq x_0$ and $x_n \to x_0$ as $n \to \infty$ such that $\lim_{n\to\infty} f(x_n) \neq A$ which contradicts our assumption. Hence $\lim\limits_{x\to x_0} f(x) = A$. ∎

3.1.1 One-sided Limits, Infinite Limits, and Limits at Infinity

Definition 3.3 (a) We are writing

$$\lim_{x\to x_0+0} f(x) = A$$

if for all sequences (x_n) with $x_n > x_0$ and $\lim\limits_{n\to\infty} x_n = x_0$, we have $\lim\limits_{n\to\infty} f(x_n) = A$. Sometimes we use the notation $f(x_0+0)$ in place of $\lim\limits_{x\to x_0+0} f(x)$. We call $f(x_0+0)$ the *right-hand limit* of f at x_0 or we say "A is the limit of f as x approaches x_0 from above (from the right)."
Similarly one defines the *left-hand limit* of f at x_0, $\lim\limits_{x\to x_0-0} f(x) = A$ with $x_n < x_0$ in place of $x_n > x_0$. Sometimes we use the notation $f(x_0 - 0)$.
(b) We are writing

$$\lim_{x\to+\infty} f(x) = A$$

if for all sequences (x_n) with $\lim\limits_{n\to\infty} x_n = +\infty$ we have $\lim\limits_{n\to\infty} f(x_n) = A$. Sometimes we use the notation $f(+\infty)$. In a similar way we define $\lim\limits_{x\to-\infty} f(x) = A$.
(c) Finally, the notions of (a) and (b) still make sense in case $A = +\infty$ and $A = -\infty$. For example,

$$\lim_{x\to x_0-0} f(x) = -\infty$$

if for all sequences (x_n) with $x_n < x_0$ and $\lim\limits_{n\to\infty} x_n = x_0$ we have $\lim\limits_{n\to\infty} f(x_n) = -\infty$.

Remark 3.1 All notions in the above definition can be given in ε-δ or ε-D or E-δ or E-D languages using inequalities. For example, $\lim\limits_{x\to x_0-0} f(x) = -\infty$ if and only if

$$\forall E > 0\ \exists \delta > 0\ \forall x \in D(f) : 0 < x_0 - x < \delta \implies f(x) < -E.$$

Similarly, $\lim\limits_{x\to+\infty} f(x) = +\infty$ if and only if

$$\forall E > 0\ \exists D > 0\ \forall x \in D(f) : x > D \implies f(x) > E.$$

The proves are along the lines of Proposition 3.1

Example 3.2 (a) Let $p(x)$ and $q(x)$ be polynomials and $a \in \mathbb{R}$. Then

$$\lim_{x \to a} p(x) = p(a).$$

This immediately follows from Example 2.3 (c) and Proposition 3.1. Suppose moreover that $q(a) \neq 0$. Let $x_n \to a$ as $n \to \infty$; then $q(x_n) \to q(a)$ and the limit laws for sequences give

$$\lim_{n\to\infty} \frac{p(x_n)}{q(x_n)} = \frac{p(a)}{q(a)}.$$

By Proposition 3.1 this means

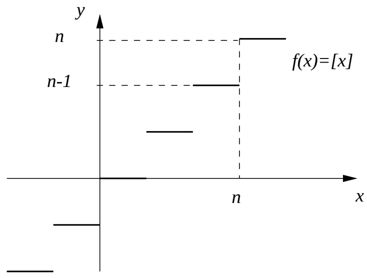
$$\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

Hence, the limit of a rational function $f(x)$ as x approaches a point a of the domain of f is $f(a)$.

(b) $\lim\limits_{x\to+\infty} \dfrac{1}{x} = 0$. For, let $\varepsilon > 0$; choose $D = 1/\varepsilon$. Then $x > D = 1/\varepsilon$ implies $0 < 1/x < \varepsilon$. This proves the claim.

(c) Consider the entier function $f(x) = [x]$, defined in Example 2.6 (b). If $n \in \mathbb{Z}$, $\lim\limits_{x\to n-0} f(x) = n - 1$ whereas $\lim\limits_{x\to n+0} f(x) = n$.

Proof. We use the ε-δ definition of the one-sided limits to prove the first claim. Let $\varepsilon > 0$. Choose $\delta = \frac{1}{2}$ then $0 < n - x < \frac{1}{2}$ implies $n - \frac{1}{2} < x < n$ and therefore $f(x) = n - 1$. In particular $|\, f(x) - (n-1)\,| = 0 < \varepsilon$. Similarly one proves $\lim\limits_{x\to n+0} f(x) = n$. ∎



Since the one-sided limits are different, $\lim\limits_{x\to n} f(x)$ does not exist.

Definition 3.4 Suppose we are given two functions f and g, both defined on (a, b). By $f + g$ we mean the function which assigns to each point x of (a, b) the number $f(x) + g(x)$. Similarly, we define the difference $f - g$, the product fg, and the quotient f/g, with the understanding that the quotient is defined only at those points x at which $g(x) \neq 0$.

Proposition 3.2 *Suppose that f and g are functions on (a, b), $a < x_0 < b$, and*

$$\lim_{x\to x_0} f(x) = A, \qquad \lim_{x\to x_0} g(x) = B.$$

Then

(a) $\lim\limits_{x\to x_0} f(x) = A'$ *implies* $A' = A$.

(b) $\lim\limits_{x\to x_0} (f + g)(x) = A + B$;

(c) $\lim\limits_{x\to x_0} (fg)(x) = AB$;

(d) $\lim\limits_{x\to x_0} \dfrac{f}{g}(x) = \dfrac{A}{B}$, *if* $B \neq 0$.

(e) $\lim\limits_{x\to x_0} |\, f(x)\,| = |\, A\,|$.

Proof. In view of Proposition 3.1, all these assertions follow immediately from the analogous properties of sequences, see Proposition 2.3. ■

Remark 3.2 The above Proposition remains true if we replace (at the same time in all places) $x \to x_0$ by $x \to x_0 + 0$, $x \to x_0 - 0$, $x \to +\infty$, or $x \to -\infty$. Moreover we can replace A or B by $+\infty$ or by $-\infty$ provided the right members of (b), (c), (d) and (e) are defined.
Note that $+\infty + (-\infty)$, $0 \cdot \infty$, ∞/∞, and $A/0$ are not defined.
The *extended real number system* consists of the real field $\mathbb{R}$ and two symbols, $+\infty$ and $-\infty$. We preserve the original order in $\mathbb{R}$ and define

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.
It is the clear that $+\infty$ is an upper bound of every subset of the extended real number system, and every nonempty subset has a least upper bound. If, for example, E is a set of real numbers which is not bounded above in $\mathbb{R}$, then $\sup E = +\infty$ in the extended real system. Exactly the same remarks apply to lower bounds.
The extended real system does not form a field, but it is customary to make the following conventions:

(a) If x is real then

$$x + \infty = +\infty, \quad x - \infty = -\infty, \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

(b) If $x > 0$ then $x \cdot (+\infty) = +\infty$ and $x \cdot (-\infty) = -\infty$.
(c) If $x < 0$ then $x \cdot (+\infty) = -\infty$ and $x \cdot (-\infty) = +\infty$.

When it is desired to make the distinction between the real numbers on the one hand and the symbols $+\infty$ and $-\infty$ on the other hand quite explicit, the real numbers are called *finite*.
In Homework 9.3 (a) and (b) you are invited to give explicit proves in two special cases.

Example 3.3 Let $f(x) = p(x)/q(x)$ be a rational function with polynomials $p(x) = \sum_{k=0}^{r} a_k x^k$ and $q(x) = \sum_{k=0}^{s} b_k x^k$ with real coefficients a_k and b_k and of degree r and s, respectively. Then

$$\lim_{x\to+\infty} f(x) = \begin{cases} 0, & \text{if} \quad r < s, \\ \frac{a_r}{b_s}, & \text{if} \quad r = s, \\ +\infty, & \text{if} \quad r > s \quad \text{and} \quad \frac{a_r}{b_s} > 0, \\ -\infty, & \text{if} \quad r > s \quad \text{and} \quad \frac{a_r}{b_s} < 0. \end{cases}$$

The first two statements ($r \geq s$) follow from Example 3.2 (b) together with Proposition 3.2. Namely, $a_k x^{k-r} \to 0$ as $x \to +\infty$ provided $0 \leq k < r$. The statements for $r > s$

follow from $x^{r-s} \to +\infty$ as $x \to +\infty$ and the above remark.
Note that

$$\lim_{x\to-\infty} f(x) = (-1)^{r+s} \lim_{x\to+\infty} f(x)$$

since

$$\frac{p(-x)}{q(-x)} = \frac{(-1)^r a_r x^r + \dots}{(-1)^s b_s x^s + \dots} = (-1)^{r+s} \frac{a_r x^r + \dots}{b_s x^s + \dots}.$$

3.2 Continuous Functions

Definition 3.5 Let f be a function and $x_0 \in D(f)$. We say that f *is continuous at* x_0 if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in D(f) : |\, x - x_0 \,| < \delta \Longrightarrow |\, f(x) - f(x_0) \,| < \varepsilon. \tag{3.1}$$

We say that f is continuous in $A \subset D(f)$ if f is continuous at all points $x_0 \in A$.

Proposition 3.1 shows that the above definition of continuity in x_0 is equivalent to: For all sequences (x_n), $x_n \in D(f)$, with $\lim\limits_{n\to\infty} x_n = x_0$, $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$. In other words, f is continuous at x_0 if $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

Example 3.4 (a) In example 3.2 we have seen that every polynomial is continuous in $\mathbb{R}$ and every rational functions f is continuous in their domain $D(f)$.
$f(x) = |\, x \,|$ is continuous in $\mathbb{R}$.
(b) Continuity is a *local* property: If two functions $f, g \colon D \to \mathbb{R}$ coincide in a neighborhood $U_\varepsilon(x_0) \subset D$ of some point x_0, then f is continuous at x_0 if and only if g is continuous at x_0.
(c) $f(x) = [x]$ is continuous in $\mathbb{R} \setminus \mathbb{Z}$. If x_0 is not an integer, then $n < x_0 < n+1$ for some $n \in \mathbb{N}$ and $f(x) = n$ coincides with a constant function in a neighborhood $x \in U_\varepsilon(x_0)$. By (b), f is continuous at x_0. If $x_0 = n \in \mathbb{Z}$, $\lim_{x\to n}[x]$ does not exist; hence f is not continuous at n.
(d) $f(x) = \dfrac{x^2 - 1}{x - 1}$ if $x \neq 1$ and $f(1) = 1$. Then f is not continuous at $x_0 = 1$ since

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1}(x + 1) = 2 \neq 1 = f(1).$$

There are two reasons for a function not being continuous at x_0. First, $\lim_{x\to x_0} f(x)$ does not exist. Secondly, f has a limit at x_0 but $\lim_{x\to x_0} f(x) \neq f(x_0)$.

Proposition 3.3 *Suppose $f, g \colon D \to \mathbb{R}$ are continuous at $x_0 \in D$. Then $f + g$ and fg are also continuous at x_0. If $g(x_0) \neq 0$, then f/g is continuous at x_0.*

The proof is obvious from Proposition 3.2.
The set $\mathrm{C}(D)$ of continuous function on $D \subset \mathbb{R}$ form a commutative algebra with 1.

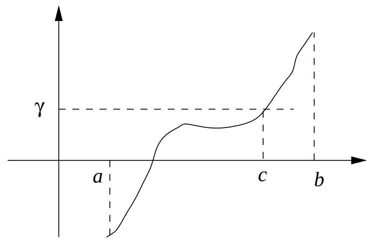
Proposition 3.4 *Let $f\colon D \to \mathbb{R}$ and $g\colon E \to \mathbb{R}$ functions with $f(D) \subset E$. Suppose f is continuous at $a \in D$, and g is continuous at $b = f(a) \in E$. Then the composite function $g \circ f\colon D \to \mathbb{R}$ is continuous at a.*

Proof. Let (x_n) be a sequence with $x_n \in D$ and $\lim_{n\to\infty} x_n = a$. Since f is continuous at a, $\lim_{n\to\infty} f(x_n) = b$. Since g is continuous at b, $\lim_{n\to\infty} g(f(x_n)) = g(b)$; hence $g \circ f(x_n) \to g \circ f(a)$. This completes the proof. ∎

3.2.1 The Intermediate Value Theorem

In this paragraph, $[a,b] \subset \mathbb{R}$ is a closed, bounded interval, $a, b \in \mathbb{R}$.
The intermediate value theorem is the basis for several existence theorems in analysis. It is again equivalent to the order completeness of $\mathbb{R}$.

Theorem 3.5 (Intermediate Value Theorem) *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function and γ a real number between $f(a)$ and $f(b)$. Then there exists $c \in [a,b]$ such that $f(c) = \gamma$.*



The statement is clear from the graphical presentation. Nevertheless, it needs a proof since pictures do not prove anything. The statement is wrong for rational numbers. For example, let $D = \{x \in \mathbb{Q} \mid 1 \le x \le 2\}$ and $f(x) = x^2 - 2$. Then $f(1) = -1$ and $f(2) = 2$ but there is no $p \in D$ with $f(p) = 0$ since 2 has no rational square root.

Proof. Without loss of generality suppose $f(a) \le f(b)$. Starting with $[a_1, b_1] = [a,b]$, we successively construct a nested sequence of intervals $[a_n, b_n]$ such that $f(a_n) \le \gamma \le f(b_n)$. As in the proof of Proposition 2.10, the $[a_n, b_n]$ is one of the two halfintervals $[a_{n-1}, m]$ and $[m, b_{n-1}]$ where $m = (a_{n-1} + b_{n-1})/2$ is the midpoint of the $(n-1)$st interval. By Proposition 2.9 the monotonic sequences (a_n) and (b_n) both converge to a common point c. Since f is continuous,

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(c) = f(\lim_{n\to\infty} b_n) = \lim_{n\to\infty} f(b_n).$$

By Proposition 2.14 (2.14), $f(a_n) \le \gamma \le f(b_n)$ implies

$$\lim_{n\to\infty} f(a_n) \le \gamma \le \lim_{n\to\infty} f(b_n);$$

Hence, $\gamma = f(c)$. ∎

Example 3.5 (a) We again show the existence of the nth root of a positive real number $a > 0$, $n \in \mathbb{N}$. By Example 3.2, the polynomial $p(x) = x^n - a$ is continuous in $\mathbb{R}$. We find $p(0) = -a < 0$ and by Bernoulli's inequality

$$p(1+a) = (1+a)^n - a \ge 1 + (n-1)a \ge 1 > 0.$$

Theorem 3.5 shows that p has a root in the interval $(0, 1+a)$.
(b) A polynomial p of odd degree with real coefficients has a real zero. Namely, by Example 3.3, if the leading coefficient a_r of p is positive, $\lim_{x\to-\infty} p(x) = -\infty$ and $\lim_{x\to\infty} p(x) = +\infty$. Hence there are a and b with $a < b$ and $p(a) < 0 < p(b)$. Therefore, there is a $c \in (a, b)$ such that $p(c) = 0$.

There are polynomials of even degree having no real zeros. For example $f(x) = x^{2k} + 1$.

3.2.2 Continuous Functions on Compact Sets — The Theorem about Maximum and Minimum

Continuous functions on compact domains have special properties of fundamental significance. For instance, the theorem about maximum and minimum guarantees the existence of solutions in many extremal problems.
Before we can start we need some topological notions—closed and compact sets.

Definition 3.6 A subset $A \subset \mathbb{R}$ or $A \subset \mathbb{C}$ is called *closed* if for every convergent sequence (a_n) of elements of A, $a_n \in A$, the limit of (a_n) also belongs to A. The empty set is also closed.

Examples for closed subsets in $\mathbb{R}$ are the closed intervals $[a, b]$, $(-\infty, b]$, $[a, +\infty)$, and $\mathbb{R}$ itself. This follows from the fact that for every convergent sequence (x_n) with $a \le x_n \le b$, $a \le \lim x_n \le b$. The open interval (a, b) and the half-closed interval $(a, b]$ are not closed, since the sequence $x_n = a + (b-a)/n$ converges to $a \notin (a, b]$.
Examples for closed subsets in $\mathbb{C}$ are half-planes $\{z \in \mathbb{C} \mid \operatorname{Re} z \ge a\}$, $\{z \in \mathbb{C} \mid \operatorname{Im} z \le a\}$, $a \in \mathbb{R}$, closed discs $\{z \in \mathbb{C} \mid |z - z_0| \le r\}$, $z_0 \in \mathbb{C}$, $r > 0$, and the unit circle $\mathrm{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Remark 3.3 If $E \subset \mathbb{R}$ is closed and $\alpha = \sup E$ exists in $\mathbb{R}$ then $\alpha \in E$ and hence $\alpha = \max E$.
For, by the properties of the supremum, for every $1/n$ there exists $x_n \in E$ such that

$$\alpha - \frac{1}{n} < x_n \le \alpha.$$

This implies $0 < \alpha - x_n < 1/n$ which shows that x_n tends to α. Since E is closed, $\alpha \in E$.

Lemma 3.6 (a) *The union of finitely many closed sets is closed.*
(b) *The intersection of any family of closed sets is closed.*

Proof. (a) It suffices to prove the statement for two closed sets A and B, the general case is by induction. Suppose (x_n) is a convergent to x sequence with $x_n \in A \cup B$. At least one of the given sets, say A, contains infinitely many elements x_{n_k} forming a subsequence. By Proposition 2.7 (2.7) (x_{n-k}) also converges to x. Since A is closed $x \in A$, and moreover $x \in A \cup B$.

(b) A *family* $\mathcal{A}$ of sets is just a set of sets. We define the *union* and the *intersection* of $\mathcal{A}$ as

$$\bigcup \mathcal{A} := \{x \mid \exists A \in \mathcal{A} : x \in A\},$$
$$\bigcap \mathcal{A} := \{x \mid \forall A \in \mathcal{A} : x \in A\}.$$

If the family $\mathcal{A}$ is finite, say $\mathcal{A} = \{A_1, \dots, A_n\}$ we write

$$\bigcup \mathcal{A} = \bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cup \dots \cup A_n.$$

If the family $\mathcal{A}$ is indexed by the positive integers $I = \mathbb{N}$ or an arbitrary set I, $\mathcal{A} = \{A_\alpha \mid \alpha in I\}$ we write

$$\bigcap \mathcal{A} = \bigcap_{\alpha \in I} A_\alpha.$$

If $x_n \in \bigcap_\alpha A_\alpha$ converges to x, then $x_n \in A_\alpha$ for every α. Since A_α is closed, $x \in A_\alpha$ for every α. Hence $x \in \bigcap_\alpha A_\alpha$. ■

Definition 3.7 A subset $K \subset \mathbb{C}$ is called *compact* if every sequence (z_n) of elements of K has a subsequence converging to some point z of K.

The next proposition gives a nice criterion for compactness in $\mathbb{R}$ and $\mathbb{C}$. It is also true in $\mathbb{R}^n$ and $\mathbb{C}^n$ but not in infinit-dimensional spaces.

Proposition 3.7 *A subset $K \subset \mathbb{C}$ is compact if and only if it is bounded and closed.*

The closed finite intervals $[a, b]$ is compact. The closed disc and the unit circle are compact. All finite unions of these sets are compact. The half-lines and half-planes are not compact.
Proof. Suppose first that K is closed and bounded and that (z_n) is a sequence of elements of K. Since K and therefore (z_n) is bounded Bolzano–Weierstraß implies that (z_n) has a convergent to some point z subsequence. Since K is closed, $z \in K$. Hence, K is compact.
Conversely, suppose first that K is not closed. Then there exists a convergent to some z sequence (z_n) of elements of K with $z \not\in K$. However, every subsequence of (z_n) also converges to $z \not\in K$. Hence, K is not compact.
Suppose now that K is not bounded. Then there exists a sequence (z_n) in K with $\mid z_n \mid \geq n$. This sequence has no convergent subsequence; K is not compact. ■

Lemma 3.8 (a) *The union of finitely many compact sets is compact.*
(b) *The intersection of an arbitrary family of compact sets is compact.*
(c) *The intersection $A \cap K$ of a closed set A and a compact set K is compact.*

Proof. All statements are immediate from the Proposition 3.7 and Lemma 3.6. ■

Proposition 3.9 *If f is continuous and K is compact then $f(K)$ is compact.*

Proof. Let $(f(x_n))$ be any sequence in $f(K)$. Since (x_n) is a sequence in the compact set K, by definition, it has a subsequence (x_{n_k}) converging to $x \in K$. Since f is continuous, $(f(x_{n_k})$ converges to $f(x) \in f(K)$. Hence, $f(K)$ is compact. ■

Theorem 3.10 (Theorem of Weierstraß about Maximum and Minimum) *Let K be a bounded and closed subset of $\mathbb{R}$ and $f\colon K \to \mathbb{R}$ a continuous function. Put $M := \sup_{x\in K} f(x)$ and $m := \inf_{x\in K} f(x)$.*
Then f is bounded, and there exist points $p, q \in K$ with $f(p) = M$ and $f(q) = m$.

In other words: A continuous function on a compact set attains its maximum and its minimum.
Proof. The image $f(K)$ of the compact set K is compact; inparticular $f(K)$ is bounded. Hence, both M and m are finite real numbers. Since $f(K)$ is closed, $M \in f(K)$ by Remark 3.3. Hence there exist $p \in K$ such that $f(p) = M$. The same argument works for m. ■

Proposition 3.11 *Suppose that $f\colon K \to Y$ is a continuous bijective mapping of a compact set K onto the set $Y = f(K) \subset \mathbb{R}$. Then the inverse mapping $f^{-1}\colon Y \to K$ defined by*

$$f^{-1}(f(x)) = x, \quad x \in K$$

is a continuous function.

Proof. Suppose (y_n) is a sequence in Y converging to $y \in Y$. We will show that $f^{-1}(y_n) \underset{n\to\infty}{\longrightarrow} f^{-1}(y)$. Set $x_n := f^{-1}(y_n)$ and. Since (x_n) is a sequence inside a compact set K it has subsequence (x_{n_k}) converging to some point limit point $x_0 \in K$. Since f is continuous,

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0).$$

On the other hand (y_{n_k}) converges to y. Hence $x_0 = f^{-1}(y)$ by the uniqueness of the limit of (y_{n_k}). This shows that $f^{-1}(y)$ is the only limit point of (x_n); therefore (x_n) converges to x. ■

Remark 3.4 Here is an application of the above proposition. Since the power function $p(x) = x^n$, $n \in \mathbb{N}$, is continuous and bijective on the compact set $[0, a^2]$, $a \in \mathbb{R}_+$, the inverse function $q(y) = \sqrt[n]{y}$ is continuous on $[0, a]$. Together with Proposition 3.4 we obtain

$$\lim_{n\to\infty} x_n^s = \left(\lim_{n\to\infty} x_n\right)^s, \qquad \text{for} \quad s \in \mathbb{Q}.$$

Definition 3.8 A function $f\colon D \to \mathbb{R}$ is called *uniformly continuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, x' \in D$ $|\, x - x' \,| < \delta$ implies $|\, f(x) - f(x') \,| < \varepsilon$.

Let us consider the differences between the concepts of continuity and uniform continuity. First, uniform continuity is a property of a function on a set, whereas continuity can be defined in a single point. To ask whether or not a given function is uniformly continuous at a certain point is meaningless. Second, if f is continuous on X, then it is possible to find, for each $\varepsilon > 0$ and for each point p, a number $\delta > 0$ having the property specified in Definition 3.5. This δ depends on ε *and* on p. If f is, however, uniformly continuous on X, then it is possible, for each $\varepsilon > 0$ to find *one* $\delta > 0$ which will do for all possible points p of X.
Evidently, every uniformly continuous function is continuous. That the two concepts are equivalent on compact set follows from the next proposition.

Proposition 3.12 *Let $f\colon K \to \mathbb{R}$ be a continuous function on a compact set $K \subset \mathbb{R}$. Then f is uniformly continuous on K.*

Proof. Suppose to the contrary that f is not uniformly continuous. Then there exists $\varepsilon_0 > 0$ without matching $\delta > 0$; for every positive integer $n \in \mathbb{N}$ there exists a pair of points x_n, x'_n with $|\, x_n - x'_n \,| < 1/n$ but $|\, f(x_n) - f(x'_n) \,| \geq \varepsilon_0$. By the compactness of K, (x_n) has a subsequence converging to some point $\xi \in K$. Since $|\, x_n - x'_n \,| < 1/n$, the sequence (x'_n) also converges to ξ. Hence

$$\lim_{k\to\infty} f(x_{n_k}) = f(\xi) = \lim_{k\to\infty} f(x'_{n_k})$$

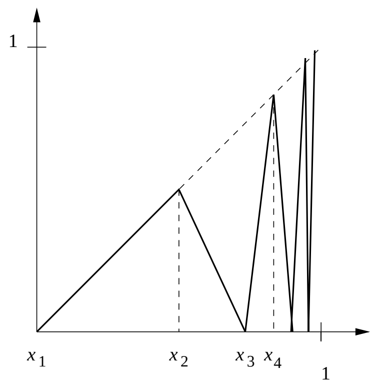
which contradicts $|\, f(x_n) - f(x'_n) \,| \geq \varepsilon_0$ for all n. ∎

Example 3.6 We give an example of a continuous bounded function f on the interval $[0, 1)$ not being uniformly continuous. We define f to be picewise linear. For, let (x_n) be strictly increasing sequence with $x_1 = 0$, $x_2 = 2/3$, and $\lim x_n = \sup x_n = 1$. Define $f(x_{2n-1}) = 0$ and $f(x_{2n}) = x_{2n}$ for all positive integers $n \in \mathbb{N}$.
This function f is not uniformly continuous. Indeed, let $\varepsilon = \frac{1}{2}$. To every $\delta > 0$ choose $n \in \mathbb{N}$ such that

$$x_{2n+1} - x_{2n} < \delta.$$

Then

$$|\, f(x_{2n+1}) - f(x_{2n}) \,| = x_{2n} > \frac{1}{2}.$$



This shows that f is not uniformly continuous.

Discontinuities

If x is a point in the domain of a function f at which f is not continuous, we say f is *discontinuous* at x or f has a *discontinuity* at x. It is customary to divide discontinuities into two types.

Definition 3.9 Let $f\colon (a,b) \to \mathbb{R}$ be a function which is discontinuous at a point x_0. If the one-sided limits $\lim_{x\to x_0+0} f(x)$ and $\lim_{x\to x_0-0} f(x)$ exist, then f is said to have a *simple* discontinuity or a discontinuity of the *first kind.* Otherwise the discontinuity is said to be of the *second kind.*

Example 3.7 (a) $f(x) = \operatorname{sign}(x)$ is continuous on $\mathbb{R}\setminus\{0\}$ since it is locally constant. Moreover, $f(0+0) = 1$ and $f(0-0) = -1$. Hence, $\operatorname{sign}(x)$ has a simple discontinuity at $x_0 = 0$. (b) Define $f(x) = 0$ if x is rational, and $f(x) = 1$ if x is irrational. Then f has a discontinuity of the second kind at every point x since neither $f(x+0)$ nor $f(x-0)$ exists.
(c) Define

$$f(x) = \begin{cases} \sin\frac{1}{x}, & \text{if} \quad x \neq 0; \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Consider the two sequences

$$x_n = \frac{1}{\frac{\pi}{2} + n\pi} \quad \text{and} \quad y_n = \frac{1}{n\pi},$$

Then both sequences (x_n) and (y_n) approach 0 from above but $\lim_{n\to\infty} f(x_n) = 1$ and $\lim_{n\to\infty} f(y_n) = 0$; hence $f(0+0)$ does not exist. Therefore f has a discontinuity of the second kind at $x = 0$. We have not yet shown that $\sin x$ is a continuous function. This will be done in Section 3.5.

3.3 Monotonic Functions

Definition 3.10 Let f be a real function on the interval (a,b). Then f is said to be *monotonically increasing* on (a,b) if $a < x < y < b$ implies $f(x) \leq f(y)$. If the last inequality is reversed, we obtain the definition of a *monotonically decreasing* function. The class of *monotonic functions* consists of both the increasing and the decreasing functions. If $a < x < y < b$ implies $f(x) < f(y)$, the function is said to be *strictly increasing.* Similarly, *strictly decreasing* functions are defined.

Theorem 3.13 *Let f be a monotonically increasing function on (a,b). Then $f(x+0)$ and $f(x-0)$ exist at every point x of (a,b). More precisely,*

$$\sup_{t\in(a,x)} f(t) = f(x-0) \leq f(x) \leq f(x+0) = \inf_{t\in(x,b)} f(t). \tag{3.2}$$

Furthermore, if $a < x < y < b$, then

$$f(x+0) \le f(y-0). \tag{3.3}$$

Analogous results evidently hold for monotonically decreasing functions.

Proof. See Appendix B to this chapter. ■

Corollary 3.14 *Monotonic functions have no discontinuities of the second kind.*

Lemma 3.15 *A strictly increasing function $f\colon X \to \mathbb{R}$ is injective. Thus, it has an inverse function $f^{-1}\colon f(X) \to \mathbb{R}$ which is also strictly increasing.*
A similar statement holds for strictly decreasing functions.

Proof. If $x_1 \ne x_2$ then $x_1 < x_2$ or $x_1 > x_2$ so that $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, respectively. Hence, $f(x_1) \ne f(x_2)$, and f is injective. Therefore f^{-1} exists, and f^{-1} is also strictly increasing, by the above inequalities. ■

Combining Lemma 3.15 and Proposition 3.11 we obtain

Corollary 3.16 *Let $f\colon X \to \mathbb{R}$ be a strictly increasing continuous function. Then $f^{-1}\colon f(X) \to \mathbb{R}$ is also a strictly increasing continuous function.*

Q 19. Suppose, f is a real valued function defined on $\mathbb{R}$ which satisfies

$$\lim_{h\to 0}(f(x+h) - f(x-h)) = f(x)$$

for every $x \in \mathbb{R}$. Does this imply that f is continuous?

3.4 The Exponential and the Logarithmic Functions

In this section we are dealing with the exponential function which is one of the most important in analysis. We use the exponential series to define the function. We will see that this definition is consistent with the definition e^x using Definition 1.9.

Definition 3.11 For $z \in \mathbb{C}$ put

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{3.4}$$

Proposition 3.17 *The radius of convergence of the exponential series* (3.4) *is $R = +\infty$, i. e. the series converges absolutely for all $z \in \mathbb{C}$. We can estimate the remainder term $r_n := \sum_{k=n}^{\infty} z^k/k!$ as follows*

$$|\, r_n(z) \,| \le \frac{2\,|\, z\,|^n}{n!} \quad \textit{if} \quad |\, z\,| \le \frac{n+1}{2}. \tag{3.5}$$

Proof. The ratio test gives convergence for all $z \in \mathbb{C}$, see Example 2.12. We have

$$\begin{aligned} |\, r_n(z)\,| &\leq \sum_{k=n}^{\infty} \left| \frac{z^k}{k!} \right| = \frac{|\,z\,|^n}{n!} \left(1 + \frac{|\,z\,|}{n+1} + \frac{|\,z\,|^2}{(n+1)(n+2)} + \cdots + \frac{|\,z\,|^k}{(n+1)\cdots(n+k)} + \cdots \right) \\ &\leq \frac{|\,z\,|^n}{n!} \left(1 + \frac{|\,z\,|}{n+1} + \frac{|\,z\,|^2}{(n+1)^2} + \cdots + \frac{|\,z\,|^k}{(n+1)^k} + \cdots \right). \end{aligned}$$

$|\,z\,| \leq (n+1)/2$ implies,

$$|\, r_n(z)\,| \leq \frac{|\,z\,|^n}{n!} \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \cdots \right) \leq \frac{2\,|\,z\,|^n}{n!}.$$

∎

Applying Proposition 2.29 (Cauchy product) on multiplication of absolutely convergent series, we obtain

$$\begin{aligned} E(z)E(w) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k\, w^{n-k}}{k!\,(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}, \end{aligned}$$

which gives us the important addition formula

$$E(z+w) = E(z)E(w), \quad z, w \in \mathbb{C}. \tag{3.6}$$

One consequence is that

$$E(z)E(-z) = E(0) = 1, \quad z \in \mathbb{C}. \tag{3.7}$$

This shows that $E(z) \neq 0$ for all z. By (3.4), $E(x) > 0$ if $x > 0$; hence (3.7) shows $E(x) > 0$ for all real x. By (3.4), $\lim_{x\to\infty} E(x) = +\infty$; hence (3.7) shows that $\lim_{x\to-\infty} E(x) = 0$. By (3.4), $0 < x < y$ implies that $E(x) < E(y)$; by (3.7), it follows that $E(-y) < E(-x)$; hence, E is strictly increasing on the whole real axis.
The addition formula also shows that

$$\lim_{h\to 0}(E(z+h) - E(z)) = E(z) \lim_{h\to 0}(E(h) - 1) = E(z) \cdot 0 = 0, \tag{3.8}$$

where $\lim_{h\to 0} E(h) = 1$ directly follows from (3.5) namely

$$|\, E(h) - 1\,| = \left| \sum_{n=1}^{\infty} \frac{h^n}{n!} \right| = |\, r_1(h)\,| \leq 2\,|\,h\,| \quad \text{if} \quad |\,h\,| \leq 1.$$

Hence, $E(z)$ is continuous for all z.
Iteration of (3.6) gives

$$E(z_1 + \cdots + z_n) = E(z_1) \cdots E(z_n). \tag{3.9}$$

Let us take $z_1 = \cdots = z_n = 1$. Since $E(1) = \mathrm{e}$ by (2.18), we obtain

$$E(n) = \mathrm{e}^n, \quad n \in \mathbb{N}. \tag{3.10}$$

If $p = m/n$, where m, n are positive integers, then

$$E(p)^n = E(pn) = E(m) = \mathrm{e}^m, \tag{3.11}$$

so that

$$E(p) = \mathrm{e}^p, \quad p \in \mathbb{Q}_+. \tag{3.12}$$

It follows from (3.7) that $E(-p) = \mathrm{e}^{-p}$ if p is positive and rational. Thus (3.12) holds for all rational p. In Definition 1.9 we suggested the definition

$$x^y = \sup\{x^p \mid p < y, p \in \mathbb{Q}\}, \tag{3.13}$$

where $x > 1$ and y is any real number. Since E is continuous and increasing, (3.12) shows that

$$E(x) = \mathrm{e}^x. \tag{3.14}$$

for all real x. The notation $\exp(x)$ is often used in place of e^x. (3.14) is a much more convenient starting point for the investigation of the properties of e^x. We now revert to the customary notation, e^x, in place of $E(x)$ and summarize what we have proved so far.

Proposition 3.18 *Let* e^x *be defined on* $\mathbb{R}$ *by* (3.4) *and* (3.14). *Then*

(a) e^x *is continuous for all* x.
(b) e^x *is a strictly increasing function and* $\mathrm{e}^x > 0$.
(c) $\mathrm{e}^{x+y} = \mathrm{e}^x \mathrm{e}^y$.
(d) $\lim_{x \to +\infty} \mathrm{e}^x = +\infty, \quad \lim_{x \to -\infty} \mathrm{e}^x = 0.$
(e) $\lim_{x \to +\infty} \dfrac{x^n}{\mathrm{e}^x} = 0$ *for every* $n \in \mathbb{N}$.

Proof. We have already proved (a) to (d); (3.4) shows that

$$\mathrm{e}^x > \frac{x^{n+1}}{(n+1)!}$$

for $x > 0$, so that

$$\frac{x^n}{\mathrm{e}^x} < \frac{(n+1)!}{x},$$

and (e) follows. Part (e) shows that e^x tends faster to $+\infty$ than any power of x, as $x \to +\infty$. ∎

Since e^x, $x \in \mathbb{R}$, is a strictly increasing continuous function, by Corollary 3.16 e^x has an strictly increasing continuous inverse function $\log y$, $\log\colon \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$. log is defined by

$$\mathrm{e}^{\log y} = y, \quad y > 0, \tag{3.15}$$

or, equivalently, by

$$\log(\mathrm{e}^x) = x, \quad x \in \mathbb{R}. \tag{3.16}$$

Writing $u = \mathrm{e}^x$ and $v = \mathrm{e}^y$, (3.6) gives

$$\log(uv) = \log(\mathrm{e}^x \mathrm{e}^y) = \log(\mathrm{e}^{x+y}) = x + y,$$

such that

$$\log(uv) = \log u + \log v, \quad u > 0, v > 0. \tag{3.17}$$

This shows that log has the familiar property which makes the logarithm useful for computations. Another customary notation for $\log x$ is $\ln x$. Proposition 3.18 shows that

$$\lim_{x \to +\infty} \log x = +\infty, \quad \lim_{x \to 0+0} \log x = -\infty.$$

We summarize what we have proved so far.

Proposition 3.19 *Let the logarithm* $\log\colon \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$ *be the inverse function to the exponential function* e^x. *Then*

(a) log *is continuous on* $\mathbb{R}_+ \setminus \{0\}$.
(b) log *is strictly increasing.*
(c) $\log(uv) = \log u + \log v$ *for* $u, v > 0$.
(d) $\lim\limits_{x \to +\infty} \log x = +\infty, \quad \lim\limits_{x \to 0+0} \log x = -\infty.$

It is easily seen from (3.15) that

$$x^n = \mathrm{e}^{n \log x} \tag{3.18}$$

if $x > 0$ and n is an integer. Similarly, if m is a positive integer, we have

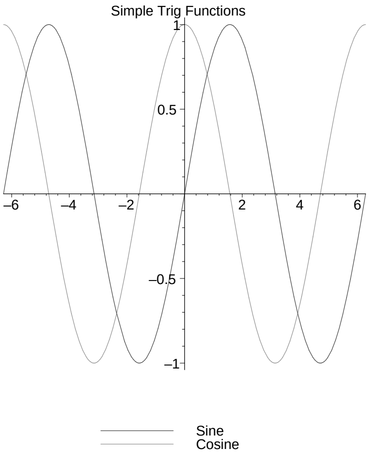
$$x^{1/m} = \mathrm{e}^{\frac{\log x}{m}} \tag{3.19}$$

Combining (3.18) and (3.19), we obtain

$$x^\alpha = \mathrm{e}^{\alpha \log x}. \tag{3.20}$$

for any rational α. We now define x^α for any real α and $x > 0$, by (3.20). The continuity and monotonicity of e^x and $\log x$ show that this definition leads to the same result as the previously suggested one.

3.5 Trigonometric Functions



In this section we define the trigonometric functions using the exponential function e^z.

Definition 3.12 For $z \in \mathbb{C}$ define

$$\cos z = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\,z} + \mathrm{e}^{-\mathrm{i}\,z}\right), \quad \sin z = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\,z} - \mathrm{e}^{-\mathrm{i}\,z}\right) \tag{3.21}$$

such that

$$\mathrm{e}^{\mathrm{i}\,z} = \cos z + \mathrm{i}\sin z \quad \text{(Euler formula)} \tag{3.22}$$

Proposition 3.20 (a) *The functions* $\sin z$ *and* $\cos z$ *can be considered as power series which absolutely converge for all* $z \in \mathbb{C}$*:*

$$\begin{aligned}\cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, z^{2n} = 1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + - \cdots \\ \sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, z^{2n+1} = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - + \cdots .\end{aligned} \tag{3.23}$$

(b) $\sin x$ *and* $\cos x$ *are real valued and continuous on* $\mathbb{R}$*, where* $\cos x$ *is an even and* $\sin x$ *is an odd function, i. e.* $\cos(-x) = \cos x$*,* $\sin(-x) = -\sin x$*. We have*

$$\sin^2 x + \cos^2 x = 1; \tag{3.24}$$

$$\begin{aligned}\cos(x+y) &= \cos x \cos y - \sin x \sin y; \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y.\end{aligned} \tag{3.25}$$

Proof. (a) Inserting $\mathrm{i}z$ into (3.4) in place of z and using $(\mathrm{i}^n) = (\mathrm{i}, -1, -\mathrm{i}, 1, \mathrm{i}, -1, \dots)$, we have

$$\mathrm{e}^{\mathrm{i}\,z} = \sum_{n=0}^{\infty} \mathrm{i}^n \frac{z^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} + \mathrm{i}\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}.$$

Inserting $-\mathrm{i}z$ into (3.4) in place of z we have

$$\mathrm{e}^{-\mathrm{i}\,z} = \sum_{n=0}^{\infty} \mathrm{i}^n \frac{z^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} - \mathrm{i}\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}.$$

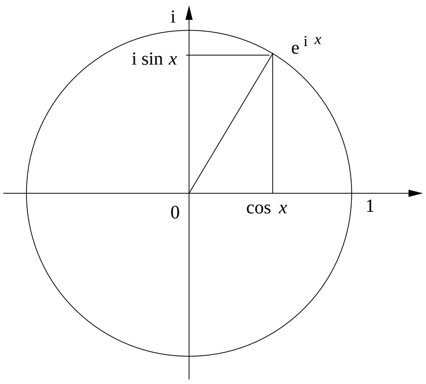
Inserting this into (3.21) proves (a).
(b) Since the exponential function is continuous on $\mathbb{C}$, $\sin z$ and $\cos z$ are also continuous on $\mathbb{C}$. In particular, their restrictions to $\mathbb{R}$ are continuous. Now let $x \in \mathbb{R}$, then $\overline{\mathrm{i}\,x} = -\mathrm{i}\,x$. By Homework 11.3 and (3.21) we obtain

$$\cos x = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{\overline{\mathrm{i}\,x}}\right) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\,x} + \overline{\mathrm{e}^{\mathrm{i}\,x}}\right) = \operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\,x}\right)$$

and similarly

$$\sin x = \operatorname{Im}\left(e^{ix}\right).$$

Hence, $\sin x$ and $\cos x$ are real for real x.



For $x \in \mathbb{R}$ we have $\left| e^{ix} \right| = 1$. Namely by (3.7) and Homework 11.3

$$\left| e^{ix} \right|^2 = e^{ix}\,\overline{e^{ix}} = e^{ix}e^{-ix} = e^0 = 1,$$

so that for $x \in \mathbb{R}$

$$\left| e^{ix} \right| = 1. \tag{3.26}$$

On the other hand, the Euler formula and the fact that $\cos x$ and $\sin x$ are real give

$$1 = \left| e^{ix} \right| = \left| \cos x + i \sin x \right| = \cos^2 x + \sin^2 x.$$

Hence, $e^{ix} = \cos x + i \sin x$ is a point on the unit circle in the complex plane, and $\cos x$ and $\sin x$ are its coordinates.

It is trivial from the definition that $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$ for all $z \in \mathbb{C}$. The addition laws for $\sin x$ and $\cos x$ follow from (3.6) applied to $e^{i(x+y)}$. This completes the proof of (b). ■

Lemma 3.21 *There exists a unique number $\tau \in (0, 2)$ such that $\cos \tau = 0$. We define the number π by*

$$\pi = 2\tau. \tag{3.27}$$

The proof is based on the following Lemma.

Lemma 3.22

$$\text{(a)} \quad 0 < x < \sqrt{6} \quad \textit{implies} \quad \sin x > 0. \tag{3.28}$$

$$\text{(b)} \quad 0 < x < \sqrt{2} \quad \textit{implies} \quad 0 < \cos x, \tag{3.29}$$

$$0 < \sin x < x < \frac{\sin x}{\cos x}, \tag{3.30}$$

$$\cos^2 x < \frac{1}{1 + x^2}. \tag{3.31}$$

(c) $\cos x$ *is strictly decreasing on* $[0, \pi]$*; whereas* $\sin x$ *is strictly increasing on* $[-\pi/2, \pi/2]$.

The proof of both lemmas is in the Appendix B to this chapter.

By definition, $\cos(\pi/2) = 0$; and (3.24) shows $\sin(\pi/2) = \pm 1$. By (3.28), $\sin \pi/2 = 1$. Thus $e^{i\pi/2} = i$, and the addition formula for e^z gives

$$e^{\pi i} = -1, \quad e^{2\pi i} = 1; \tag{3.32}$$

hence,

$$e^{z+2\pi i} = e^z, \quad z \in \mathbb{C}. \tag{3.33}$$

Proposition 3.23 (a) *The function* e^z *is periodic with period* $2\pi i$.
We have $e^{ix} = 1$, $x \in \mathbb{R}$, *if and only if* $x = 2k\pi$, $k \in \mathbb{Z}$.
(b) *The functions* $\sin z$ *and* $\cos z$ *are periodic with period* 2π.
The real zeros of the sine and cosine functions are $\{k\pi \mid k \in \mathbb{Z}\}$ *and* $\{\pi/2 + k\pi \mid k \in \mathbb{Z}\}$, *respectively.*

Proof. We have already proved (a). (b) follows from (a) and (3.21). ■

Proposition 3.24 *For real* x *we have*

$$\cos x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + r_{2n+2}(x) \tag{3.34}$$

$$\sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+3}(x), \tag{3.35}$$

where

$$|\, r_{2n+2}(x) \,| \le \frac{|\, x \,|^{2n+2}}{(2n+2)!} \quad \textit{if} \quad |\, x \,| \le 2n+3, \tag{3.36}$$

$$|\, r_{2n+3}(x) \,| \le \frac{|\, x \,|^{2n+3}}{(2n+3)!} \quad \textit{if} \quad |\, x \,| \le 2n+4. \tag{3.37}$$

Proof. Let

$$r_{2n+2}(x) = \pm \frac{x^{2n+2}}{(2n+2)!} \left(1 - \frac{x^2}{(2n+3)(2n+4)} \pm \cdots \right).$$

Put

$$a_k := \frac{x^{2k}}{(2n+3)(2n+4)\cdots(2n+2(k+1))}.$$

Then we have, by definition

$$r_{2n+2}(x) = \pm \frac{x^{2n+2}}{(2n+2)!} \left(1 - a_1 + a_2 - + \cdots \right).$$

Since

$$a_k = a_{k-1} \frac{x^2}{(2n+2k+1)(2n+2k+2)},$$

$|x| \le 2n+3$ implies

$$1 > a_1 > a_2 > \cdots > 0$$

and finally as in the proof of the Leibniz criterion

$$0 \le 1 - a_1 + a_2 - a_3 + - \cdots \le 1.$$

Hence, $|r_{2n+2}(x)| \le |x|^{2n+2}/(2n+2)!$. The estimate for the remainder of the sine series is similar. ∎

Corollary 3.25

$$\text{(a)} \quad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}; \tag{3.38}$$

$$\text{(b)} \quad \lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{3.39}$$

Proof. (a) Consider the remainder term of order 3

$$\sin x = x + r_3(x), \quad \text{where} \quad |x| \le \frac{|x|^3}{3!} \quad \text{if} \quad |x| \le 4.$$

That is

$$\begin{aligned} |\sin x - x| &\le \frac{|x|^3}{6} \quad \text{if} \quad |x| \le 4 \\ \Longrightarrow \left| \frac{\sin x}{x} - 1 \right| &\le \frac{|x|^2}{6} \quad \text{if} \quad |x| \le 4. \end{aligned}$$

Now (a) follows.
(b) Consider the remainder term of order 4 in the expansion of $\cos x$

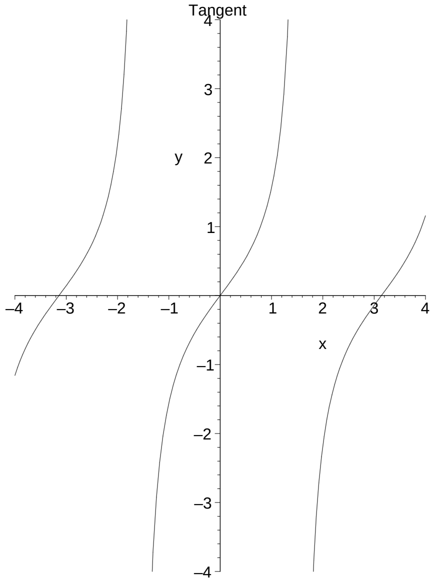
$$\cos x = 1 - \frac{x^2}{2} + r_4(x) \quad \text{with} \quad |r_4(x)| \le \frac{|x|^4}{24} \quad \text{if} \quad |x| \le 5.$$

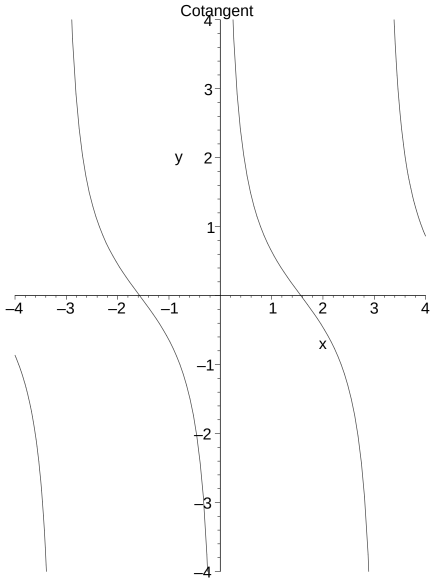
We obtain

$$\left| \frac{1-\cos x}{x^2} \right| = \left| \frac{1}{2} - \frac{r_4}{x^2} \right| \le \frac{1}{2} + \frac{|x|^2}{24} \quad \text{if} \quad |x| \le 5;$$

and (b) follows. ∎

3.5.1 The Tangent and Cotangent Functions





For $x \neq \pi/2 + k\pi$, $k \in \mathbb{Z}$, define

$$\tan x = \frac{\sin x}{\cos x}. \tag{3.40}$$

For $x \neq k\pi$, $k \in \mathbb{Z}$, define

$$\cot x = \frac{\cos x}{\sin x}. \tag{3.41}$$

Lemma 3.26 (a) $\tan x$ *is continuous at* $x \in \mathbb{R}\setminus\{\pi/2 + k\pi \mid k \in \mathbb{Z}\}$, *and* $\tan(x + \pi) = \tan x$*;*
(b) $\lim\limits_{x\to\frac{\pi}{2}-0} \tan x = +\infty$, $\lim\limits_{x\to\frac{\pi}{2}+0} \tan x = -\infty$*;*
(c) $\tan x$ *is strictly increasing on* $(-\pi/2, \pi/2)$*;*

Proof. (a) is clear by Proposition 3.3 since $\sin x$ and $\cos x$ are continuous. We show only (c) and let (b) as an exercise. Let $0 < x < y < \pi/2$. Then $0 < \sin x < \sin y$ and $\cos x > \cos y > 0$. Therefore

$$\tan x = \frac{\sin x}{\cos x} < \frac{\sin y}{\cos y} = \tan y.$$

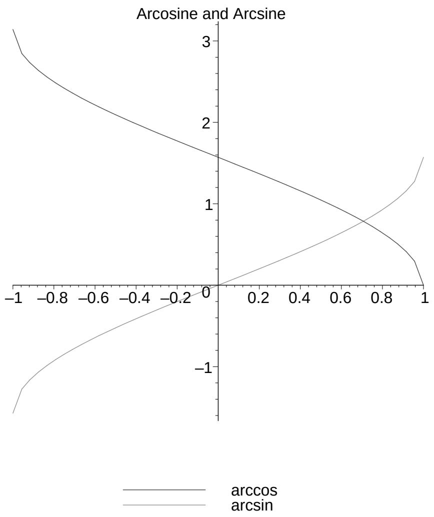
Hence, tan is strictly increasing on $(0, \pi/2)$. Since $\tan(-x) = -\tan(x)$, tan is strictly increasing on the whole interval $(-\pi/2, \pi/2)$. ■

Similarly as Lemma 3.26 one proves the next lemma.

Lemma 3.27 (a) $\cot x$ *is continuous at* $x \in \mathbb{R}\setminus\{k\pi \mid k \in \mathbb{Z}\}$, *and* $\cot(x + \pi) = \cot x$*;*
(b) $\lim\limits_{x\to 0-0} \cot x = -\infty$, $\lim\limits_{x\to 0+0} \cot x = +\infty$;
(c) $\cot x$ *is strictly decreasing on* $(0, \pi)$.

3.6 Inverse Trigonometric Functions

We have seen in Lemma 3.22 that $\cos x$ is strictly decreasing on $[0,\pi]$ and $\sin x$ is strictly increasing on $[-\pi/2,\pi/2]$. Obviously, the images are $\cos[0,\pi] = \sin[-\pi/2,\pi/2] = [-1,1]$. Using Corollary 3.16 we obtain the inverse functions.



Proposition 3.28 (and Definition) *There exists the inverse function to* cos

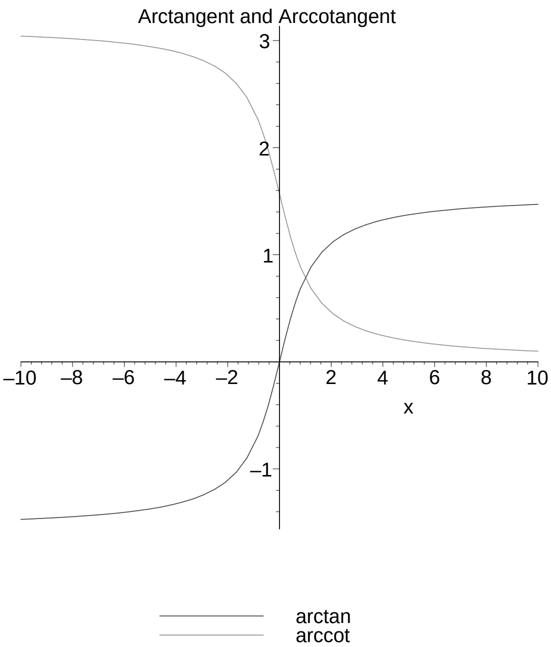
$$\arccos\colon [-1,1] \to [0,\pi] \tag{3.42}$$

given by $\arccos(\cos x) = x$, $x \in [0,\pi]$ *or* $\cos(\arccos y) = y$, $y \in [-1,1]$. *The function* $\arccos x$ *is strictly decreasing and continuous.*
There exists the inverse function to sin

$$\arcsin\colon [-1,1] \to [-\pi/2,\pi/2] \tag{3.43}$$

given by $\arcsin(\sin x) = x$, $x \in [-\pi/2,\pi/2]$ *or* $\sin(\arcsin y) = y$, $y \in [-1,1]$. *The function* $\arcsin x$ *is strictly increasing and continuous.*

Note that $\arcsin x + \arccos x = \pi/2$ if $x \in [-1,1]$. Indeed, let $y = \arcsin x$; then $x = \sin y = \cos(\pi/2 - y)$. Since $y \in [0,\pi]$, $\pi/2 - y \in [-\pi/2,\pi/2]$, and we have $\arccos x = \pi/2 - y$. Therefore $y + \arccos x = \pi/2$.



By Lemma 3.26, $\tan x$ is strictly increasing on $(-\pi/2,\pi/2)$. Therefore, there exists the inverse function on the image $\tan(-\pi/2,\pi/2) = \mathbb{R}$.

Proposition 3.29 (and Definition) *There exists the inverse function to* tan

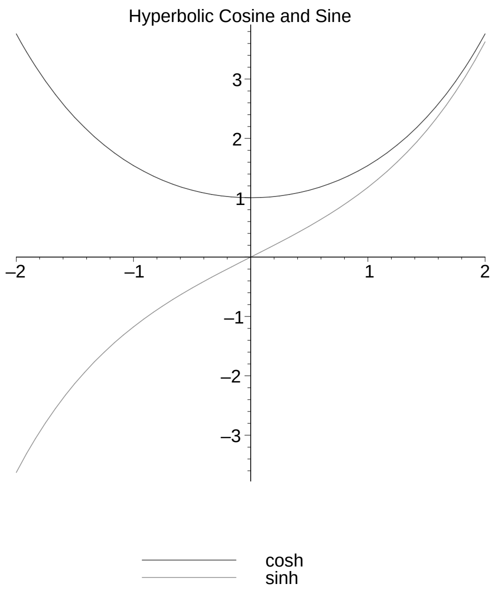
$$\arctan\colon \mathbb{R} \to (-\pi/2,\pi/2) \tag{3.44}$$

given by $\arctan(\tan x) = x$, $x \in (-\pi/2,\pi/2)$ *or* $\tan(\arctan y) = y$, $y \in \mathbb{R}$. *The function* $\arctan x$ *is strictly increasing and continuous.*
There exists the inverse function to $\cot x$

$$\operatorname{arccot}\colon \mathbb{R} \to (0,\pi) \tag{3.45}$$

given by $\operatorname{arccot}(\cot x) = x$, $x \in (0,\pi)$ *or* $\cot(\operatorname{arccot} y) = y$, $y \in \mathbb{R}$. *The function* $\operatorname{arccot} x$ *is strictly decreasing and continuous.*

3.7 Hyperbolic Functions



The functions

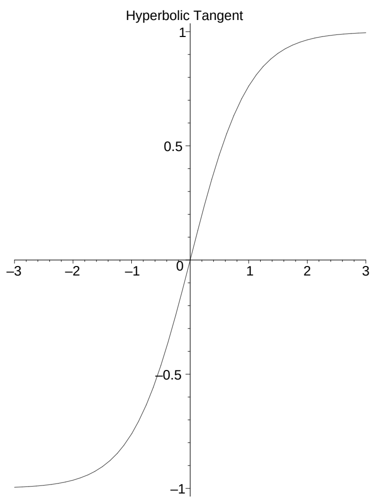
$$\sinh x = \frac{e^x - e^{-x}}{2}, \tag{3.46}$$

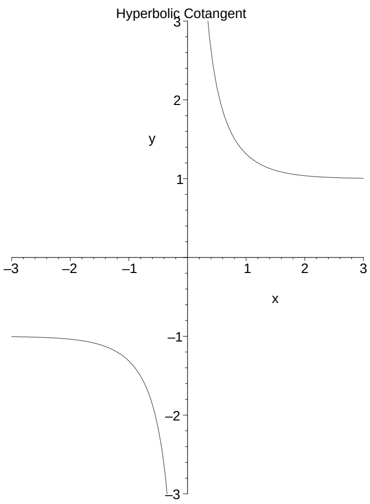
$$\cosh x = \frac{e^x + e^{-x}}{2}, \tag{3.47}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x} \tag{3.48}$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{\cosh x}{\sin x} \tag{3.49}$$

are called *hyperbolic sine*, *hyperbolic cosine*, *hyperbolic tangent*, and *hyperbolic cotangent*, respectively. There are many analogies between these functions and their ordinary trigonometric counterparts.





The functions $\sinh x$ and $\tanh x$ are strictly increasing with $\sinh(\mathbb{R}) = \mathbb{R}$ and $\tanh(\mathbb{R}) = (-1, 1)$. Hence, their inverse functions are defined on $\mathbb{R}$ and on $(-1, 1)$, respectively, and are also strictly increasing and continuous. The function

$$\operatorname{arsinh} : \mathbb{R} \to \mathbb{R} \tag{3.50}$$

is given by $\operatorname{arsinh}(\sinh(x)) = x$, $x \in \mathbb{R}$ or $\sinh(\operatorname{arsinh}(y)) = y$, $y \in \mathbb{R}$.
The function

$$\operatorname{artanh} : (-1, 1) \to \mathbb{R} \tag{3.51}$$

is defined by $\operatorname{artanh}(\tanh(x)) = x$, $x \in \mathbb{R}$ or $\tanh(\operatorname{artanh}(y)) = y$, $y \in (-1, 1)$.
The function cosh is strictly increasing on the half line $\mathbb{R}_+$ with $\cosh(\mathbb{R}_+) = [1, \infty)$. Hence, the inverse function is defined on $[1, \infty)$ taking values in $\mathbb{R}_+$. It is also strictly increasing and continuous.

$$\operatorname{arcosh} : [1, \infty) \to \mathbb{R}_+ \tag{3.52}$$

is defined via $\operatorname{arcosh}(\cosh(x)) = x$, $x \geq 0$ or by $\cosh(\operatorname{arcosh}(y)) = y$, $y \geq 1$.
The function coth is strictly decreasing on the $x < 0$ and on $x > 0$ with $\coth(\mathbb{R} \setminus 0) = \mathbb{R} \setminus [-1, 1]$. Hence, the inverse function is defined on $\mathbb{R} \setminus [-1, 1]$ taking values in $\mathbb{R} \setminus 0$. It is also strictly decreasing and continuous.

$$\operatorname{arcoth} : \mathbb{R} \setminus [-1, 1] \to \mathbb{R} \tag{3.53}$$

is defined via $\operatorname{arcoth}(\coth(x)) = x$, $x \neq 0$ or by $\coth(\operatorname{arcoth}(y)) = y$, $y < -1$ or $y > 1$.

3.8 Appendix B

Proof of Theorem 3.13. By hypothesis, the set $\{f(t) \mid a < t < x\}$ is bounded above by $f(x)$, and therefore has a least upper bound which we shall denote by A. Evidently $A \leq f(x)$. We have to show that $A = f(x - 0)$.
Let $\varepsilon > 0$ be given. It follows from the definition of A as a least upper bound that there exists $\delta > 0$ such that $a < x - \delta < x$ and

$$A - \varepsilon < f(x - \delta) \leq A. \tag{3.54}$$

Since f is monotonic, we have

$$f(x - \delta) < f(t) \leq A, \quad \text{if} \quad x - \delta < t < x. \tag{3.55}$$

Combining (3.54) and (3.55), we see that

$$|\, f(t) - A \,| < \varepsilon \quad \text{if} \quad x - \delta < t < x.$$

Hence $f(x - 0) = A$.
The second half of (3.2) is proved in precisely the same way. Next, if $a < x < y < b$, we see from (3.2) that

$$f(x + 0) = \inf_{x<t<b} f(t) = \inf_{x<t<y} f(t). \tag{3.56}$$

The last equality is obtained by applying (3.2) to (a, y) instead of (a, b). Similarly,

$$f(y - 0) = \sup_{a<t<y} f(t) = \sup_{x<t<y} f(t). \tag{3.57}$$

Comparison of the (3.56) and (3.57) gives (3.3). ∎

Proof of Lemma 3.22. (a) By (3.23)

$$\cos x = \left(1 - \frac{1}{2!}x^2\right) + x^4 \left(\frac{1}{4!} - \frac{1}{6!}x^2\right) + \cdots .$$

$0 < x < \sqrt{2}$ implies $1 - x^2/2 > 0$ and, moreover $1/(2n)! - x^2/(2n + 2)! > 0$ for all $n \in \mathbb{N}$; hence $C(x) > 0$.
By (3.23),

$$\sin x = x \left(1 - \frac{1}{3!}x^2\right) + x^5 \left(\frac{1}{5!} - \frac{1}{7!}x^2\right) + \cdots .$$

Now,

$$1-\frac{1}{3!}x^2>0 \iff x<\sqrt{6}, \quad \frac{1}{5!}-\frac{1}{7!}x^2>0 \iff x<\sqrt{42}, \dots.$$

Hence, $S(x)>0$ if $0<x<\sqrt{6}$. This gives (3.28). Similarly,

$$x-\sin x = x^3\left(\frac{1}{3!}-\frac{1}{5!}x^2\right)+x^7\left(\frac{1}{7!}-\frac{1}{9!}x^2\right)+\cdots,$$

and we obtain $\sin x < x$ if $0<x<\sqrt{20}$. Finally we have to check whether $\sin x - x\cos x > 0$; equivalently

$$0 \overset{?}{<} x^3\left(\frac{1}{2!}-\frac{1}{3!}\right)-x^5\left(\frac{1}{4!}-\frac{1}{5!}\right)+x^7\left(\frac{1}{6!}-\frac{1}{7!}\right)-+\cdots$$
$$0 \overset{?}{<} x^3\left(\frac{2}{3!}-x^2\frac{4}{5!}\right)+x^7\left(\frac{6}{7!}-x^2\frac{8}{9!}\right)+\cdots$$

Now $\sqrt{10}>x>0$ implies

$$\frac{2n}{(2n+1)!}-\frac{2n+2}{(2n+3)!}x^2>0$$

for all $n\in\mathbb{N}$. This completes the proof of (a)
(b) Using (3.24), we get

$$0<x\cos x<\sin x \Longrightarrow 0<x^2\cos^2 x<\sin^2 x$$
$$\Longrightarrow x^2\cos^2 x+\cos^2 x<1 \Longrightarrow \cos^2 x<\frac{1}{1+x^2}.$$

(c) In the proof of Lemma 3.21 (see below) we will see that $\cos x$ is strictly decreasing in $(0,\pi/2)$. By (3.24), $\sin x = \sqrt{1-\cos^2 x}$ is strictly increasing. Since $\sin x$ is an odd function, $\sin x$ is strictly increasing on $[-\pi/2,\pi/2]$. Since $\cos x = -\sin(x-\pi/2)$, the statement for $\cos x$ follows. ∎

Proof of Lemma 3.21. $\cos 0 = 1$. By the Lemma 3.22, $\cos^2 1 < 1/2$. By the double angle formula for cosine, $\cos 2 = 2\cos^2 1 - 1 < 0$. By continuity of $\cos x$ and Theorem 3.5, cos has a zero τ in the interval $(0,2)$.
By addition laws,

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

So that by Lemma 3.22 $0<x<y<2$ implies $0<\sin((x+y)/2)$ and $\sin((x-y)/2)<0$; therefore $\cos x > \cos y$. Hence, $\cos x$ is strictly decreasing on $(0,2)$. The zero τ is therefore unique. ∎

3.8.1 Estimates for π

This is an application of Proposition 3.24. For numerical calculations it is convenient to use the following order of operations

$$\cos x = \left(\cdots\left(\left(\left(\frac{-x^2}{2n(2n-1)}+1\right)\frac{-x^2}{(2n-2)(2n-3)}+1\right)\frac{-x^2}{(2n-4)(2n-5)}+1\right)\cdots \right.$$
$$\left.\cdots\right)\frac{-x^2}{2}+1+r_{2n+2}(x).$$

First we compute $\cos 1.5$ and $\cos 1.6$. Choosing $n=7$ we obtain

$$\cos x = \left(\left(\left(\left(\left(\left(\frac{-x^2}{182}+1\right)\frac{-x^2}{132}+1\right)\frac{-x^2}{90}+1\right)\frac{-x^2}{56}+1\right)\frac{-x^2}{30}+1\right)\frac{-x^2}{12}+1\right)\frac{-x^2}{2}+$$
$$+1+r_{16}(x).$$

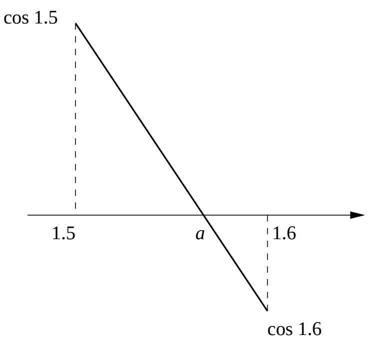
By Proposition 3.24

$$|\, r_{16}(x)\,| \leq \frac{|\, x\,|^{16}}{16!} \leq 0.9\cdot 10^{-10} \quad \text{if} \quad |\, x\,| \leq 1.6.$$

The calculations give

$$\cos 1.5 = 0.07073720163 \pm 20\cdot 10^{-11} > 0, \cos 1.6 \; = -0.02919952239 \pm 20\cdot 10^{-11} < 0.$$

By the itermediate value theorem, $1.5 < \pi/2 < 1.6$.



Now we compute $\cos x$ for two values of x which are close to the linear interpolation

$$a = 1.5 + 0.1\frac{\cos 1.5}{\cos 1.5 - \cos 1.6} = 1.57078\ldots$$

$$\cos 1.5707 = 0.000096326273 \pm 20\cdot 10^{-11} > 0,$$
$$\cos 1.5708 = -0.00000367326 \pm 20\cdot 10^{-11} < 0.$$

Hence, $1.5707 < \pi/2 < 1.5708$.

The next linear interpolation gives

$$b = 1.5707 + 0.00001\frac{\cos 1.5707}{\cos 1.707 - \cos 1.708} = 1.570796326\ldots$$
$$\cos 1.570796326 = 0.00000000073 \pm 20\cdot 10^{-11} > 0,$$
$$\cos 1.570796327 = -0.00000000027 \pm 20\cdot 10^{-11} < 0.$$

Therefore $1.570796326 < \pi/2 < 1.570796327$ so that

$$\pi = 3.141592653 \pm 10^{-9}.$$