APPLICATIONS OF ELEMENTARY SUBMODELS IN GENERAL TOPOLOGY

STEFAN GESCHKE

Abstract. Elementary submodels of some initial segment of the set-theoretic universe are useful in order to prove certain theorems in general topology as well as in algebra. As an illustration we give proofs of two theorems due to Arkhangel'skii concerning cardinal invariants of compact spaces.

1. Introduction

Several theorems in general topology, especially inequalities between certain cardinal invariants of a topological space, can be proved in the following way:

For a given topological space X consider a space X_0 which is small in some sense and approximates X sufficiently well. Calculate the cardinal invariant in question for X_0 and show that this cardinal invariant is the same for X_0 and X since X_0 is a good approximation of X.

The method of elementary submodels provides a uniform approach for generating small approximations of topological spaces as well as of other structures.

Typically, mathematics takes place in the set-theoretic universe $(V, \in)$, i.e., the class of all sets together with the usual $\in$-relation. For a given topological space X we would like to use the Löwenheim-Skolem Theorem to get a small elementary submodel $(M, \in \cap M^2)$ of the universe $(V, \in)$ such that X and its topology are contained in M and then consider the space X_0 which is what M thinks that X is. However, there are two problems. First of all, working in V we cannot get elementary submodels of $(V, \in)$. This follows from Gödel's Second Incompleteness Theorem. Moreover, it is not clear what it really is that M *thinks that* X *is.*

The first problem can be solved by taking elementary submodels not of the whole universe but of a sufficiently large initial segment of the universe. The solution of the second problem depends on the application we have in mind.

Much of the material presented here is contained in [JuWee97]. Much more on this topic can be found in [Dow88].

2. Models of set theory

Let us briefly recall some basics from logic. [Ku80] and [ChKei90] provide an excellent background in set theory and model theory, respectively.

The *language* $\mathcal{L}$ *of set theory* is the first-order language with the binary relation-symbol $\in$. That is, the language $\mathcal{L}$ consists of the formulae over the alphabet $\{\wedge, \neg, \exists, (,), \in, =\} \cup \mathrm{Var}$, where Var is a countably infinite set of variables. As usual, we will freely use abbreviations like '$\subseteq$', '$\Rightarrow$', and '$\exists x \in y$' inside a formula.

An $\mathcal{L}$*-structure* is a pair (N, E) where N is a set and E is a binary relation on N, i.e., $E \subseteq N^2$. By the usual abuse of notation, sometimes we will identify (N, E) and N. If $\varphi(x_1, \dots, x_n)$ is a formula and $a_1, \dots, a_n \in N$, then $\varphi[a_1, \dots, a_n]$ is the word obtained from φ by replacing x_i by a_i whereever x_i occurs freely, i.e., not in the scope of a quantifier $\exists x_i$, in φ for $i = 1, \dots, n$. Abusing notation, we will refer to $\varphi[a_1, \dots, a_n]$ as a formula as well. As usual, by induction on the length of φ, one defines when (N, E) satisfies $\varphi[a_1, \dots, a_n]$. We write $(N, E) \models \varphi[a_1, \dots, a_n]$ for '(N, E) satisfies $\varphi[a_1, \dots, a_n]$'.

(N, E) is a *model* of $\varphi[a_1, \dots, a_n]$ if it satisfies $\varphi[a_1, \dots, a_n]$. If Φ is a set of formulae, (N, E) satisfies Φ or is a model of Φ if it satisfies all formulae in Φ.

An $\mathcal{L}$-structure (M, F) is called an *elementary submodel* of (N, E) if $M \subseteq N$ and for all formulae $\varphi[x_1, \dots, x_n]$ and all $a_1, \dots, a_n \in M$,

$$(M, F) \models \varphi[a_1, \dots, a_n] \text{ if, and only if, } (N, E) \models \varphi[a_1, \dots, a_n].$$

We write $(M, F) \preccurlyeq (N, E)$ if (M, F) is an elementary submodel of (N, E). Note that $(M, F) \preccurlyeq (N, E)$ implies $F = E \cap M^2$. In the following, we will simply write (M, E), respectively M, instead of $(M, E \cap M^2)$.

The well-known Löwenheim-Skolem Theorem guarantees the existence of many small elementary submodels of a given structure (N, E).

Theorem 2.1 (Löwenheim-Skolem). *Let N be an $\mathcal{L}$-structure and $A \subseteq N$ infinite. Then there is $M \subseteq N$ such that $A \subseteq M$, $|M|=|A|$, and $M \preccurlyeq N$.* □

Intuitively, $(V, \in)$ satisfies all the axioms of our standard set theory ZFC. However, V is a proper class and not a set. Thus we cannot get an elementary submodel of $(V, \in)$ from the Löwenheim-Skolem Theorem. Moreover, as mentioned in the introduction, Gödel's Second Incompleteness Theorem implies that we cannot hope to get a model of ZFC at all working in ZFC alone. However, every proof of a theorem of ZFC uses only finitely many axioms. (Note that ZFC contains infinitely many axioms.) And we can get models of every finite part of ZFC.

Recall the use of $\bigcup$ in set theory: For a set x, $\bigcup x := \{z : z \in y \text{ for some } y \in x\}$. For every $n \in \mathbb{N}$ we define $\bigcup^n x$ recursively. Let $\bigcup^0 x := x$ and $\bigcup^n x := \bigcup^{n-1} \bigcup x$ for every $n \geq 1$. The *transitive closure* $\mathrm{tc}(x)$ of x is the set $\bigcup_{n \in \mathbb{N}} \bigcup^n x$. For a cardinal χ let $H_\chi := \{x : |\mathrm{tc}(x)| < \chi\}$. Each H_χ is a set and $(H_\chi, \in)$ satisfies some quite large part of ZFC. Moreover, the following holds:

Theorem 2.2 (Reflection Principle). *If $\varphi_i(x_1, \dots, x_n)$, $i \in \{1, \dots, m\}$, are formulae and ρ is a cardinal, then there is $\chi > \rho$ such that for all $a_1, \dots, a_n \in H_\chi$ and all $i \in \{1, \dots, m\}$,*

$$(H_\chi, \in) \models \varphi_i[a_1, \dots, a_n] \textit{ if, and only if, } \varphi_i[a_1, \dots, a_n] \textit{ holds in the universe.} \quad \square$$

Suppose $\varphi_i(x_1, \dots, x_n)$, $i \in \{1, \dots, m\}$, and χ are as in the Reflection Principle and we have $M \preccurlyeq (H_\chi, \in)$. Then for all $a_1, \dots, a_n \in M$ and all $i \in \{1, \dots, m\}$,

$$M \models \varphi_i[a_1, \dots, a_n] \text{ if, and only if, } \varphi_i[a_1, \dots, a_n] \text{ holds in the universe.}$$

Thus, with respect to a given set of finitely many formulae, M looks like an elementary submodel of the universe. In the situation above, the formulae φ_i are said to *absolute* over M.

3. Topological spaces

Let us recall a few basic notions from topology. See [En77] for a lot of information on general topology.

Let X be a topological space. X is *Hausdorff* if for any two distinct points $x, y \in X$ there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$. X is *compact* if it is Hausdorff and for every family C of open sets with $X = \bigcup C$ there is a finite subfamily F of C such that $X = \bigcup F$. It is an easy exercise to show that a continuous image of a compact space is compact, provided it is Hausdorff.

If Y is a subset of X, a subset U of Y is open in Y if it is of the form $O \cap Y$ for some open subset O of X. The topology on Y we have just defined is the *subspace topology* on Y with respect to X. Note that Y is compact with respect to this topology if X is compact and Y is a closed subset of X.

Two topological spaces X and Y are topologically the same if they are *homeomorphic*, that is, if there is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous. f is called a *homeomorphism*.

The topology of a space X can be described by giving a *base* for the topology. Let τ be the topology of X, i.e., the collection of open subsets of X. $B \subseteq \tau$ is a base of X if for all $x \in X$ and all $U \in \tau$ with $x \in U$ there is $V \in B$ such that $x \in V \subseteq U$. If we do not require B to be a subset of τ, we get the notion of a *network*. $N \subseteq \mathcal{P}(X)$ is a network of X if for all $x \in X$ and all $U \in \tau$ there is $V \in N$ such that $x \in V \subseteq U$. If B is a base of X, then the open subsets of X are precisely the unions of elements of B.

Example 3.1. Consider the set $\mathbb{R}$ of real numbers with the usual topology, i.e., the topology consisting of unions of open intervals. The set $\{(p,q) : p,q \in \mathbb{Q}, p < q\}$ is a base of $\mathbb{R}$. The set $\{[p,q] : p,q \in \mathbb{Q}, p < q\}$ fails to be a base since it consists of sets which are not open. However, this set is a network of $\mathbb{R}$. □

3.1. **Cardinal invariants.** A *cardinal invariant* of topological spaces is a mapping i assigning a cardinal $i(X)$ to each space X such that $i(X) = i(Y)$ if X and Y are homeomorphic. An easy example is the cardinality of a space. Clearly, any two spaces which are homeomorphic have the same cardinality. Two less trivial cardinal invariants are the weight and the network-weight.

The *weight* $\mathrm{w}(X)$ of a topological space X is the least infinite cardinal κ such that X has a base of size at most κ.

The *network-weight* $\mathrm{nw}(X)$ of a topological space X is the least infinite cardinal κ such that X has a network of size at most κ.

Since every base of X is a network, $\mathrm{nw}(X) \leq \mathrm{w}(X)$. Example 3.1 shows $\mathrm{nw}(\mathbb{R}) = \mathrm{w}(\mathbb{R}) = \aleph_0$.

4. Arhangel'skii's Theorem

Theorem 4.1 (Arhangel'skii, see [En77])**.** *Let X and Y be compact. If there is a continuous mapping $f : X \to Y$ which is onto, then* $\mathrm{w}(Y) \leq \mathrm{w}(X)$.

For the proof let us first observe that the corresponding statement for network-weight holds for all topological spaces.

Lemma 4.2. *Let X and Y be topological spaces and let $f : X \to Y$ be continuous and onto. Then* $\mathrm{nw}(Y) \leq \mathrm{nw}(X)$.

Proof. Let N be a network for X. Then $N' := \{f[V] : V \in N\}$ is a network for Y:

Let $U \subseteq Y$ be open and non-empty. Let $y \in U$. Let $x \in X$ be such that $f(x) = y$. Since f is continuous, $f^{-1}[U]$ is open. Clearly, $x \in f^{-1}[U]$. Thus, there is $V \in N$ such that $x \in V \subseteq f^{-1}[U]$. Now $f[V] \in N'$ and $y \in f[V] \subseteq U$. □

Theorem 4.1 follows from this lemma once we know that for compact spaces weight and network-weight are the same. We give a proof of this fact using elementary submodels.

Lemma 4.3. *If X is compact, then* $\mathrm{w}(X) = \mathrm{nw}(X)$.

Proof. Let N be a network of X with $\kappa := |N| = \mathrm{nw}(X)$ and let τ be the topology of X. Let χ be sufficiently large. That is, let χ be large enough for H_χ to contain X and τ and such that all those finitely many formulae are absolute over H_χ we want to be absolute in the following proof. We could write down these formulae having a close look at the rest of the proof. But this it not necessary since the Reflection Principle says that a suitable χ exists for any finite set of formulae. Now pick $M \preccurlyeq H_\chi$ such that $N \cup \{N, X, \tau\} \subseteq M$ and $|M| = \kappa$. We *claim* that $\tau \cap M$ is a base of X.

Let $U \in \tau$ be non-empty and let $x \in U$. For $y \in X \setminus U$ there are disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Since N is a network, there are sets $A_y, B_y \in N$ such that $x \in A_y \subseteq U_y$ and $y \in B_y \subseteq V_y$. Since χ is sufficiently large,

$$H_\chi \models \exists U'_y, V'_y \in \tau (U'_y \cap V'_y = \emptyset \wedge A_y \subseteq U'_y \wedge B_y \subseteq V'_y).$$

Therefore,

$$M \models \exists U'_y, V'_y \in \tau (U'_y \cap V'_y = \emptyset \wedge A_y \subseteq U'_y \wedge B_y \subseteq V'_y).$$

Let $U'_y, V'_y \in M$ be such that $U'_y, V'_y \in \tau$, $A_y \subseteq U'_y$, $B_y \subseteq V'_y$, and $U'_y \cap V'_y = \emptyset$. Now $U'_y, V'_y \in \tau \cap M \subseteq \tau$, $x \in U'_y$, and $y \in V'_y$. Clearly, $X \setminus U \subseteq \bigcup_{y \in X \setminus U} V'_y$. Since $X \setminus U$ is compact, there is a finite set $F \subseteq X \setminus U$ such that $X \setminus U \subseteq \bigcup_{y \in F} V'_y$. Note that F does not have to be a subset of M. However, $\{U'_y : y \in F\}$ is a subset of M. Since this set is finite, it can be defined in H_χ. More precisely, suppose $\{U'_y : y \in F\} = \{U_1, \dots, U_n\}$. Let $\varphi(z, x_1, \dots, x_n)$ be the formula saying that the elements of z are precisely $x_1, \dots, x_n$. Now $\varphi[W, U_1, \dots, U_n]$ holds in H_χ if, and only if, $W = \{U_1, \dots, U_n\}$. Here we could argue as follows: φ is one of those formulae we want to be absolute over H_χ and thus, by the choice of χ, we have

$$H_\chi \models \varphi[W, U_1, \dots, U_n] \text{ if, and only if, } W = \{U_1, \dots, U_n\}.$$

However, it turns out that formulae which are as simple as φ are absolute over every H_θ, no matter what cardinal θ is. This is due to the fact that for every $b \in H_\theta$ and every $a \in b$ we have $a \in H_\theta$. This property of the H_θ's is called *transitivity*. Since

$$H_\chi \models (\exists z \varphi)[U_1, \dots, U_n],$$

$$M \models (\exists z \varphi)[U_1, \dots, U_n].$$

Thus $\{U'_y : y \in F\} \in M$. Therefore $U' := \bigcap_{y \in F} U'_y \in M \cap \tau$. Clearly, $x \in U' \subseteq X \setminus \bigcup_{y \in F} V'_y \subseteq U$. This finishes the proof of the claim and thus the proof of the lemma. □

Proof of the theorem. Let X and Y be compact and $f : X \to Y$ continuous and onto. Then by Lemma 4.3 and Lemma 4.2,

$$\mathrm{w}(X) = \mathrm{nw}(X) \geq \mathrm{nw}(Y) = \mathrm{w}(Y).$$

□

Theorem 4.1 does not hold if the spaces are only assumed to be Hausdorff.

Example 4.4 (Alexandroff and Niemytzki, McAuley, see [En77])**.** Consider $X := \mathbb{R}^2$ and let the topology τ on X be generated by the base B which is defined as follows:

For each point $(x, y) \in X$ and $i \in \mathbb{N}$ let $U_i(x, y) := \{(a, b) \in \mathbb{R}^2 : |(a, b) - (x, y)| < \frac{1}{i+1}\}$ and $C_i(x, y) := \{(a, b) \in \mathbb{R}^2 : |(a, b) - (x, y)| \leq \frac{1}{i+1}\}$. Let

$$\begin{aligned} B := &\{U_i(x, y) : x, y \in \mathbb{R},\ y \neq 0,\ \text{and } i \in \mathbb{N}\} \\ &\cup \{\{(x, 0)\} \cup U_i(x, 0) \setminus (C_i(x, \frac{1}{i+1}) \cup C_i(x, -\frac{1}{i+1})) : x \in \mathbb{R} \text{ and } i \in \mathbb{N}\}. \end{aligned}$$

This space is Hausdorff. It turns out that $\mathrm{nw}(X) = \aleph_0$ while $\mathrm{w}(X) = 2^{\aleph_0}$. Let N be a countable network of X. Let σ be the topology on X which is generated by N as a base. By the properties of a network, $\tau \subseteq \sigma$. Thus (X, σ) is Hausdorff and the identity $\mathrm{id}_X : (X, \sigma) \to (X, \tau); x \mapsto x$ is continuous. However, by definition of σ,

$$\aleph_0 = \mathrm{w}(X, \sigma) < \mathrm{w}(X, \tau) = 2^{\aleph_0}.$$

□

5. Better submodels

In the proof of Lemma 4.3, we used the fact that for every elementary submodel M of H_χ a finite subset of M already is an element of M. On the other hand, for every $M \preccurlyeq H_\chi$, $\mathbb{N} = \aleph_0 \in M$. Clearly, $\mathbb{N} \subseteq M$. Thus, if $x \in M$ is countable, then M contains a bijection $f : \mathbb{N} \to x$. But now $x = f[\mathbb{N}] \subseteq M$. The same argument shows that $x \subseteq M$ if $x \in M$ has size κ and $\kappa \subseteq M$. However, typically we do not know whether for some $y \subseteq M$, y is an element of M. The following lemma comes in handy.

Lemma 5.1. *Let δ be an ordinal and χ a cardinal. Suppose $(M_\alpha)_{\alpha<\delta}$ is a* chain *of elementary submodels of H_χ, i.e., for all $\alpha < \delta$, $M_\alpha \preccurlyeq H_\chi$ and for all $\alpha, \beta < \delta$ with $\alpha < \beta$, $M_\alpha \preccurlyeq M_\beta$. Then $M := \bigcup_{\alpha<\delta} M_\alpha \preccurlyeq H_\chi$.* □

This lemma allows it to construct elementary submodels of H_χ with various closure properties.

Lemma 5.2. *Let χ, κ, and λ be infinite cardinals such that $\kappa^\lambda = \kappa$ and $\lambda < \chi$. Then for every $A \subseteq H_\chi$ with $|A| \leq \kappa$ there is $M \preccurlyeq H_\chi$ such that $A \subseteq M$, $|M| \leq \kappa$, and for $x \subseteq M$ with $|x| \leq \lambda$, $x \in M$.*

Proof. By the Löwenheim-Skolem Theorem, there is $M_0 \preccurlyeq H_\chi$ with $A \subseteq M_0$ and $|M_0| \leq \kappa$. By induction on $\alpha < \lambda^+$, construct a chain $(M_\alpha)_{\alpha<\lambda^+}$ of elementary submodels of H_χ of size $\leq \kappa$ as follows:

For a limit ordinal $\alpha < \lambda^+$ let $M_\alpha := \bigcup_{\beta<\alpha} M_\beta$. By Lemma 5.1, $M_\alpha \preccurlyeq H_\chi$. By $\kappa^\lambda = \kappa$, $\lambda < \kappa$ and thus $\alpha < \kappa$. Since $|M_\beta| \leq \kappa$ for all $\beta < \alpha$, $|M_\alpha| \leq \kappa$.

If α is a successor, say $\alpha = \beta + 1$, let $M_\alpha \preccurlyeq H_\chi$ be such that $|M_\alpha| \leq \kappa$, $M_\beta \subseteq M_\alpha$, and for each $x \subseteq M_\beta$ with $|x| \leq \lambda$, $x \in M_\alpha$. This is possible by the Löwenheim-Skolem Theorem together with the fact that M_β has not more than $\kappa^\lambda = \kappa$ subsets of size λ.

$M := \bigcup_{\alpha<\lambda^+} M_\alpha$ works for the lemma: Clearly, $A \subseteq M$. Since $\lambda^+ \leq \kappa$, $|M| \leq \kappa$. By Lemma 5.1, $M \preccurlyeq H_\chi$. Let $x \subseteq M$ be of size λ. For each $y \in x$ let $\alpha_y < \lambda^+$ be such that $y \in M_{\alpha_y}$. Let $\alpha := \sup\{\alpha_y : y \in x\}$. Note that $\alpha < \lambda^+$ since $|x| \leq \lambda$. Now $x \subseteq M_\alpha$ and thus $x \in M_{\alpha+1} \subseteq M$. □

Using this lemma, we can give an easy proof of another famous theorem of Arhangel'skii. For a topological space X and $x \in X$, a family B of open subsets of X is called a *local base* at x if every element of B contains x and for every open set $O \subseteq X$ containing x there is $U \in B$ with $U \subseteq O$. X is *first countable* if for every $x \in X$ there is an at most countable local base at x.

Theorem 5.3 (Arhangel'skii, see [En77])**.** *Let X be compact and first countable. Then $|X| \leq 2^{\aleph_0}$.*

Proof. Let τ be the topology of X. Let χ be large enough and pick $M \preccurlyeq H_\chi$ such that $X, \tau \in M$, $|M| \leq 2^{\aleph_0}$, and for all countable $a \subseteq M$, $a \in M$. M exists by Lemma 5.2.

Claim 1. $X \cap M$ is a closed subspace of X.

Let $x \in X$ be in the closure of $X \cap M$. Since X is first countable, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $X \cap M$ converging to x. Since every countable subset of M is an element of M and $(x_n)_{n\in\mathbb{N}}$ can be considered as a subset of $\mathbb{N} \times (X \cap M) \subseteq M$, $(x_n)_{n\in\mathbb{N}} \in M$. Since X is Hausdorff, x is the unique limit of $(x_n)_{n\in\mathbb{N}}$. By compactness of X, M thinks that $(x_n)_{n\in\mathbb{N}}$ has a limit. Thus $x \in M$. This finishes the proof of the claim.

Clearly, the lemma follows from

Claim 2. $X \subseteq M$.

Suppose there is $x \in X \setminus M$. Since M knows that X is first countable, for every $y \in X \cap M$, M contains a countable local base B_y at y. Since $\mathbb{N} \in M$, $B_y \subseteq M$ for all $y \in X \cap M$. For each $y \in X \cap M$ pick $U_y \in B_y$ such that $x \notin U_y$. Clearly, $X \cap M \subseteq \bigcup_{y\in X\cap M} U_y$ and $x \in X \setminus \bigcup_{y\in X\cap M} U_y$. Since $X \cap M$ is compact by Claim 1, there is a finite set $F \subseteq X \cap M$ such that $X \cap M \subseteq \bigcup_{y\in F} U_y$. Since $F \subseteq M$ is finite, $\{U_y : y \in F\} \in M$. Now

$$H_\chi \models X \setminus \bigcup \{U_y : y \in F\} \neq \emptyset.$$

Thus

$$M \models X \setminus \bigcup \{U_y : y \in F\} \neq \emptyset.$$

Therefore $X \cap M \not\subseteq \bigcup \{U_y : y \in F\}$. A contradiction. □

Note that for every M that contains all its subsets of size $\leq \lambda$, we have $|M| \geq 2^\lambda$. And even $2^{\aleph_0}$ can be large. Sometimes it is sufficient to consider models $M \preccurlyeq H_\chi$ with the property that all subsets of M of size $\leq \lambda$ are covered by elements of M of size $\leq \lambda$. For example, for every $n \in \mathbb{N}$ and every set $A \subseteq H_\chi$ of size $\aleph_n$ there is $M \preccurlyeq H_\chi$ such that $A \subseteq M$, $|M| = \aleph_n$, and for every countable subset x of M there is a countable set $y \in M$ with $x \subseteq y$. Recently, models of this kind have been very useful in [FuGeSou∞]. Another application of such models, more closely connected to the topic of this article, is Dow's proof ([Dow88]) of the result of Hajnal and Juhasz ([Juh80]) that a topological space X has countable weight if every subspace of size at most $\aleph_1$ has countable weight. Another important class of models are the *internally approachables*. One instance of internal approachability is *V_κ-likeness*.

For a cardinal κ an elementary submodel M of some H_χ, $\chi > \kappa$, is called V_κ-like if there is a chain $(M_\alpha)_{\alpha<\kappa}$ of elementary submodels of H_χ with $M = \bigcup_{\alpha<\kappa} M_\alpha$ such that for each α, M_α has size less than κ and $(M_\beta)_{\beta\leq\alpha} \in M_{\alpha+1}$. Note that if $(M_\beta)_{\beta\leq\alpha} \in M_{\alpha+1}$, then $M_\alpha \in M_{\alpha+1}$ since $M_\alpha \in H_\chi$, M_α can be defined in H_χ as the last element of the sequence $(M_\beta)_{\beta\leq\alpha}$, and $M_{\alpha+1} \preccurlyeq H_\chi$. Also note that every

V_κ-like model has size κ. Among other nice properties, if κ is regular, every subset of a V_κ-like model M of size less than κ is covered by an element of M of size less than κ. For let $a \subseteq M$ be of size less than κ. By regularity of κ, there is $\alpha < \kappa$ such that $a \subseteq M_\alpha$. Now $M_\alpha \in M_{\alpha+1}$ covers a, has size less than κ, and is contained in M.

Various kinds of internally approachable models have been successfully used by Shelah and others. For example, some socalled *black box principles* are formulated using internally approachable models. These principles hold in ZFC and can be used to construct structures with certain second order properties (e.g. in [MeSh93] or in [EkMe90]). That is, while constructing a structure by induction, one can keep track of its later endomorphisms. More on this topic will be contained in [Sh∞]. V_κ-like models were used in [FuSou97] in order to characterize partial orderings with the socalled *weak Freese-Nation property.*

6. How to get approximations of X from M

What we really did in the proof Lemma 4.3 was to consider the the topology on X which is generated by the open subsets of X that are contained in the elementary submodel M. We basically showed that this topology coincides with the original topology on X.

Similar things happened in the proof of Theorem 5.3. Here we concidered the space $X \cap M$ and showed that $X \cap M$ already coincides with X. However, there are many proofs using elementary submodels M of some H_χ where the approximation of a topological space X given by M is really smaller than X. All we did so far was to consider a space obtained from X by thinning out the topology or by passing to a subspace of X. Another method, which is especially useful if the spaces under consideration are compact, is to pass to a quotient.

Let X be compact and χ sufficiently large. For $M \preccurlyeq H_\chi$ define an equivalence relation $\sim_M$ on X as follows:

$$x \sim_M y \text{ if, and only if, for all continuous } f : X \to \mathbb{R} \text{ with } f \in M,\ f(x) = f(y).$$

$X/{\sim_M}$ is compact and in some sense the most reasonable approximation of X we can obtain from M.

For a set A let $[A]^{\aleph_0}$ denote the set of countably infinite subsets of X. $\mathcal{C} \subseteq [A]^{\aleph_0}$ is *closed and unbounded* in $[A]^{\aleph_0}$ if every countable $B \subseteq A$ is included in some element of $\mathcal{C}$ and the union of every countable chain in $\mathcal{C}$ is again an element of $\mathcal{C}$. Note that by the Löwenheim-Skolem Theorem together with Lemma 5.1, for every infinite cardinal χ the set $\{M \in [H_\chi]^{\aleph_0} : M \preccurlyeq H_\chi\}$ is closed and unbounded in $[H_\chi]^{\aleph_0}$.

Bandlow ([Ba91]) proposed the following type of characterization of a class $\mathcal{K}$ of compact spaces by a class $\mathcal{F}$ of continuous mappings:

> A compact space X is in the class $\mathcal{K}$ if, and only if, for every sufficiently large χ there is a closed and unbounded subset $\mathcal{C}$ of $[H_\chi]^{\aleph_0}$ consisting of elementary submodels of H_χ such that for every $M \in \mathcal{C}$ the quotient map $q : X \to X/\sim_M$ belongs to $\mathcal{F}$.

For example, this works well in the case of *openly generated* compact spaces, which were studied by Ščepin ([Šč81]). Among other things, Ščepin proved that every openly generated compact space X satisfies the countable chain condition (c.c.c.), i.e., every family pairwise disjoint non-empty open subsets of X is at most countable. Bandlow ([Ba91]) characterized openly generated compact spaces in terms of elementary submodels of H_χ's using the $\sim_M$-approach and gave a simple proof Ščepin's result on the c.c.c. of openly generated spaces using his characterization. The class $\mathcal{F}$ of mappings used to characterize open generatedness is the class of *open* mappings. A continuous mapping is called open if the images of open sets under this mapping are again open.

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Freie Universität Berlin, II. Mathematisches Institut, Arnimallee 3, D—14195 Berlin
E-mail address: geschke@math.fu-berlin.de

[1]This is number 375 in Shelah's publication list. It can be found at Shelah's archive, which is located at http:/www.math.rutgers.edu/FTP_DIR/shelah.