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On the weak Freese-Nation property of $\mathcal{P}(\omega)$

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Abstract Continuing [6], [8] and [16], we study the consequences of the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$. Under this assumption, we prove that most of the known cardinal invariants including all of those appearing in Cichoń's diagram take the same value as in the corresponding Cohen model. Using this principle we could also strengthen two results of W. Just about cardinal sequences of superatomic Boolean algebras in a Cohen model. These results show that the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$ captures many of the features of Cohen models and hence may be considered as a principle axiomatizing a good portion of the combinatorics available in Cohen models.

Key words weak Freese-Nation property - Cohen models - Cichoń's diagram - superatomic Boolean algebras

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1 Introduction

A quasi-ordering $(P, \leq)$ is said to have *the weak Freese-Nation property* if there is a mapping $f : P \to [P]^{\leq \aleph_0}$ such that:

> For any p, $q \in P$ with $p \leq q$ there is $r \in f(p) \cap f(q)$ such that $p \leq r \leq q$.

A mapping f as above is called a *weak Freese-Nation mapping* on P.

The weak Freese-Nation property was introduced in Chapter 4 of [11] as a weakening of a notion of almost freeness of Boolean algebras. The property was further studied in [6] and [8]. Some consequences of the weak Freese-Nation property of $\mathcal{P}(\omega)$ on the algebraic behavior of $\mathcal{P}(\omega)/fin$ were studied in [16].

Every quasi-ordering P of cardinality $\leq \aleph_1$ has the weak Freese-Nation property: with a fixed injective enumeration $\langle p_\alpha : \alpha < \lambda \rangle$ of P for some $\lambda \leq \aleph_1$, the mapping $f : P \to [P]^{\aleph_0}$ defined by $f(p_\alpha) = \{p_\beta : \beta \leq \alpha\}$ witnesses the weak Freese-Nation property of P. In particular, $(\mathcal{P}(\omega), \subseteq)$ has the weak Freese-Nation property under CH. Under $\neg$CH the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$ is known to be independent: e.g. $(\mathcal{P}(\omega), \subseteq)$ does not have the weak Freese-Nation property if $\mathfrak{b} > \aleph_1$ (see [6]; see also section 2 below where the weaker condition $shr(meager) > \aleph_1$ is shown to be enough for this). On the other hand, if we start from a model of ZFC + CH and add κ Cohen reals for $\kappa < \aleph_\omega$, then $(\mathcal{P}(\omega), \subseteq)$ has the weak Freese-Nation property in the resulting model (see [6]). If we start from a model with enough covering property (e.g. L) then addition of *any* number of Cohen reals makes the extension a model of the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$ (see [8]) — quite recently it was shown that we do need some additional assumption to assure the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$ after adding arbitrary number of Cohen reals (see [9]).

In this paper, we show that the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$ captures a good deal of the features of Cohen models — i.e. models obtained by starting from a model of ZFC + CH and adding arbitrary (sometimes regular) number of Cohen reals — and hence may be considered as one of the principles axiomatizing some portion of the combinatorics available in Cohen models along with the principles studied in [12], [13] or [4]. Indeed, we prove in Section 3 that a weakening $C^s_2(\kappa)$ of one of the principles (called $C^s(\kappa)$) considered in [12], follows from the weak Freese-Nation property of $\mathcal{P}(\omega)$. The principle $C^s_2(\kappa)$ however is still strong enough to drive most of the consequences of the original principle $C^s(\kappa)$.

In section 2, we show that, under the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$, most of the known cardinal invariants including all of those which appear in Cichoń's diagram take the same value as in a corresponding Cohen model.

In section 4, we prove generalizations of two results by W. Just on non-existence of certain superatomic Boolean algebras under weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$.

While the combinatorial principles studied in the papers cited above are known to hold in various models, Cohen models and their slight variants are the only models of the weak Freese-Nation property of $\mathcal{P}(\omega)$ we know at the present.

Problem 1.1 *Is there a model of "ZFC + $\neg$CH + the weak Freese-Nation property of $\mathcal{P}(\omega)$" which is essentially different from a Cohen model?*

We believe, even in the case that the problem above may be solved negatively, the results of the present paper remain interesting on their own, since they give a uniform treatment of various assertions known to hold in Cohen models.

The weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$ is equivalent to the weak Freese-Nation property of some other quasi-orderings. For $x, y \in \mathcal{P}(\omega)$, $x \subseteq^* y \Leftrightarrow |x \setminus y| < \aleph_0$. $(\mathcal{P}(\omega)/fin, \subseteq^*)$ is the quotient structure of $\mathcal{P}(\omega)$ modulo the ideal $fin = [\omega]^{<\aleph_0}$ with the partial ordering induced from $\subseteq^*$ which will be also denoted by $\subseteq^*$. For $f, g \in {}^{\omega}\omega$, $f \leq g \Leftrightarrow \forall n \in \omega(f(n) \leq g(n))$ and $f \leq^* g \Leftrightarrow |\{n \in \omega : f(n) > g(n)\}| < \aleph_0$. The following can be easily checked:

Lemma 1.2 *The following are equivalent:*

(*a*) $(\mathcal{P}(\omega), \subseteq)$ *has the weak Freese-Nation property;*
(*b*) $(\mathcal{P}(\omega), \subseteq^*)$ *has the weak Freese-Nation property;*
(*c*) $(\mathcal{P}(\omega)/fin, \subseteq^*)$ *has the weak Freese-Nation property;*
(*d*) $({}^{\omega}\omega, \leq)$ *has the weak Freese-Nation property;*
(*e*) $({}^{\omega}\omega, \leq^*)$ *has the weak Freese-Nation property.* □

The following characterization of the weak Freese-Nation property is a fundamental tool and used throughout. For a quasi-ordering P and $Q \subseteq P$, Q is said to be a σ-subordering of P (notation: $Q \leq_\sigma P$) if, for every $p \in P$, $Q \restriction p = \{q \in Q : q \leq p\}$ has a countable cofinal subset and $Q \uparrow p = \{q \in Q : q \geq p\}$ has a countable coinitial subset.

Theorem 1.3 ([6]). *For any quasi-ordering P, the following are equivalent:*

(1) *P has the weak Freese-Nation property;*
(2) *$\{Q \in [P]^{\aleph_1} : Q \leq_\sigma P\}$ contains a club subset of $[P]^{\aleph_1}$;*

(3) *For any sufficiently large regular cardinal χ and for any $M \prec \mathcal{H}(\chi)$ with $P \in M$, $P \cap M$ is a σ-subordering of P.* □

2 Cardinal invariants

In this section we show that the weak Freese-Nation property of $\mathcal{P}(\omega)$ implies that all the cardinal invariants of reals appearing in [3] or [15] behave just as in a Cohen model.

We shall begin with a brief review of definitions and basic facts of some of these cardinal invariants.

cov(*meager*) and *non*(*meager*) denote the covering number and the uniformity number of the ideal of meager sets respectively:

$$\begin{aligned} \mathit{cov}(\mathit{meager}) &= \min\{|\mathcal{F}| : \forall X \in \mathcal{F}(X \subseteq \mathbb{R} \wedge X \text{ is meager}) \\ &\qquad\qquad \wedge \bigcup \mathcal{F} = \mathbb{R}\}, \\ \mathit{non}(\mathit{meager}) &= \min\{|X| : X \subseteq \mathbb{R} \wedge X \text{ is non-meager}\}. \end{aligned}$$

The following characterization of *cov*(*meager*) will be used:

Lemma 2.1 (for a proof see e.g. [3])

$$\begin{aligned} \mathit{cov}(\mathit{meager}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^{\omega}\omega \wedge \\ \forall f \in {}^{\omega}\omega \exists g \in \mathcal{F} \forall n \in \omega(f(n) \neq g(n))\}. \end{aligned}$$

□

In a Cohen model, $\mathit{cov}(\mathit{meager}) = 2^{\aleph_0}$ and $\mathit{non}(\mathit{meager}) = \aleph_1$, and the values of cardinal invariants in Cichoń's diagram are decided from these equations:

$$\begin{tikzcd} \mathit{cov}(\mathit{null}) \arrow[d] & \mathit{non}(\mathit{meager}) \arrow[l] \arrow[d] & \mathit{cof}(\mathit{meager}) \arrow[l] \arrow[d] & \mathit{cof}(\mathit{null}) \arrow[l] \arrow[d] \\ {} & \mathfrak{b} \arrow[d] & \mathfrak{d} \arrow[l] \arrow[d] & {} \\ \mathit{add}(\mathit{null}) & \mathit{add}(\mathit{meager}) \arrow[l] & \mathit{cov}(\mathit{meager}) \arrow[l] & \mathit{non}(\mathit{null}) \arrow[l] \\ \Leftarrow \aleph_1 & & & 2^{\aleph_0} \Rightarrow \end{tikzcd}$$

The equations $\mathit{cov}(\mathit{meager}) = 2^{\aleph_0}$ and $\mathit{non}(\mathit{meager}) = \aleph_1$ also imply that $\mathtt{s} = \mathtt{e} = \aleph_1$ and $\mathtt{r} = \mathtt{u} = \mathtt{i} = 2^{\aleph_0}$ (for the definition of these cardinals see e.g. [3]).

The following variant of *non*(I) for an ideal I over $\mathbb{R}$ was considered in the papers [5], [15], [20] and [21]:

$$\begin{aligned} \mathit{shr}(I) = \min\{\kappa : \forall X \subseteq \mathbb{R}\,(X \notin I \\ \rightarrow \exists Y \subseteq X\,(|Y| \leq \kappa \wedge Y \notin I))\} \end{aligned}$$

— here, "*shr*" stands for "shrinkability". Historically, $shr(I)$ for $I = K_\sigma$ (= the ideal of σ-compact subsets of ${}^\omega\omega$) was introduced earlier under the name $\mathtt{b}^*$

$$\mathtt{b}^* = \min\{\kappa : \forall \mathcal{F} \subseteq {}^\omega\omega\,(\mathcal{F} \text{ is unbounded} \\ \rightarrow \exists \mathcal{G} \subseteq \mathcal{F}\,(|\,\mathcal{G}\,| \leq \kappa \wedge \mathcal{G} \text{ is unbounded}))\}$$

and studied in [5] and [15]. $shr(meager)$ was studied in [15], [20], [21]. Clearly $\mathtt{b} \leq \mathtt{b}^* \leq \mathtt{d}$ and $non(meager) \leq shr(meager) \leq cof(meager)$. A proof of the equation $\mathtt{b}^* \leq shr(meager)$ can be found in [20].

Cichoń's diagram in Cohen models with these new cardinal invariants looks like this (see [15] and [20]):

$$\begin{array}{ccccccc:ccc}
cov(null) & \longleftarrow & non(meager) & \longleftarrow & shr(meager) & \longleftarrow & & cof(meager) & \longleftarrow & cof(null) \\
\Big\downarrow & & \downarrow & & \downarrow & & & \downarrow & & \Big\downarrow \\
 & & \mathtt{b} & \longleftarrow & \mathtt{b}^* & \longleftarrow & & \mathtt{d} & & \\
 & & \downarrow & & & & & \downarrow & & \\
add(null) & \longleftarrow & add(meager) & & \longleftarrow & & & cov(meager) & \longleftarrow & non(null) \\
 & & \Leftarrow & \aleph_1 & & & & 2^{\aleph_0} & \Rightarrow &
\end{array}$$

A family $G \subseteq [\omega]^{\aleph_0}$ is said to be groupwise dense if it is downward closed with respect to $\subseteq^*$ and for every strictly increasing $f \in {}^\omega\omega$, there is an infinite $X \subseteq \omega$ such that $\bigcup_{n\in X}[f(n), f(n+1)) \in G$. The groupwise density number g is defined by

$$\mathtt{g} = \min\{|\,\mathcal{G}\,| : \forall G \in \mathcal{G}(G \subseteq [\omega]^{\aleph_0} \wedge G \text{ is groupwise dense}) \wedge \bigcap \mathcal{G} = \emptyset\}$$

(see [2]). It is known that $G \subseteq [\omega]^{\aleph_0}$ is groupwise dense if, and only if, G is downward closed with respect to $\subseteq^*$ and G is non-meager, under the identification of $[\omega]^{\aleph_0}$ with the subspace $\{f \in {}^\omega 2 : |\,\{n \in \omega : f(n) = 1\}\,| = \aleph_0\}$ of ${}^\omega 2$ in the canonical way (see [3]). In [2], it is shown that $\mathtt{g} = \aleph_1$ in a Cohen model.

In the earlier versions of this paper, (1) of the following proposition claimed the weaker equation: $non(meager) = \aleph_1$. We thank M. Kada and S. Kamo for pointing out that our proof actually shows the stronger equation as below and that this simplifies calculation of some of the cardinal invariants under the weak Freese-Nation property of $\mathcal{P}(\omega)$.

Theorem 2.2 *Assume that $\mathcal{P}(\omega)$ has the weak Freese-Nation property. Then*

(1) $shr(meager) = \aleph_1$. *Hence* $non(meager) = \mathtt{b}^* = \aleph_1$ *and* $\mathtt{s} = \mathtt{e} = \aleph_1$*;*

(2) $\mathtt{a} = \aleph_1$*;*

(3) $\mathfrak{g} = \aleph_1$.

Proof (1): Suppose that $S \subseteq \mathbb{R}$ is non-meager.

Let χ be large enough and $M \prec \mathcal{H}(\chi)$ be such that $|M| = \aleph_1$, $S \in M$ and that $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ with respect to $\subseteq$. We show that $S \cap M$ is non-meager. This suffices as $|S \cap M| \leq \aleph_1$.

Let $Q = \{(q,r)^2 \in \mathbb{Q} : q < r\}$ and let $f \in M$ be a weak Freese-Nation mapping on $(\mathcal{P}(Q), \subseteq)$. For each $x \in \mathcal{P}(Q)$, let $o(x)$ denote the union of open intervals corresponding to each element of x. Thus $o(x) = \{t \in \mathbb{R} : \exists (q,r) \in x\, (q < t < r)\}$.

Let $D \in [\mathcal{P}(Q)]^{\aleph_0}$ be such that $o(d)$ is a dense subset of $\mathbb{R}$ for all $d \in D$. We show below that $\bigcap\{o(d) : d \in D\} \cap S \cap M \neq \emptyset$. Since every dense open subset of $\mathbb{R}$ can be obtained as $o(d)$ for some $d \in D$, it follows readily that $S \cap M$ is non-meager.

For each $d \in D$, let $C(d) = f(d) \cap M \cap \{x \in \mathcal{P}(Q) : d \subseteq x\}$. Then $C(d) \in [M]^{\aleph_0}$ and for all $x \in C(d)$, $o(x)$ is a dense subset of $\mathbb{R}$.

Claim 2.2.1 *$o(d) \cap M = \bigcap\{o(x) : x \in C(d)\} \cap M$.*

$\vdash$ "$\subseteq$" is clear. For "$\supseteq$", let $t \in \mathbb{R} \cap M$ be such that $t \notin o(d)$. Then $d^* = \{(q,r) \in Q : t \leq q \vee t \geq r\} \in M$ and $d \subseteq d^*$. There is $x \in C(d)$ such that $d \subseteq x \subseteq d^*$. Since $t \notin o(d^*)$ and $o(x) \subseteq o(d^*)$, we have $t \notin o(x) \cap M$. Hence $t \notin \bigcap\{o(x) : x \in C(d)\} \cap M$. $\dashv$ (Claim 2.2.1)

Let $\tilde{C}(d) \in [M]^{\aleph_0} \cap M$ be such that $C(d) \subseteq \tilde{C}(d)$ and that, for all $x \in \tilde{C}(d)$, $o(x)$ is a dense subset of $\mathbb{R}$. Let $\tilde{o}(d) = \bigcap\{o(x) : x \in \tilde{C}(d)\}$. Then $\tilde{o}(d) \in M$, $\tilde{o}(d)$ is co-meager and $\tilde{o}(d) \subseteq o(d)$. Let $\mathcal{F} \in [M]^{\aleph_0} \cap M$ be such that $\{\tilde{o}(d) : d \in D\} \subseteq \mathcal{F}$ and that each $X \in \mathcal{F}$ is a co-meager set $\subseteq \mathbb{R}$. Then, since $\bigcap \mathcal{F} \in M$ is co-meager, there is an $r \in S \cap M \cap \bigcap \mathcal{F}$. But $r \in \bigcap \mathcal{F} \cap M \subseteq \bigcap\{\tilde{o}(d) : d \in D\} \cap M \subseteq \bigcap\{o(d) : d \in D\} \cap M$. Hence $\bigcap\{o(d) : d \in D\} \cap S \cap M \neq \emptyset$.

(2): Let χ and $M \prec \mathcal{H}(\chi)$ be as in the proof of (1) and let $f \in M$ be a weak Freese-Nation mapping on $(\mathcal{P}(\omega), \subseteq^*)$. Let $F : \mathcal{P}(\omega) \to [\mathcal{P}(\omega)]^{\leq\aleph_0}$ be defined by $F(x) = f(x) \cup f(\omega \setminus x)$. Then F is again an element of M. Let $\langle S_\alpha : \alpha < \omega_1\rangle$ be an enumeration of $[[\omega]^{\aleph_0}]^{\aleph_0} \cap M$. By induction we can choose a sequence $\langle a_\alpha : \alpha < \omega_1\rangle$ of almost disjoint elements of $[\omega]^{\aleph_0} \cap M$ such that:

($*$) for any $x \in S_\alpha$, if $|x \setminus \bigcup_{\beta \in u} a_\beta| = \aleph_0$ for all $u \in [\alpha]^{<\aleph_0}$, then $|x \cap a_\alpha| = \aleph_0$.

We show that $\{a_\alpha : \alpha < \omega_1\}$ is maximal almost disjoint. Otherwise, there is a $b \in [\omega]^{\aleph_0}$ almost disjoint from all a_α, $\alpha < \omega_1$. Let $\alpha^* < \omega_1$ be such that $F(b) \cap M \subseteq S_{\alpha^*}$. Since b and a_{α^*} are almost disjoint, there is $x \in F(b) \cap F(a_{\alpha^*})$ such that (i) $b \subseteq^* x$ and (ii) $|x \cap a_{\alpha^*}| < \aleph_0$.

Since $x \in F(b) \cap M \subseteq S_{\alpha^*}$, (i) implies that x satisfies the if-clause in $(*)$ for α^*. But then (ii) contradicts the choice of a_{α^*}.

(3): Let M be as before and let $f \in M$ be a weak Freese-Nation mapping on $(\mathcal{P}(\omega), \subseteq^*)$. For $C \in [[\omega]^{\aleph_0}]^{\aleph_0} \cap M$ let

$$G_C = \{x \in [\omega]^{\aleph_0} : \exists y \in [\omega]^{\aleph_0} \cap M(\forall c \in C\ (c \not\subseteq^* y) \wedge x \subseteq y)\}.$$

G_C is groupwise dense: Clearly G_C is downward-closed with respect to $\subseteq^*$. To show that G_C is non-meager, let $G'_C = \{y \in [\omega]^{\aleph_0} \cap M : \forall c \in C\,(c \not\subseteq^* y)\}$. Then $G'_C = G_C \cap M = s[\omega]^{\aleph_0} \cap M \setminus \bigcup_{c \in C, n \in \omega}\{y \in [\omega]^{\aleph_0} : c \setminus n \subseteq y\}$. Since $[\omega]^{\aleph_0} \cap M$ is non-meager by the proof of (1) and each $\{y \in [\omega]^{\aleph_0} : c \setminus n \subseteq y\}$ is nowhere dense, it follows that G'_C is non-meager. By $G'_C \subseteq G_C$, G_C is non-meager as well.

Let $\mathcal{G} = \{G_C : C \in [[\omega]^{\aleph_0}]^{\aleph_0} \cap M\}$. Since $|\,\mathcal{G}\,| = \aleph_1$, it is enough to show $\bigcap \mathcal{G} = \emptyset$.

Let $x \in [\omega]^{\aleph_0}$ arbitrary and we show that x is not an element of $\cap\mathcal{G}$. Let $C \in [[\omega]^{\aleph_0}]^{\aleph_0} \cap M$ be such that $f(x) \cap [\omega]^{\aleph_0} \cap M \subseteq C$. Then for any $y \in [\omega]^{\aleph_0} \cap M$ with $x \subseteq^* y$, there is $c \in C$ such that $x \subseteq^* c \subseteq^* y$. It follows that such y is not in G'_C. Thus $x \notin G_C$ and hence $x \notin \cap\mathcal{G}$. □ (Theorem 2.2)

The following proposition implies that, in the most of the cases, the right half of Cichoń's diagram takes the value $2^{\aleph_0}$ under the weak Freese-Nation property of $\mathcal{P}(\omega)$:

Proposition 2.3 *Suppose that χ is a sufficiently large regular cardinal and $M \prec \mathcal{H}(\chi)$ be such that*

- (i) *$[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ with respect to $\subseteq$;*
- (ii) *for $h \in {}^{\omega}\omega$, there is $f \in {}^{\omega}\omega \cap M$ such that $h(n) \neq f(n)$ for every $n \in \omega$;*
- (iii) *$\mathbb{R} \setminus M$ is not empty.*

Then $\mathcal{P}(\omega) \cap M$ is not a σ-subordering of $\mathcal{P}(\omega)$.

Proof Towards a contradiction, assume that $\mathcal{P}(\omega) \cap M \leq_\sigma \mathcal{P}(\omega)$. Fix $x \in \mathbb{R} \setminus M$.

Claim 2.3.1 *There is a countable set $C \in M$ of infinite closed subsets of $\mathbb{R}$ such that, for any closed set $c \in M$ containing x, there is $c' \in M$ such that $x \in c' \subseteq c$.*

$\vdash$ Let Q be as in the proof of Theorem 2.2, (1). Let $s = \{(q,r) \in Q : x \leq q \vee r \leq x\}$. Then, by assumption, there is a countable $D \subseteq \mathcal{P}(Q) \cap M$ cofinal in $\{y \in \mathcal{P}(Q) \cap M : y \subseteq s\}$ with respect to $\subseteq$. By (i), let $D' \in [\mathcal{P}(Q)]^{\aleph_0} \cap M$ be such that $D \subseteq D'$. Then $C = \{\mathbb{R} \setminus o(d) : d \in D', \mathbb{R} \setminus o(d)$ is infinite$\}$ is as desired where $o(d)$

is defined as in the proof of Theorem 2.2, (1). ⊣ (Claim 2.3.1)

Let C be as in Claim 2.3.1 and let $\langle c_n : n \in \omega \rangle$ be an enumeration of C in M and let $\{u_n^m : m \in \omega, n \in \omega\} \in M$ be a family of open subsets of $\mathbb{R}$ such that for each $n \in \omega$, u_n^m, $m \in \omega$ are pairwise disjoint and, for every $m \in \omega$, $u_n^m \cap c_n \neq \emptyset$. Let $f \in {}^\omega\omega$ be such that $x \in u_n^m$ implies $f(n) = m$. This is possible as u_n^m, $m \in \omega$ are pairwise disjoint. By (ii), there is a $g \in {}^\omega\omega \cap M$ such that $f(n) \neq g(n)$ for every $n \in \omega$. Let

$$c^* = \bigcap_{n \in \omega} \mathbb{R} \setminus u_n^{g(n)}.$$

Then $x \in c^*$ and by definition of c^*, $c \not\subseteq c^*$ for all $c \in C$. This contradicts the choice of C. □ (Proposition 2.3)

Using Borel coding, we can also prove the variant of Proposition 2.3 for sufficiently absolute inner models M of models of some fragment of ZFC.

The following corollary together with Theorem 2.2 establishes that, under the conditions as in the corollary, the weak Freese-Nation property of $\mathcal{P}(\omega)$ implies that the values of cardinal invariants of the reals are just the same as their values in a Cohen model with the same value of $2^{\aleph_0}$.

Corollary 2.4 *Suppose that $\{\kappa < 2^{\aleph_0} : cf([\kappa]^{\aleph_0}, \subseteq) = \kappa\}$ is cofinal in $\{\kappa < 2^{\aleph_0} : cf(\kappa) > \omega\}$ (This is the case if e.g. $2^{\aleph_0} < \aleph_\omega$ or if $\neg 0^\#$ holds). If $\mathcal{P}(\omega)$ has the weak Freese-Nation property then $cov(meager) = 2^{\aleph_0}$ and hence also $\mathtt{r} = \mathtt{u} = \mathtt{i} = 2^{\aleph_0}$.*

Proof Suppose that $cov(meager) < 2^{\aleph_0}$. By the assumptions, we can find a cardinal κ such that $cov(meager) \leq \kappa < 2^{\aleph_0}$ and $cf([\kappa]^{\aleph_0}, \subseteq) = \kappa$. Then there is $M \prec \mathcal{H}(\chi)$ of size κ satisfying (i) of Proposition 2.3. By Lemma 2.1, (ii) of Proposition 2.3 also holds, and (iii) is clear since $|\,M\,| = \kappa < 2^{\aleph_0}$. By Proposition 2.3, it follows that $\mathcal{P}(\omega) \cap M \not\leq_\sigma \mathcal{P}(\omega)$. Hence, by Theorem 1.3, $\mathcal{P}(\omega)$ does not have the weak Freese-Nation property. □ (Corollary 2.4)

The weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$ leaves some of the cardinal invariants undecided: in a Cohen model the stick number is equal to $2^{\aleph_0}$. On the other hand, in [7], a model of the club principle was constructed where $\mathcal{P}(\omega)$ has the weak Freese-Nation property. Of course the stick number in the model is equal to $\aleph_1$.

3 Combinatorial principles

In this section we show that the weak Freese-Nation property of $\mathcal{P}(\omega)$ implies a weak form of the axiom $C^s(\kappa)$ introduced in [12] for every regular $\kappa > \aleph_1$. For a set S, we denote by $(S)^2$ the set $\{(a,b) \in S^2 : a \neq b\}$. For $T, U \subseteq S$ let $(T,U) = \{(a,b) \in T \times U : a \neq b\}$. Let $C^s_2(\kappa)$ be the following assertion:

$C^s_2(\kappa)$: *For any $T \subseteq \omega^2$ and any matrix $\langle a(\alpha,n) : \alpha < \kappa, n \in \omega\rangle$ of subsets of ω, either*

- $(c0)$ *there is a stationary $S \subseteq \kappa$ such that for each $t = (t_0,t_1) \in T$ and $s = (s_0,s_1) \in (S)^2$ we have $a(s_0,t_0) \cap a(s_1,t_1) \neq \emptyset$; or*
- $(c1)$ *there are $t = (t_0,t_1) \in T$ and stationary D_0, $D_1 \subseteq \kappa$ such that for every $s = (s_0,s_1) \in (D_0,D_1)$ we have $a(s_0,t_0) \cap a(s_1,t_1) = \emptyset$.*

Theorem 3.1 *If $\mathcal{P}(\omega)$ has the weak Freese-Nation property, then $C^s_2(\kappa)$ holds for every regular $\kappa \geq \aleph_2$.*

Proof If $\kappa > 2^{\aleph_0}$, then clearly $C^s_2(\kappa)$ holds. Hence we may assume that $\kappa \leq 2^{\aleph_0}$.

Let $f : \mathcal{P}(\omega) \to [\mathcal{P}(\omega)]^{\leq\aleph_0}$ be a weak Freese-Nation mapping on $(\mathcal{P}(\omega), \subseteq)$ and let $F : \mathcal{P}(\omega) \to [\mathcal{P}(\omega)]^{\leq\aleph_0}$ be defined by $F(x) = f(x) \cup f(\omega \setminus x)$ for $x \in \mathcal{P}(\omega)$.

Suppose that $T \subseteq \omega^2$ and $A = \langle a(\alpha,n) : \alpha < \kappa, n \in \omega\rangle$ is a matrix of subsets of ω. Assume that $(c0)$ of $C^s_2(\kappa)$ fails for these T and A. To show that $(c1)$ holds, let χ be a sufficiently large regular cardinal and $\langle N_\beta : \beta < \omega_1\rangle$, $\langle C_\beta : \beta < \omega_1\rangle$ be sequences such that:

(1) $\langle N_\beta : \beta < \omega_1\rangle$ is an increasing sequence of elementary submodels of $\mathcal{H}(\chi)$; $\kappa, F, A, T \in N_0$; $|N_\beta| < \kappa$; $N_\beta \cap \kappa \in \kappa$ and $\langle N_\gamma : \gamma < \beta\rangle \in N_\beta$ for all $\beta < \omega_1$;
(2) $\langle C_\beta : \beta < \omega_1\rangle$ is a decreasing sequence of club subsets of κ; $C_\beta \in N_\beta$ for all $\beta < \omega_1$; and $C_\beta \subseteq C$ for all club $C \in \bigcup_{\alpha<\beta} N_\alpha$.

Let $N = \bigcup_{\beta<\omega_1} N_\beta$ and $\alpha^* = \sup(\kappa \cap N)$.

Now, let $\langle S_\beta : \beta < \omega_1\rangle$ be a sequence of subsets of κ such that for all $\beta < \omega_1$:

(a) $S_\beta \in \mathcal{P}(\kappa \setminus \bigcup_{\gamma<\beta} S_\gamma) \cap N_\beta$;
(b) $S_\beta \subseteq C_\beta$;
(c) $a(s_0,t_0) \cap a(s_1,t_1) \neq \emptyset$ for every $t = (t_0,t_1) \in T$ and $s = (s_0,s_1) \in (S_\beta)^2$;
(d) S_β is so that if $\alpha \in C_\beta \setminus \bigcup_{\gamma\leq\beta} S_\gamma$ then there is $\alpha' \in S_\beta \cap \alpha$ and $(t_0,t_1) \in T$ such that $a(\alpha',t_0) \cap a(\alpha,t_1) = \emptyset$.

By assumption, S_β is a non-stationary subset of κ for every $\beta < \omega_1$ and, since α^* is contained in every club set C of κ with $C \in N$, $\alpha^* \notin S_\beta$. Hence by (d) and since $N \cap \kappa \in \kappa$ by (1), there is $\alpha_\beta \in S_\beta \cap N$

and $t_\beta = (t_0^\beta, t_1^\beta) \in T$, such that $a(\alpha_\beta, t_0^\beta) \cap a(\alpha^*, t_1^\beta) = \emptyset$. By moving to a subsequence of $\langle N_\beta : \beta < \omega_1 \rangle$ if necessary, we may assume that $t_\beta = t$ for all $\beta < \omega_1$ for some $t = (t_0, t_1) \in T$.

For each $\beta \in \omega_1$ there is an $x_\beta \in F(a(\alpha_\beta, t_0)) \cap F(a(\alpha^*, t_1))$ such that $a(\alpha_\beta, t_0) \subseteq x_\beta$ and $a(\alpha^*, t_1) \cap x_\beta = \emptyset$. Since $F(a(\alpha^*, t_1))$ is countable, we may assume that $x_\beta = x$ for all $\beta < \omega_1$ for some $x \in N \cap F(a(\alpha^*, t_1))$.

Let $D_0 = \{\alpha < \kappa : a(\alpha, t_0) \subseteq x\}$ and $D_1 = \{\alpha < \kappa : a(\alpha, t_1) \cap x = \emptyset\}$. Clearly D_0, $D_1 \in N$. The following two claims show that these D_0 and D_1 are as in $(c1)$.

Claim 3.1.1 *D_0 is a stationary subset of κ.*

$\vdash$ By elementarity, it is enough to show that this holds in N. Let $C \in N$ be a club subset of κ. Then there is $\beta < \omega_1$ such that $C_\beta \subseteq C$ by (2). Then $\alpha_\beta \in S_\beta \subseteq C_\beta \subseteq C$ and, since $\alpha_\beta \in D_0$, it follows that $C \cap D_0 \neq \emptyset$. $\dashv$ (Claim 3.1.1)

Claim 3.1.2 *D_1 is a stationary subset of κ.*

$\vdash$ Again, it is enough to show that the assertion holds in N. Let $C \in N$ be a club subset of κ then $\alpha^* \in C$. Hence, by elementarity, we have $N \models$ "$C \cap D_1 \neq \emptyset$. $\dashv$ (Claim 3.1.2)

□ (Theorem 3.1)

For a regular cardinal κ an almost disjoint family $A \subseteq [\omega]^{\aleph_0}$ is called *κ-Lusin gap* if $|A| = \kappa$ and there is no $x \in [\omega]^{\aleph_0}$ such that $|\{a \in A : |a \setminus x| < \aleph_0\}| = \kappa$ and $|\{a \in A : |a \cap x| < \aleph_0\}| = \kappa$.

Corollary 3.2 *If $(\mathcal{P}(\omega), \subseteq)$ has the weak Freese-Nation property, then there is no κ-Luzin gap for all regular $\kappa \geq \aleph_2$.*

Proof In [12], it is shown that the assertion of the corollary holds under $C^s(\kappa)$. But actually the proof there uses only $C_2^s(\kappa)$. Hence, by Theorem 3.1, the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$ implies the non-existence of κ-Lusin gaps for $\kappa \geq \aleph_2$. □ (Corollary 3.2)

The converse of Theorem 3.1 does not hold:

Proposition 3.3 *$C^s(\kappa)$ for all $\kappa \geq \aleph_2$ of cofinality $\geq \omega_2$ does not imply the weak Freese-Nation property of $(\mathcal{P}(\omega), \subseteq)$.*

Proof In [4], it is shown that $C^s(\kappa)$ for all $\kappa \geq \aleph_2$ of cofinality $\geq \omega_2$ holds in the model obtained by starting from a model of CH and adding e.g. $\aleph_2$ random reals side-by-side. In [10], it is shown that $(\mathcal{P}(\omega), \subseteq)$ does not have the weak Freese-Nation property in such a model. □ (Proposition 3.3).

4 Superatomic Boolean algebras

A superatomic Boolean algebra B is called *thin very tall* if $wd(B) = \aleph_0$ and $ht(B) \geq \aleph_2$; B is called *very thin thick* if $wd_\alpha(B) = \aleph_0$ for $\alpha < \omega_1$ and $wd_{\omega_1}(B) = \aleph_2$ ([19]). The thin very tall Boolean algebras and thin thick Boolean algebras are known to exist in some models of ZFC (see [1], [18]). W. Just showed that such superatomic Boolean algebras do not exist in a Cohen model ([14]). The following theorem generalizes this non-existence theorem. For more about superatomic Boolean algebras and for the notation used below, see [19].

Theorem 4.1 *Suppose that $\mathcal{P}(\omega)$ has the weak Freese-Nation property. Then:*

(1) *There are no superatomic Boolean algebras B with $\mathrm{wd}_0(B) = \aleph_0$, $\mathrm{wd}_\alpha(B) \leq \aleph_1$ for all $\alpha < \omega_1$ and $\mathrm{wd}_{\omega_1}(B) \geq \aleph_2$.*
(2) *There are no superatomic Boolean algebras B with $\mathrm{wd}_0(B) = \aleph_0$, and $0 < \mathrm{wd}_\alpha(B) \leq \aleph_1$ for all $\alpha < \omega_2$.*

Proof (1): Towards a contradiction, assume that there is a superatomic Boolean algebra with $\mathrm{wd}_0(B) = \aleph_0$, $\mathrm{wd}_\alpha(B) \leq \aleph_1$ for all $\alpha < \omega_1$ and $\mathrm{wd}_{\omega_1}(B) \geq \aleph_2$. Let $(X, \mathcal{O})$ be the topological dual of such a Boolean algebra. So $|X_\alpha| \leq \aleph_1$ for all $\alpha < \omega_1$ and $|X_{\omega_1}| \geq \aleph_2$. Without loss of generality we may assume that $X_0 = \omega$.

For each $x \in X$ let $u(x)$ be a clopen neighborhood of x such that, for $\alpha = \mathrm{ht}(x)$, $u(x) \setminus X_{<\alpha} = \{x\}$. Let

$$\begin{aligned}\mathcal{S} = \{u \cap \omega \,:\, & u \subseteq X \text{ is closed and covered by finitely many} \\ & u(x)\text{'s such that } \mathrm{ht}(x) < \omega_1\}\end{aligned}$$

and

$$\mathcal{L} = \{A \subseteq \omega \,:\, \sup\{\mathrm{ht}(x) \,:\, x \in \overline{A}\} = \omega_1\}.$$

Claim 4.1.1 *For $A \in \mathcal{L}$, $\overline{A} \cap X_{\omega_1} \neq \emptyset$.*

$\vdash$ Suppose that $\overline{A} \cap X_{\omega_1} = \emptyset$. Then, by compactness of $(X, \mathcal{O})$, there is $S \in [\omega_1]^{<\aleph_0}$, such that $\overline{A} \subseteq \bigcup_{\alpha \in S} X_{<\alpha}$. $\dashv$ (Claim 4.1.1)

Claim 4.1.2 *For any $A \in \mathcal{L}$ there exists at most one $y \in X_{\omega_1}$ such that $A \subseteq u(y)$.*

$\vdash$ By the previous claim $\overline{A} \cap X_{\omega_1} \neq \emptyset$. If $A \subseteq u(y)$ then $\overline{A} \subseteq u(y)$ and hence $\overline{A} \cap X_{\omega_1} = \{y\}$. $\dashv$ (Claim 4.1.2)

Now let χ be sufficiently large and let $M \prec \mathcal{H}(\chi)$ be such that $X \in M$, $|M| = \aleph_1$ and $\omega_1 \subseteq M$. Note that $X_\alpha \subseteq M$ for all $\alpha < \omega_1$. Let $F \in M$ be a weak Freese-Nation mapping on $(\mathcal{P}(\omega), \subseteq)$.

By the claim above, and since $|M| = \aleph_1$ and $|X_{\omega_2}| \geq \aleph_2$, there exists an $x^* \in X_{\omega_1}$ such that, for any $A \in \mathcal{P}(\omega) \cap M$, if $A \subseteq u(x^*) \cap \omega$, then ht "$\overline{A}$ is bounded. Let $\langle A_n : n \in \omega \rangle$ be an enumeration of $\{A \in \mathcal{P}(\omega) \cap M : A \in F(u(x^*) \cap \omega),\ A \subseteq u(x^*) \cap \omega\}$. Then $\beta_n =$ sup ht "$\overline{A_n}$ is less than ω_1 for each $n \in \omega$. Let $\beta = \sup_{n \in \omega} \beta_n + 1$ and let $y \in X_\beta \cap u(x^*)$. Then there is a $B \in [X_{<\beta}]^{<\aleph_0}$ such that $V = u(y) \setminus \bigcup_{t \in B} u(t) \subseteq u(x^*)$. Since $V \in M$ there should be an $n \in \omega$ such that $V \cap \omega \subseteq A_n$. But this is a contradiction as sup ht "$V = \beta >$ sup ht "$\overline{A_n}$.

(2): Assume that there is a superatomic Boolean algebra with $\mathrm{wd}_0(B) = \aleph_0$, and $0 < \mathrm{wd}_\alpha(B) \leq \aleph_1$ for all $\alpha < \omega_2$. Let $(X, \mathcal{O})$ be the topological dual of such a Boolean algebra. Without loss of generality we may assume that $X_0 = \omega$.

Let χ be sufficiently large and let $M \prec \mathcal{H}(\chi)$ be such that $X \in M$, $\omega_1 \subseteq M$, $\omega_2 \cap M \in \omega_2$ and $\mathit{cof}(\gamma) = \omega_1$ for $\gamma = \omega_2 \cap M$.

Let $F \in M$ be a weak Freese-Nation mapping on $(\mathcal{P}(\omega), \subseteq)$. Take $x^* \in X_\gamma$ and let $\langle A_n : n \in \omega \rangle$ be an enumeration of $\{A \in \mathcal{P}(\omega) \cap M : A \in F(u(x^*) \cap \omega),\ A \subseteq u(x^*) \cap \omega\}$. By elementarity, sup ht "$\overline{A_n} < \gamma$. Hence the same argument as in the proof of (1) leads to a contradiction. □ (Theorem 4.1)

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