FUNCTIONS FOR WHICH ALL POINTS ARE A LOCAL MINIMUM OR MAXIMUM

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Abstract. Let X be a connected separable linear order, a connected separable metric space or a connected, locally connected complete metric space. We show that every continuous function $f : X \to \mathbb{R}$ with the property that every $x \in X$ is a local maximum or minimum of f is in fact constant. We provide an example of a compact connected linear order X and a continuous function $f : X \to \mathbb{R}$ that is not constant and yet every point of X is a local minimum or maximum of f.

The following question was recently asked by M. R. Wojcik [1]:

Question 1. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that every point in $[0, 1]$ is a local maximum or minimum of f. Is it true that f has to be constant?

The answer is clearly yes if f is assumed to be differentiable, but the question is about continuous functions. Still, the answer to Question 1 is yes, and this was shown by a number of people independently. However, we are not aware of any published proof of this fact. In this note we give two elementary proofs showing that a continuous function from $[0, 1]$ to $\mathbb{R}$ for which every point in $[0, 1]$ is a local minimum or maximum indeed has to be constant. The first proof only uses the most basic topological properties of $\mathbb{R}$. We actually get the following theorem:

Theorem 2. *Let X be a connected separable metric space. Then every continuous function $f : X \to \mathbb{R}$ for which every $x \in X$ is a local minimum or maximum is constant.*

The second proof uses the linear order on the reals.

Theorem 3. *Let X be a connected separable linearly ordered space. Then every continuous function $f : X \to \mathbb{R}$ for which every $x \in X$ is a local minimum or maximum is constant.*

Note, however, that Theorem 3 is weaker than it looks at first sight. Every connected separable linear order is actually isomorphic to some interval of the real line. But see Remark 4.

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Proof of Theorem 2. Let $f : X \to \mathbb{R}$ be continuous and such that f has a local extremum at every $x \in X$. Since X is a separable metric space, the topology on X has a countable base $\{B_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ let

$$D_n^{\min} = \{x \in B_n : \forall y \in B_n(f(x) \leq f(y))\}$$

and

$$D_n^{\max} = \{x \in B_n : \forall y \in B_n(f(x) \geq f(y))\}.$$

Clearly, f is constant on each $D_n^{\min}$ and each $D_n^{\max}$.

If $x \in X$, then by our assumptions on f, x is a local minimum or maximum of f. Assume it is a local mininum. Then there is some $n \in \mathbb{N}$ such that $x \in B_n$ and for all $t \in B_n$, $f(t) \geq f(x)$. In particular, $x \in D_n^{\min}$. Similarly, if x is a local maximum, then for some $n \in \mathbb{N}$, $x \in D_n^{\max}$. In summary, we have

$$X = \bigcup_{n \in \mathbb{N}} (D_n^{\min} \cup D_n^{\max}).$$

It follows that $f[X]$ is countable. Since X is connected, so is $f[X]$. But the only nonempty countable and connected subsets of the real line are the singletons. It follows that f is constant. $\square$

Proof of Theorem 3. Let $<$ denote the order on X. Suppose that f is not constant. For simplicity assume that there are $x, y \in X$ such that $x < y$ and $f(x) < f(y)$. Since X is connected, $[x, y]$ is connected. Since f is continuous, $f[[x, y]]$ is connected. It follows that $[f(x), f(y)] \subseteq f[[x, y]]$.

Since X is separable, every family of pairwise disjoint open intervals in X is countable. It follows that there is some $z \in (f(x), f(y))$ such that $f^{-1}(z)$ does not contain a nonempty open interval. Since X is connected, every bounded subset of X has a supremum in X. Let

$$a = \sup\{b \in (x, y) : \forall c \in (x, b)(f(c) \leq z)\}.$$

By the continuity of f, $f(a) = z$. By the definition of a and by the connectedness of X, for every $b > a$ there is $c \in (a, b)$ such that $f(c) > z$. Since a is a local extremum of f, it follows that a is a local minimum.

This implies that there is $b < a$ such that for all $c \in (b, a)$, $f(c) \geq z$. But by the definition of a, for all $c \in (b, a)$, $f(c) = z$. Hence $f^{-1}(z)$ contains a non-empty open interval after all, contradicting the choice of z. $\square$

Remark 4. A closer analysis of the proofs of Theorem 2 and Theorem 3 shows that in both cases the separability assumption can be weakened.

a) Let X be a connected topological space that has a base of its topology of size $< |\mathbb{R}|$. If $f : X \to \mathbb{R}$ is continuous and such that every $x \in X$ is a local extremum of f, then f is constant. This holds in particular if X is a connected metric space such that every family of pairwise disjoint open sets is of size $<|\mathbb{R}|$.

b) Let X be a connected linear order such that every family of pairwise disjoint open intervals is of size $< |\mathbb{R}|$. If $f : X \to \mathbb{R}$ is continuous and such that every $x \in X$ is a local extremum of f, then f is constant.

A questions that arises naturally is this:

Question 5. Let X be a connected topological space such that every family of pairwise disjoint open sets is of size $<|\mathbb{R}|$. If $f : X \to \mathbb{R}$ is continuous and such that every $x \in X$ is a local extremum of f, does f have to be constant?

Remark 4 tells us where we should look if we want to find a connected space X and a continuous function $f : X \to \mathbb{R}$ that is not constant but such that every $x \in X$ is a local minimum or maximum.

Example 6. Let I denote the closed unit interval. Consider the set $X = I \times I$ ordered lexicographically, i.e., for $a, b, c, d \in I$ let $(a, b) < (c, d)$ if $a < c$ or ($a = c$ and $b < d$). The linear order X can be considered as obtained from I by replacing every point of I by a copy of I.

It is easily checked that X is a connected linear order. It is even compact. The projection $f : X \to \mathbb{R}; (a, b) \mapsto a$ is continuous and obviously not constant. However, every $x \in X$ is a local extremum of f.

It it worth pointing out that X is not metrizable, which follows from the fact that X is compact but not separable. This brings up the following question:

Question 7. Is there an example of a connected metric space X with a continuous function $f : X \to \mathbb{R}$ that is not constant but such that every point in X is a local minimum or maximum of f?

We can provide a partial answer to this question:

Theorem 8. *Suppose X is a connected, locally connected complete metric space. If $f : X \to \mathbb{R}$ is a continuous function and every $x \in X$ is a local extremum of f, then f is constant.*

The proof of this theorem is based on the following lemma:

Lemma 9. *Let X be a metric space that is Baire, i.e., in which no nonempty open set is the union of countably many nowhere dense sets. If $f : X \to \mathbb{R}$ is continuous and such that every $x \in X$ is a local extremum of f, then $V = \bigcup_{y \in \mathbb{R}} \operatorname{int}(f^{-1}(y))$ is dense in X.*

Proof. Since X is metric, by Bing's Metrization Theorem it has a σ-discrete base $\mathcal{B}$. Let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ with each $\mathcal{B}_n$ discrete. For each $x \in X$ fix $B_x \in \mathcal{B}$ such that $f(x) \leq f(x')$ for all $x' \in B_x$ if x is a local minimum of f or $f(x) \geq f(x')$ for all $x' \in B_x$ if x is a local maximum. For every $n \in \mathbb{N}$ let $X_n^{\min}$ denote the set of all $x \in X$ that are local minima of f with $B_x \in \mathcal{B}_n$. Similarly, let $X_n^{\max}$ denote the set of all $x \in X$ that are local maxima with $B_x \in \mathcal{B}_n$.

Now let $G \subseteq X$ be nonempty and open. Since X is Baire, there is $n \in \mathbb{N}$ such that $X_n^{\min}$ or $X_n^{\max}$ is dense in some nonempty open set $G_0 \subseteq G$. Assume that $X_n^{\min}$ is dense in G_0 and fix $x \in X_n^{\min} \cap G_0$. Then $H = G_0 \cap B_x$ is nonempty and open, and $X_n^{\min} \cap H$ is dense in H. Since $\mathcal{B}_n$ is discrete, for every $x' \in B_x \cap X_n^{\min}$ we have $B_x = B_{x'}$ and thus $f(x) = f(x')$. It follows that f is constant on $H \cap X_n^{\min}$. Since f is continuous, f is constant on all of H. Therefore $H \subseteq V$ and hence $G \cap V \neq \emptyset$. $\square$

Proof of Theorem 8. Suppose f is not constant. For every $y \in \mathbb{R}$ let $V_y = \operatorname{int}(f^{-1}(y))$. Let $V = \bigcup_{y\in\mathbb{R}} V_y$ and $F = X \setminus V$. Note that $F = \bigcup_{y\in\mathbb{R}} \operatorname{bd}(f^{-1}(y))$. Since X is connected and f is not constant, for every $y \in f[X]$ we have $\operatorname{bd}(f^{-1}(y)) \neq \emptyset$. In particular, $F \neq \emptyset$.

Since F is closed in X, F is a complete metric space. By Lemma 9, the space F has a nonempty open subset on which f is constant. In other words, there is an open subset U of X such that $U \cap F \neq \emptyset$ and f is constant on $U \cap F$. Since X is locally connected, we may assume that U is connected.

Since $U \not\subseteq V$, f is not constant on U. Let $y \in f[U]$ be different from the unique value of f on $U \cap F$. Now $\operatorname{bd}(V_y) \subseteq F$. Since $y \notin f[U \cap F]$, $\operatorname{bd}(V_y) \cap U = \emptyset$. But this implies that $V_y \cap U = \operatorname{int}(V_y) \cap U$ is a proper clopen subset of U, contradicting the assumption that U is connected. $\square$

References

[1] M.R. Wojcik, problem session, 34th Winter School in Abstract Analysis, Lhota nad Rohanovem, Czech Republik (2006)

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