Symmetric Topological Complexity and Embedding Problems for Projective Spaces

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This note describes the main components for characterizing the relation of the symmetric topological complexity and the embedding dimension of r-dimensional projective spaces $\mathbb{R}\mathrm{P}^r$. The key property relating these concepts are $\mathbb{Z}_2$-equivariant maps defined on $\mathbb{R}\mathrm{P}^r$, together with fundamental ideas of the Haefliger's metastable range theorem. These results have been announced in [2] and are in relation to previous investigations, discussed in [1], relating the nonsymmetric topological complexity and the immersion dimension of projective spaces.

We first recall some basic definitions. Given a fibration $p : E \to B$, the *Schwarz genus* of p, denoted as genus(p), is defined as the smallest number of open sets $\{U_i\}$ covering B such that p admits a continuous section on each U_i. The *topological complexity* of a topological space X is defined as the Schwarz genus of the endpoints evaluation map $\mathrm{ev} : P(X) \to X \times X$, $\mathrm{ev}(\gamma) = (\gamma(0), \gamma(1))$, $\gamma \in P(X)$, where $P(X)$ is the path space $X^{[0,1]}$ with compact open topology. The *symmetric topological complexity* of X is defined as $\mathrm{TC}^{\mathrm{S}}(X) := \mathrm{genus}(\mathrm{ev}_2) + 1$, where $\mathrm{ev}_2 : P_2(X) \to B(X, 2)$, is a fibration with $P_2(X) := P_1(X)/\mathbb{Z}_2$, and $B(X, 2) := (X \times X - \Delta_X)/\mathbb{Z}_2$, $\Delta_X := \{(x, x), x \in X\}$. Here, we use the fibration $\mathrm{ev}_1 : P_1(X) \to X \times X - \Delta_X$, $\mathrm{ev}_1 = \mathrm{ev}|_{P_1(X)}$, with $P_1(X) := \{\text{paths } \gamma \in X^{[0,1]}, \gamma(0) \neq \gamma(1)\}$. The orbit spaces $P_2(X)$ and $B(X, 2)$ are defined with the actions of $\mathbb{Z}_2$ which reverse the direction of the paths $\gamma \in P_1(X)$, and interchange the coordinates of the elements in $X \times X - \Delta_X$. We discuss now the following theorem:

Theorem (Gonzáles and Landweber, 2009, [2])**.** The symmetric topological complexity of the r dimensional projective space $\mathbb{R}\mathrm{P}^r$, denoted as $\mathrm{TC}^{\mathrm{S}}(\mathbb{R}\mathrm{P}^r)$, is related to $E(r)$, the Euclidean embedding dimension of $\mathbb{R}\mathrm{P}^r$, as $\mathrm{TC}^{\mathrm{S}}(\mathbb{R}\mathrm{P}^r) = E(r)+1$, $r \in \{1, 2, 4, 8, 9, 13\}$, $r > 15$.

In the following, we sketch the two main components in the proof strategy. On the one hand, a relation is established between the symmetric topological complexity of $\mathbb{R}\mathrm{P}^r$ with the level of an involution defined by considering $\mathbb{Z}_2$-equivariant maps using $\mathbb{R}\mathrm{P}^r$. On the other hand, we use the identification, as described in the Haefliger's metastable theorem, between isotopy classes of smooth embeddings of a manifold $M \subset \mathbb{R}^m$ and homotopy classes of $\mathbb{Z}_2$-equivariant maps $M \times M - \Delta_M \to \mathbb{S}^{m-1}$.

The level of an involution given by a $\mathbb{Z}_2$-action on X is denoted as $\mathrm{level}(X, \mathbb{Z}_2)$, and is defined as the minimum $\ell > 0$, such that there exists an $\mathbb{Z}_2$-equivariant map $X \to \mathbb{S}^{\ell-1}$. The theorem that relates the level of an involution to the symmetric topological complexity for projective spaces has been presented in [2], and ensures that for all values of r, $\mathrm{TC}^{\mathrm{S}}(P^r) = \mathrm{level}(P^r \times P^r - \Delta_{P^r}, \mathbb{Z}_2) + 1$. There are three main components for proving this result. First, we need a fundamental property, presented in [3], of the Schwarz genus of a canonical projection which guarantees that for an $\mathbb{Z}_2$-action on X which admits a $\mathbb{Z}_2$-equivariant map $X \to \mathbb{S}^{n-1}$, and for the canonical projection $p : X \to X/\mathbb{Z}_2$, we have $\mathrm{genus}(p) = \mathrm{level}(X, \mathbb{Z}_2)$.

The second property characterizes genus(ev_i) for $\text{ev}_1 : P_1(X) \to X \times X - \Delta_X$. Finally, we characterize also genus(ρ) for $\rho : \mathbb{R}\text{P}^r \times \mathbb{R}\text{P}^r - \Delta_{\mathbb{R}\text{P}^r} \to B(\mathbb{R}\text{P}^r, 2)$ the canonical projection. More precisely, we consider the property that for $i \in \{1, 2\}$ genus(ev_i) = genus(π_i), defined by constructing (commutative) diagrams:

$$\begin{array}{ccc} P(\mathbb{R}\text{P}^r) & \xrightarrow{f} & \mathbb{S}^r \times_{\mathbb{Z}_2} \mathbb{S}^r \\ & {\scriptstyle \text{ev}} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & \mathbb{R}\text{P}^r \times \mathbb{R}\text{P}^r & \end{array} \qquad \begin{array}{ccc} P_1(\mathbb{R}\text{P}^r) & \xrightarrow{f_1} & E_1 \\ & {\scriptstyle \text{ev}_1} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & \mathbb{R}\text{P}^r \times \mathbb{R}\text{P}^r - \Delta_{\mathbb{R}\text{P}^r} & \end{array}$$

$$\begin{array}{ccc} P_2(\mathbb{R}\text{P}^r) & \xrightarrow{f_2} & E_2 \\ & {\scriptstyle \text{ev}_2} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & B(\mathbb{R}\text{P}^r, 2) & \end{array}$$

For defining the map f (and proving the commutativity of the diagrams), we consider a path $\gamma \in P(\mathbb{R}\text{P}^r)$, and $\hat{\gamma} : [0, 1] \to \mathbb{S}^r$ any lifting through the canonical projection $\mathbb{S}^r \to \mathbb{R}\text{P}^r$, then $f(\gamma)$ is the class of $(\hat{\gamma}(0), \hat{\gamma}(1))$ in the Borel construction $\mathbb{S}^r \times_{\mathbb{Z}_2} \mathbb{S}^r := (\mathbb{S}^r \times \mathbb{S}^r)/(-x, y) \sim (x, -y)$. Now, the commutativity of these diagrams ensures that part of the equalities genus(ev_i) = genus(π_i) are valid. In order to analyze the missing inequalities, we use similar ideas by constructing additional commutative diagrams, using an $\mathbb{Z}_2$-equivariant map $g_1 : E_1 \to P_1(\mathbb{R}\text{P}^r)$, where g_1 run backwards with respect to f_1. The explicit construction of g_1 uses a model for E_1 as the set $(\mathbb{S}^r \times \mathbb{S}^r - \tilde{\Delta})/(x, y) \sim (-x, -y)$, $\tilde{\Delta} := \{(x, y) \in \mathbb{S}^r \times \mathbb{S}^r | x \neq \pm y\}$, and g_1 maps the class of a pair (x_1, x_2) into the curve $[0, 1] \to \mathbb{S}^r \to \mathbb{R}\text{P}^r$ with the first map given by $t \mapsto v(tx_1 + (1 - t)x_2)$, and v is the normalization map.

Using similar ideas, we can also characterize genus(ρ), for $\rho : \mathbb{R}\text{P}^r \times \mathbb{R}\text{P}^r - \Delta_{\mathbb{R}\text{P}^r} \to B(\mathbb{R}\text{P}^r, 2)$ the canonical projection, with the property genus(ρ) = genus(π_2). With all these steps, we have a rough synthesis of some basic ideas for proving the theorem:

Theorem. For all values of r, $\text{TC}^{\text{S}}(\mathbb{R}\text{P}^r) = \text{level}(\mathbb{R}\text{P}^r \times \mathbb{R}\text{P}^r - \Delta_{\mathbb{R}\text{P}^r}, \mathbb{Z}_2) + 1$.

The second part of the proof of the property relating the symmetric topological complexity and the embedding dimension uses the celebrated Haefliger's metastable range theorem:

Theorem (Haefliger's metastable range)**.** Let M be a smooth n-dimensional manifold and $2m \geq 3(n + 1)$, then there is a surjective map from the set of isotopy classes of smooth embeddings $M \subset \mathbb{R}^m$ onto the set of $\mathbb{Z}_2$-equivariant homotopy classes of maps $M^* \to \mathbb{S}^{m-1}$, $M^* := M \times M - \Delta_M$.

In our particular case, we only use from the Haefliger's metastable range the fact that the existence of a smooth embedding $M \subset \mathbb{R}^m$ is equivalent to the existence of a $\mathbb{Z}_2$-equivariant map $M^* \to \mathbb{S}^{m-1}$. Notice that the we have an explicit construction for the surjective map used in the Haefliger's metastable range by considering for any embedding $g : \mathbb{R}\text{P}^r \to \mathbb{R}^d$, a $\mathbb{Z}_2$-equivariant map $\tilde{g} : \mathbb{R}\text{P}^r \times \mathbb{R}\text{P}^r - \Delta_{\mathbb{R}\text{P}^r} \to \mathbb{S}^{d-1}$: $\tilde{g}(a, b) := (g(a) - g(b))/(||g(a) - g(b)||)$.

The components required for relating the embedding dimension of a projective space with the level of the involution for $\mathbb{Z}_2$-equivariant map $\tilde{g} : \mathbb{R}\mathrm{P}^r \times \mathbb{R}\mathrm{P}^r - \Delta_{\mathbb{R}\mathrm{P}^r} \to \mathbb{S}^{d-1}$ are the following two properties from Gonzáles and Landweber, complementing results from Haefliger and Hirsch (1961, 1962):

Proposition (Gonzáles and Landweber, 2009)**.** For $r \in \{8, 9, 13\}$ or $r > 15$, an axial map $\mathbb{R}\mathrm{P}^r \times \mathbb{R}\mathrm{P}^r \to \mathbb{R}\mathrm{P}^s$ can exist only when $2s \geq 3(r+1)$.

Theorem (Gonzáles and Landweber, 2009)**.** The existence of a symmetric axial map $\mathbb{R}\mathrm{P}^r \times \mathbb{R}\mathrm{P}^r \to \mathbb{R}\mathrm{P}^s$ implies the existence of a smooth embedding $\mathbb{R}\mathrm{P}^r \subset \mathbb{R}^{s+1}$ provided $2s > 3r$.

Theorem (Haefliger, Hirsch, 1961, 1962)**.** The existence of a smooth embedding $\mathbb{R}\mathrm{P}^r \subset \mathbb{R}^s$ implies the existence of a symmetric axial map $\mathbb{R}\mathrm{P}^r \times \mathbb{R}\mathrm{P}^r \to \mathbb{R}\mathrm{P}^s$.

In order to analyze the missing cases outside the metastable range, $r \leq 15$, we consider lower and upper bounds of the symmetric topological complexity. This can be achieved by considering the inequalities $\mathrm{TC}(\mathbb{R}\mathrm{P}^r) \leq \mathrm{TC}^{\mathrm{S}}(\mathbb{R}\mathrm{P}^r) \leq E(\mathbb{R}\mathrm{P}^r)+1$, and $\mathrm{TC}(\mathbb{R}\mathrm{P}^r) \leq \mathrm{TC}^{\mathrm{S}}(\mathbb{R}\mathrm{P}^r) \leq E_{\mathrm{TOP}}(\mathbb{R}\mathrm{P}^r)+1$, where E_{TOP} is defined for embeddings which are non necessarily smooth. These inequalities can be proved by considering the property we already discussed $\mathrm{TC}^{\mathrm{S}}(\mathbb{R}\mathrm{P}^r) = \mathrm{level}(\mathbb{R}\mathrm{P}^r \times \mathbb{R}\mathrm{P}^r - \Delta_{\mathbb{R}\mathrm{P}^r}, \mathbb{Z}_2) + 1$.

We finally remark that the corresponding result for complex projective spaces is significantly simpler to proof that the real case. The main property is $\mathrm{TC}^{\mathrm{S}}(\mathbb{C}\mathrm{P}^n) = 2n+1$. As it is know from [1], $\mathrm{TC}(\mathbb{C}\mathrm{P}^n) = 2n+1$, and therefore, we need to verify that $\mathrm{TC}^{\mathrm{S}}(\mathbb{C}\mathrm{P}^n) \leq 2n+1$. This inequality can be verified with the following diagram of pullback squares

$$\begin{array}{ccccc}
P(\mathbb{C}\mathrm{P}^r) & \longleftarrow & P_1(\mathbb{C}\mathrm{P}^n) & \longrightarrow & P_2(\mathbb{C}\mathrm{P}^n) \\
\downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_2} \\
\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n & \longleftarrow & \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n - \Delta_{\mathbb{C}\mathrm{P}^n} & \longrightarrow & B(\mathbb{C}\mathrm{P}^n, 2)
\end{array}$$

which guarantees that a common fiber for the fibrations $\mathrm{ev}, \mathrm{ev}_1$ and ev_2 is the path connected loop space $\Omega\mathbb{C}\mathrm{P}^n$. Now, with the Theorem 5 in [3], which estimates the genus of a fibration using the homotopy type of the base and connectivity of the fiber, we obtain the following inequality: $\mathrm{TC}^{\mathrm{S}}(\mathbb{C}\mathrm{P}^n) = \mathrm{genus}(ev_2) + 1 \leq \dim(Y)/2 + 2$, where Y is a CW-complex with the same homotopy type of $B(\mathbb{C}\mathrm{P}^n, 2)$. We can conclude our remark using an observation by Farber and Grant that for M being a smooth closed m-dimensional manifold, $B(M, 2)$ has the homotopy type of a $(2m-1)$-dimensional CW-complex.

References

[1] M. Farber, S. Tabachnikov, and S. Yuzvinsky, *Topological robotics: motion planning in projective spaces*, Int. Math. Res. Not., no. 34, 1853-1870 (2003).

[2] J. González and P. Landweber, *Symmetric topological complexity of projective and lens spaces*, Algebraic & Geometric Topology, no. 9 473-494 (2009).

[3] A. S. Schwarz, *The genus of a fiber space*, Amer. Math. Soc. Transl., Ser. 2, vol. 55 49-140 (1966).