ON THE INFINITE LUCCHESI-YOUNGER CONJECTURE

J. PASCAL GOLLIN AND KARL HEUER

ABSTRACT. A *dicut* in a directed graph is cut for which all of its edges a directed to a common side of the cut. A famous theorem of Lucchesi and Younger states that in every finite digraph the least size of an edge set meeting every dicut is equal to the maximum number of disjoint dicuts in that digraph.

In this paper, we conjecture an version of this theorem using a more structural description of this min-max property for finite dicuts in infinite digraphs. We show that this conjecture can be reduced to countable digraphs where the underlying undirected graph is 2-connected, and we prove several special cases of the conjecture.

§1. INTRODUCTION

In finite graph theory there exist a lot of theorems which relate the maximum number of disjoint substructures of a certain type in a graph with the minimal size of another substructure in that graph, which bounds the number of disjoint objects of the first type that can exist. Often there is no gap between such numbers. Some results of this type even have a reformulation in the language of linear programming.

Probably the most well-known example of such a result is the theorem of Menger for finite graphs. In order to state the theorem more easily let us make the following definition. For two vertex sets $A, B \subseteq V(G)$ in a graph G we call a path P an *A–B path* in G if one endvertex of P lies in A, the other in B and except from these two vertices P is disjoint from the set $A \cup B$. Note that a vertex in $A \cap B$ is also an A–B path.

Theorem 1.1. [3, Thm. 3.3.1] *Let G be a finite graph and $A, B \subseteq V(G)$. Then the maximum number of disjoint A–B paths in G equals the minimum size of a vertex set separating A from B in G.*

This theorem has the following immediate corollary.

Corollary 1.2. *Let G be a finite graph and $A, B \subseteq V(G)$. Then there exists a tuple $(S, \mathcal{P})$ such that the following statements hold.*

(i) $\mathcal{P}$ is a set of disjoint A–B paths in G.
(ii) $S \subseteq V(G)$ separates A from B in G.
(iii) $S \subseteq \bigcup \mathcal{P}$.

(iv) $|S \cap P| = 1$ *for every* $P \in \mathcal{P}$.

However, this corollary is not weaker than Theorem 1.1 because Theorem 1.1 is conversely implied by Corollary 1.2. The crucial point of Corollary 1.2 is that the elements of the tuple $(S, \mathcal{P})$ make certain optimality assertions about each other: The set $\mathcal{P}$ and the way it interacts with S proves that the separator S has minimum size. Conversely, the size of S bounds the size of any set of disjoint A–B paths. Hence, S and its interaction with $\mathcal{P}$ shows that $\mathcal{P}$ is of maximum size.

The benefit of the formulation of Corollary 1.2 is that it avoids talking about maximality and minimality in terms of sizes or cardinalities. In infinite graphs this now becomes much more meaningful. An extension of Theorem 1.1 which only asks for the existence of κ many disjoint A–B paths and a set of size κ separating A from B for some cardinality κ is quite easy to prove. In contrast to this, the extension of Corollary 1.2 which asks for the same tuple but in a graph of arbitrary cardinality, is probably one of the deepest theorems in infinite graph theory and due to Aharoni and Berger [1]. While the proof of this theorem is already challenging for countable graphs, it becomes much more complicated in graphs of higher cardinality.

We want to consider a theorem about finite digraphs which has similar formulations as Theorem 1.1. To state the theorem we have do give some definitions first. In a weakly connected directed graph D we call a cut of D directed, or a *dicut* of D, if all of its edges have their head in a common side of the cut. Now we call a set of edges a *dijoin* of D if it meets every non-empty dicut of D. Now we can state the mentioned theorem, which is due to Lucchesi and Younger.

Theorem 1.3. [8, Thm.] *In every weakly connected finite digraph, the maximum number of disjoint dicuts equals the minimum size of a dijoin.*

Beside the proof Theorem 1.3 of Lucchesi and Younger [8, Thm.], further ones appeared by Lovász [7, Thm. 2] and Frank [4, Thm. 9.7.2]. As for Theorem 1.1 we state a reformulation of Theorem 1.3 which avoids talking about maximality and minimality in terms of sizes or cardinalities.

Corollary 1.4. *Let* D *be a finite weakly connected digraph. Then there exists a tuple* $(F, \mathcal{B})$ *such that the following statements hold.*

(i) $\mathcal{B}$ *is a set of disjoint dicuts of* D.
(ii) $F \subseteq E(D)$ *is a dijoin of* D.
(iii) $F \subseteq \bigcup \mathcal{B}$.
(iv) $|F \cap B| = 1$ *for every* $B \in \mathcal{B}$.

Now we consider the question whether Corollary 1.4 extends to infinite digraphs as Corollary 1.2 did for infinite graphs. Let us first show that a direct extension of this formulation to arbitrary infinite digraphs fails. To do this we define a *double ray* to be an undirected two-way infinite path. Now consider the digraph depicted in Figure 1.1. Its underlying graph is the Cartesian product of a double ray with an edge. Then we orient all edges corresponding to one copy of the double ray in one direction and all edges of the other copy in the different direction. Finally, we direct all remaining edges such that they have their tail in the same copy of the double ray.

This digraph does not have any finite dicut, but infinite ones. Note that every dicut of this digraph contains at most one horizontal edge, which corresponds to a oriented one of some copy of the double ray, and all vertical edges left to some vertical edge. So we cannot even find two disjoint dicuts. Next let us look at dijoins of the digraph depicted in Figure 1.1. In order to hit every dicut which contains a horizontal edge, a dijoin must contain infinitely many vertical edges left to some vertical edge. So we obtain that each dijoin hits every dicut infinitely often in this digraph. Therefore, neither the statement of Corollary 1.4 nor the statement of Theorem 1.3 using cardinalities remains true if we consider arbitrary dicuts in infinite digraphs.

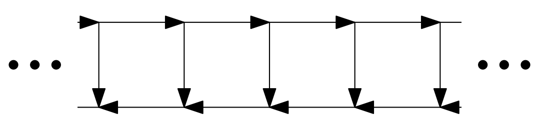

FIGURE 1.1. A counterexample to an extension of Corollary 1.4 to infinite digraphs where infinite dicuts are considered too.

In order to overcome the problem of this example let us again consider the situation in Corollary 1.2. There, all elements of the set $\mathcal{P}$ are just finite paths. So we might need to restrict our attention to finite dicuts when extending Corollary 1.4 to infinite digraphs. Hence, we make the following definitions. In a weakly connected digraph D we call an edge set $F \subseteq E(D)$ a *finitary dijoin* of D if it intersects every non-empty finite dicut of D. Building up on this definition we call a tuple $(F, \mathcal{B})$ as in Corollary 1.4 but where F is now only a finitary dijoin and $\mathcal{B}$ a set of disjoint finite dicuts of D, an *optimal pair* for D. Furthermore, we call an optimal pair *nested* if the elements of $\mathcal{B}$ are pairwise nested, i.e., any two finite dicuts $E(X_1, X_2), E(Y_1, Y_2) \in \mathcal{B}$ either satisfy $X_1 \subseteq Y_1$ or $Y_1 \subseteq X_1$.

Not in contradiction to the example given above we make following conjecture, which we call the Infinite Lucchesi-Younger Conjecture.

Conjecture 1.5. *There exists an optimal pair for every weakly connected digraph.*

Apparently, an extension of Theorem 1.3 as in Conjecture 1.5 has independently been conjectured by Aharoni [6].

The three mentioned proofs [8, Thm.] [7, Thm. 2] [4, Thm. 9.7.2] of Theorem 1.3 even show a slightly stronger result.

Theorem 1.6. [8, Thm.] *There exists a nested optimal pair for every weakly connected finite digraph.*

Hence, we also make the following conjecture.

Conjecture 1.7. *There exists a nested optimal pair for every weakly connected digraph.*

An indication why Conjecture 1.7 might be properly stronger than Conjecture 1.5 is the following. Different from finite digraphs, not every finitary dijoin that is part of an optimal for a given weakly connected infinite digraph can also feature as part of some nested optimal pair for that digraph. As an example for this consider the infinite digraph depicted twice in Figure 1.2. Its underlying graph consists of a ray R together with an additional vertex $v \notin V(R)$ which is precisely adjacent to every second vertex along R, beginning with the unique vertex on R with degree 1. Then we orient all edges incident with v towards v. Each remaining edge is oriented towards the unique neighbour of v to which it is incident with.

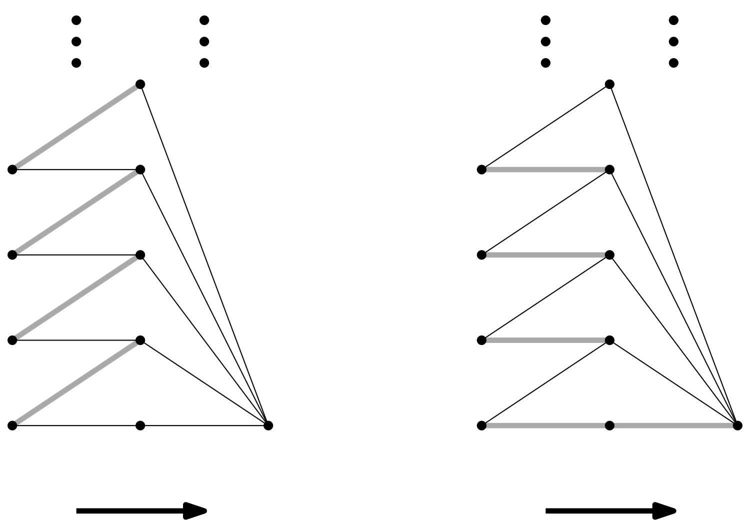

Figure 1.2. All edges are meant to be directed from left to right. The grey edges in the left picture feature in a finitary dijoin of a nested optimal pair. The grey edges in the right picture feature in a finitary dijoin of an optimal pair, but not in any finitary dijoin of a nested optimal pair.

Considering Figure 1.2 it is easy to check that the grey edges F_L in the left instance of the digraph are a finitary dijoin. Furthermore, we can easily find a nested optimal pair in which F_L features. In the right instance of the digraph the grey edges F_R also form a finitary dijoin and we can also easily find an optimal pair in which F_R features. However, no matter which finite dicut we choose on which the rightmost grey edge lies, it cannot be nested with all the finite dicuts we choose for all the other edges of F_R.

One of the main results here is that we verify Conjecture 1.7 for several classes of digraphs. We gather all these results in the following theorem. Before we can state the theorem we have to give some further definitions. We call a minimal non-empty dicut of a digraph a *dibond*. Furthermore, we call an undirected one-way infinite path a *ray*. An undirected multigraph which does not contain a ray, is called *rayless*.

Theorem 1.8. *Conjecture 1.7 holds for a weakly connected digraph D if it has any of the following properties:*

(i) There exists a finitary dijoin of D of finite size.
(ii) There is a finite maximal number of disjoint finite dicuts of D.
(iii) There is a finite maximal number of disjoint and pairwise nested finite dicuts of D.
(iv) Every edge of D lies in only finitely many finite dibonds of D.
(v) D has no infinite dibond.
(vi) The underlying multigraph of D is rayless.

The other main result of this paper is that we can reduce Conjecture 1.5 and Conjecture 1.7 to countable digraphs with a certain separability property and whose underlying multigraph is 2-connected. In order to state the theorem, we have to make a further definition. We call a digraph D *finitely diseparable* if for any two vertices $v, w \in V(D)$ there is a finite dicut of D such that v and w lie in different sides of that finite dicut.

Theorem 1.9. *(i) If Conjecture 1.5 holds for all countable finitely diseparable digraphs whose underlying multigraph is 2-connected, then Conjecture 1.5 holds for all weakly connected digraphs.*
(ii) If Conjecture 1.7 holds for all countable finitely diseparable digraphs whose underlying multigraph is 2-connected, then Conjecture 1.7, respectively, holds for all weakly connected digraphs.

The structure of this paper is as follows. In Section 2 we introduce our needed notation. Furthermore, we state and prove several lemmas we shall need to prove the main theorems. Section 3 is dedicated to the proof of Theorem 1.9. In the last part, Section 4, shall prove Theorem 1.8 via several lemmas.

§2. Preliminaries

For basic facts about finite and infinite graphs we refer the reader to [3]. Several proofs, especially in Section 4, base on certain compactness arguments using the compactness principle in combinatorics. We omit stating it here but refer to [3, Appendix A]. Especially for facts about directed graphs we refer to [2].

In general, we allow our digraphs to have parallel edges, but no loops if we do not explicitly mention them. Similarly, all undirected multigraphs we consider do not have loops if nothing else is explicitly stated.

Throughout this section let $D = (V, E)$ denote a digraph. Similarly as in undirected graphs we shall call the elements of E just *edges*. We view the edges of D as ordered pairs (u, v) of vertices $u, v \in V$ and shall write uv instead of (u, v), although this might not uniquely determine an edge. In parts where a finer distinction becomes important we shall clarify the situation. For an edge $uv \in E$ we furthermore denote the vertex u as the *tail* of uv and v as the *head* of uv. We denote the underlying multigraph of D by $\mathrm{Un}(D)$.

In an undirected non-trivial path we call the vertices incident with just one edge the *endvertices* of that path. For the trivial path consisting just of one vertex, we call that vertex also an *endvertex* of that path. If P is an undirected path with endvertices v and w, we call P a *v–w path*. For a path P containing two vertices $x, y \in V(P)$ we write xPu for the x–u subpath contained in P. Should P additionally be a directed path where v has out-degree 1, then we call P a *directed v–w path*. We also allow to call the trivial path with endvertex v a *directed v–v path*. For two vertex sets $A, B \subseteq V$ we call an undirected path $P \subseteq D$ an *A–B path* if P is an a–b path for some $a \in A$ and $b \in B$ but is disjoint from $A \cup B$ except from its endvertices. Similarly, we call an directed path that is an A–B path a *directed A–B path*.

We call an undirected graph a *star* if it is isomorphic to the complete bipartite graph $K_{1,\kappa}$ for some cardinal κ, where the vertices of degree 1 are its *leaves* and the vertex of degree κ is its *centre*.

We define a *ray* to be an undirected one-way infinite path. Any subgraph of a ray R that is itself a ray is called a *tail* of R. An undirected multigraph that does not contain a ray is called *rayless*.

A *comb* C is an undirected graph that is the union of a ray R together with infinitely many disjoint undirected finite paths each of which has precisely one vertex in common with R, which has to be an endvertex of that path. The endvertices of the finite paths that are not on R together with the endvertices of the trivial paths are the *teeth* of C.

For two vertex sets $X, Y \subseteq V$ we define $E(X,Y) \subseteq E$ as the set of those edges that have their head in $X \setminus Y$ and their tail in $Y \setminus X$, or their head in $Y \setminus X$ and their tail in $X \setminus Y$. Further we define $\vec{E}(X,Y) := \{uv \in E(X,Y)\ ;\ u \in X \text{ and } v \in Y\}$. If $X \cup Y = V$ and $X \cap Y = \varnothing$, we call $E(X,Y)$ a *cut* of D and refer to X and Y as the *sides* of the cut. Moreover, by writing $E(M,N)$ and calling it a cut of D we implicitly assume M and N to be the sided of that cut. We call two cuts $E(X_1,Y_1)$ and $E(X_2,Y_2)$ of D *nested* if either $X_1 \subseteq X_2$ and $Y_1 \supseteq Y_2$ holds or $X_2 \subseteq X_1$ and $Y_2 \supseteq Y_1$ is true. Moreover, we call a set or sequence of cuts of D nested if its elements are pairwise nested. If two cuts of D are not nested, we call them *crossing* (or say that they *cross*). A cut is said to *separate* two vertices $v, w \in V$ if v and w lie on different sides of that cut. We call a cut $E(X,Y)$ *directed*, or briefly a *dicut*, if all edges of $E(X,Y)$ have their head in one common side of the cut. We call D *finitely separable* if for any two different vertices $v, w \in V$ there exists a finite cut of D such that v and w are separated by that cut. If furthermore any two different vertices $v, w \in V$ can even be separated by a finite dicut of D, we call D *finitely diseparable*. A minimal non-empty cut is called a *bond*. Note that the induced subdigraphs $D[X]$ and $D[Y]$ are weakly connected digraphs for a bond $E(X,Y)$. A bond that is also a dicut is called a *dibond*. For a vertex set $Y \subseteq V$ we define $\delta^-(Y) = \vec{E}(V \setminus Y, Y)$. Analogously, we set $\delta^+(Y) = \vec{E}(Y, V \setminus Y)$. Given a dicut $B = \vec{E}(X,Y)$ with sides $X, Y \in V$, we call Y the *in-shore* of B and X the *out-shore* of B. We shall writ $\text{in}(B)$ for the in-shore of the dicut B and $\text{out}(B)$ for the out-shore of B.

For undirected multigraphs *cuts*, *bonds*, *sides*, the notion of being *nested* and the notion of *separating* two vertices are analogously defined. Hence, we call an undirected multigraph *finitely separable* if any two vertices can be separated by a finite cut of the multigraph. Furthermore, in an undirected multigraph G with $X, Y \subseteq V(G)$ we write $E(X,Y)$ for the set of those edges of G that have one endvertex in $X \setminus Y$ and the other in $Y \setminus X$.

Let us mention two very basic but important observations with respect to dicuts.

Remark 2.1. Let D be a digraph and let X_n be an in-shore of a dicut of D for each $n \in \mathbb{N}$ such that $\bigcap_{n\in\mathbb{N}} X_n \neq \varnothing$. Then $\bigcap_{n\in\mathbb{N}} X_n$ and $\bigcup_{n\in\mathbb{N}} X_n$ are in-shores of dicuts of D as well. □

Note that $\bigcap_{n\in\mathbb{N}} X_n$ and $\bigcup_{n\in\mathbb{N}} X_n$ might be infinite dicuts of D, even if each X_n is finite. Furthermore, note that if X_1 and X_2 are in-shores of dibonds, $X_1 \cap X_2$ does not need to be an in-shore of a dibond, even if $X_1 \cap X_2 \neq \varnothing$.

Remark 2.2. Let D be a digraph and let X_1 and X_2 be in-shores of dicuts of D such that $X_1 \cap X_2 \neq \varnothing$. Then $\delta^-(X_1) \cup \delta^-(X_2) = \delta^-(X_1 \cup X_2) \cup \delta^-(X_1 \cap X_2)$.

Moreover, if $\delta^-(X_1)$ and $\delta^-(X_2)$ are disjoint, then $\delta^-(X_1 \cup X_2)$ and $\delta^-(X_1 \cap X_2)$ are disjoint as well. □

For a set $N \subseteq E$ let D/N denote the contraction minor of D which is obtained by contracting inside D all edges of N and deleting all loops that might occur. Similar, we define $D.N := D/(E \setminus N)$. For a vertex $v \in V$ and any contraction minor D/N with $N \subseteq E$ let $\dot{v}$ denote the vertex in D/N which corresponds to the contracted (possibly trivial) weak component of $D[N]$ containing v.

We state the following basic remark without proof.

Remark 2.3. Let D be a digraph and $v, w \in V(D)$. Then the following statements hold.

(i) If B is a cut or dicut in D, then it is also a cut or dicut, respectively, in $D.N$ for every $N \supseteq B$.

(ii) If B is a cut or dicut in $D.N$ for some $N \supseteq B$, then it is also a cut or dicut, respectively, in D.

(iii) If B is a cut or dicut in $D.M$ for some $M, N \subseteq E(D)$ with $N \supseteq M \supseteq B$, then it is also a cut or dicut, respectively, in $D.N$.

(iv) If B is a cut in D and separates v and w in D, then B separates $\dot{v}$ and $\dot{w}$ in $D.N$ for every $N \supseteq B$.

(v) If B is a cut in $D.N$ and separates $\dot{v}$ and $\dot{w}$ in $D.N$ for some $N \supseteq B$, then B separates v and w in D.

(vi) If B is a cut in $D.M$ and separates $\dot{v}$ and $\dot{w}$ in $D.M$ for some $M, N \subseteq E(D)$ with $N \supseteq M \supseteq B$, then B separates $\dot{v}$ and $\dot{w}$ in $D.N$. □

For a multigraph G we call a subgraph $X \subseteq G$ a *2-block* of G if X either consists of a set of pairwise parallel edges in G or is a maximal 2-connected subgraph of G. In a digraph D we call a subdigraph X a *2-block* of D if $\mathrm{Un}(X)$ is a 2-block of $\mathrm{Un}(D)$.

We call an edge set $F \subseteq E$ a *dijoin* of D if $F \cap B \neq \emptyset$ holds for every non-empty dicut B of D. Similarly, we call an edge set $F \subseteq E$ a *finitary dijoin* of D if $F \cap B \neq \emptyset$ holds for every non-empty finite dicut B of D. Note that an edge set $F \subseteq E$ is already a (finitary) dijoin if $F \cap B \neq \emptyset$ holds for every (finite) dibond of D since every (finite) dicut is a disjoint union of (finite) dibonds. We call a pair $(F, \mathcal{B})$ consisting of a finitary dijoin F and a set of disjoint finite dicuts $\mathcal{B}$ an *optimal pair* for D if $F \subseteq \bigcup \mathcal{B}$ and $|F \cap B| = 1$ for every $B \in \mathcal{B}$. Furthermore, we call an optimal pair $(F, \mathcal{B})$ for D *nested*, if $\mathcal{B}$ is nested.

We state a basic remark about optimal pairs.

Remark 2.4. If $(F, \mathcal{B})$ is an optimal pair for a weakly connected digraph D, then each $B \in \mathcal{B}$ is a finite dibond of D.

Proof. Suppose for a contradiction that there is some $B \in \mathcal{B}$ such that B is not a finite dibond of D. Since B is the disjoint union of finite dibonds of D, we find two finite dibonds B_1 and B_2 of D such that $B_1, B_2 \subseteq B$. By the property of $(F, \mathcal{B})$ is an optimal pair for D, we know that $|F \cap B| = 1$. This, however, implies that $F \cap B_j = \varnothing$ for some $j \in \{1, 2\}$. Now we have a contradiction to F being a finitary dijoin of D. □

The following lemma is a basic tool in infinite graph theory. We shall only apply it for vertex sets of cardinality $\aleph_0$ and $\aleph_1$ in this section.

Lemma 2.5. [5, Lemma 2.5] *Let G be an infinite connected undirected multigraph and let $U \subseteq V(G)$ be such that $|U| = \kappa$ for some regular cardinal κ. Then there exists a set $U' \subseteq U$ with $|U'| = |U|$ such that G either contains a comb whose set of teeth is U' or a subdivided star whose set of leaves is U'.*

Using Lemma 2.5 let us now prove the following lemma.

Lemma 2.6. *In a finitely separable rayless multigraph all 2-blocks are finite.*

Proof. Let G be a finitely separable rayless multigraph and suppose for a contradiction that there exists a 2-block X of G such that $V(X)$ is infinite. Let $U \subseteq V(X)$ be such that $|U| = \aleph_0$. Applying Lemma 2.5 to U in X, we obtain a subdivided star S_1 in X whose set of leaves L_1 satisfies $|L_1| = |U|$ since G is rayless. Let c_1 be the centre of S_1. Using that X is 2-connected, we now apply Lemma 2.5 to L_1 in $G - c_1$, which is still a connected rayless multigraph. Hence, we obtain a subdivided star S_2 in $G - c_1$ whose set of leaves L_2 satisfies $|L_2| = |L_1| = \aleph_0$ and $L_2 \subseteq L_1$. Let c_2 denote the centre of S_2. Now we get a contradiction to G being finitely separable because S_1 and S_2 have infinitely many common leaves in L_2. So $G[V(S_1) \cup V(S_2)]$ contains infinitely many disjoint c_1–c_2 paths, witnessing that c_1 and c_2 cannot be separated by a finite cut of G.

To complete the proof we still need to consider for a contradiction a 2-block X of G whose vertex set is finite but whose edge set is infinite. Since there are only finitely many two-element subsets of $V(X)$, we find by the pigeonhole principle two vertices $x, y \in V(X)$ such that infinitely many edges of X have x and y as their endvertices. Now these infinitely many edges witness that x and y cannot be separated by a finite cut in G, contradicting again that G in finitely separable. □

We obtain the following immediate corollary.

Corollary 2.7. *A finitely separable rayless multigraph has no infinite bond.*

Proof. By considering the 2-block-cutvertex tree (cf. [3, Lemma 3.1.4]) of a given multigraph we can easily deduce that each bond of that multigraph is contained in precisely one of its 2-blocks. Hence, the statement follows from Lemma 2.6. □

The following lemma makes a similar assertion as Lemma 2.6 but without the assumption of being rayless. The proof strategy is the same as in Lemma 2.6: We apply Lemma 2.5 twice and use our assumption to ensure that we do not get a comb by the application of Lemma 2.5. We state the proof for the sake of completeness here.

Lemma 2.8. *Every* 2*-block of a finitely separable multigraph is countable.*

Proof. Let G be a finitely separable multigraph. Suppose for a contradiction that X is a 2-block of some finitely separable multigraph such that $V(X)$ is uncountable. We obtain that X is also finitely separable, and by definition that X is 2-connected. Let $U \subseteq V(X)$ be a set of cardinality $\aleph_1$. By applying Lemma 2.5 with U in X we have to find a subdivided star S_1 whose set of leaves is some $U' \subseteq U$ with $|U'| = \aleph_1$. Let c_1 denote the centre of S_1. Using the 2-connectedness of X we know that $X - c_1$ is still connected. So we can again apply Lemma 2.5, this time with U' in $X - c_1$. We obtain a subdivided star S_2 whose set of leaves is some $U'' \subseteq U'$ with $|U''| = \aleph_1$. Let c_2 be the centre of S_2. Since X is finitely separable, there exists a finite dicut B of X which separates c_1 from c_2. However, the subdivided stars S_1 and S_2, which have uncountably many common leaves in U'', witness that B cannot be finite. This is a contradiction.

It remains to consider for a contradiction a 2-block X of some finitely separable multigraph such that $V(X)$ is countable but $E(X)$ in uncountable. As before we know that X is finitely separable. Since there exist only countably many two-element subsets of $V(X)$, we have to find uncountably many edges in X that have pairwise the same endvertices, say x and y. Now we have again a contradiction to X being finitely separable since any dicut separating x and y would need to contain uncountably many edges. □

2.1. **Quotients.** For G being a digraph or a multigraph with $v, w \in V(G)$ let us write $v \equiv w$ if and only if we cannot separate v from w by a finite cut in G. It is easy to check that $\equiv$ defines an equivalence relation. For $v \in V(G)$ we shall write $[v]_\equiv$ for the equivalence class with respect to $\equiv$ containing v.

Let $G/\!\equiv$ denote the di- or multigraph which is formed from G by identifying for each equivalence class of $\equiv$ all vertices contained in it while keeping all edges that did not become loops. For any vertex $v \in V(G)$ let (v) denote the vertex of $V(G/\!\equiv)$ corresponding to $[v]_\equiv$. Furthermore, let $\hat{X} := \{(x)\ ;\ x \in X\}$ for every set $X \subseteq V(D)$.

The proofs for the statements (i)–(iv) in the following proposition work analogously to those for the statements in Proposition 2.12. Hence we only carry out the proof of Proposition 2.12. The proof of statement (v) in the following proposition works via a proof by contradiction and using a straightforward inductive construction. Therefore, we omit to state it as well.

Proposition 2.9. *Let G be a digraph or a multigraph. Then the following statements hold.*

(i) $G/\equiv$ *is (weakly) connected if* G *is (weakly) connected.*

(ii) For every finite cut $E(X,Y)$ *of* G *we get that* $E(\hat{X},\hat{Y})$ *is a finite cut of* $G/\equiv$ *with* $E(X,Y) = E(\hat{X},\hat{Y})$.

(iii) For every finite cut $E(M,N)$ *of* $G/\equiv$ *we get that* $M = \hat{X}$ *and* $N = \hat{Y}$ *for some finite cut* $E(X,Y)$ *of* G *with* $E(\hat{X},\hat{Y}) = E(X,Y)$.

(iv) $G/\equiv$ *is finitely separable.*

(v) $G/\equiv$ *is rayless if* $\mathrm{Un}(G)$ *or* G*, respectively, is rayless.* □

Let D be any digraph. We define a relation $\sim$ on $V(D)$ by saying that $v \sim w$ for $v, w \in V(D)$ if and only if there is no finite dicut separating v and w. It is easy to check that $\sim$ defines an equivalence relation and so we omit a proof for this statement. Let $[v]_\sim$ denote the equivalence class of $\sim$ containing v.

We define the digraph $D/\sim$ in the same way as we defined the quotient $D/\equiv$ but now with respect to the relation $\sim$. For any vertex $v \in V(D)$ let $[v]$ denote the vertex of $V(D/\sim)$ which corresponds to $[v]_\sim$. Further, set $\tilde{X} = \{[x] \; ; \; x \in X\}$ for every set $X \subseteq V(D)$.

Next we prove some basic lemmas about the relation $\sim$ that we shall need later. The first lemma will help us to work with the relation $\sim$ more easily. More precisely, the lemma characterises the relation $v \sim w$ for any two vertices v, w of the digraph by the existence of a certain edge set working as a witness. For any finite cut separating v and w it will be enough to consider this edge set to see that this cut is not a dicut.

Lemma 2.10. *Let* D *be a digraph and* $v, w \in V(D)$*. Then* $v \sim w$ *if and only if there is an edge set* $C \subseteq E(D)$ *such that* $|C \cap \overrightarrow{E}(X,Y)| = |C \cap \overrightarrow{E}(Y,X)|$ *holds, with* $C \cap \overrightarrow{E}(X,Y) \neq \varnothing$ *if* $E(X,Y)$ *separates* v *and* w*, for every finite cut* $E(X,Y)$ *of* D.

Moreover, $C = \varnothing$ *is satisfies the properties for* $v \sim w$ *precisely when* $v \equiv w$.

Proof. If an edge set C as in the statement of the lemma exists, then obviously $v \sim w$ holds.

For the converse we assume $v \sim w$. We prove the existence of the desired set C via a compactness argument. Let $\mathcal{B}$ be a finite set of finite cuts of D. Now we consider the finite contraction minor $D.(\bigcup \mathcal{B})$. Since $v \sim w$ and using Remark 2.3, there is no dicut in $D.(\bigcup \mathcal{B})$

separating $\dot{v}$ and $\dot{w}$. This, however, implies the existence of a directed $\dot{v}$–$\dot{w}$ path and a directed $\dot{w}$–$\dot{v}$ path in $D.(\bigcup \mathcal{B})$. So the union of these paths yields an edge set $C_{\mathcal{B}} \subseteq \bigcup \mathcal{B}$ such that for each cut $E(X_{\mathcal{B}}, Y_{\mathcal{B}})$ of $D.(\bigcup \mathcal{B})$ we have $|C_{\mathcal{B}} \cap \overrightarrow{E}(X_{\mathcal{B}}, Y_{\mathcal{B}})| = |C_{\mathcal{B}} \cap \overrightarrow{E}(Y_{\mathcal{B}}, X_{\mathcal{B}})|$ with $C_{\mathcal{B}} \cap \overrightarrow{E}(X_{\mathcal{B}}, Y_{\mathcal{B}}) \neq \varnothing$ if $E(X_{\mathcal{B}}, Y_{\mathcal{B}})$ separates $\dot{v}$ and $\dot{w}$.

Now let $\mathcal{B}' \subseteq \mathcal{B}$ and let $C_{\mathcal{B}}$ be any subset of $\bigcup \mathcal{B}$ with the properties mentioned above. Then we get that $C_{\mathcal{B}'} := C_{\mathcal{B}} \cap \bigcup \mathcal{B}'$ satisfies the properties mentioned above as well but with respect to $D.(\bigcup \mathcal{B}')$ by Remark 2.3.

By the compactness principle there exists an edge set $C \subseteq E(D)$ such that the equation $|C \cap \overrightarrow{E}(X, Y)| = |C \cap \overrightarrow{E}(Y, X)|$ holds for every finite cut $E(X, Y)$ of D as $E(X, Y)$ is also a cut of the finite contraction minor $D.(E(X, Y))$ by Remark 2.3. Similarly, $C \cap \overrightarrow{E}(X, Y) \neq \varnothing$ if $E(X, Y)$ separates v and w, because $E(X, Y)$ separates $\dot{v}$ and $\dot{w}$ in the finite contraction minor $D.(E(X, Y))$ again by Remark 2.3. Hence, C is as desired in the statement of the lemma.

For the last assertion of the lemma let us first assume $v \equiv w$. Then there is no finite dicut of D separating v and w by definition of $\equiv$. Therefore, $C = \varnothing$ satisfies all desired conditions and $v \sim w$.

For the converse we assume that $C = \varnothing$ satisfies all desired conditions and $v \sim w$. This implies that there is no finite cut of D separating v and w. Hence, we know $v \equiv w$. □

For two vertices $v, w \in V(D)$ such that $v \sim w$ let us call any edge set $C \subseteq E(D)$ with the properties as in Lemma 2.10 a *witness for* $v \sim w$. Note that there exists always an inclusion-minimal witness for $v \sim w$ by Zorn's Lemma.

The following lemmas tells us that given a minimal witness C for $v \sim w$, all vertices incident with an edge of C are also equivalent to v with respect to $\sim$.

Lemma 2.11. *Let D be a digraph and $v \sim w$ for two vertices $v, w \in V(D)$. Then a minimal edge set C of D witnessing $v \sim w$ does also witness $v \sim y$ for any $y \in V(D[C])$.*

Proof. Let C be as in the statement of the lemma. Now suppose for a contradiction that there is a $y \in V(D[C])$ which is separated from v by a finite dicut $B = E(X, Y)$ of D and $C \cap B = \varnothing$. Without loss of generality let $y \in Y$. Since C witnesses $v \sim w$, both vertices v and w have to lie on the same side of B, namely X. We claim that $C' := C \cap E(D[X])$ does also witness $v \sim w$. This would be a contradiction to the minimality of C as y is incident with an edge of C both of which endvertices lie in Y because $C \cap B = \varnothing$.

Let us first consider any finite cut $E(M, N)$ of D. Since $E(X \cap M, Y \cup N)$ is also a finite cut, but $C \cap E(X \cap M, Y) = \varnothing$, we obtain the desired equation $|C' \cap \overrightarrow{E}(M, N)| = |C' \cap \overrightarrow{E}(N, M)|$.

Especially, if $E(M, N)$ separates v and w, then $E(X \cap M, Y \cup N)$ does so as well. Hence, the same argument yields $C' \cap \overrightarrow{E}(M, N) \neq \varnothing$. □

We continue by collecting some properties of $D/{\sim}$ in the following proposition. The proof of statement (v) needs a bit more preparation. Therefore, we shall postpone it until we have proved two further lemmas.

Proposition 2.12. *Let D be a digraph. Then the following statements hold.*

(i) *$D/{\sim}$ is weakly connected if D is weakly connected.*
(ii) *For every finite dicut $E(X, Y)$ of D we get that $E(\tilde{X}, \tilde{Y})$ is a finite dicut of $D/{\sim}$ with $E(X, Y) = E(\tilde{X}, \tilde{Y})$.*
(iii) *For every finite dicut $E(M, N)$ of $D/{\sim}$ we get that $M = \tilde{X}$ and $N = \tilde{Y}$ for some finite dicut $E(X, Y)$ of D with $E(\tilde{X}, \tilde{Y}) = E(X, Y)$.*
(iv) *$D/{\sim}$ is finitely diseparable.*
(v) $\mathrm{Un}(D/{\sim})$ *is rayless if* $\mathrm{Un}(D)$ *is rayless.*

Proof of statements (i)–(iv). Statement (i) is immediate.

If $E(X, Y)$ is a finite dicut of D, then for every $x \in X$ all vertices of $[x]_\sim$ are contained in X by definition of $\sim$. Analogously, all vertices of $[y]_\sim$ lie in Y for each $y \in Y$. Hence, $E(\tilde{X}, \tilde{Y})$ is a finite dicut of $D/{\sim}$ proving statement (ii).

Next let us verify statement (iii). Let $E(M, N)$ be a finite dicut of $D/{\sim}$. Then set $X = \bigcup\{m \in V(D)\ ;\ [m] \in M\}$ and $Y = \bigcup\{n \in V(D)\ ;\ [n] \in N\}$. By definition of $\sim$ we obtain that $E(X, Y)$ is a finite dicut of D as well as $M = \tilde{X}$ and $N = \tilde{Y}$ yielding $E(X, Y) = E(\tilde{X}, \tilde{Y})$.

In order to show statement (iv), let $[v]$ and $[w]$ be two different vertices of $V(D/{\sim})$. Since v and w are not contained in the same equivalence class, there must exist a finite dicut $E(X, Y)$ of D separating them. By statement (ii) we get that $E(\tilde{X}, \tilde{Y})$ is a finite dicut of $D/{\sim}$ and it separates $[v]$ from $[w]$ by definition of $\sim$. □

Before we can complete the proof of Proposition 2.12, we have to prepare some lemmas. The first is about inclusion-minimal edge sets witnessing the equivalence of two vertices with respect to $\sim$ in digraphs whose underlying multigraph is rayless.

Lemma 2.13. *Let D be a digraph such that* $\mathrm{Un}(D)$ *is rayless and let $v \sim w$ for two vertices $v, w \in V(D)$. Then any inclusion-minimal edge set of D witnessing $v \sim w$ is finite.*

Proof. Let $C \subseteq E(D)$ be an inclusion-minimal edge set witnessing that $v \sim w$. Due to the minimality of C we know that each element of C lies on a finite cut of D separating v and w. As each cut is a disjoint union of bonds, each edge in C is contained in a finite bond of D separating v and w. Using Proposition 2.9 we get that $C \subseteq E(D/{\equiv})$ where each edge in C lies on a finite bond of $D/{\equiv}$ separating (v) and (w).

Next we consider the 2-block-cutvertex tree T of $D/{\equiv}$ (cf. [3, Lemma 3.1.4]). Let P denote the finite path in T whose endvertices are the 2-blocks of $D/{\equiv}$ containing (v) and (w), respectively. Now we use the basic fact that each bond of a di- or multigraph is also a bond of a unique 2-block of that di- or multigraph, respectively, and therefore completely contained in that 2-block. Hence, each bond of $D/{\equiv}$ separating (v) and (w) is a bond of the finitely many 2-blocks corresponding to the vertices of P. This implies that all edges in C are contained in the finitely many 2-blocks which correspond to vertices of P. However, each 2-block of $D/{\equiv}$ is finite because $\mathrm{Un}(D/{\equiv})$ is finitely separable and rayless by Proposition 2.9 and such multigraphs do not have infinite 2-blocks by Lemma 2.6. So C is contained in a finite set and thus itself finite. □

The next lemma builds up on Lemma 2.13 and is the last one we shall need to complete the proof of Proposition 2.12.

Lemma 2.14. *Let D be a digraph such that $\mathrm{Un}(D)$ is rayless and let $v \sim w$ for two vertices $v, w \in V(D)$. Then any minimal edge set of D witnessing $v \sim w$ is in $D/{\equiv}$ a strongly connected finite edge-disjoint union of directed cycles.*

Proof. Let C be a minimal edge set of D witnessing $v \sim w$. Since $\mathrm{Un}(D)$ is rayless, we know by Lemma 2.13 that C is finite. Let $v_1, v_2, \ldots, v_n$ be the endvertices of all edges in C where $n \in \mathbb{N}$. By Lemma 2.11 we know that $v_i \sim v$ holds for all i with $1 \leqslant i \leqslant n$. Next consider the set $M = \{(v_i)\ ;\ 1 \leqslant i \leqslant n\} \subseteq V(D/{\equiv})$, whose size is at most n. Because of Remark 2.3 we get that C is also an inclusion-minimal witness for $(v) \sim (w)$ and a witness for $(v) \sim (v_i)$ for every $(v_i) \in M$. We fix for each pair of vertices in M a cut of $D/{\equiv}$ that separates these two vertices, which is possible since $D/{\equiv}$ is finitely separable by Proposition 2.9. Let $\mathcal{B}$ denote the set of all these cuts. As C witnesses $(v_i) \sim (v_j)$ for all $(v_i), (v_j) \in M$, we obtain that C intersects each cut in $\mathcal{B}$. Especially, $C \subseteq \bigcup \mathcal{B}$ as each edge in C has two vertices of M as its endvertices.

Next we consider the finite contraction minor $K := (D/{\equiv}).(\bigcup \mathcal{B})$. We observe, similarly as in the proof of Lemma 2.10, that $K[C]$ is a finite edge-disjoint union of directed cycles. Furthermore, it contains a directed $(\dot{v})$–$(\dot{v_i})$ path and a directed $(\dot{v_i})$–$(\dot{v})$ path for every $(v_i) \in M$. Therefore, $K[C]$ is also strongly connected. Due to our choice of $\mathcal{B}$ we know that C is still a strongly connected finite edge-disjoint union of directed cycles in $D/{\equiv}$. □

We are now able to prove the last statement of Proposition 2.12.

Proof of statement (v) of Proposition 2.12. Suppose for a contradiction that $\mathrm{Un}(D)$ is rayless but $\mathrm{Un}(D/{\sim})$ contains a ray $R = [v_0][v_1]\ldots$ with vertices $[v_i] \in V(D/{\sim})$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ let $v_i' \in [v_i]_\sim$ and $v_{i+1}'' \in [v_{i+1}]_\sim$ be the endvertices of the edge $[v_i][v_{i+1}]$ seen in D. Furthermore, let C_i be an inclusion-minimal edge set witnessing $v_i'' \sim v_{i+1}'$ for every $i \in \mathbb{N}$ with $i \geqslant 1$. We know by Lemma 2.11 that each C_i is completely contained in $[v_i]_\sim$.

Next we consider the graph $D/\equiv$. Since $\mathrm{Un}(D)$ is rayless, we obtain that $\mathrm{Un}(D/\equiv)$ is rayless as well by statement (v) of Proposition 2.9. Therefore, we know by Lemma 2.14 that each C_i is a strongly connected finite edge-disjoint union of directed cycles in $D/\equiv$. Since each C_i is completely contained in $[v_i]_\sim$, we get that $(D/\equiv[C_i]) \cap (D/\equiv[C_j]) = \varnothing$ holds for all $i, j \in \mathbb{N}$ with $i \neq j$. Similarly, v_i' and v_{i+1}'' lie in different equivalence classes with respect to $\equiv$ for every $i \in \mathbb{N}$ because they do so as well with respect to $\sim$. Let $P_i \subseteq D/\equiv$ be a directed (v_i'')–(v_{i+1}') path that is contained in C_i for every $i \in \mathbb{N}$ with $i \geqslant 1$. We define the edge set $M := \bigcup_{i \geqslant 1} E(P_i) \cup \bigcup_{i \geqslant 0} (v_i')(v_{i+1}'') \subseteq E(D/\equiv)$. Now we derive a contradiction because the graph $D/\equiv[M]$ is a ray in $\mathrm{Un}(D/\equiv)$. □

Let us close this subsection with the following observation.

Lemma 2.15. *For every digraph D each 2-block of $D/\sim$ is countable.*

Proof. We know by Proposition 2.12 that each 2-block of $D/\sim$ is finitely diseparable. Hence, $\mathrm{Un}(X)$ is a 2-connected finitely separable multigraph. So Lemma 2.8 implies the statement of this lemma. □

§3. Reductions for the Infinite Lucchesi-Younger Conjecture

In this section we prove some reductions for Conjecture 1.5 and Conjecture 1.7 in the sense that it suffices to solve these conjectures on a smaller class of digraphs. We begin by reducing these conjectures to finitely diseparable digraphs via the following lemma.

Lemma 3.1. *Let D be a weakly connected digraph. Then the following statements are true:*

(i) If $(F, \mathcal{B})$ is a (nested) optimal pair for D, then $(F, \tilde{\mathcal{B}})$ is a (nested) optimal pair, respectively, for $D/\sim$, where $\tilde{\mathcal{B}} := \{E(\tilde{X}, \tilde{Y}) \; ; \; E(X, Y) \in \mathcal{B}\}$.

(ii) If $(F, \mathcal{B}')$ is a (nested) optimal pair for $D/\sim$, then there is a (nested) optimal pair $(F, \mathcal{B})$, respectively, for D such that $\mathcal{B}' = \tilde{\mathcal{B}} := \{E(\tilde{X}, \tilde{Y}) \; ; \; E(X, Y) \in \mathcal{B}\}$.

Proof. Note first that by Proposition 2.12 $D/\sim$ is weakly connected because D is so. We now start with the proof of statement (i). Since $\mathcal{B}$ is a set of disjoint finite dicuts of D we obtain by Proposition 2.12 that $\tilde{\mathcal{B}}$ is a set of disjoint finite dicuts of $D/\sim$. Furthermore, if $\mathcal{B}$ is nested, then so is $\tilde{\mathcal{B}}$ since the definition of $D/\sim$ ensures that we never identify two vertices of D that lie on different sides of a finite dicut of D. We also obtain $F \subseteq \bigcup \tilde{\mathcal{B}}$ and $|F \cap B'| = 1$ for every $B' \in \tilde{\mathcal{B}}$ because $(F, \mathcal{B})$ is a optimal pair for D and because of the definition of $\tilde{\mathcal{B}}$. In order to see that F is a finitary dijoin of $D/\sim$, consider any finite dicut

$E(M, N)$ of $D/{\sim}$. We know by Proposition 2.12 that $M = \tilde{X}$ and $N = \tilde{Y}$ holds for some finite dicut $E(X, Y)$ of D. Since F is a finitary dijoin of D, we know that F intersects with $E(X, Y)$. So F intersects with $E(M, N)$ as well.

Now we prove statement (ii). By Proposition 2.12 we know that for each finite dicut $E(M, N)$ of $D/{\sim}$ we have $M = \tilde{X}$ and $N = \tilde{Y}$ for some finite dicut $E(X, Y)$ of D. Hence, $\mathcal{B}' = \tilde{\mathcal{B}}$ for some set $\mathcal{B}$ of finite dicuts of D. Since the elements of $\tilde{\mathcal{B}}$ are pairwise disjoint, we know that the elements of $\mathcal{B}$ are also pairwise disjoint. Furthermore, if $\tilde{\mathcal{B}}$ is nested, then $\mathcal{B}$ is nested as well. We directly obtain that $F \subseteq \bigcup \mathcal{B}$ and $|F \cap B| = 1$ holds for every $B \in \mathcal{B}$ since $(F, \tilde{\mathcal{B}})$ is an optimal pair for $D/{\sim}$. It remains to verify that F is a finitary dijoin of D. Using Proposition 2.12 again we know that for any finite dicut $E(X, Y)$ of D the set $E(\tilde{X}, \tilde{Y})$ is a finite dicut of $D/{\sim}$. Since F intersects $E(\tilde{X}, \tilde{Y})$ as F is a finitary dijoin of $D/{\sim}$, we get that F intersects $E(X, Y)$ as well. So F is a finitary dijoin of D. □

The next reduction of Conjecture 1.5 and Conjecture 1.7 tells us that we can restrict our attention also to digraphs whose underlying multigraphs is 2-connected.

Lemma 3.2. *Let D be a weakly connected digraph. Then the following statements are true.*

(i) If $(F, \mathcal{B})$ is a (nested) optimal pair for D, then $(F \cap E(X), \mathcal{B}{\restriction}X)$ defines a (nested) optimal pair, respectively, for every 2-block X of D, where we set $\mathcal{B}{\restriction}X := \{B \in \mathcal{B}\ ;\ B \subseteq E(X)\}$.

(ii) If $(F_X, \mathcal{B}_X)$ is a (nested) optimal pair for every $X \in \mathcal{X}$ of D, then $(\bigcup_{X \in \mathcal{X}} F_X, \bigcup_{X \in \mathcal{X}} \mathcal{B}_X)$ is a (nested) optimal pair, respectively, for D, where $\mathcal{X}$ denotes the set of all 2-blocks of D.

Proof. We first prove statement (i). Let X be a 2-block of D. We assume that $(F, \mathcal{B})$ is an optimal pair for D. This implies that each element of $\mathcal{B}$ is a finite dibond of D by Remark 2.4. By considering the 2-block-cutvertex tree of D (cf. [3, Lemma 3.1.4]) we can easily deduce that for any dibond $B = E(M, N)$ of D we have either $B \cap E(X) = \emptyset$ or $B \subseteq E(X)$. In the later case B is also a dibond of X, but with sides $M \cap V(X)$ and $N \cap V(X)$. Hence, if $\mathcal{B}$ is a set of disjoint dibonds of D, we get that $\mathcal{B}{\restriction}X$ is a set of disjoint dibonds of X. Furthermore, $\mathcal{B}{\restriction}X$ is nested if $\mathcal{B}$ is nested. We also directly obtain from our observation that $F \cap E(X) \subseteq \bigcup \mathcal{B}{\restriction}X$ and $|(F \cap E(X)) \cap B| = 1$ for every $B \in \mathcal{B}{\restriction}X$ since $(F, \mathcal{B})$ is an optimal pair for D. What remains is to check that $(F \cap E(X)$ is a finitary dijoin of X. It is easy to see using the 2-block-cutvertex tree of D (cf. [3, Lemma 3.1.4]) that any dibond of X is also a dibond of D, although with adapted sides. Hence, F intersects every finite dibond of X as F is a finitary dijoin of D. So F is also a finitary dijoin of X.

Now we show that statement (ii) is true. So let us assume that $(F_X, \mathcal{B}_X)$ is a optimal pair for every $X \in \mathcal{X}$. We know by Remark 2.4 that all elements of some $\mathcal{B}_X$ are finite dibonds of the 2-block X of D. As noted in the proof of statement (i) all these dibonds are also finite dibonds of D. Hence, $\bigcup_{X\in\mathcal{X}} \mathcal{B}_X$ is a set of disjoint dibonds of D. Furthermore, if each $\mathcal{B}_X$ is nested, then it is easy to deduce that $\bigcup_{X\in\mathcal{X}} \mathcal{B}_X$ is a set of nested dibonds of D using the 2-block-cutvertex tree of D (cf. [3, Lemma 3.1.4]). Using that for each $X \in \mathcal{X}$ the pair $(F_X, \mathcal{B}_X)$ is an optimal one, we immediately get $\bigcup_{X\in\mathcal{X}} F_X \subseteq \bigcup\bigcup_{X\in\mathcal{X}} \mathcal{B}_X$ and $|B \cap \bigcup_{X\in\mathcal{X}} F_X| = 1$ for every $B \in \bigcup_{X\in\mathcal{X}} \mathcal{B}_X$. To see that $\bigcup_{X\in\mathcal{X}} F_X$ is a finitary dijoin of D let B be any finite dicut of D. Then B contains a finite dibond B' of D, which needs to intersect with some 2-block of D, say with $X \in \mathcal{X}$. As noted in the proof of statement (i), we know that B' is also a finite dibond of X. Since $(F_X, \mathcal{B}_X)$ is an optimal pair for X, we get that $\bigcup_{X\in\mathcal{X}} F_X$ intersects B' and, therefore, also B. □

We can now close this section by proving Theorem 1.9. In order to do this we basically only need to combine Lemma 3.1 and Lemma 3.2. Let us restate the theorem.

Theorem 1.9. *(i) If Conjecture 1.5 holds for all countable finitely diseparable digraphs whose underlying multigraph is* 2*-connected, then Conjecture 1.5 holds for all weakly connected digraphs.*

(ii) If Conjecture 1.7 holds for all countable finitely diseparable digraphs whose underlying multigraph is 2*-connected, then Conjecture 1.7, respectively, holds for all weakly connected digraphs.*

Proof. Let us prove statement (i) and assume that Conjecture 1.5 holds for all countable finitely diseparable digraphs whose underlying multigraph is 2-connected. Now let D be any weakly connected digraph. We know by Proposition 2.12 that $D/\!\sim$ is a weakly connected finitely diseparable digraph, and so is every 2-block of it. Furthermore, Lemma 2.15 yields that each 2-block of $D/\!\sim$ is countable. By our assumption we know that Conjecture 1.5 holds for every countable 2-block of $D/\!\sim$. So using Lemma 3.2 we obtain an optimal pair for $D/\!\sim$. Then we also obtain an optimal pair for D by Lemma 3.1.

The proof for statement (ii) works completely analogously to the one for statement (i). □

§4. Special cases

In this section we prove some special cases of Conjecture 1.7. Before we come to the first special case, we state a basic observation.

Lemma 4.1. *In a weakly connected digraph D the following are equivalent:*

(i) There is finitary dijoin of D of finite size.

(ii) There is a finite maximal number of disjoint finite dicuts of D.

(iii) There is a finite maximal number of disjoint and pairwise nested finite dicuts of D.

Proof. We start by proving the implication from (i) to (ii). Let F be a finitary dijoin of D of finite size. Then, by definition, we can find at most $|F|$ many disjoint finite dicuts of D.

The implication from (ii) to (iii) is immediate.

Finally, we assume statement (iii) and prove statement (i). Let $\mathcal{B}$ be a finite set of maximal size containing disjoint and pairwise nested finite dicuts of D. We claim that $F := \bigcup \mathcal{B}$ is a finite finitary dijoin of D.

Suppose this is not the case. Then there exists a finite dicut B_0 of D which is disjoint to F. By our choice of $\mathcal{B}$ we know that B_0 is not nested with each element of $\mathcal{B}$. Let $\mathcal{B}'_0 = \{B'_0, \dots, B'_k\}$ with $k \in \mathbb{N}$ be the set of those elements of $\mathcal{B}$ which are crossing with B_0. Further, let B'_i with $i \in \{0, \dots, k\}$ be such that either $\text{in}(B'_i)$ or $\text{out}(B'_i)$ is inclusion-minimal among all sides of the dicuts $B'_j \in \mathcal{B}'_0$. If $\text{in}(B'_i)$ is inclusion-minimal among all sides of the elements $B'_j \in \mathcal{B}'_0$, set $B''_i := \delta^-(\text{in}(B'_i) \cap \text{in}(B_0))$ and $B_1 := \delta^-(\text{in}(B'_i) \cup \text{in}(B_0))$. Otherwise, define $B''_i := \delta^-(\text{in}(B'_i) \cup \text{in}(B_0))$ as well as $B_1 := \delta^-(\text{in}(B'_i) \cap \text{in}(B_0))$. We also define $\mathcal{B}'_1 = \mathcal{B}'_0 \setminus \{B'_i\}$. By Remark 2.1 and Remark 2.2 we know that B_1 and B''_i are nested finite dicuts of D and the elements of the set $\{B_1, B''_i\} \cup \mathcal{B}'_1$ are pairwise disjoint. Furthermore, B''_i is nested with each element of $\mathcal{B}$ and B_1 is nested with each element of $\mathcal{B} \setminus \mathcal{B}'_1$.

We can repeat the argument with B_1 instead of B_0 and with $\mathcal{B}'_1$ instead of $\mathcal{B}'_0$. Iterating this procedure we obtain after $k+1$ steps the set $\mathcal{B}'' = \{B''_0, \dots, B''_k\}$ and the finite dicut B_k of D such that $(\mathcal{B} \setminus \mathcal{B}'_0) \cup \mathcal{B}'' \cup \{B_k\}$ is a nested set of disjoint finite dicuts of D. This, however, is a contradiction to the maximality of the set $\mathcal{B}$. Hence, F is a finite finitary dijoin of D. □

Now we prove a first special case for Conjecture 1.7 about digraphs that admit a finitary dijoin of finite size.

Lemma 4.2. *Let D be a weakly connected digraph with one of the following properties:*

(i) D has a finitary dijoin of finite size.

(ii) There is a finite maximal number of disjoint finite dicuts of D.

(iii) There is a finite maximal number of disjoint and pairwise nested finite dicuts of D.

Then Conjecture 1.7 holds for D.

Proof. We know by Lemma 4.1 that properties (i), (ii) and (iii) are equivalent. So let us fix a set $\mathcal{B}$ of maximum size which consists of pairwise nested and disjoint finite dicuts of D. By assumption $|\mathcal{B}|$ is finite.

Let $N \subseteq E(D)$ be a finite set of edges such that $\bigcup \mathcal{B} \subseteq N$ holds and $D.N$ is weakly connected. Since $D.N$ is a finite weakly connected digraph, there exists a nested optimal pair $(F_N, \mathcal{B}_N)$ for $D.N$ by Theorem 1.6. By the choice of N we know that each element of $\mathcal{B}$ is also a finite dicut of $D.N$. Furthermore, each finite dicut in $D.N$ is also one in D and, thus, $\mathcal{B}_N$ is a set of disjoint finite dicuts in D. Hence, $|\mathcal{B}| = |\mathcal{B}_N| = |F_N|$. Using that the elements in $\mathcal{B}$ are pairwise nested and disjoint finite dicuts, we get that $(F_N, \mathcal{B})$ is a nested optimal pair for $D.N$ as well. Given a finite edge set $M \supseteq N$ with a nested optimal pair $(F_M, \mathcal{B}_M)$ in $D.M$ we obtain that $(F_M, \mathcal{B})$ is also a nested optimal pair for $D.N$.

Note that for any finite edge set $N \subseteq E(D)$ satisfying $\bigcup \mathcal{B} \subseteq N$ there are only finitely many possible edge sets $F_N \subseteq \bigcup \mathcal{B}$ such that $(F_N, \mathcal{B})$ is a nested optimal pair for $D.N$. Hence, we get via the compactness principle an edge set $F \subseteq \bigcup \mathcal{B}$ with $|F \cap B| = 1$ for every $B \in \mathcal{B}$ such that $(F, \mathcal{B})$ is a nested optimal pair for $D.M$ for every finite edge set $M \subseteq E(D)$ satisfying $\bigcup \mathcal{B} \subseteq M$.

We claim that $(F, \mathcal{B})$ is a nested optimal pair for D. We already know by definition that $\mathcal{B}$ is a nested set of disjoint finite dicuts of D and that $F \subseteq \bigcup \mathcal{B}$ with $|F \cap B| = 1$ for every $B \in \mathcal{B}$. It remains to check that F is a finitary dijoin of D. So let B' be any finite dicut of D. Then the set $N' := B' \cup \bigcup \mathcal{B}$ is also finite and B' is a finite dicut of $D.N'$. Since $(F, \mathcal{B})$ is also a nested optimal pair for $D.N'$, we know that $F \cap B' \neq \emptyset$ holds, which proves that F is a finitary dijoin of D. □

We continue with another special case. Its proof is also based on a compactness argument. However, we need to choose the set up for the argument more carefully.

Lemma 4.3. *Conjecture 1.7 holds for weakly connected digraphs in which every edge lies in only finitely many finite dibonds.*

Proof. Let D be a weakly connected digraph where every edge lies in only finitely many finite dibonds. For an edge $e \in E(D)$ let $\mathcal{B}_e$ denote the set of finite dibonds of D that contain e. Our assumption on D implies that $\mathcal{B}_e$ is a finite set. For a finite set $\mathcal{B}$ of finite dibonds of D we define $\hat{\mathcal{B}} = \bigcup \{\mathcal{B}_e \,;\, e \in \bigcup \mathcal{B}\}$. Again our assumption on D implies that $\hat{\mathcal{B}}$ is finite. Note that $\mathcal{B} \subseteq \hat{\mathcal{B}}$ holds.

Given a finite set $\mathcal{B}$ of finite dibonds of D, we call $(F_{\mathcal{B}}, \mathcal{B}')$ a *nested pre-optimal pair* for $D.(\bigcup \mathcal{B})$ if the following hold:

(1) $F_{\mathcal{B}}$ intersects every element of $\mathcal{B}$,
(2) $\mathcal{B}' \subseteq \hat{\mathcal{B}}$,
(3) the elements of $\mathcal{B}'$ are pairwise nested,
(4) $F_{\mathcal{B}} \subseteq \bigcup \mathcal{B}'$, and
(5) $|F_{\mathcal{B}} \cap B'| = 1$ for every $B' \in \mathcal{B}'$.

We know that for every finite set $\mathcal{B}$ of finite dibonds of D there exists a nested pre-optimal pair for $D.(\bigcup\mathcal{B})$, since a nested optimal pair for $D.(\bigcup\hat{\mathcal{B}})$ is one and it exists by Theorem 1.6. However, there can only be finitely many nested pre-optimal pairs for $D.(\bigcup\mathcal{B})$ as $\bigcup\hat{\mathcal{B}}$ is finite.

Now let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two finite sets of finite dibonds of D with $\mathcal{B}_1 \subseteq \mathcal{B}_2$, and let $(F_{\mathcal{B}_2}, \mathcal{B}_2')$ be a nested pre-optimal pair for $D.(\bigcup\mathcal{B}_2)$. Then $(F_{\mathcal{B}_2} \cap \bigcup\mathcal{B}_1, \mathcal{B}_2' \cap \hat{\mathcal{B}_\infty})$ is a nested pre-optimal pair for $D.(\bigcup\mathcal{B}_1)$. Now we get by the compactness principle an edge set $F_D' \subseteq E(D)$ and a set $\mathcal{B}_D$ of finite dibonds of D such that $(F_D' \cap \bigcup\mathcal{B}, \mathcal{B}_D \cap \hat{\mathcal{B}})$ is a nested pre-optimal pair for $D.(\bigcup\mathcal{B})$ for every finite set $\mathcal{B}$ of finite dibonds of D. Further let $F_D \subseteq F_D'$ be such that each element of F_D lies on a finite dibond of D and $(F_D \cap \bigcup\mathcal{B}, \mathcal{B}_D \cap \hat{\mathcal{B}})$ is still a nested pre-optimal pair for $D.(\bigcup\mathcal{B})$ for every finite set $\mathcal{B}$ of finite dibonds of D.

We claim that $(F_D, \mathcal{B}_D)$ is a nested optimal pair for D. First we verify that F_D is a finitary dijoin of D. Let B be any finite dibond of D. Then F_D intersects B, because $(F_D \cap B, \mathcal{B}_D \cap \widehat{\{B\}})$ is a nested pre-optimal pair for $D.B$. So F_D is a finitary dijoin of D.

Next consider any element $e \in F_D$. By definition of F_D we know that $e \in B_e$ holds for some finite dibond B_e of D. Using again that $(F_D \cap B_e, \mathcal{B}_D \cap \widehat{\{B_e\}})$ is a nested pre-optimal pair for $D.B_e$, we get that $e \in \bigcup\mathcal{B}_D$. So the inclusion $F_D \subseteq \bigcup\mathcal{B}_D$ is valid.

Given any $B_D \in \mathcal{B}_D$ we know that $(F_D \cap B_D, \mathcal{B}_D \cap \widehat{\{B_D\}})$ is a nested pre-optimal pair for $D.B_D$. Hence, $|F_D \cap B| = 1$ holds for every $B \in \mathcal{B}_D \cap \widehat{\{B_D\}}$. Especially, $|F_D \cap B_D| = 1$ is true because $B_D \in \mathcal{B}_D \cap \widehat{\{B_D\}}$.

Finally, let us consider two arbitrary but different elements B_1 and B_2 of $\mathcal{B}_D$. We know that $(F_D \cap (B_1 \cup B_2), \mathcal{B}_D \cap \widehat{\{B_1, B_2\}})$ is a nested pre-optimal pair for $D.(B_1 \cup B_2)$. Therefore, B_1 and B_2 are disjoint and nested. This shows that $(F_D, \mathcal{B}_D)$ is a nested optimal pair for D and completes the proof of this lemma. □

Before we can continue proving further special cases of Conjecture 1.7, we have to state the following lemma, which is due to Thomassen and Woess. This lemma is a helpful tool in infinite graph theory. For us it will be especially useful in connection with Lemma 4.3.

Lemma 4.4. [9, Prop. 4.1] *Let G be a connected graph, $e \in E(G)$ and $k \in \mathbb{N}$. Then there are only finitely many bonds of G of size k that contain e.*

The next lemma can be used together with Lemma 4.3 to deduce that Conjecture 1.7 holds for weakly connected digraphs without infinite dibonds.

Lemma 4.5. *In a weakly connected digraph without infinite dibonds each edge lies in only finitely many finite dibonds.*

Proof. Let D be a weakly connected digraph and $e \in E(D)$ such that it lies on infinitely many finite dibonds. We shall prove that e lies on some infinite dibond of D. Since e lies on only finitely many dibonds of D with size k for every $k \in \mathbb{N}$ by Lemma 4.4, we can pick a sequence $(B''_n)_{n\in\mathbb{N}}$ of finite dibonds of D all containing e such that $|B''_n| < |B''_{n+1}|$ holds for every $n \in \mathbb{N}$. Iteratively using Remark 2.1 we can obtain a nested sequence $(B'_n)_{n\in\mathbb{N}}$ of finite dibonds of D all containing e such that $|B'_n| < |B'_{n+1}|$ holds for every $n \in \mathbb{N}$, where the inequality can again be achieved due to Lemma 4.4. Now we can find an infinite set $I \subseteq \mathbb{N}$ such that either $\text{in}(B'_i) \supseteq \text{in}(B'_j)$ holds for all $i, j \in I$ with $i \leqslant j$ or $\text{in}(B'_i) \subseteq \text{in}(B'_j)$ is true for all $i, j \in I$ with $i \leqslant j$. Since the following argument is symmetric with respect to in- or out-shores, we assume without loss of generality that the first case holds for the sequence $(B'_i)_{i\in I}$.

We inductively find an edge set $E^* = \{e_i \in E(D) \; ; \; i \in \mathbb{N}\}$ together with a subsequence $(B_n)_{n\in\mathbb{N}}$ of $(B'_i)_{i\in I}$ in the sense that there is an order preserving bijection $\sigma : I \longrightarrow \mathbb{N}$ such that $B'_i = B_{\sigma(i)}$ holds for every $i \in I$, such that the following properties are fulfilled:

(1) $e_0 = e$ holds.

(2) $E^*_n := \{e_0, \dots, e_n\} \subseteq B_n$ holds for every $n \in \mathbb{N}$.

(3) $\text{in}(B_n)$ contains an undirected tree T_n that covers all heads of edges in E^*_n and satisfies $E(T_n) \cap B_m = \emptyset$ for all $n, m \in \mathbb{N}$ with $m \geqslant n$.

We start by setting $B_0 := B'_{k_0}$ for some $k_0 \in I$ such that B_0 contains an edge $e'_1 \notin E^*_0$. This is possible since $|B'_i| < |B'_j|$ holds for all $i, j \in I$. Further, set T_0 as the head of e_0. Let v'_1 be the head of e'_1. Now let P'_1 be an undirected $\{v'_1\}$–$V(T_0)$ path in $D[\text{in}(B_0)]$. Such a path exists since B_0 is a dibond of D and so $D[\text{in}(B_0)]$ is weakly connected. We now define e_1 to be the last edge on P'_1 in the direction from v'_1 to T_0 which lies in infinitely many dibonds in $(B'_i)_{i\in I}$ if it exists, and $e_1 := e'_1$ otherwise. Note that there needs to be an edge in $E(P'_1) \cup \{e'_1\}$ which lies in infinitely many dibonds in $(B'_i)_{i\in I}$ because $\text{in}(B'_i) \subseteq \text{in}(B'_{k_0})$ holds for all $i \in I$ with $i \geqslant k_0$ and so $B'_i \cap (E(P'_1) \cup \{e'_1\}) \neq \emptyset$ holds for all $i \geqslant k_0$. Let v_1 be the head of e_1. Now we set $I_1 \subseteq I \setminus \{k_0\}$ to be an infinite index set such that $e_1 \in B'_i$ for all $i \in I_1$ and $E(P_1) \cap B'_i = \emptyset$ for all $i \in I_1$ where P_1 is the $\{v_1\}$–$V(T_0)$ path contained in P'_1. Also we set $T_1 := T_0 \cup P_1$ and $B_1 := B'_{k_1}$ for some $k_1 \in I_1$ such that B_1 contains an edge $e'_2 \notin E^*_1$. Note that $T_1 \subseteq D[B'_i]$ for each $i \in I_1$ by construction. Now we repeat the argument with k_1 instead of k_0 and with T_1 instead of T_0, etc. Iterating this construction infinitely often yields our desired sequence $(B_n)_{n\in\mathbb{N}}$ of finite dibonds of D.

Let B' be the dicut of D whose in-shore is defined via $\text{in}(B') := \bigcap_{n\in\mathbb{N}} \text{in}(B_n)$. Remark 2.1 ensures that B' is in fact a dicut of D. Note that the equality $\text{out}(B') = \bigcup_{n\in\mathbb{N}} \text{out}(B_n)$ holds and each $\text{out}(B_n)$ induces a weakly connected subdigraph of D as B_n is a dibond of D. So we know that $D[\text{out}(B')]$ is weakly connected as well. Now we set $T := \bigcup_{n\in\mathbb{N}} T_n$.

Property 3 ensures that $V(T) \subseteq \text{in}(B')$ holds. Let K be the vertex set of the weak component of $D[\text{in}(B')]$ that contains $V(T)$. Then $B := E(V(D) \setminus K, K)$ is a dicut of D whose in-shore is K. By definition $D[K]$ is a weakly connected subdigraph of D and $B \subseteq B'$ holds. Let $\mathcal{C}$ be the set of weak components of $D[\text{in}(B')]$. Since each element of $\mathcal{C}$ is adjacent with $\text{out}(B')$, we obtain that $D[\text{out}(B)]$ is also a weakly connected digraph. Hence, B is a dibond of D. Finally, property 2 together with property 3 ensure that $E^* \subseteq B$ holds. Especially, $e = e_0 \in B$ by property 1. So B is an infinite dibond of D containing e. □

As noted before, we obtain the following corollary.

Corollary 4.6. *Conjecture 1.7 holds for weakly connected digraphs without infinite dibonds.* □

We close this section with a last special case where we can show that Conjecture 1.7 holds.

Lemma 4.7. *Conjecture 1.7 holds for weakly connected digraphs whose underlying multigraph is rayless.*

Proof. Let D be a weakly connected digraph such that $\text{Un}(D)$ is rayless. We know by Proposition 2.12 that $\text{Un}(D/\!\sim)$ is rayless as well, and that $D/\sim$ is weakly connected and finitely diseparable. So we obtain from Corollary 2.7 that $D/\sim$ has no infinite dibond. Now Corollary 4.6 implies that Conjecture 1.7 is true in the digraph $D/\sim$. Using again that $D/\sim$ is finitely diseparable, any nested optimal pair for $D/\sim$ directly translates to one for D by Lemma 3.1. Hence, Conjecture 1.7 is true for D as well. □

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