CONVEX GEOMETRY, CONTINUOUS RAMSEY THEORY, AND IDEALS GENERATED BY GRAPHS OF FUNCTIONS

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Abstract. This is a survey of the results in [1, 2, 3, 4, 5, 6] on convexity numbers of closed sets in $\mathbb{R}^n$, homogeneity numbers of continuous colorings on Polish spaces and families of functions covering $\mathbb{R}^n$.

1. Convexity numbers of closed sets in $\mathbb{R}^2$

A natural way to measure the degree of non-convexity of a subset S of a real vector space is the *convexity number* $\gamma(S)$, the least size of a family $\mathcal{C}$ of convex sets such that $\bigcup\mathcal{C} = S$. We will almost exclusively consider closed subsets of $\mathbb{R}^n$. Moreover, we concentrate on sets whose convexity numbers are uncountable. Such sets will be called *uncountably convex*. Sets with a countable convexity number are *countably convex*.

A simple reason for a set $S \subseteq \mathbb{R}^n$ to be uncountably convex is the existence of an uncountable m-clique $C \subseteq S$ for some $m \in \omega$. $C \subseteq S$ is an *m-clique* if for all $F \in [C]^m$ the convex hull of F is not a subset of S. It can be shown that every closed set $S \subseteq \mathbb{R}^n$ that has an uncountable m-clique for some m in fact has a (nonempty) perfect m-clique. In particular, we have $\gamma(S) = 2^{\aleph_0}$ for such a set. Let us point out that Caratheodory's theorem implies that a subset of $\mathbb{R}^n$ has an uncountable m-clique for some $m \in \omega$ iff it has an uncountable $(n+1)$-clique. Therefore, if we are speaking about subsets of $\mathbb{R}^n$, clique will always mean $(n+1)$-clique.

It is natural to ask whether the existence of a perfect clique is the only reason for a closed set $S \subseteq \mathbb{R}^n$ to have $\gamma(S) = 2^{\aleph_0}$. Indeed, closed subsets of $\mathbb{R}$ are either countably convex or have a perfect clique.

But this already fails in dimension 2. Kubiś constructed a closed subset of $\mathbb{R}^2$ that does not have an uncountable clique but is still uncountably convex [6, Section 2, Theorem 1]. In particular, under CH the convexity number of the Kubiś set is $2^{\aleph_0}$. From the construction it is clear that the convexity number of the Kubiś set can be characterized as follows.

For $\{x, y\} \in [\omega^\omega]^2$ let

$$\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}.$$

For $\{x, y\} \in 2^\omega$ let

$$c_{\min}(x, y) = \Delta(x, y) \mod 2.$$

A set $H \subseteq [2^\omega]^2$ is *$c_{\min}$-homogeneous* iff $c_{\min}$ is constant on $[H]^2$. Now the convexity number of the Kubiś set is precisely $\mathfrak{hm}$, the least size of a family of $c_{\min}$-homogeneous subsets of 2^ω that covers 2^ω.

As it turns out, the construction of the Kubiś set is in some sense the only way to construct an uncountably convex, closed subset of $\mathbb{R}^2$ that does not have an uncountable clique.

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For a Polish space X the set $[X]^2$ of all two-element subsets of X carries a natural topology, the one generated by the basic open sets of the form

$$\{\{x, y\} : x \in U \wedge y \in V\}$$

for disjoint open subsets U and V of X. A continuous function $c : [X]^2 \to 2$ is a *continuous pair coloring* on X. A set $H \subseteq X$ is *c-homogeneous* if c is constant on $[H]^2$. The *homogeneity number* $\mathfrak{hm}(c)$ is the least size of a family of homogeneous subsets of X that covers X. Note that $\mathfrak{hm} = \mathfrak{hm}(c_{\min})$.

Kubiś proved that for every closed set $S \subseteq \mathbb{R}^2$ that does not have an uncountable clique there are sets U_n, $n \in \omega$, such that $S = \bigcup_{n\in\omega} U_n$, the U_n are Polish spaces and for all $n \in \omega$ there is a continuous pair coloring c_n on U_n such that a subset H of U_n is c_n-homogeneous iff the convex hull of H is a subset of S [6, Section 4, Theorem 15].

This implies that convexity numbers of closed subsets of $\mathbb{R}^2$ without uncountable cliques can be calculated by calculating homogeneity numbers of continuous pair-colorings on Polish spaces.

2. Continuous pair colorings on Polish spaces

It was pointed out by Kojman that for every continuous pair-coloring c on a Polish space X with $\mathfrak{hm}(c) > \aleph_0$ we have $\mathfrak{hm}(c) \geq \mathfrak{hm}$. More precisely, if c is a continuous pair coloring on a Polish space X such that $\mathfrak{hm}(c) > \aleph_0$, then there is a topological embedding $e : 2^\omega \to X$ such that for all $\{x, y\} \in [2^\omega]^2$, $c_{\min}(x, y) = c(e(x), e(y))$. In particular, for every c-homogeneous set $H \subseteq X$, $e^{-1}[H]$ is $c_{\min}$-homogeneous. Thus, a family of c-homogeneous subsets of X that covers X gives rise to a covering family of the same size of $c_{\min}$-homogeneous subsets of 2^ω.

This implies that $\mathfrak{hm}$, the convexity number of the Kubiś set, is minimal among the convexity numbers of closed, uncountably convex subsets of $\mathbb{R}^2$. But what is $\mathfrak{hm}$? It is easily checked that $\mathfrak{hm}$ is at least as big as the covering numbers of the ideals of measure zero sets and of meager sets on the real line. Another important information was my observation that $\mathfrak{hm}^+ \geq 2^{\aleph_0}$ [6, Section 3, Lemma 8]. The argument for this can be used to show that $\mathfrak{hm}$ is $2^{\aleph_0}$ in every model of set theory that was obtained by forcing with a large side-by-side product over a model of GCH [6, Section 6, Lemma 43].

On the other hand, Schipperus and I were able to show $\mathfrak{hm} = \aleph_1$ in the Sacks model ([6, Section 5, Theorem 38] is a slightly more general statement, due to myself). Later I generalized our argument to show that in the Sacks model, for every continuous pair coloring c on a Polish space X, $\mathfrak{hm}(c) < 2^{\aleph_0}$ [6, Section 5, Theorem 40].

This implies the following dichotomy for closed sets $S \subseteq \mathbb{R}^2$: either S has a perfect clique or there is a forcing extension of the universe in which $\gamma(S) < 2^{\aleph_0}$ [6, Section 5, Theorem 41]. The presentation of the non-geometrical part of [6] is mostly due to myself.

We have already observed that $\mathfrak{hm}$ is minimal among all uncountable homogeneity numbers of continuous pair colorings on Polish spaces. As it turns out, there is also a continuous pair coloring $c_{\max}$ on 2^ω whose homogeneity number is maximal among all homogeneity number of continuous pair-colorings on Polish spaces [4, Corollary 2.2]. The definition of $c_{\max}$ and the proof of the maximality of $\mathfrak{hm}(c_{\max})$ are due to myself, modulo the inequality $\mathfrak{hm} \geq \mathfrak{d}$.

The proof of the consistency of $\mathfrak{hm}(c_{\min}) < \mathfrak{hm}(c_{\max})$ [4, Corollary 4.13] was developed jointly with Goldstern and Kojman, except for a theorem about finite graphs due to Alon [4, Lemma 4.4]. The presentation of the forcing part of [4] is due to myself, the combinatorial part was written jointly by Kojman and myself.

3. Covering squares by graphs of functions

The first proof of the inequality $\mathfrak{hm} \geq \mathfrak{d}$ was developed jointly with Goldstern and used [4, Lemma 2.10], which is due to Kojman. The proof used a connection between homogeneous sets and graphs of functions from ω^ω to itself. The cardinal $\mathfrak{hm}$ is the least size of a family $\mathcal{F}$ of functions from ω^ω to ω^ω such that for all $(x, y) \in (\omega^\omega)^2$ there is a function $f \in \mathcal{F}$ such that $f(x) = y$ and the first n coordinates of the input of f determine the first n coordinates of the output or such that $f(y) = x$ and the first n coordinates of the input of f determine the first $n+1$ coordinates of the output. In terms of the usual metric on ω^ω the functions are Lipschitz of constant 1, respectively Lipschitz of constant $\frac{1}{2}$.

This characterization can be used to show that $\mathfrak{hm}$ is not smaller than the cofinality of the ideal of measure zero sets on the real line, the greatest cardinal in Cichoń's diagram [3, Theorem 4.1].

A characterization of $\mathfrak{hm}$ that looks a bit more attractive was obtained by varying the Lipschitz constants. Let us say that a function $f : X \to X$ covers a point $(x, y) \in X^2$ if $f(x) = y$ or $f(y) = x$. I was able to prove that $\mathfrak{hm}$ is precisely the least size of a family of Lipschitz functions of arbitrary constant that covers all of $(\omega^\omega)^2$, respectively $(2^\omega)^2$ [4, Theorem 3.6].

After the inequality $\mathfrak{hm} \geq \mathfrak{d}$ was established, I was able to show that the least size of a family $\mathcal{F}$ of continuous functions from 2^ω to 2^ω that covers all of $(2^\omega)^2$ is at least $\mathfrak{d}$ [4, Theorem 3.9], which easily implies $\mathfrak{hm} \geq \mathfrak{d}$. The proof of this theorem on continuous functions presented in [4] is a streamlined version of my original proof and was found jointly with Kojman.

I used this result to show that least sizes of families of continuous functions that cover $(\omega^\omega)^2$, $(2^\omega)^2$ and $\mathbb{R}^2$ are all the same [4, Theorem 3.11]. I also showed that these cardinal invariants can be strictly smaller than $\mathfrak{hm}$ [4, Theorem 5.1]. However, a minor gap in the proof of this theorem in the first submitted version of the paper was pointed out by the anonymous referee. The gap has been fixed in the current version of [4].

4. Finite open pair covers

As mentioned before, the least sizes of families of Lipschitz functions that cover $(2^\omega)^2$ and $(\omega^\omega)^2$ are the same, namely $\mathfrak{hm}$. I showed that the corresponding cardinal invariant for $\mathbb{R}^2$ is at least $\mathfrak{hm}$ [4, Remark 3.8]. However, it might be bigger than $\mathfrak{hm}$ in some models of set theory.

This cardinal invariant motivated a slight generalization of continuous pair colorings. Let X be a Polish space. $C = (U_1, \ldots, U_n)$ is a *finite open pair cover* on X iff the U_i are open and $[X]^2 = U_1 \cup \cdots \cup U_n$. $H \subseteq X$ is *C-homogeneous* if for some i, $[H]^2 \subseteq U_i$. We define the *homogeneity number* $\mathfrak{hm}(C)$ in the same way as for continuous pair colorings.

It is not difficult to show that there is a finite open pair cover C on $\mathbb{R}^2$ such that the C-homogeneous sets are graphs of Lipschitz functions or reflections of graphs of Lipschitz functions on the diagonal. In other words, the least size of a family of Lipschitz functions that covers $\mathbb{R}^2$ is not bigger than the homogeneity number of a certain open pair cover on a Polish space [3, Example 2.4].

Surprisingly, uncountable homogeneity numbers of finite open pair covers can be smaller than $\mathfrak{hm}$ (consistently) [3, Example 2.2], but they are not bigger than $\mathfrak{hm}(c_{\max})$ [3, Theorem 3.4]. This implies that the least size of a family of Lipschitz functions that covers $\mathbb{R}^2$ is not bigger than $\mathfrak{hm}(c_{\max})$.

5. Covering X^n by graphs of functions

All of the more interesting consistency results mentioned so far use models of $2^{\aleph_0} = \aleph_2$. One of the most interesting question in this field is the question whether $\mathfrak{hm}(c_{\max}) < 2^{\aleph_0}$ or even $\mathfrak{hm} < 2^{\aleph_0}$ is consistent with $2^{\aleph_0} > \aleph_2$. This seems to be wide open.

However, if we are more modest, we can get positive results in this direction. Abraham and I showed that the least size of a family of continuous functions that covers $\mathbb{R}^2$ can be smaller than $2^{\aleph_0}$, even if the continuum is big. I later generalized this result to higher dimensions:

A point $(x_0, \dots, x_n) \in X^{n+1}$ is covered by a function $f : X^n \to X$ iff there is a permutation σ of $n + 1$ such that $f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = x_{\sigma(0)}$. By an ancient theorem of Kuratowski, for every infinite cardinal κ the number of n-ary functions on κ needed to cover the $(n + 1)$-th power of the n-th successor of κ, i.e., the set $(\kappa^{+n})^{n+1}$, is exactly κ.

Now for every infinite cardinal κ with $\kappa^{+n} \geq 2^{\aleph_0}$ there is a c.c.c.-forcing extension of the universe in which $2^{\aleph_0} = \kappa^{+n}$ and $\mathbb{R}^{n+1}$ is covered by κ continuous n-ary functions [1, Theorem 4.1]. The other results in [1] are due to myself.

6. Convexity numbers in higher dimensions

So far we have only discussed convexity numbers of closed subsets of $\mathbb{R}$ and $\mathbb{R}^2$. In higher dimensions it is easier to construct uncountably convex closed sets.

In [5] for every $n > 2$ an uncountably convex, closed subset S_n of $\mathbb{R}^n$ has been constructed whose convexity number can be small (i.e., $= \aleph_1$) while the continuum is arbitrarily large. For $m \geq n > 2$ we have $\gamma(S_m) \leq \gamma(S_n)$.

For every finite set $F \subseteq \mathbb{N} \setminus 3$ the convexity numbers of the sets S_n, $n \in F$, can be made pairwise different in a forcing extension of the universe.

Since a closed subset of $\mathbb{R}^m$ can be embedded into $\mathbb{R}^n$ for every $n \geq m$, this shows that in dimension $n > 2$ we can simultaneously have $n - 1$ different uncountable convexity numbers of closed subsets of $\mathbb{R}^2$. The geometric construction is essentially due to Kojman, the forcing construction, the related combinatorics, and the presentation are due to myself.

Two things are missing here: It would be nice to have a reasonable description of uncountably convex subsets of $\mathbb{R}^n$ for $n > 2$. Recall that such a description is available in dimension 2. Moreover, it would be nice if in dimension n we could realize at least n different uncountable convexity numbers of closed sets. The problem here is that the forcing that takes care about the convexity numbers in dimensions > 2 makes $\mathfrak{hm}$, the convexity number of the Kubiś set in $\mathbb{R}^2$, equal to $2^{\aleph_0}$.

While we do know that in dimension 1 we only have one uncountable convexity number of a closed set and in dimension 2 we have at most two such numbers, nothing is known about upper bounds for the number of uncountable convexity numbers in higher dimensions. The best result concerning such questions is that for all $n \geq 2$ there is an uncountably convex closed set $S \subseteq \mathbb{R}^{n+1}$ whose convexity number is consistently strictly smaller than any convexity number of an uncountably convex, closed subset of $\mathbb{R}^n$ [2, Corollary 2.5]. In this sense, there are more possibilities for uncountable convexity numbers in $\mathbb{R}^{n+1}$ than in $\mathbb{R}^n$.

Another natural question is whether in dimension $n > 2$ the existence of a perfect clique is the only (absolute) reason for a closed set $S \subseteq \mathbb{R}^n$ to have $\gamma(S) = 2^{\aleph_0}$. In other words, do we have a forcing dichotomy (as in dimension 2) saying that S either has a perfect clique or its convexity number can be forced to be smaller than $2^{\aleph_0}$? The answer is no. There is a closed set $S \subseteq \mathbb{R}^3$ without uncountable cliques whose convexity number is $2^{\aleph_0}$ in every forcing extension of the universe [2, Corollary 3.5].

References

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