NEGATIVE INDUCED RAMSEY THEOREMS

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We give the proofs, due to Shelah, of two negative partition results. A slightly weaker form of Theorem 1 b) has been claimed by Hajnal and Komjath [2], but their proof was incorrect. Hajnal and Komjath then gave a corrected version of their proof in [3].

Later, Komjath [4] showed that it is consistent that there is a graph G of size $\aleph_1$ such that for every graph H and every coloring of the edges of H with $\aleph_1$ colors, every induced copy of G in H has edges of any of the $\aleph_1$ colors. Komjath proves his result by generically adding an uncountable graph, not just a single Cohen real as we do below.

Theorem 1. *Let V be the ground model and $g : \omega \to 2$ a Cohen real over V.*

a) In $V[g]$ there is a bipartite graph $(\omega_1 \dot\cup \omega_1, E_{\omega_1,\omega_1})$ such that for all graphs (λ, E),

$$(\lambda, E) \not\to (\omega_1 \dot\cup \omega_1, E_{\omega_1,\omega_1})^2_2.$$

b) In $V[g]$ there is a bipartite graph $(\omega \dot\cup \mathfrak{b}^V, E_{\omega,\mathfrak{b}})$ such that for all graphs (λ, E),

$$(\lambda, E) \not\to (\omega \dot\cup \mathfrak{b}^V, E_{\omega,\mathfrak{b}})^2_2.$$

Note that adding a single Cohen real can decrease $\mathfrak{b}$ [1]. That is why we write $\mathfrak{b}^V$ in b).

The proofs of the two parts of the theorem are very similar. They rely on two simple combinatorial lemmas that can be considered as somewhat trivial instances of a much deeper result due to Todorčević [5]. Todorcevic's result plays an important role in Komjath's argument in [4].

Lemma 2. *There is a map $c_{\omega_1,\omega_1} : [\omega_1]^2 \to \omega$ such that for every uncountable $S \subseteq \omega_1$, $c[[S]^2]$ is infinite.*

Proof. For each $\alpha < \omega_1$ fix a 1-1 map $f_\alpha : \alpha \to \omega$. For $\alpha < \beta < \omega_1$ let $c_{\omega_1,\omega_1}(\alpha, \beta) = f_\beta(\alpha)$.

Now suppose that $S \subseteq \omega_1$ is uncountable. Let $\beta \in S$ be such that $S \cap \beta$ is infinite. Since f_β is 1-1,

$$\{c_{\omega_1,\omega_1}(\alpha, \beta) : \alpha \in S \cap \beta\} = f_\beta[S \cap \beta]$$

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is infinite. It follows that $c_{\omega_1,\omega_1}[[S]^2]$ is infinite. □

Lemma 3. *Let $\mathfrak{b}$ denote the unboundedness number. Then there is a mapping $c_{\omega,\mathfrak{b}} : \omega \times \mathfrak{b} \to \omega$ such that for every uncountable $S \subseteq \mathfrak{b}$ there is $n \in \omega$ such that the set*

$$\{c_{\omega,\mathfrak{b}}(n,\alpha) : \alpha \in S\}$$

is infinite.

Proof. Let $(f_\alpha)_{\alpha<\mathfrak{b}}$ be a $\leq^*$-increasing unbounded sequence in ω^ω. For $n \in \omega$ and $\alpha < \mathfrak{b}$ let $c_{\omega,\mathfrak{b}}(n,\alpha) = f_\alpha(n)$.

Now suppose that $S \subseteq \mathfrak{b}$ is unbounded in $\mathfrak{b}$. Since $(f_\alpha)_{\alpha<\mathfrak{b}}$ is $\leq^*$-increasing and unbounded, $(f_\alpha)_{\alpha\in S}$ is also unbounded in ω^ω. Assume that for all $n \in \omega$ the set

$$\{c_{\omega,\mathfrak{b}}(n,\alpha) : \alpha \in S\} = \{f_\alpha(n) : \alpha \in S\}$$

is finite. Then the function $f : \omega \to \omega$ defined by $f(n) = \max\{f_\alpha(n) : \alpha \in S\}$ is an upper bound of $(f_\alpha)_{\alpha\in S}$, a contradiction.

It follows that for some $n \in \omega$, $\{c_{\omega,\mathfrak{b}}(n,\alpha) : \alpha \in S\}$ is infinite. □

Remark 4. A partial converse of Lemma 3 is also true: If κ regular and uncountable and there is a mapping $c : \omega \times \kappa \to \omega$ such that for all unbounded sets $S \subseteq \kappa$ there is $n \in \omega$ such that $\{c(n,\alpha) : \alpha \in S\}$ is infinite, then $\kappa \geq \mathfrak{b}$:

For $\alpha < \kappa$ and $n \in \omega$ let $f_\alpha(n) = c(n,\alpha)$. We claim that $(f_\alpha)_{\alpha<\kappa}$ is unbounded in ω^ω. If not, then there is a function $f : \omega \to \omega$ that is $\leq^*$-above all f_α. For each $\alpha < \kappa$ fix n_α such that for all $n \geq n_\alpha$, $f(n) \leq f_\alpha(n)$. Now for some $m \in \omega$, the set $S = \{\alpha < \kappa : n_\alpha = m\}$ is unbounded in κ. By thinning out S further, we may assume that all f_α, $\alpha \in S$, agree on all $n < m$.

By the properties of c, for some $n \in \omega$, $F_n = \{f_\alpha(n) : \alpha \in S\}$ is infinite. However, if $n < m$, then F_n is a singleton by the choice of S. If $n \geq m$, then F_n is bounded by $f(n)$, a contradiction.

Proof of Theorem 1. a) In V, let $c_{\omega_1,\omega_1} : [\omega_1]^2 \to \omega$ be as in Lemma 2. For $\alpha, \beta < \omega_1$ we let $\{\alpha,\beta\} \in E_{\omega_1,\omega_1}$ if and only if $\alpha \neq \beta$ and $g(c_{\omega_1,\omega_1}(\alpha,\beta)) = 1$. Fix a name $\dot{E}_{\omega_1,\omega_1}$ for E_{ω_1,ω_1} using the maximality pinciple. As usual, for ground model elements we do not distinguish between the actual sets and their canonical names.

Now let (λ, E) be a graph on some cardinal λ. Choose a name $\dot{E}$ for E. We may assume that every Cohen condition forces $\dot{E}$ to be a subset of $[\lambda]^2$. We define a coloring $c : E \to 2$ as follows:

For $\{\sigma,\tau\} \in E$ let n be minimal with the property that the Cohen condition $g \restriction n$ forces $\{\sigma,\tau\} \in \dot{E}$. Let $c(\sigma,\tau) = g(n)$. We claim that (λ, E) does not contain an induced monochromatic copy of $(\omega_1 \dot{\cup} \omega_1, E_{\omega_1,\omega_1})$.

For suppose that $h_0, h_1 : \omega_1 \to \lambda$ induce an embedding of $(\omega_1 \dot\cup \omega_1, E_{\omega_1,\omega_1})$ into (λ, E) such that all edges in E_{ω_1,ω_1} are mapped to edges of the same color $i \in 2$. Let $\dot h_0$ and $\dot h_1$ be names for h_0 and h_1, respectively, and let $\dot c$ be a name for c. There is some $n \in \omega$ such that the condition $g \restriction n$ forces that $\dot h_0$ and $\dot h_1$ induce an embedding of $(\omega_1 \dot\cup \omega_1, E_{\omega_1,\omega_1})$ into (λ, E) such that all edges in E_{ω_1,ω_1} are mapped to edges of color i and moreover, $g \restriction n$ decides $\dot h_0(\alpha)$ and $\dot h_1(\alpha)$ for all α in some uncountable set $S \subseteq \omega_1$. Note that S can be chosen in the ground model. We can also choose n so that $g \restriction n$ forces $\dot c$ to satisfy the definition of c using the parameter $(\lambda, \dot E)$. By Lemma 2, the set $c_{\omega_1,\omega_1}[[S]^2]$ is infinite. Hence, there are $\alpha, \beta \in S$ such that $\alpha \neq \beta$ and $m = c_{\omega_1,\omega_1}(\alpha, \beta) \geq n$.

Let $p : m + 1 \to 2$ be an extension of $g \restriction n$ such that $p(m) = 1$. Now, if $\dot g$ is a name for the Cohen real, then p forces that $m + 1$ is the minimal k such that $\dot g \restriction k$ forces $\{\alpha, \beta\}$ to be in $\dot E_{\omega_1,\omega_1}$. Since $p \restriction m$ already decides $\dot h_0(\alpha)$ and $\dot h_1(\beta)$ to be $h_0(\alpha)$ and $h_1(\beta)$, respectively, p also forces that $m + 1$ is the minimal k such that $\dot g \restriction k$ forces $\{h_0(\alpha), h_1(\beta)\} \in E$. Now the condition $p^\frown(1 - i)$ forces that $\dot c(\dot h_0(\alpha), \dot h_1(\beta)) = 1 - i$, contradicting our assumption that $g \restriction n$ and hence p force that $\{\alpha, \beta\}$ is mapped to an edge of color i.

b) In V, let $c_{\omega,\mathfrak{b}} : \omega \times \mathfrak{b} \to \omega$ be as in Lemma 3. For $n \in \omega$ and $\alpha < \mathfrak{b}^V$ we let $\{n, \alpha\} \in E_{\omega,\mathfrak{b}}$ if and only if $g(c_{\omega,\mathfrak{b}^V}(n, \alpha)) = 1$ and fix a name $\dot E_{\omega,\mathfrak{b}}$ for $E_{\omega,\mathfrak{b}}$.

Now let (λ, E) be a graph on some cardinal λ. Choose a name $\dot E$ for E. We may assume that every Cohen condition forces $\dot E$ to be a subset of $[\lambda]^2$. We define a coloring $c : E \to 2$ exactly as in the proof of a) and claim that (λ, E) does not contain an induced monochromatic copy of $(\omega \dot\cup \mathfrak{b}^V, E_{\omega,\mathfrak{b}})$.

Suppose that $h_0 : \omega \to \lambda$ and $h_1 : \mathfrak{b}^V \to \lambda$ induce an embedding of $(\omega \dot\cup \mathfrak{b}^V, E_{\omega,\mathfrak{b}})$ into (λ, E) such that all edges in $E_{\omega,\mathfrak{b}}$ are mapped to edges of the same color $i \in 2$. Let $\dot h_0$ and $\dot h_1$ be names for h_0 and h_1, respectively, and let $\dot c$ be a name for c. There is some $n \in \omega$ such that the condition $g \restriction n$ forces that $\dot h_0$ and $\dot h_1$ induce an embedding of $(\omega \dot\cup \mathfrak{b}^V, E_{\omega,\mathfrak{b}})$ into $(\lambda, \dot E)$ such that all edges in $E_{\omega,\mathfrak{b}}$ are mapped to edges of color i and moreover, $g \restriction n$ decides $\dot h_1(\alpha)$ for all α in some unbounded set $S \subseteq \mathfrak{b}^V$. The set S exists since $\mathfrak{b}^V$ is regular and hence of uncountable cofinality. Note that S can be chosen in the ground model. We can also choose n so that $g \restriction n$ forces $\dot c$ to satisfy the definition of c using the parameter $(\lambda, \dot E)$. By Lemma 3, for some $a \in \omega$ the set $\{c_{\omega,\mathfrak{b}}(a, \beta) : \beta \in S\}$ is infinite. By enlarging n if necessary, we may assume that $g \restriction n$ already decides $\dot h_0(a)$.

By the choice of a, there is $\beta \in S$ such that $m = c_{\omega,\mathfrak{b}}(a, \beta) \geq n$. Let $p : m+1 \to 2$ be an extension of $g \restriction n$ such that $p(m) = 1$. Now p forces that $m+1$ is the minimal k such that $\dot g \restriction k$ forces $\{a, \beta\}$ to be in $\dot E_{\omega,\mathfrak{b}}$. Since $p \restriction m$ already decides $\dot h_0(a)$ and $\dot h_1(\beta)$ to be $h_0(a)$ and $h_1(\beta)$, respectively, p also forces that $m + 1$ is the minimal k such that $\dot g \restriction k$ forces $\{h_0(a), h_1(\beta)\} \in E$. Now the condition $p^\frown(1 - i)$ forces that

$\dot{c}(\dot{h}_0(a), \dot{h}_1(\beta)) = 1 - i$, contradicting our assumption that $g \restriction n$ and hence p force that $\{a, \beta\}$ is mapped to an edge of color i. $\square$

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