FUNCTIONS FOR WHICH ALL POINTS ARE A LOCAL MINIMUM OR MAXIMUM

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Abstract. Let $X$ be a connected separable linear order, a connected separable metric space or a connected, locally connected complete metric space. We show that every continuous function $f : X \to \mathbb{R}$ with the property that every $x \in X$ is a local maximum or minimum of $f$ is in fact constant. We provide an example of a compact connected linear order $X$ and a continuous function $f : X \to \mathbb{R}$ that is not constant and yet every point of $X$ is a local minimum or maximum of $f$.

The following question was recently asked by M. R. Wojcik [1]:

**Question 1.** Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that every point in $[0, 1]$ is a local maximum or minimum of $f$. Is it true that $f$ has to be constant?

The answer is clearly yes if $f$ is assumed to be differentiable, but the question is about continuous functions. Still, the answer to Question 1 is yes, and this was shown by a number of people independently. However, we are not aware of any published proof of this fact. In this note we give two elementary proofs showing that a continuous function from $[0, 1]$ to $\mathbb{R}$ for which every point in $[0, 1]$ is a local minimum or maximum indeed has to be constant. The first proof only uses the most basic topological properties of $\mathbb{R}$. We actually get the following theorem:

**Theorem 2.** Let $X$ be a connected separable metric space. Then every continuous function $f : X \to \mathbb{R}$ for which every $x \in X$ is a local minimum or maximum is constant.

The second proof uses the linear order on the reals.

**Theorem 3.** Let $X$ be a connected separable linearly ordered space. Then every continuous function $f : X \to \mathbb{R}$ for which every $x \in X$ is a local minimum or maximum is constant.

Note, however, that Theorem 3 is weaker than it looks at first sight. Every connected separable linear order is actually isomorphic to some interval of the real line. But see Remark 4.

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Proof of Theorem 2. Let \( f : X \to \mathbb{R} \) be continuous and such that \( f \) has a local extremum at every \( x \in X \). Since \( X \) is a separable metric space, the topology on \( X \) has a countable base \( \{ B_n : n \in \mathbb{N} \} \). For each \( n \in \mathbb{N} \) let
\[
D_n^{\min} = \{ x \in B_n : \forall y \in B_n (f(x) \leq f(y)) \}
\]
and
\[
D_n^{\max} = \{ x \in B_n : \forall y \in B_n (f(x) \geq f(y)) \}.
\]
Clearly, \( f \) is constant on each \( D_n^{\min} \) and each \( D_n^{\max} \).

If \( x \in X \), then by our assumptions on \( f \), \( x \) is a local minimum or maximum of \( f \). Assume it is a local minimum. Then there is some \( n \in \mathbb{N} \) such that \( x \in B_n \) and for all \( t \in B_n \), \( f(t) \geq f(x) \). In particular, \( x \in D_n^{\min} \). Similarly, if \( x \) is a local maximum, then for some \( n \in \mathbb{N} \), \( x \in D_n^{\max} \). In summary, we have
\[
X = \bigcup_{n \in \mathbb{N}} (D_n^{\min} \cup D_n^{\max}).
\]
It follows that \( f[X] \) is countable. Since \( X \) is connected, so is \( f[X] \). But the only nonempty countable and connected subsets of the real line are the singletons. It follows that \( f \) is constant. \( \square \)

Proof of Theorem 3. Let \( < \) denote the order on \( X \). Suppose that \( f \) is not constant. For simplicity assume that there are \( x, y \in X \) such that \( x < y \) and \( f(x) < f(y) \). Since \( X \) is connected, \([x, y]\) is connected. Since \( f \) is continuous, \([f([x, y])]\) is connected. It follows that \([f(x), f(y)] \subseteq f([x, y])\).

Since \( X \) is separable, every family of pairwise disjoint open intervals in \( X \) is countable. It follows that there is some \( z \in (f(x), f(y)) \) such that \( f^{-1}(z) \) does not contain a nonempty open interval. Since \( X \) is connected, every bounded subset of \( X \) has a supremum in \( X \). Let
\[
a = \sup \{ b \in (x, y) : \forall c \in (x, b) (f(c) \leq z) \}.
\]
By the continuity of \( f \), \( f(a) = z \). By the definition of \( a \) and by the connectedness of \( X \), for every \( b > a \) there is \( c \in (a, b) \) such that \( f(c) > z \). Since \( a \) is a local extremum of \( f \), it follows that \( a \) is a local minimum.

This implies that there is \( b < a \) such that for all \( c \in (b, a) \), \( f(c) \geq z \). But by the definition of \( a \), for all \( c \in (b, a) \), \( f(c) = z \). Hence \( f^{-1}(z) \) contains a non-empty open interval after all, contradicting the choice of \( z \). \( \square \)

Remark 4. A closer analysis of the proofs of Theorem 2 and Theorem 3 shows that in both cases the separability assumption can be weakened.

a) Let \( X \) be a connected topological space that has a base of its topology of size \( < |\mathbb{R}| \). If \( f : X \to \mathbb{R} \) is continuous and such that every \( x \in X \) is a local extremum...
of $f$, then $f$ is constant. This holds in particular if $X$ is a connected metric space such that every family of pairwise disjoint open sets is of size $<|\mathbb{R}|$.

b) Let $X$ be a connected linear order such that every family of pairwise disjoint open intervals is of size $<|\mathbb{R}|$. If $f : X \to \mathbb{R}$ is continuous and such that every $x \in X$ is a local extremum of $f$, then $f$ is constant.

A question that arises naturally is this:

**Question 5.** Let $X$ be a connected topological space such that every family of pairwise disjoint open sets is of size $<|\mathbb{R}|$. If $f : X \to \mathbb{R}$ is continuous and such that every $x \in X$ is a local extremum of $f$, does $f$ have to be constant?

Remark 4 tells us where we should look if we want to find a connected space $X$ and a continuous function $f : X \to \mathbb{R}$ that is not constant but such that every $x \in X$ is a local minimum or maximum.

**Example 6.** Let $I$ denote the closed unit interval. Consider the set $X = I \times I$ ordered lexicographically, i.e., for $a, b, c, d \in I$ let $(a, b) < (c, d)$ if $a < c$ or $(a = c$ and $b < d)$. The linear order $X$ can be considered as obtained from $I$ by replacing every point of $I$ by a copy of $I$.

It is easily checked that $X$ is a connected linear order. It is even compact. The projection $f : X \to \mathbb{R}; (a, b) \mapsto a$ is continuous and obviously not constant. However, every $x \in X$ is a local extremum of $f$.

It it worth pointing out that $X$ is not metrizable, which follows from the fact that $X$ is compact but not separable. This brings up the following question:

**Question 7.** Is there an example of a connected metric space $X$ with a continuous function $f : X \to \mathbb{R}$ that is not constant but such that every point in $X$ is a local minimum or maximum?

We can provide a partial answer to this question:

**Theorem 8.** Suppose $X$ is a connected, locally connected complete metric space. If $f : X \to \mathbb{R}$ is a continuous function and every $x \in X$ is a local extremum of $f$, then $f$ is constant.

The proof of this theorem is based on the following lemma:

**Lemma 9.** Let $X$ be a metric space that is Baire, i.e., in which no nonempty open set is the union of countably many nowhere dense sets. If $f : X \to \mathbb{R}$ is continuous and such that every $x \in X$ is a local extremum of $f$, then $V = \bigcup_{y \in \mathbb{R}} \text{int}(f^{-1}(y))$ is dense in $X$. 
Proof. Since $X$ is metric, by Bing’s Metrization Theorem it has a $\sigma$-discrete base $\mathcal{B}$. Let $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ with each $\mathcal{B}_n$ discrete. For each $x \in X$ fix $B_x \in \mathcal{B}$ such that $f(x) \leq f(x')$ for all $x' \in B_x$ if $x$ is a local minimum of $f$ or $f(x) \geq f(x')$ for all $x' \in B_x$ if $x$ is a local maximum. For every $n \in \mathbb{N}$ let $X_n^{\text{min}}$ denote the set of all $x \in X$ that are local minima of $f$ with $B_x \in \mathcal{B}_n$. Similarly, let $X_n^{\text{max}}$ denote the set of all $x \in X$ that are local maxima with $B_x \in \mathcal{B}_n$.

Now let $G \subseteq X$ be nonempty and open. Since $X$ is Baire, there is $n \in \mathbb{N}$ such that $X_n^{\text{min}}$ or $X_n^{\text{max}}$ is dense in some nonempty open set $G_0 \subseteq G$. Assume that $X_n^{\text{min}}$ is dense in $G_0$ and fix $x \in X_n^{\text{min}} \cap G_0$. Then $H = G_0 \cap B_x$ is nonempty and open, and $X_n^{\text{min}} \cap H$ is dense in $H$. Since $B_n$ is discrete, for every $x' \in B_x \cap X_n^{\text{min}}$ we have $B_x = B_{x'}$ and thus $f(x) = f(x')$. It follows that $f$ is constant on $H \cap X_n^{\text{min}}$. Since $f$ is continuous, $f$ is constant on all of $H$. Therefore $H \subseteq V$ and hence $G \cap V \neq \emptyset$. \hfill \Box

Proof of Theorem 8. Suppose $f$ is not constant. For every $y \in \mathbb{R}$ let $V_y = \text{int}(f^{-1}(y))$. Let $V = \bigcup_{y \in \mathbb{R}} V_y$ and $F = X \setminus V$. Note that $F = \bigcup_{y \in \mathbb{R}} \text{bd}(f^{-1}(y))$. Since $X$ is connected and $f$ is not constant, for every $y \in f[X]$ we have $\text{bd}(f^{-1}(y)) \neq \emptyset$. In particular, $F \neq \emptyset$.

Since $F$ is closed in $X$, $F$ is a complete metric space. By Lemma 9, the space $F$ has a nonempty open subset on which $f$ is constant. In other words, there is an open subset $U$ of $X$ such that $U \cap F \neq \emptyset$ and $f$ is constant on $U \cap F$. Since $X$ is locally connected, we may assume that $U$ is connected.

Since $U \not\subseteq V$, $f$ is not constant on $U$. Let $y \in f[U]$ be different from the unique value of $f$ on $U \cap F$. Now $\text{bd}(V_y) \subseteq F$. Since $y \not\in f[U \cap F]$, $\text{bd}(V_y) \cap U = \emptyset$. But this implies that $V_y \cap U = \text{int}(V_y) \cap U$ is a proper clopen subset of $U$, contradicting the assumption that $U$ is connected. \hfill \Box

References


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