

THE NUMBER OF OPENLY GENERATED BOOLEAN ALGEBRAS

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ABSTRACT. This article is devoted to two different generalizations of projective Boolean algebras: openly generated Boolean algebras and tightly σ -filtered Boolean algebras.

We show that for every uncountable regular cardinal κ there are 2^κ pairwise non-isomorphic openly generated Boolean algebras of size $\kappa > \aleph_1$ provided there is an almost free non-free abelian group of size κ . The openly generated Boolean algebras constructed here are almost free.

Moreover, for every infinite regular cardinal κ we construct 2^κ pairwise non-isomorphic Boolean algebras of size κ that are tightly σ -filtered and c.c.c.

These two results contrast nicely with Koppelberg's theorem in [13] that for every uncountable regular cardinal κ there are only $2^{<\kappa}$ isomorphism types of projective Boolean algebras of size κ .

1. INTRODUCTION

Projectivity is usually defined as a universal property. A Boolean algebra A is projective if and only if for every Boolean algebra B and every epimorphism $f : B \rightarrow A$ there is a homomorphism $g : A \rightarrow B$ such that $f \circ g = \text{id}_A$. However, theorems of Haydon, Koppelberg, and Ščepin provide an internal characterization of projectivity for Boolean algebras (see [13, Section 2]).

Using her characterization of projectivity, Koppelberg [13, Section 4] showed that for every uncountable regular cardinal κ there are only $2^{<\kappa}$ isomorphism types of projective Boolean algebras of size κ . She also showed that for every singular cardinal μ there are 2^μ pairwise non-isomorphic projective Boolean algebras of size μ .

For Boolean algebras, there are two natural generalizations of projectivity: open generatedness and tight σ -filteredness.

Openly generated Boolean algebras are studied to a great extent in [11], under the name *rc-filtered Boolean algebras*, respectively *Boolean algebras with the Freese-Nation property*. We only note that openly generated Boolean algebras seem to be quite close to projective Boolean algebras. Every openly generated Boolean algebra of size $\leq \aleph_1$ is projective and it is a non-trivial task to find an openly generated Boolean algebra which is not projective. Examples of openly generated, non-projective Boolean algebras were provided by Ščepin (see [11, Chapter 3, Section 4]). A generalization of Ščepin's argument shows that there are openly generated Boolean algebras that are not even tightly σ -filtered [10, Section 4].

Tight σ -filteredness was introduced in [12] and studied systematically in [10]. It turns out that several properties of projective Boolean algebras can be generalized to tightly σ -filtered Boolean algebras.

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However, in the present article we point out a fundamental difference between projective Boolean algebras and tightly σ -filtered Boolean algebras. Namely, we show that for each infinite regular cardinal κ there are 2^κ pairwise non-isomorphic tightly σ -filtered Boolean algebras of size κ , contrasting Koppelberg's result on the number of projective Boolean algebras. The construction is fairly elementary and even yields Boolean algebras which are c.c.c. and not openly generated. This result is contained in the second author's PhD thesis [9].

With openly generated Boolean algebras the situation is more complicated. Fuchino showed that if κ is a regular cardinal that has a non-reflecting stationary subset consisting of ordinals of cofinality \aleph_1 , then there are 2^κ pairwise non-isomorphic openly generated Boolean algebras of size κ (see [11, Chapter 6, Section 4]). Fuchino's assumption, i.e., the existence of a non-reflecting stationary set of ordinals of cofinality \aleph_1 , implies the existence of an almost free, non-free abelian group of size κ [3, Chapter VI, Lemma 2.2 and Theorem 2.3]. Here an abelian group is almost free if every subgroup of strictly smaller cardinality is free.

In the present article we show that for an infinite regular cardinal κ there are 2^κ pairwise non-isomorphic openly generated Boolean algebras if there exists an almost free, non-free abelian group of size κ .

While every almost free abelian group of singular cardinality is in fact free by Shelah's singular compactness theorem (see [3, Theorem IV.3.5]), there are almost free, non-free abelian groups of various regular cardinalities. Let \mathcal{C} denote the class of cardinals κ for which there is an almost free, non-free abelian group of size κ . It is relatively easy to show that $\aleph_1 \in \mathcal{C}$ [3, Chapter IV, Section 0].

Magidor and Shelah [14, Section 1, Corollary 2] showed that every regular cardinal below \aleph_{ω^2} belongs to \mathcal{C} . More generally, \mathcal{C} is closed under the operation $(\lambda, \kappa) \mapsto \lambda^{+(\kappa+1)}$.

In [2, Section 10] Cummings, Foreman and Magidor show that, assuming the consistency of infinitely many supercompact cardinals, it is consistent that no stationary subset of $\aleph_{\omega+1}$ reflects. In particular, Fuchino's assumption, the existence of a non-reflecting stationary subset consisting of ordinals of cofinality \aleph_1 , can fail at $\aleph_{\omega+1}$, while our assumption, the existence of an almost free non-free abelian group, is always satisfied at $\aleph_{\omega+1}$.

Magidor and Shelah also proved, assuming the consistency of infinitely many supercompact cardinals, that it is consistent that \mathcal{C} contains no cardinal above the first cardinal fixed point, i.e., the first κ with $\kappa = \aleph_\kappa$. However, if $V = L$, then every regular cardinal κ which is not weakly compact belongs to \mathcal{C} . A proof of the latter fact and much more on this topic can be found in the Chapters IV and VII of the book [3] by Eklof and Mekler. Note that there is a new, revised edition of this book [4].

2. BASIC DEFINITIONS AND PRELIMINARY LEMMAS

Open generatedness and tight σ -filteredness are both defined in terms of nicely embedded subalgebras. Openly generated Boolean algebras have many relatively complete subalgebras and tightly σ -filtered Boolean algebras have many σ -subalgebras.

Definition 2.1. Let A and B be Boolean algebras such that $A \leq B$. Then for $b \in B$ the ideal $\{a \in A : a \leq b\}$ of A is denoted as $A \upharpoonright b$. A is called a *relatively complete subalgebra* (*rc-subalgebra*) of B if for each $b \in B$ the ideal $A \upharpoonright b$ is principal. In this case we write $A \leq_{\text{rc}} B$. If $A \leq_{\text{rc}} B$, then lpr_A^B denotes the mapping from B to A assigning to each $b \in B$ the generator of $A \upharpoonright b$, the lower projection of b in A .

A is called a σ -subalgebra of B if for every $x \in B$ the ideal $A \upharpoonright x$ is countably generated. In this case we write $A \leq_\sigma B$. If C is a Boolean algebra and $e : C \rightarrow B$

an embedding, then e is an *rc-embedding* if $e[C] \leq_{\text{rc}} B$ and an *σ -embedding* if $e[C] \leq_{\sigma} B$.

Note that $A \leq_{\sigma} B$ if and only if $A \leq B$ and for every ideal I of B which is countably generated, $I \cap A$ is countably generated as well.

Definition 2.2. Let A be a Boolean algebra and δ an ordinal. A sequence $(A_{\alpha})_{\alpha < \delta}$ of subalgebras of A is a *filtration* of A if

- a) $\bigcup_{\alpha < \delta} A_{\alpha} = A$,
- b) $A_{\alpha} \leq A_{\beta}$ for $\alpha < \beta < \delta$, and
- c) the sequence $(A_{\alpha})_{\alpha < \delta}$ is continuous, i.e., for every limit ordinal $\beta < \delta$, $A_{\beta} = \bigcup_{\alpha < \beta} A_{\alpha}$.

A filtration $(A_{\alpha})_{\alpha < \delta}$ of A is *tight* if there is a sequence $(x_{\alpha})_{\alpha < \delta}$ in A such that for all $\beta < \delta$, $A_{\beta} = \langle \{x_{\alpha} : \alpha < \beta\} \rangle$. Here for a set $X \subseteq A$, $\langle X \rangle$ denotes the subalgebra of A generated by X . If $B \leq A$ and $x \in A$, we write $B(x)$ for $\langle B \cup \{x\} \rangle$.

A is *tightly σ -filtered* if it has a tight filtration $(A_{\alpha})_{\alpha < \delta}$ such that for all $\alpha < \delta$, $A_{\alpha} \leq_{\sigma} A$. $(A_{\alpha})_{\alpha < \delta}$ is called a *tight σ -filtration* of A .

A is *openly generated* if the set of rc-subalgebras of A includes a club of $[A]^{\leq \aleph_0}$.

Koppelberg's characterization of projective Boolean algebras is obtained by replacing \leq_{σ} by \leq_{rc} in the definition of tight σ -filteredness. The term "openly generated" comes from the fact that rc-embeddings correspond to open mappings via Stone duality.

Why are projective algebras openly generated? For a set X let $\text{Fr}(X)$ denote the free Boolean algebra over the set X . If $X \subseteq Y$, we regard $\text{Fr}(X)$ as a subalgebra of $\text{Fr}(Y)$ in the obvious way. It is easy to see that $\text{Fr}(X) \leq_{\text{rc}} \text{Fr}(Y)$ for $X \subseteq Y$. It follows that free Boolean algebras are openly generated.

By abstract nonsense, a Boolean algebra A is projective if and only if it is a retract of a free Boolean algebra, i.e., if there are a free Boolean algebra B and homomorphisms $e : A \rightarrow B$ and $p : B \rightarrow A$ such that $p \circ e = \text{id}_A$. It is more or less straight forward to see that open generatedness is hereditary with respect to retracts. It follows that projective Boolean algebras are openly generated.

We collect some facts on rc-embeddings, σ -embeddings, and open generatedness.

Lemma 2.3. Let $A \leq B$ and $x \in B$. Then the following hold:

- a) $A \leq_{\text{rc}} A(x)$ if and only if $A \upharpoonright x$ and $A \upharpoonright -x$ are principal.
- b) $A \leq_{\sigma} A(x)$ if and only if $A \upharpoonright x$ and $A \upharpoonright -x$ are countably generated.

Proof. We show b) only. The direction from the left to the right is trivial. For the other direction let $E \subseteq A$ and $F \subseteq A$ be countable sets which generate $A \upharpoonright x$ and $A \upharpoonright -x$ respectively. Without loss of generality we may assume that E and F are closed under finite joins. Suppose $y \in A(x)$. Then there are $v, w \in A$ such that $y = (v+x) \cdot (w+(-x))$. Let $z \in A$ such that $z \leq y$. Then $z-v \leq x$ and $z-w \leq -x$. Hence $z-v \leq a$ and $z-w \leq b$ for some $a \in E$ and some $b \in F$. It follows that $z \leq (v+a) \cdot (w+b)$. Clearly, $(v+a) \cdot (w+b) \leq y$ for every $a \in E$ and every $b \in F$. Hence $A \upharpoonright y$ is generated by $\{(v+a) \cdot (w+b) : a \in E \wedge b \in F\}$. \square

Lemma 2.4. a) $A \leq_{\text{rc}} B \leq_{\text{rc}} C \Rightarrow A \leq_{\text{rc}} C$,

$$A \leq_{\sigma} B \leq_{\sigma} C \Rightarrow A \leq_{\sigma} C.$$

- b) $A \leq B$, $B = \bigcup_{\alpha < \lambda} B_{\alpha}$, $A \leq_{\text{rc}} B_{\alpha}$ for every $\alpha < \lambda \Rightarrow A \leq_{\text{rc}} B$,

$$A \leq B$$
, $B = \bigcup_{\alpha < \lambda} B_{\alpha}$, $A \leq_{\sigma} B_{\alpha}$ for every $\alpha < \lambda \Rightarrow A \leq_{\sigma} B$.

- c) $A \leq_{\text{rc}} B$, $C \leq B$ and $\text{lpr}_A^B[C] \subseteq C \Rightarrow A \cap C \leq_{\text{rc}} C$.

Proof. Easy. \square

Lemma 2.5. [11, Proposition 2.2.4] If δ is an ordinal and A is the union of an increasing continuous chain $(A_{\alpha})_{\alpha < \delta}$ of rc-subalgebras that are openly generated, then A itself is openly generated.

3. THE NUMBER OF TIGHTLY σ -FILTERED BOOLEAN ALGEBRAS

We show that Koppelberg's result on the number of projective Boolean algebras cannot be extended to tightly σ -filtered Boolean algebras.

Theorem 3.1. *For every infinite cardinal κ there are 2^κ pairwise non-isomorphic tightly σ -filtered c.c.c. Boolean algebras of size κ .* \square

The proof of the theorem uses the following lemma, which says that stationary sets consisting of ordinals of countable cofinality can be coded by tightly σ -filtered Boolean algebras.

Lemma 3.2. Let κ be an uncountable regular cardinal and let S be a subset of κ consisting of ordinals of cofinality \aleph_0 . Then there are a Boolean algebra A of size κ and a tight σ -filtration $(A_\alpha)_{\alpha < \kappa}$ of A such that the following hold:

- a) $A_\alpha \not\leq_{\text{rc}} A$ for all $\alpha \in S$
- b) $A_\alpha \leq_{\text{rc}} A$ for all $\alpha \in \kappa \setminus S$.

Proof. For every $\alpha \in S$ let $(\delta_n^\alpha)_{n \in \omega}$ be a strictly increasing sequence of ordinals with least upper bound α and $S \cap \{\delta_n^\alpha : n \in \omega\} = \emptyset$. We will construct $(A_\alpha)_{\alpha < \kappa}$ together with a sequence $(x_\alpha)_{\alpha < \kappa}$ such that

- (i) $A_0 = 2$,
- (ii) $A_{\alpha+1} = A_\alpha(x_\alpha)$ for all $\alpha < \kappa$,
- (iii) x_α is independent over A_α whenever $\alpha \notin S$,
- (iv) $A_\alpha \upharpoonright x_\alpha$ is generated by $\{x_{\delta_n^\alpha} : n \in \omega\}$ and $A_\alpha \upharpoonright -x_\alpha = \{0\}$ whenever $\alpha \in S$,
- (v) $A_\beta = \bigcup_{\alpha < \beta} A_\alpha$ holds for all limit ordinals $\beta < \kappa$.

Clearly, the construction can be done and is uniquely determined. We have to show that a) and b) of the lemma hold for $(A_\alpha)_{\alpha < \kappa}$.

For a) let $\alpha \in S$. Then $A_\alpha \upharpoonright x_\alpha$ is non-principal. For suppose $a \in A_\alpha$ is such that $a \leq x_\alpha$. Since $(\delta_n^\alpha)_{n \in \omega}$ is cofinal in α , there is $n \in \omega$ such that $a \in A_{\delta_n^\alpha}$. Since $\delta_n^\alpha \notin S$, $x_{\delta_n^\alpha}$ is independent over $A_{\delta_n^\alpha}$ by construction. Hence $a + x_{\delta_n^\alpha}$ is strictly larger than a , but still smaller than x_α . So a does not generate $A_\alpha \upharpoonright x_\alpha$.

For b) let $\alpha \notin S$. By induction on $\gamma < \kappa$, we show that $A_\alpha \leq_{\text{rc}} A_\gamma$ holds for every $\gamma \geq \alpha$. $A_\alpha \leq_{\text{rc}} A_\alpha$ holds trivially. Suppose γ is a limit ordinal and $A_\alpha \geq_{\text{rc}} A_\beta$ holds for all $\beta < \gamma$ such that $\alpha \leq \beta$. Then $A_\alpha \leq_{\text{rc}} A_\gamma$ follows from Lemma 2.4. Now suppose $\gamma = \beta + 1$ for some $\beta \geq \alpha$. There are two cases:

- I. $\beta \notin S$. In this case $A_\beta \leq_{\text{rc}} A_\gamma$ by construction. By hypothesis, $A_\alpha \leq_{\text{rc}} A_\beta$. By Lemma 2.4, this implies $A_\alpha \leq_{\text{rc}} A_\gamma$.
- II. $\beta \in S$. This is the non-trivial case. We **claim** that $A_\delta \leq_{\text{rc}} A_\delta(x_\beta)$ holds for every $\delta < \beta$. This can be seen as follows: By Lemma 2.3, it is sufficient to show that both $A_\delta \upharpoonright x_\beta$ and $A_\delta \upharpoonright -x_\beta$ are principal. But $A_\delta \upharpoonright -x_\beta \subseteq A_\beta \upharpoonright -x_\beta = \{0\}$ by construction. Let $a \in A_\delta$ be such that $a \leq x_\beta$. Let $m := \{n \in \omega : x_{\delta_n^\beta} \in A_\delta\}$. Clearly $m \in \omega$. Let $T \in [\omega]^{< \aleph_0}$ be such that $a \leq \sum \{x_{\delta_n^\beta} : n \in T\}$. Then

$$a \leq \sum \{x_{\delta_n^\beta} : n \in T \cap m\} + \sum \{x_{\delta_n^\beta} : n \in T \setminus m\}.$$

Since $\sum \{x_{\delta_n^\beta} : n \in T \setminus m\}$ is independent over A_δ by construction,

$$a \leq \sum \{x_{\delta_n^\beta} : n \in T \cap m\} \leq \sum \{x_{\delta_n^\beta} : n < m\} \leq x_\beta.$$

This shows that $A_\delta \upharpoonright x_\beta$ is generated by $\sum \{x_{\delta_n^\beta} : n < m\}$ and the claim follows.

Now $A_\gamma = A_\beta(x_\beta) = \bigcup_{\alpha \leq \delta < \beta} A_\delta(x_\beta)$. Hence, $A_\alpha \leq_{\text{rc}} A_\gamma$ follows from the claim together with Lemma 2.4.

This shows b). \square

In order to show that the Boolean algebra A constructed in the lemma above is c.c.c., we use an argument which was used in an early version of [7] to prove that, assuming the consistency of the existence of a supercompact cardinal, it is consistent with ZFC+GCH that there is a complete c.c.c. Boolean algebra without the so-called WFN, a property whose definition is obtained by replacing \leq_{rc} by \leq_σ and $[A]^{\leq \aleph_0}$ by $[A]^{\leq \aleph_1}$ in the definition of open generatedness.

Lemma 3.3. The Boolean algebra A constructed in the proof of Lemma 3.2 is c.c.c.

Proof. Assume A is not c.c.c. Let $C \subseteq A$ be an uncountable antichain. Let $X := \{x_\alpha : \alpha < \kappa\}$. For $x \in X$ let $x^0 := x$ and $x^1 := -x$. We may assume that each $a \in C$ is an elementary product of elements of X , i.e., there is $X_a \in [X]^{< \aleph_0}$ and $f_a : X_a \rightarrow 2$ such that $a = \prod_{x \in X_a} x^{f_a(x)}$. After thinning out C if necessary, we may assume that $\{X_a : a \in C\}$ is a Δ -system with root R , there is $f : R \rightarrow 2$ such that $f_a \upharpoonright R = f$ for all $a \in C$, and all X_a are of the same size, say n .

Claim. Let $Y \in [X]^{< \omega}$ and $g : Y \rightarrow 2$ be such that $\prod_{x \in Y} x^{g(x)} = 0$. Then there are $\alpha \in S$ and $i \in \omega$ with $x_\alpha, x_{\delta_i^\alpha} \in Y$ such that $g(x_\alpha) = 1$ and $g(x_{\delta_i^\alpha}) = 0$.

First note that for $y, z \in X$, $y^{g(y)} \cdot z^{g(z)} = 0$ holds if and only if there are $\alpha \in S$ and $i \in \omega$ with $\{y, z\} = \{x_\alpha, x_{\delta_i^\alpha}\}$ such that $g(x_\alpha) = 1$ and $g(x_{\delta_i^\alpha}) = 0$. Now we show the claim by induction on $\max\{\alpha < \lambda : x_\alpha \in Y\}$. The case $|Y| < 3$ is trivial.

Assume the claim has been proved for $\max\{\alpha < \lambda : x_\alpha \in Y\} < \beta$. Suppose $\max\{\alpha < \lambda : x_\alpha \in Y\} = \beta$ and for no two elements $y, z \in Y$, $y^{g(y)} \cdot z^{g(z)} = 0$. For $\beta \notin S$ the argument is easy. By assumption, $b := \prod_{x \in Y \setminus \{x_\beta\}} x^{g(x)} \neq 0$. By construction, x_β and b are independent. Thus $\prod_{x \in Y} x^{g(x)} \neq 0$.

Now suppose $\beta \in S$ and $\prod_{x \in Y} x^{g(x)} = 0$. By construction, $A_\beta \upharpoonright -x_\beta = \{0\}$. Thus $b := \prod_{x \in Y \setminus \{x_\beta\}} x^{g(x)} \not\leq -x_\beta$. Therefore $g(x_\beta) = 1$ and $b \leq x_\beta$. By construction, there is $m \in \omega$ such that $b \leq \sum_{i < m} x_{\delta_i^\beta}$. It follows from the inductive hypothesis that $b \cdot \prod_{i < m} -x_{\delta_i^\beta} \neq 0$. This contradicts the choice of m and the claim is proved.

For each $a \in C$ let $X_a = \{x_{a,i} : i < n\}$. Clearly, we may assume that C has size \aleph_1 . Let \leq be a wellordering on C of ordertype ω_1 . For each $\{a, b\} \in [C]^2$ choose a color $c(\{a, b\}) \in n^2$ such that

$$\forall (i, j) \in n^2 (c(\{a, b\}) = (i, j) \wedge a \leq b \Rightarrow x_{a,i}^{f_a(x_{a,i})} \cdot x_{b,j}^{f_b(x_{b,j})} = 0).$$

It follows from the claim that c can be defined. Clearly, for all $\{a, b\} \in [C]^2$, if $c(\{a, b\}) = (i, j)$ and $a \leq b$, then $x_{a,i}, x_{b,j} \notin R$. In [1] Baumgartner and Hajnal established the following partition result:

$$\forall m \in \omega \forall \alpha < \omega_1 (\omega_1 \rightarrow (\alpha)_m^2).$$

In particular, $\omega_1 \rightarrow (\omega + 2)_{n^2}^2$ holds. That is, there are $(i, j) \in n^2$ and a subset C' of C of ordertype $\omega + 2$ such that for all $\{a, b\} \in [C']^2$, $c(\{a, b\}) = (i, j)$. Let a and b be the last two elements of C' . Assume $x_{a,j} = x_\alpha$ for some $\alpha \in S$. By construction of A , for all $c \in C' \setminus \{a, b\}$, $x_{c,i} = x_{\delta_k^\alpha}$ for some $k \in \omega$. By the Δ -system assumption, all the $x_{c,i}$'s are different. This implies $x_{a,j} = x_{b,j}$, contradicting the Δ -system assumption.

Now assume that for all $\alpha \in S$, $x_{a,j} \neq x_\alpha$. In this case, for all $c \in C' \setminus \{a, b\}$, $x_{c,i} = x_\alpha$ for some $\alpha \in S$. Let d and e be the first two elements of C' . Now for all $c \in C' \setminus \{d, e\}$, $x_{c,j} = x_{\delta_k^\alpha}$ for some $k \in \omega$. By the Δ -system assumption, all the $x_{c,j}$'s are different. This implies $x_{d,i} = x_{e,i}$, contradicting the Δ -system assumption. This finishes the proof of the lemma. \square

Proof of Theorem 3.1. Let κ be an infinite cardinal. If $\kappa = \aleph_0$, then there are 2^κ pairwise non-isomorphic Boolean algebras of size κ and all of them are projective, hence tightly σ -filtered. Also, if κ is singular, then there are 2^κ pairwise non-isomorphic projective Boolean algebras by the result of Koppelberg mentioned before. Projective Boolean algebras are c.c.c.

If κ is regular and uncountable, let \mathcal{P} be a disjoint family of stationary subsets of $\{\alpha < \kappa : \text{cf}(\alpha) = \aleph_0\}$ of size κ . Such a family exists by the wellknown results of Ulam and Solovay. For every subset \mathcal{T} of \mathcal{P} let $A^\mathcal{T}$ be the Boolean algebra which is constructed in Lemma 3.2 from the set $S := \bigcup \mathcal{T}$ and let $(A_\alpha^\mathcal{T})_{\alpha < \kappa}$ be its associated tight σ -filtration. Then for $\mathcal{T}, \mathcal{T}' \subseteq \mathcal{P}$ with $\mathcal{T} \neq \mathcal{T}'$ the Boolean algebras $A^\mathcal{T}$ and $A^{\mathcal{T}'}$ are non-isomorphic.

For suppose $h : A^\mathcal{T} \rightarrow A^{\mathcal{T}'}$ is an isomorphism. Without loss of generality we may assume that $\mathcal{T} \setminus \mathcal{T}'$ is nonempty. The set $\{\alpha < \kappa : h[A_\alpha^\mathcal{T}] = A_\alpha^{\mathcal{T}'}\}$ is club in λ . Since $\bigcup(\mathcal{T} \setminus \mathcal{T}')$ is stationary, there is $\alpha \in \bigcup(\mathcal{T} \setminus \mathcal{T}')$ such that $h[A_\alpha^\mathcal{T}] = A_\alpha^{\mathcal{T}'}$. But $A_\alpha^\mathcal{T} \not\leq_{\text{rc}} A^\mathcal{T}$ and $A_\alpha^{\mathcal{T}'} \leq_{\text{rc}} A^{\mathcal{T}'}$, a contradiction.

By Lemma 3.3, the Boolean algebras $A^\mathcal{T}$ are c.c.c. \square

The two lemmas above give even more:

Theorem 3.4. *Let κ be an uncountable and regular cardinal. Then there is a family of size 2^κ of tightly σ -filtered c.c.c. Boolean algebras of size κ such that no member of this family is embeddable into another one as an rc-subalgebra.*

Proof. Suppose \mathcal{T} and \mathcal{T}' are subsets of \mathcal{P} , where \mathcal{P} is as in the proof of the theorem above. Assume there is an embedding $e : A^\mathcal{T} \rightarrow A^{\mathcal{T}'}$ such that $e[A^\mathcal{T}] \leq_{\text{rc}} A^{\mathcal{T}'}$.

Let $C \subseteq \kappa$ be a club such that $e[A_\alpha^\mathcal{T}] = A_\alpha^{\mathcal{T}'} \cap e[A^\mathcal{T}]$ and $\text{lpr}_{e[A^\mathcal{T}]}^{A^{\mathcal{T}'}}[A_\alpha^{\mathcal{T}'}] \subseteq A_\alpha^{\mathcal{T}'}$ hold for every $\alpha \in C$.

Let $\alpha \in C \cap \bigcup \mathcal{T}$. Then $e[A_\alpha^\mathcal{T}] \not\leq_{\text{rc}} e[A^\mathcal{T}]$ and hence $e[A_\alpha^\mathcal{T}] \not\leq_{\text{rc}} A^{\mathcal{T}'}$. Since $A_\alpha^{\mathcal{T}'}$ is closed under $\text{lpr}_{e[A^\mathcal{T}]}^{A^{\mathcal{T}'}}$, $e[A_\alpha^\mathcal{T}] \leq_{\text{rc}} A_\alpha^{\mathcal{T}'}$.

Hence $A_\alpha^{\mathcal{T}'} \not\leq_{\text{rc}} A^{\mathcal{T}'}$. Therefore $C \cap \bigcup \mathcal{T} \subseteq C \cap \bigcup \mathcal{T}'$. Thus, since \mathcal{P} consists of stationary sets, $\mathcal{T} \subseteq \mathcal{T}'$. Now let I be an independent family of subsets of \mathcal{P} of size 2^λ . In particular, the elements of I are pairwise \subseteq -incomparable. Thus the family $\{A^\mathcal{T} : \mathcal{T} \in I\}$ consists of pairwise non-rc-embeddable tightly σ -filtered c.c.c. Boolean algebras of size λ . \square

As mentioned in the introduction, Lemma 3.2 gives the existence of tightly σ -filtered Boolean algebras that are not openly generated.

Theorem 3.5. *Let κ be an uncountable regular cardinal. Then there is a tightly σ -filtered Boolean algebra that is c.c.c. and not openly generated.*

Proof. Let $S \subseteq \kappa$ be a stationary set consisting of ordinals of countable cofinality. Let A be the Boolean algebra constructed in Lemma 3.2 and suppose that A is openly generated. By [11, Theorem 2.2.3], there is a function $f : A \rightarrow [A]^{<\aleph_0}$ such that every subalgebra B of A that is closed under f is an rc-subalgebra of A . It follows that A has a filtration $A = \bigcup_{\alpha < \kappa} B_\alpha$ consisting of rc-subalgebras of size $< \kappa$.

It is easily checked that there is a club $C \subseteq \kappa$ such that for all $\alpha \in C$, $A_\alpha = B_\alpha$. Let $\alpha \in C \cap S$. Then by a) of Lemma 3.2, A_α is not an rc-subalgebra of A , contradicting the fact that A_α equals B_α , which is an rc-subalgebra of A . \square

4. OPENLY GENERATED BOOLEAN ALGEBRAS

4.1. Almost free Boolean algebras. The openly generated Boolean algebras we are going to construct will be almost free. Unlike in the case of abelian groups, subalgebras of free Boolean algebras do not have to be free. Thus we have to use a

slightly generalized definition of almost freeness for Boolean algebras. We use the definition given in [3].

Definition 4.1. Let A be a Boolean algebra of size κ . A filtration $(A_\alpha)_{\alpha < \kappa}$ of A is a κ -filtration if for every $\alpha < \kappa$, $|A_\alpha| < \kappa$.

A is *almost free* if it has a κ -filtration $(A_\alpha)_{\alpha < \kappa}$ consisting of free Boolean algebras.

In the proof of Theorem 3.1 we coded stationary sets by tightly σ -filtered Boolean algebras using the difference between rc-subalgebras and σ -subalgebras. If there is an almost free abelian group of size κ , we can code certain stationary subsets of κ by openly generated Boolean algebras. The coding will be more subtle as in the case of tightly σ -filtered Boolean algebras.

Definition 4.2. Let A and B be Boolean algebras such that $B \leq A$. Then $B \leq_{\text{free}} A$ if there is a free Boolean algebra F such that A is isomorphic to $B \oplus F$ over B . Here \oplus denotes the coproduct (free product) in the category of Boolean algebras. Equivalently, $B \leq_{\text{free}} A$ if there is a set $X \subseteq A$ such that X is independent over B and $A = \langle B \cup X \rangle$.

Suppose A is almost free of size κ and let $(A_\alpha)_{\alpha < \kappa}$ be a κ -filtration of A . Let

$$E := \{\alpha < \kappa : \{\beta \in (\alpha, \kappa) : A_\alpha \not\leq_{\text{free}} A_\beta\} \text{ is stationary in } \kappa\}.$$

For $X \subseteq \kappa$ let

$$\tilde{X} := \{Y \subseteq \kappa : \exists C \subseteq \kappa (C \text{ is club and } Y \cap C = X \cap C)\}.$$

Let $\Gamma(A) := \tilde{E}$. For $X, Y \subseteq \kappa$ let $\tilde{X} \leq \tilde{Y}$ if $X \setminus Y$ is non-stationary.

It is routine matter to check that $\Gamma(A)$ does not depend on the choice of $(A_\alpha)_{\alpha < \kappa}$. Therefore $\Gamma(A) = \Gamma(B)$ if A and B are isomorphic. Thus $\Gamma(A)$ is an invariant of A . Also, it is not difficult to see that A is free if and only if $\Gamma(A) = \emptyset$. See [3, Chapter IV] for more information on the Γ -invariant.

4.2. Almost free families of countable sets and the strong construction principle (CP+). Shelah has given an exact translation of the algebraic question whether there is an almost free, non-free abelian group of size κ into a set-theoretic one. We use the representation given in [3, Chapter VII, Sections 3 and 3A], where all the missing proofs can be found.

Let $\mathcal{S} = \{s_i : i \in \mathcal{S}\}$ be a family of countably infinite sets. A transversal for \mathcal{S} is a one-one function $T : I \rightarrow \bigcup \mathcal{S}$ such that for all $i \in I$, $T(i) \in s_i$. We say that \mathcal{S} is *free* if it has a transversal. \mathcal{S} is *almost free* if for each $J \subseteq I$ with $|J| < |I|$, $\{s_i : i \in J\}$ has a transversal. Let $\text{NPT}(\lambda, \aleph_0)$ denote the statement “there is an almost free, non-free family of size λ of countable sets”. (We use the notation of [15].)

Shelah proved the following:

Theorem 4.3. *For every uncountable cardinal λ there is an almost free, non-free abelian group of size λ if and only if $\text{NPT}(\lambda, \aleph_0)$ holds.*

We mimic the proof of one direction of this theorem and construct a Boolean algebra $A(\mathcal{S})$ for any given family \mathcal{S} of countable sets. For this we use the strong construction principle (CP+) for Boolean algebras. If the family \mathcal{S} is sufficiently good, $A(\mathcal{S})$ will be openly generated.

(CP+) (for arbitrary varieties) was introduced by Eklof and Mekler [3, Chapter VII, Section 3A], who proved

Theorem 4.4. *$\text{NPT}(\lambda, \aleph_0)$ implies the existence of almost free, non-free objects of size λ in every variety \mathcal{V} which satisfies (CP+).*

Shelah showed

Lemma 4.5. (See [6]) The variety of Boolean algebras satisfies the following *strong construction principle* (CP+):

For each $n \in \omega \setminus 1$ there are countably generated free Boolean algebras $H \leq K \leq L$ and a partition of ω into n infinite blocks s^1, \dots, s^n such that

- (i) H is freely generated by $\{h_m : m \in \omega\}$ and for each $J \subseteq \omega$, if for some $k \in \{1, \dots, n\}$, $J \cap s^k$ is finite, then $\langle \{h_m : m \in J\} \rangle \leq_{\text{free}} L$ and
- (ii) $L = K \oplus \text{Fr}(\omega)$ and $H \not\leq_{\text{free}} L$.

For the convenience of the reader we include a proof of this lemma. The proof uses

Lemma 4.6. (Sirota's Lemma, see [11, Theorem 1.4.10]) Let A and B be Boolean algebras such that $A \leq_{\text{rc}} B$ and B is countably generated over A . If for all $b_1, \dots, b_n \in B$ there is $u \in B$ such that u is independent over $A(b_1, \dots, b_n)$, then $A \leq_{\text{free}} B$.

Proof of Lemma 4.5. Let $n \in \omega \setminus 1$ and let $(x_{k,l})_{k \in \{1, \dots, n\}, l \in \omega}$ be a family of pairwise distinct sets. Let H be the free Boolean algebra $\text{Fr}(X)$ over the set $X := \{x_{k,l} : k \in \{1, \dots, n\} \wedge l \in \omega\}$. Let $(h_m)_{m \in \omega}$ be a 1-1-enumeration of X . For $k \in \{1, \dots, n\}$ let $s^k := \{m \in \omega : \exists l \in \omega (h_m = x_{k,l})\}$. Let I be the ideal of H generated by $\{\prod_{k \in \{1, \dots, n\}} x_{k,l} : l \in \omega\}$. Let K be a Boolean algebra of the form $H(x)$ where $H \upharpoonright x = I$ and $H \upharpoonright -x = \{0\}$. Finally, let $L := K \oplus \text{Fr}(\omega)$.

We show that H , K , and L are as required in the definition of (CP+). Clearly, the three Boolean algebras are countable and atomless. Therefore, they are free. By the choice of K and x , $H \not\leq_{\text{rc}} K$ and thus $H \not\leq_{\text{rc}} L$. In particular, $H \not\leq_{\text{free}} L$.

Now suppose that $Y \subseteq X$ is such that for some $k_0 \in \{1, \dots, n\}$ the set $Y \cap \{x_{k_0,l} : l \in \omega\}$ is finite. Let H' be the subalgebra of H generated by Y . We have to show $H' \leq_{\text{free}} L$. By Lemma 4.6 it is enough to show $H' \leq_{\text{rc}} L$. Since $K \leq_{\text{free}} L$ and thus $K \leq_{\text{rc}} L$, it is in fact sufficient to show $H' \leq_{\text{rc}} K$.

For every $m \in \omega$ let H_m be the subalgebra of H generated by

$$\{x_{k,l} : k \in \{1, \dots, n\} \wedge l \in \omega \wedge (k = k_0 \Rightarrow l \leq m)\}.$$

Let $a \in K = H(x)$. Then for some $m \in \omega$, $H' \subseteq H_m$ and $a \in H_m(x)$. Clearly, $H' \leq_{\text{rc}} H_m$. The ideal $I \cap H_m$ is generated by $\{\prod_{k \in \{1, \dots, n\}} x_{k,l} : l \leq m\}$, a finite set. Therefore $H_m \leq_{\text{rc}} H_m(x)$. It follows that $H' \leq_{\text{rc}} H_m(x)$. This implies that $H' \upharpoonright a$ is principal. This shows $H' \leq_{\text{rc}} H(x)$, finishing the proof of the lemma. \square

The construction of $A(\mathcal{S})$ from a family \mathcal{S} of countable sets is the following.

Definition 4.7. Let $\mathcal{S} = \{s_i : i \in I\}$ be a family of countable sets. Suppose for all $i \in I$, s_i is the disjoint union of the infinite sets s_i^1, \dots, s_i^n . Let H , K , L , $\{h_m : m \in \omega\}$, and s^1, \dots, s^n be as in Lemma 4.5. For each $i \in I$ fix an enumeration $(x_{i,m})_{m \in \omega}$ of s_i such that for each $k \in \{1, \dots, n\}$, $s_i^k = \{x_{i,m} : m \in s^k\}$.

For each $i \in I$ choose a copy L_i of L . Assume that all the copies are disjoint. Let K_i , H_i , and $\{h_{i,m} : m \in \omega\}$ denote the corresponding copies of K , H , and $\{h_m : m \in \omega\}$ in L_i .

Let $G := \bigoplus_{i \in I} L_i$ and let Θ be the smallest congruence on G identifying $h_{i,m}$ and $h_{j,l}$ for all i, m, j, l with $x_{i,m} = x_{j,l}$. Let $A(\mathcal{S}) := G/\Theta$.

4.3. λ -systems and the strong reshuffling property. We can only show that the algebra $A(\mathcal{S})$ constructed in Definition 4.7 is openly generated if the family \mathcal{S} has some special properties. We need \mathcal{S} to be *based on a λ -system Λ of height n* for some $n > 1$ such that (\mathcal{S}, Λ) has the *strong reshuffling property*.

Shelah used λ -systems and the (weak) reshuffling property for proving Theorem 4.3. The proof of Theorem 4.4 uses the strong reshuffling property. We use λ -systems to construct many openly generated Boolean algebras in the same way as

they are used in the proof of Theorem 4.4. Let us start with the definition of a λ -system.

Definition 4.8. The set $\lambda^{<\omega}$ ordered by set inclusion is a tree. Let (β) denote the sequence of length one with value β and let \frown denote concatenation of sequences. If S is a subtree of $\lambda^{<\omega}$, an element η of S is called a *final node* of S if in S there is no proper extension of η . Let S_f denote the set of final nodes of S .

1. A λ -set S is a subtree of $\lambda^{<\omega}$ together with a cardinal λ_η for every $\eta \in S$ such that $\lambda_\emptyset = \lambda$ and

- a) for all $\eta \in S$, $\eta \in S_f$ if and only if $\lambda_\eta = \aleph_0$ and
- b) if $\eta \in S \setminus S_f$, then $\eta \frown (\beta) \in S$ implies $\beta \in \lambda_\eta$ and $\lambda_{\eta \frown (\beta)} < \lambda_\eta$ and $E_\eta := \{\beta < \lambda_\eta : \eta \frown (\beta) \in S\}$ is stationary in λ_η .

2. A λ -system is a λ -set together with a set B_η for each $\eta \in S$ such that $B_\emptyset = \emptyset$ and for all $\eta \in S \setminus S_f$

- a) for all $\beta \in E_\eta$, $\lambda_{\eta \frown (\beta)} \leq |B_{\eta \frown (\beta)}| < \lambda_\eta$ and
- b) $(B_{\eta \frown (\beta)})_{\beta \in E_\eta}$ is increasing and continuous, that is, if $\sigma \in E_\eta$ is a limit point of E_η , then $B_{\eta \frown (\sigma)} = \bigcup \{B_{\eta \frown (\beta)} : \beta \in E_\eta \cap \sigma\}$.

For any λ -system $\Lambda = (S, \lambda_\eta, B_\eta)_{\eta \in S}$ and any $\eta \in S$ let $\overline{B}_\eta := \bigcup \{B_{\eta \upharpoonright m} : m \leq \text{dom}(\eta)\}$. A family \mathcal{S} of countable sets is *based on* Λ if \mathcal{S} is indexed by S_f and for every $\eta \in S_f$, $s_\eta \subseteq \overline{B}_\eta$.

A subtree S of $\lambda^{<\omega}$ has *height* n if all final nodes of S have domain n . A λ -set or λ -system has height n if its associated tree S has height n . It is not difficult to see that every λ -set has a sub- λ -set which has a height.

If \mathcal{S} is a family of countable sets based on a λ -system, then \mathcal{S} has cardinality λ and is not free. Families of countable sets based on λ -systems can be constructed from almost free, non-free abelian groups (see [3, Chapter VII, Section 3]). By Theorem 4.3, this implies

Lemma 4.9. $\text{NPT}(\lambda, \aleph_0)$ implies that there is a family of countable sets based on a λ -system.

In [15] Shelah and Väisänen provided the tools to show that for $\lambda > \aleph_1$, the λ -system in Lemma 4.9 may be chosen such that its height is at least 2, i.e., such that the underlying λ -set is really more than just a stationary subset of λ .

Lemma 4.10. If $\lambda > \aleph_1$ and $\text{NPT}(\lambda, \aleph_0)$ holds, then there is a family of countable sets based on a λ -system of height n for some $n > 1$.

Proof. We show how to extract the proof of Lemma 4.10 from [15]¹. In [15] a special kind of families of countable sets based on λ -systems is used, so-called $\text{NPT}(\lambda, \aleph_0)$ -skeletons. Moreover, so-called $\text{NRT}(\lambda, \aleph_0)$ -skeletons are used for building more complicated NPT -skeletons.

Let $\lambda > \aleph_1$ and suppose $\text{NPT}(\lambda, \aleph_0)$ holds. We show that there is an $\text{NPT}(\lambda, \aleph_0)$ -skeleton of height > 1 .

From an $\text{NRT}(\lambda, \aleph_0)$ -skeleton one can construct an $\text{NPT}(\lambda, \aleph_0)$ -skeleton of the same height [15, Corollary 4.14]. It is therefore sufficient to show the existence of an $\text{NRT}(\lambda, \aleph_0)$ -skeleton of height > 1 . By $\text{NPT}(\lambda, \aleph_0)$, there is an $\text{NRT}(\lambda, \aleph_0)$ -skeleton [15, Lemma 4.13 and Corollary 4.14]. If this $\text{NRT}(\lambda, \aleph_0)$ -skeleton is of height > 1 , we are done. So assume it is of height 1. Since the $\text{NRT}(\lambda, \aleph_0)$ -skeleton is of height 1, its type is (λ) .

As mentioned in the introduction, there is an almost free, non-free abelian group of size \aleph_1 . By Theorem 4.3, this implies $\text{NPT}(\aleph_1, \aleph_0)$. As before, it follows that there is an $\text{NRT}(\aleph_1, \aleph_0)$ -skeleton. An $\text{NRT}(\aleph_1, \aleph_0)$ -skeleton is of height 1. Since

¹The second author thanks Pauli Väisänen for explaining how Lemma 4.10 follows from the results presented in [15].

the $\text{NRT}(\lambda, \aleph_0)$ -skeleton is of type (λ) and since $\lambda > \aleph_1$, the $\text{NRT}(\lambda, \aleph_0)$ -skeleton and the $\text{NRT}(\aleph_1, \aleph_0)$ -skeleton are compatible. Therefore, the two skeletons can be combined to an $\text{NRT}(\lambda, \aleph_0)$ -skeleton of height 2 (and of type (λ, \aleph_1)). \square

To get an almost free Boolean algebra from \mathcal{S} , we need \mathcal{S} to be based on a λ -system such that (\mathcal{S}, Λ) has the strong reshuffling property.

Definition 4.11. Let $\Lambda = (S, \lambda_\eta, B_\eta)_{\eta \in S}$ be a λ -system of height n and $\mathcal{S} = (s_\eta)_{\eta \in S_f}$ a family of countable sets based on Λ . Let $<_{\text{lex}}$ denote the (strict) lexicographic order on S .

(\mathcal{S}, Λ) has the *strong reshuffling property* if the following statements hold:

- (1) If I is a subset of S_f of cardinality $< \lambda$ and $\eta_0 \in I$, then there is a wellordering $<_I$ of I such that η_0 is the $<_I$ -first element of I and for all $\eta \in I$ there is $k \in \{1, \dots, n\}$ such that $s_\eta^k \cap \bigcup_{\nu <_I \eta} s_\nu$ is finite.
- (2) For all $\mu \in S \setminus S_f$ and all $\alpha < \lambda_\mu$, if I is a subset of $\{\eta \in S_f : \mu \subseteq \eta\}$ of size $< \lambda_\mu$, then there is a wellordering $<_{\bar{I}}$ of $\bar{I} := \{\eta \in S_f : \eta <_{\text{lex}} \mu \vee \eta \in I\}$ such that for all $\eta \in \bar{I}$
 - a) there is $k \in \{1, \dots, n\}$ such that $s_\eta^k \cap \bigcup_{\nu <_{\bar{I}} \eta} s_\nu$ is finite and
 - b) if $\mu \subsetneq \eta$ and $\nu \in \bar{I}$ is such that
$$\nu <_{\text{lex}} \mu \vee (\mu \subsetneq \nu \wedge \nu(\text{dom}(\mu)) \leq \alpha < \eta(\text{dom}(\mu))),$$
then $\nu <_{\bar{I}} \eta$.

The definition of the strong reshuffling property given in Theorem 3A.7 of [3, Chapter VII] is formally weaker than ours. However, even with our version of the strong reshuffling property, the proof of that theorem shows

Lemma 4.12. If there is a family of countable sets based on a λ -system of height n , then there is a family \mathcal{S} of countable sets based on a λ -system Λ of height n such that (\mathcal{S}, Λ) has the strong reshuffling property.

Combining this with Lemma 4.10 we get

Corollary 4.13. If $\lambda > \aleph_1$ and $\text{NPT}(\lambda, \aleph_0)$ holds, then there is a family \mathcal{S} of countable sets based on a λ -system Λ of height > 1 such that (\mathcal{S}, Λ) has the strong reshuffling property.

4.4. Many openly generated Boolean algebras. In [3, Theorem VII.3A.13] it is proved that the algebra $A(\mathcal{S})$ is almost free but not free if the family \mathcal{S} of countable sets is based on a λ -system $\Lambda = (S, \lambda_\eta, B_\eta)_{\eta \in S}$ such that (\mathcal{S}, Λ) has the reshuffling property. Looking at the proof more closely, it turns out that we actually get that $\Gamma(A(\mathcal{S}))$ is \tilde{E}_\emptyset . In the case of Boolean algebras we find that $A(\mathcal{S})$ is openly generated if Λ has height > 1 and (\mathcal{S}, Λ) has the strong reshuffling property.

Lemma 4.14. Let \mathcal{S} be a family of countable sets based on a λ -system $\Lambda = (S, \lambda_\eta, B_\eta)_{\eta \in S}$ of height $n > 1$ such that (\mathcal{S}, Λ) has the strong reshuffling property. Then $\Gamma(A(\mathcal{S})) = \tilde{E}_\emptyset$ and the Boolean algebra $A(\mathcal{S})$ is almost free and openly generated.

Proof. We only have to show that $A(\mathcal{S})$ is openly generated. For this we repeat the proof that $A(\mathcal{S})$ is almost free.

Consider the λ -filtration $(A_\alpha)_{\alpha < \lambda}$ of $A(\mathcal{S})$ where

$$A_\alpha := \langle \bigcup \{L_\eta : \eta \in S_f \wedge \eta(0) < \alpha\} / \Theta \rangle$$

for all $\alpha < \lambda$.

Claim 1. $A_{\alpha+1} \leq_{\text{free}} A_\beta$ for all $\beta < \lambda$ and all $\alpha \in [-1, \beta)$.

Note that $A_\alpha = A_{\alpha+1}$ for $\alpha \notin E_\emptyset$. Thus, Claim 1 implies that

- (i) all A_α , $\alpha < \lambda$, are free and

(ii) $A_\alpha \leq_{\text{rc}} A(\mathcal{S})$ for all $\alpha \notin E_\emptyset$.

For the proof of Claim 1 let $\alpha < \beta < \lambda$. Let $I := \{\eta \in S_f : \eta(0) < \beta\}$. By the strong reshuffling property, there is a wellordering $<_I$ of I as guaranteed by the requirement (2) in the definition of the strong reshuffling property.

Now let η be the $<_I$ -first element of I such that $\eta(0) > \alpha$. For some $k \in \{1, \dots, n\}$, $s_\eta^k \cap \bigcup_{\nu <_I \eta} s_\nu$ is finite. Note that $L_\eta/\Theta \cap A_{\alpha+1} = \langle \{h_{\eta,m} : x_{\eta,m} \in s_\eta \cap \bigcup_{\nu <_I \eta} s_\nu\} \rangle / \Theta$. Choose $G_\eta \leq L_\eta$ free such that

$$L_\eta = \langle \{h_{\eta,m} : x_{\eta,m} \in s_\eta \cap \bigcup_{\nu <_I \eta} s_\nu\} \rangle \oplus G_\eta.$$

Then $\langle L_\eta/\Theta \cup A_{\alpha+1} \rangle = A_{\alpha+1} \oplus G_\eta/\Theta$ and Θ does not affect G_η , i.e., the epimorphism $\pi : G_\eta \rightarrow G_\eta/\Theta$ is an isomorphism. Thus, we may think of G_η as a subalgebra of A_β . By recursion on $<_I$ for all $\tau \in I$ with $\eta <_I \tau$ we can choose a free Boolean algebra $G_\tau \leq A_\beta$ such that $A_\beta = A_{\alpha+1} \oplus \bigoplus_{\eta \leq_I \tau} G_\tau$. This shows Claim 1.

By Lemma 2.5 it remains to show $A_\alpha \leq_{\text{rc}} A(\mathcal{S})$ for all $\alpha \in E_\emptyset$. Fix $\alpha \in E_\emptyset$. Since $A_{\alpha+1} \leq_{\text{rc}} A(\mathcal{S})$ and by the transitivity of \leq_{rc} , $A_\alpha \leq_{\text{rc}} A(\mathcal{S})$ if and only if $A_\alpha \leq_{\text{rc}} A_{\alpha+1}$. Thus, it remains to show $A_\alpha \leq_{\text{rc}} A_{\alpha+1}$.

Since Λ is of height > 1 , $(\alpha) \notin S_f$. Consider the filtration $(A_{\alpha,\beta})_{\beta < \lambda(\alpha)}$ of $A_{\alpha+1}$ where

$$A_{\alpha,\beta} := \langle \bigcup \{L_\eta : \eta \in S_f \wedge \eta(0) < \alpha \vee (\eta(0) = \alpha \wedge \eta(1) < \beta)\} \rangle / \Theta.$$

Claim 2. $A_\alpha = A_{\alpha,0} \leq_{\text{free}} A_{\alpha,\beta}$ for all $\beta < \lambda(\alpha)$.

The proof of Claim 2 is practically the same as the proof of Claim 1 and uses requirement (2) in the definition of the strong reshuffling property with $\mu := (\alpha)$ and $I := \{\eta \in S_f : \eta(0) = \alpha \wedge \eta(1) < \beta\}$.

Now let $a \in A_{\alpha+1}$. There is $\beta < \lambda_\mu$ such that $a \in A_{\alpha,\beta}$. By Claim 2, $A_\alpha \leq_{\text{free}} A_{\alpha,\beta}$. In particular, $A_\alpha \leq_{\text{rc}} A_{\alpha,\beta}$. Therefore $A_\alpha \upharpoonright a$ has a maximal element. This shows $A_\alpha \leq_{\text{rc}} A_{\alpha+1}$ and finishes the proof of the lemma. \square

Lemma 4.14 says that in certain cases we can code stationary sets by openly generated Boolean algebras. Using this coding, we can show

Theorem 4.15. *Let $\lambda > \aleph_1$ be such that there is an almost free, non-free abelian group of size λ . Then there are 2^λ pairwise non-isomorphic openly generated Boolean algebras of size λ .*

Proof. If there is an almost free, non-free abelian group of size λ , then, by Corollary 4.13, there is a family \mathcal{S} of countable sets based on a λ -system $\Lambda = (S, \lambda_\eta, B_\eta)_{\eta \in S}$ of height $n > 0$ such that (\mathcal{S}, Λ) has the strong reshuffling property. By Shelah's singular compactness theorem, λ is regular.

By the theorems of Solovay and Ulam, we can split the stationary set E_\emptyset associated with S into a disjoint family \mathcal{P} of size λ of stationary sets. For each $\mathcal{T} \subseteq \mathcal{P}$ let $E^\mathcal{T} := \bigcup \mathcal{T}$, $S^\mathcal{T} := \{\eta \in S : \eta(0) \in E^\mathcal{T}\}$, $\Lambda^\mathcal{T} := (S^\mathcal{T}, \lambda_\eta, B_\eta)_{\eta \in S^\mathcal{T}}$, $\mathcal{S}^\mathcal{T} := (s_\eta)_{\eta \in S^\mathcal{T}}$, and $A^\mathcal{T} := A(\mathcal{S}^\mathcal{T})$.

It follows immediately from the definitions that for all $\mathcal{T} \subseteq \mathcal{P}$ with $\mathcal{T} \neq \emptyset$, $\Lambda^\mathcal{T}$ is a λ -system of height n and $(\mathcal{S}^\mathcal{T}, \Lambda^\mathcal{T})$ has the strong reshuffling property. Thus, by Lemma 4.14, $A^\mathcal{T}$ is openly generated and $\Gamma(A^\mathcal{T}) = \tilde{E}^\mathcal{T}$ for each nonempty $\mathcal{T} \subseteq \mathcal{P}$. Clearly, $\tilde{E}^\mathcal{T} \neq \tilde{E}^{\mathcal{T}'}$ for $\mathcal{T} \neq \mathcal{T}'$. It follows that $(A^\mathcal{T})_{\mathcal{T} \subseteq \mathcal{P}, \mathcal{T} \neq \emptyset}$ is a family of 2^λ pairwise non-isomorphic openly generated Boolean algebras of size λ . \square

Combining this theorem with the results of Shelah and Magidor from [14] on the class of cardinals κ for which there are almost free, non-free abelian groups of size κ , we get

Corollary 4.16. a) There is a class \mathcal{C} of regular cardinals which is closed under the operations $\kappa \mapsto \kappa^+$ and $(\kappa, \lambda) \mapsto \lambda^{+(\kappa+1)}$ and contains \aleph_2 such that for each $\kappa \in \mathcal{C}$

there are 2^κ isomorphism types of openly generated Boolean algebras of size κ . In particular, for every $n > 1$ there are 2^{\aleph_n} pairwise non-isomorphic Boolean algebras of size \aleph_n .

b) Under $V = L$, for every cardinal $\kappa > \aleph_1$ there are 2^κ pairwise non-isomorphic openly generated Boolean algebras of size κ .

Note that we have to exclude \aleph_1 in this corollary since every openly generated Boolean algebra of size \aleph_1 is projective and thus, by Koppelberg's result on the number of projective Boolean algebras, there are only 2^{\aleph_0} isomorphism types of openly generated Boolean algebras of size \aleph_1 .

Recall that open generatedness follows from projectivity. By Koppelberg's results on the number of projective Boolean algebras of singular cardinality, for each singular cardinal μ there are 2^μ isomorphism types of openly generated Boolean algebras of size μ .

The main open question is

Question 4.17. Is it true that the number of isomorphism types of openly generated Boolean algebras of size κ is 2^κ for every infinite cardinal $\kappa \neq \aleph_1$?

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