

# WEAK BOREL CHROMATIC NUMBERS

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ABSTRACT. Given a graph  $G$  whose set of vertices is a Polish space  $X$ , the *weak Borel chromatic number* of  $G$  is the least size of a family of pairwise disjoint  $G$ -independent Borel sets that covers all of  $X$ . Here a set of vertices of a graph  $G$  is *independent* if no two vertices in the set are connected by an edge.

We show that it is consistent with an arbitrarily large size of the continuum that every closed graph on a Polish space either has a perfect clique or has a weak Borel chromatic number of at most  $\aleph_1$ . We observe that some weak version of Todorčević's Open Coloring Axiom for closed colorings follows from MA.

Slightly weaker results hold for  $F_\sigma$ -graphs. In particular, it is consistent with an arbitrarily large size of the continuum that every locally countable  $F_\sigma$ -graph has a Borel chromatic number of at most  $\aleph_1$ .

We refute various reasonable generalizations of these results to hypergraphs.

## 1. INTRODUCTION

Given a graph  $G = (X, E)$ , a *coloring* of  $G$  is a function  $c$  from  $X$  to some set of colors such that no two vertices that are connected by an edge get the same color. The chromatic number of a graph is the least possible size of the range of a coloring. In [6], Kechris, Solecki and Todorčević studied *Borel chromatic numbers* of analytic graphs. We give their definition of Borel chromatic numbers. If  $X$  is a Polish space and  $G = (X, E)$  is a graph on  $X$ , the *Borel chromatic number*

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of  $G$  is the least  $n \leq \omega$  such that there is a coloring  $c : X \rightarrow n$  of  $G$  that is Borel. If no such  $n$  exists, the Borel chromatic number of  $G$  is uncountable. In [10], the Borel chromatic number of a graph is defined in a slightly different way, namely as the least size of the range of a Borel coloring of the graph into a Polish space. This definition only differs from the original definition in that the value “uncountable” is replaced by  $2^{\aleph_0}$ .

We extend the definition of the Borel chromatic number by increasing the resolution in the uncountable. Given a graph  $G = (X, E)$  on a Polish space  $X$ , the *weak Borel chromatic number* of  $G$  is the least cardinal  $\kappa$  such that there is a coloring  $c : X \rightarrow \kappa$  of  $G$  whose fibers are Borel. Observe that the weak Borel chromatic number agrees with the Borel chromatic number as long as only countable numbers are considered. A graph whose Borel chromatic number is uncountable has an uncountable weak Borel chromatic number as well, only that this number can be an uncountable cardinal below  $2^{\aleph_0}$ .

Kechris, Solecki and Todorcevic showed that there is a closed graph  $\mathcal{G}_0$  on the Cantor space  $2^\omega$  such that for every analytic graph  $G = (X, E)$  on a Polish space  $X$ ,  $G$  has an uncountable chromatic number if, and only if, there is a continuous function  $f : 2^\omega \rightarrow X$  that maps edges of  $\mathcal{G}_0$  to edges of  $G$ . Clearly, the Borel chromatic number of  $\mathcal{G}_0$  itself has to be uncountable. On the other hand,  $\mathcal{G}_0$  is a tree, i.e., it does not have any cycles, and therefore its true chromatic number is 2.

B. Miller showed that the *measurable* chromatic number of  $\mathcal{G}_0$  is in fact 3 [12]. He asked whether anything can be said about the weak Borel chromatic number of  $\mathcal{G}_0$  compared to other combinatorial cardinal characteristics of the continuum [13]. The proof by Kechris et al. that the Borel chromatic number of  $\mathcal{G}_0$  is uncountable actually shows that the weak Borel chromatic number of  $\mathcal{G}_0$  is at least  $\text{cov}(\mathbf{meager})$ , the least size of a family of meager sets that covers all of  $2^\omega$ . From the universal property of  $\mathcal{G}_0$  it follows that every analytic graph with an uncountable Borel chromatic number has a weak Borel chromatic number of at least  $\text{cov}(\mathbf{meager})$ .

An obvious lower bound for the chromatic number, and therefore for the Borel chromatic number of a graph  $G$ , is the supremum of the sizes of complete subgraphs. In Section 3 we show that if  $G = (X, E)$  is a graph on a Polish space whose edge relation is closed, then the Borel chromatic number of  $G$  can be forced to be  $\aleph_1$  by some ccc forcing unless there is a perfect subset of  $X$  that supports a complete subgraph of  $G$ . This forcing dichotomy is reminiscent of Todorćević's observation that an open graph on a Polish space either has a complete subgraph whose set of vertices is perfect, or has a countable chromatic number. Todorćević's observation led to the formulation of the Open Coloring Axiom [15]. From our results it follows that the Borel chromatic number of  $\mathcal{G}_0$  can be forced to be smaller than  $2^{\aleph_0}$ .

In Section 4 we use the ideas from Section 3 and show that under Martin's Axiom, a closed graph  $G$  on a second countable Hausdorff space of size  $< 2^{\aleph_0}$  is countably chromatic unless it has large complete subgraphs. The latter statement actually is a weak version of the Open Coloring Axiom mentioned above, but for closed as opposed to open graphs. In Section 5 we repeat the treatment that closed graphs received in Sections 3 and 4 with  $F_\sigma$ -graphs and obtain slightly weaker results. The results on  $F_\sigma$ -graphs in particular apply to the Vitali equivalence relation  $E_0$  (see [4]) and to graphs of the form  $\mathcal{G}^S$  studied in [10], which are similar to  $\mathcal{G}_0$  but not closed. It follows for example that it is consistent with an arbitrarily large size of the continuum that  $2^\omega$  is the union of  $\aleph_1$  compact  $E_0$ -transversals. Here a set  $T \subseteq 2^\omega$  is an  $E_0$ -transversal if no two distinct elements of  $T$  are related by  $E_0$ .

Finally, in Section 6 we consider hypergraphs and show that practically all of the results in the previous section fail badly even for 3-uniform hypergraphs, i.e., hypergraphs whose edges are 3-element subsets of the set of vertices.

## 2. PRELIMINARIES

There are two different ways to implement graphs. In [6] a graph on a set of  $X$  vertices is a symmetric, irreflexive relation on  $X$ . If the set of vertices is a Polish space  $X$ , a graph on  $X$  is *open*, *closed*,  $F_\sigma$ ,  $G_\delta$ ,

respectively *analytic* if the relation is, as a subset of

$$X^2 \setminus \{(x, x) : x \in X\}.$$

We prefer to consider a graph  $G$  as a pair  $(X, E)$ , where  $X$  is a set, the *set of vertices*, and  $E$  is a subset of the collection  $[X]^2$  of all two-element subsets of  $X$ .  $E$  is the *set of edges* of  $G$ .

For every Hausdorff space  $X$ , the natural topology on the set  $[X]^2$  is generated by sets of the form

$$[U, V] = \{\{x, y\} : x \in U \wedge y \in V\}$$

where  $U$  and  $V$  are disjoint open subsets of  $X$ .

It turns out that with respect to this topology, the set of edges of a graph is open, closed,  $F_\sigma$ , respectively  $G_\delta$  iff the *edge relation* is with respect to the usual topology on  $X^2 \setminus \{(x, x) : x \in X\}$ . In particular, by a *closed graph on a Polish space* we mean a pair  $(X, E)$ , where  $X$  is a Polish space and  $E$  is a closed subset of  $[X]^2$ .

**Definition 1.** Let  $G = (X, E)$  be a graph on a Polish space.

- a)  $I \subseteq X$  is *G-independent* or *independent in G* iff  $[I]^2 \cap E = \emptyset$ . (*G-independent* sets are called *G-discrete* in [6].)
- b)  $C \subseteq X$  is a *G-clique* or a *clique in G* iff  $[C]^2 \subseteq E$ .
- c) The *weak Borel chromatic number* of  $G$  is the least size of a family  $\mathcal{P}$  of pairwise disjoint *G-independent* Borel subsets of  $X$  such that  $X = \bigcup \mathcal{P}$ .

### 3. A FORCING DICHOTOMY FOR CLOSED GRAPHS

We prove the following theorem:

**Theorem 2.** *Let  $G = (\omega^\omega, E)$  be a closed graph. Then either  $G$  has a perfect clique or there is a ccc forcing extension of the universe in which  $\omega^\omega$  is covered by  $\aleph_1$  compact *G-independent* sets while  $2^{\aleph_0}$  is arbitrarily large.*

Before we turn to the proof of this theorem, we derive the following corollary from it:

**Corollary 3.** *Let  $X$  be a Polish space and let  $G = (X, E)$  be a closed graph. Then either  $G$  has a perfect clique or there is a ccc forcing extension of the universe in which the weak Borel chromatic number of  $G$  is at most  $\aleph_1$  while  $2^{\aleph_0}$  is arbitrarily large.*

*Proof.* Let  $X$  and  $G$  be as in the statement of the corollary and suppose that  $X$  does not contain a perfect  $G$ -clique. After removing countably many points from  $X$ , we may assume that  $X$  has no isolated points. Since  $X$  is a perfect Polish space, there is a continuous injective map  $f$  from  $\omega^\omega$  onto  $X$ . Let

$$G^* = (\omega^\omega, \{\{x, y\} \in [\omega^\omega]^2 : \{f(x), f(y)\} \in E\}).$$

Since  $f$  is continuous and 1-1,  $G^*$  is a closed graph on  $\omega^\omega$ . If there is a perfect  $G^*$ -clique, then there is a  $G^*$ -clique  $C$  that is homeomorphic to  $2^\omega$ . Since  $f$  is 1-1,  $f[C]$  is again homeomorphic to  $2^\omega$ . Moreover,  $f[C]$  is a  $G$ -clique. But this shows that  $G$  has a perfect clique, contradicting our assumption on  $G$ .

It follows that there is no perfect  $G^*$ -clique. Hence by Theorem 2, there is a ccc forcing extension of the universe in which  $\omega^\omega$  is covered by  $\aleph_1$  compact  $G^*$ -independent sets while the continuum is arbitrarily large. The images of these  $G^*$ -independent sets under  $f$  are compact  $G$ -independent sets that cover  $X$ .

Let  $(I_\alpha)_{\alpha < \omega_1}$  be an enumeration of a collection of closed  $G$ -independent sets that cover  $X$ . For each  $\alpha < \omega_1$  let  $B_\alpha = I_\alpha \setminus \bigcup\{I_\beta : \beta < \alpha\}$ . Now  $\{B_\alpha : \alpha < \omega_1\}$  is a partition of  $X$  into  $\aleph_1$   $G$ -independent Borel sets, witnessing that the weak Borel chromatic number of  $G$  is at most  $\aleph_1$  in this generic extension of the universe.  $\square$

The main tool in the proof of Theorem 2 is the forcing notion  $\mathbb{P}(G)$  associated with every closed graph  $G$  on  $\omega^\omega$ .

**Definition 4.** For every closed graph  $G = (\omega^\omega, E)$  let  $\mathbb{P}(G)$  denote the partial order defined as follows: A pair  $p = (T_p, F_p)$  is a condition in  $\mathbb{P}(G)$  if

- (1)  $T_p$  is a finite subtree of  $\omega^{<\omega}$ ,

- (2) there is some  $m_p \in \omega$  such that all maximal elements of  $T_p$  are of length  $m_p$ ,
- (3) if  $s$  and  $t$  are two distinct maximal elements of  $T_p$ , then for all  $x, y \in \omega^\omega$  with  $s \subseteq x$  and  $t \subseteq y$  we have  $\{x, y\} \notin E$ ,
- (4)  $F_p$  is a finite subset of  $\omega^\omega$ , and
- (5) for all  $\{x, y\} \in [F_p]^2$ ,  $x \upharpoonright m_p \in T_p$  and  $x \upharpoonright m_p \neq y \upharpoonright m_p$ .

For two conditions  $p, q \in \mathbb{P}(G)$  we write  $p \leq q$  iff

- (6)  $T_q \subseteq T_p$  and  $T_q$  consists precisely of all elements of  $T_p$  of length  $\leq m_q$ .
- (7)  $F_q \subseteq F_p$ .

Observe that conditions (3) and (5) imply that  $F_p$  is a  $G$ -independent subset of  $X$ . The idea is that the tree  $T_p$  approximates a compact  $G$ -independent subset of  $\omega^\omega$  that contains all the elements of  $F_p$ . Our first goal is to show that  $\mathbb{P}(G)$  is ccc if  $G$  has no perfect cliques.

We need the following two results:

**Theorem 5** (Kubiś [7]). *Let  $Y$  be an analytic Hausdorff space. If  $F \subseteq [Y]^2$  is  $G_\delta$  and the graph  $(Y, F)$  has an uncountable clique, then it has a perfect clique.*

**Theorem 6** (Galvin, see [5, Theorem 19.7]). *Let  $Y$  be an uncountable Polish space and let  $K_0, \dots, K_{n-1} \subseteq [Y]^2$  be sets with the Baire property such that*

$$[Y]^2 = K_0 \cup \dots \cup K_{n-1}.$$

*Then there is a perfect set  $P \subseteq Y$  such that for some  $i \in n$ ,  $[P]^2 \subseteq K_i$ .*

The following lemma was conjectured by the author and proved by Conley and B. Miller. The proof presented here is due to the author.

**Lemma 7.** *If  $G$  has no perfect cliques, then  $\mathbb{P}(G)$  is ccc.*

*Proof.* Suppose that  $\mathbb{P}(G)$  has an uncountable antichain  $A$ . We can assume that all  $p \in A$  have the same first component  $T$  and that all  $F_p$ ,  $p \in A$ , are of the same size  $n$ . It follows that  $m_p$  has the same value  $m$  for all  $p \in A$ . We may also assume that

$$F_p \upharpoonright m = \{x \upharpoonright m : x \in F_p\}$$

is the same set  $S \subseteq \omega^m$  for all  $p \in A$ .

Let  $B$  be the set of all  $p \in \mathbb{P}(G)$  such that  $T_p = T$  and  $F_p \upharpoonright m = S$ . For each  $p \in B$  let  $(x_0(p), \dots, x_{n-1}(p))$  be the lexicographically increasing enumeration of  $F_p$ . The map  $p \mapsto (x_0(p), \dots, x_{n-1}(p))$  is a bijection from  $B$  onto a clopen subset of  $(\omega^\omega)^n$ . This induces a natural topology on  $B$ . Note that for all  $p, q \in B$  and all  $i < n$  we have  $x_i(p) \upharpoonright m = x_i(q) \upharpoonright m$ .

**Claim 8.** Let  $p, q \in B$ . Then  $p$  and  $q$  are compatible in  $\mathbb{P}(G)$  iff for all  $i < n$ , either  $x_i(p) = x_i(q)$  or  $\{x_i(p), x_i(q)\} \notin E$ .

If  $p$  and  $q$  are compatible, then

$$F_p \cup F_q = \{x_0(p), \dots, x_{n-1}(p), x_0(q), \dots, x_{n-1}(q)\}$$

is  $G$ -independent. But  $F_p \cup F_q$  is  $G$ -independent iff for all  $i < n$ , either  $x_i(p) = x_i(q)$  or  $\{x_i(p), x_i(q)\} \notin E$ . This is because if  $i, j < n$  are distinct, then, by condition (3) in the definition  $\mathbb{P}(G)$ ,  $\{x_i(p), x_j(q)\} \notin E$ .

On the other hand, if  $F_p \cup F_q$  is  $G$ -independent, then we can find a common extension  $r = (T_r, F_r)$  of  $p$  and  $q$  as follows: Let  $F_r = F_p \cup F_q$ . Choose  $m_r$  large enough such that for all  $x, y \in F_r$  with  $x \neq y$ ,  $x \upharpoonright m_r \neq y \upharpoonright m_r$  and for all  $x', y' \in X$  with  $x' \upharpoonright m_r = x \upharpoonright m_r$  and  $y' \upharpoonright m_r = y \upharpoonright m_r$  we have  $\{x', y'\} \notin E$ . This is possible since  $E$  is closed. We define the tree  $T_r$  in the following way:

Let

$$S_0 = (F_p \cup F_q) \upharpoonright m_r = \{y_1 \dots, y_n, z_1, \dots, z_n\}.$$

For each maximal element of  $T = T_p = T_q$  that is not an element of  $S$  we choose an extension of length  $m_r$ . Let  $S_1$  be the set of all these extensions. Let  $T_r$  be the tree consisting of all initial segments (not necessarily proper) of elements of  $S_0 \cup S_1$ , i.e., let  $T_r$  be the tree generated by  $S_0 \cup S_1$ . It is obvious that  $r = (T_r, F_r)$  is an extension of both  $p$  and  $q$  provided it is a condition at all.

Conditions (1), (2), (4), and (5) in the definition of  $\mathbb{P}(G)$  are clearly satisfied. For condition (3) let  $s$  and  $t$  be two distinct maximal elements of  $T_r$  and let  $x, y \in \omega^\omega$  be such that  $x \upharpoonright m_r = s$  and  $y \upharpoonright m_r = t$ .

If  $x \upharpoonright m \neq y \upharpoonright m$ , then  $\{x, y\} \notin E$  since  $p$  and  $q$  satisfy (3). If  $x \upharpoonright m = y \upharpoonright m$ , then  $\{x, y\} \notin E$  by the choice of  $m_r$  and of  $T_r$ . This finishes the proof of the claim.

Let

$$F = \{\{p, q\} \in [B]^2 : p \text{ and } q \text{ are not compatible in } \mathbb{P}(G)\}.$$

Since  $E$  is closed and by the claim,  $F$  is a Boolean combination of finitely many subsets of  $[B]^2$  that are open or closed. In particular,  $F$  is  $G_\delta$ . Hence Theorem 5 applies to  $(B, F)$ . Since  $A \subseteq B$  is an uncountable antichain in  $\mathbb{P}(G)$ , it is, in fact, an uncountable  $(B, F)$ -clique. Hence there is a perfect  $(B, F)$ -clique  $C \subseteq B$ .

To each pair  $\{p, q\} \in [C]^2$  we assign a color  $c(p, q) \in 3^n$  as follows: for each  $i < n$  let

$$c(p, q)(i) = \begin{cases} 0, & x_i(p) = x_i(q), \\ 1, & x_i(p) \neq x_i(q) \text{ and } \{x_i(p), x_i(q)\} \notin E, \\ 2, & x_i(p) \neq x_i(q) \text{ and } \{x_i(p), x_i(q)\} \in E. \end{cases}$$

The coloring  $c$  is clearly Borel. By Theorem 6 there is a perfect  $c$ -homogeneous set  $D \subseteq C$ . We can choose  $D$  homeomorphic to  $2^\omega$ . Let  $s$  be the constant value of  $c$  on  $[D]^2$ . Since the elements of  $D$  are pairwise incompatible and by the claim, there is some  $i < n$  such that for all  $\{p, q\} \in [D]^2$ ,  $\{x_i(p), x_i(q)\} \in E$ . Now the set  $\{x_i(p) : p \in D\}$  is a  $G$ -clique. Moreover, this set is a 1-1 continuous image of a copy of  $2^\omega$  and therefore itself a copy of  $2^\omega$ . It follows that  $G$  has a perfect clique, contradicting our assumptions on  $G$ .  $\square$

**Lemma 9.** *Let  $(G_\alpha)_{\alpha < \kappa}$  be a family of closed graphs on either  $2^\omega$  or  $\omega^\omega$ . If for all  $\alpha < \kappa$  there are no perfect  $G_\alpha$ -cliques, then the finite support product  $\mathbb{Q}$  of the forcing notions  $\mathbb{P}(G_\alpha)$  is ccc.*

*Proof.* Suppose there is an uncountable antichain  $A \subseteq \mathbb{Q}$ . Using the  $\Delta$ -System Lemma we may assume that the supports of the conditions in  $A$  form a  $\Delta$ -system with some root  $r \subseteq \kappa$ . Two conditions in  $A$  are compatible iff their restrictions to  $r$  are compatible. Since  $A$  is an antichain, no two elements of  $A$  have compatible restrictions to  $r$ .

A proof that is almost identical to the proof of Lemma 7 now shows that one of the graphs  $G_\alpha$ ,  $\alpha \in r$ , has a perfect clique, contradicting the assumptions on the graphs  $G_\alpha$ .  $\square$

**Theorem 10.** *There is a ccc forcing extension of the set-theoretic universe with the same size of  $2^{\aleph_0}$  in which for every closed graph  $G$  on  $\omega^\omega$  without perfect cliques,  $\omega^\omega$  is covered by  $\aleph_1$  compact  $G$ -independent sets.*

*Proof.* Let  $(G_\alpha)_{\alpha < 2^{\aleph_0}}$  be an enumeration of all closed graphs on  $\omega^\omega$  without perfect cliques such that every graph occurs infinitely often in the enumeration. Let  $\mathbb{Q}$  be the finite support product of the forcings  $\mathbb{P}(G_\alpha)$ ,  $\alpha < 2^{\aleph_0}$ . By Lemma 9,  $\mathbb{Q}$  is ccc. It follows that  $\mathbb{Q}$  preserves cardinals and cofinalities and that it adds exactly  $2^{\aleph_0}$  reals. In particular, forcing with  $\mathbb{Q}$  does not change the value of  $2^{\aleph_0}$ .

Since every closed graph  $G$  on  $\omega^\omega$  appears infinitely often in the enumeration  $(G_\alpha)_{\alpha < 2^{\aleph_0}}$ ,  $\mathbb{Q}$  adds a generic filter for the finite support product of countably many copies of  $\mathbb{P}(G)$ .

**Claim 11.** For every closed graph  $G$  on  $\omega^\omega$  the finite support product of  $\omega$  copies of  $\mathbb{P}_G$  adds countably many compact  $G$ -independent sets that cover the ground model elements of  $\omega^\omega$ .

For each  $n \in \omega$  let  $\Gamma_n$  be the generic filter added by the  $n$ -th component of the finite support product of  $\omega$ -many copies of  $\mathbb{P}(G)$ . Let  $T_n = \bigcup_{p \in \Gamma_n} T_p$ . By the definition of  $\mathbb{P}(G)$  and by the definition of the relation  $\leq$  on  $\mathbb{P}(G)$ ,  $T_n$  is a finitely branching tree. An easy density argument shows that  $T_n$  is a tree of height  $\omega$ . Let  $C_n = [T_n]$ . Now  $C_n$  is a compact subset of  $\omega^\omega$ . Using condition (3) of Definition 4 it is easily checked that each  $C_n$  is  $G$ -independent. If  $p = (T_p, F_p) \in \Gamma_n$  and  $x \in F_p$ , then necessarily  $x \in C_n$ .

If  $x$  is a ground model element of  $\omega^\omega$  and  $p$  is a condition in the finite support product, then there is some  $n \in \omega$  such that the  $n$ -th coordinate of  $p$  is trivial. We can extend  $p$  to a stronger condition  $q$  such that  $q(n)$  has a first component  $T_{q(n)} = \{\emptyset\}$  and a second component  $F_{q(n)} = \{x\}$ . It follows that the set of conditions that force  $x$  to be an

element of one of the sets  $C_n$  is dense in the finite support product. Hence  $\bigcup_{n \in \omega} C_n$  contains all ground model elements of  $X$ . This finishes the proof of the claim.

It follows that for every closed graph  $G$  on  $\omega^\omega$ ,  $\mathbb{Q}$  adds countably many compact  $G$ -independent sets that cover all ground model elements of  $\omega^\omega$ .

Iterating forcing with the respective versions of  $\mathbb{Q}$  of length  $\omega_1$  with finite supports yields a model of set theory in which  $2^{\aleph_0}$  is the same ordinal as in the ground model and in which for every closed graph  $G$  on  $\omega^\omega$ ,  $\omega^\omega$  is covered by  $\aleph_1$  compact  $G$ -independent sets.

This is because if the graph  $G$  appears at stage  $\alpha < \omega_1$ , all we have to do is to choose, for each  $\beta \in [\alpha, \omega_1)$ , a countable collection  $\mathcal{C}_\beta$  of compact  $G$ -independent sets in the intermediate model with index  $\beta + 1$  that covers the elements of  $\omega^\omega$  of the previous model with index  $\beta$ . In the final model, the collection  $\bigcup_{\beta \in [\alpha, \omega_1)} \mathcal{C}_\beta$  is of size at most  $\aleph_1$  and covers the underlying space of  $G$ .  $\square$

*Proof of Theorem 2.* We start by adding Cohen reals to the set-theoretic universe  $V$  in order to obtain a model  $V[\Gamma_0]$  of set theory in which  $2^{\aleph_0}$  has the desired size. Then we consider the intermediate model  $V[\Gamma_0]$  as the new ground model and form the generic extension  $V[\Gamma_0, \Gamma_1]$  of  $V[\Gamma_0]$  constructed in Theorem 10.

Observe that for a given closed graph  $G$  on  $\omega^\omega$  the statement “ $G$  has no perfect clique” is  $\Pi_2^1$  and hence absolute. That is, if  $G \in V$  has no perfect clique in  $V$ , then it has no perfect clique in any generic extension of  $V$  either and therefore the weak Borel chromatic number of  $G$  in  $V[\Gamma_0, \Gamma_1]$  is at most  $\aleph_1$ .  $\square$

#### 4. FRAGMENTS OF A “CLOSED COLORING AXIOM”

The uncountability of the Borel chromatic number of  $\mathcal{G}_0$  follows from the fact that every  $\mathcal{G}_0$ -independent subset of  $2^\omega$  with the Baire property is meager. This implies that there is a set  $A \subseteq 2^\omega$  of size  $\text{non}(\text{meager})$  that is not the union of countably many  $\mathcal{G}_0$ -independent Borel sets. Here  $\text{non}(\text{meager})$  is the least size of a nonmeager set of reals. In

particular, there are models of ZFC in which there is a subset of  $2^\omega$  of size  $\aleph_1 < 2^{\aleph_0}$  that is not the union of countably many  $\mathcal{G}_0$ -independent Borel sets. However, we have the following theorem.

**Theorem 12.** *Assume Martin's Axiom.*

a) *For every closed graph  $G$  on a Polish space  $X$  without perfect cliques, every set  $Y \subseteq X$  of size  $< 2^{\aleph_0}$  is contained in the union of countably many closed  $G$ -independent sets.*

b) *If  $X$  is a second countable Hausdorff space of size  $< 2^{\aleph_0}$  and  $G$  is a closed graph on  $X$  without infinite cliques, then  $X$  is the union of countably many closed  $G$ -independent sets.*

*Proof.* a) Let  $G$  be a closed graph on a Polish space  $X$  without perfect cliques. After removing countably many points from  $X$  if necessary, we may assume that  $X$  is perfect. Since  $X$  is a perfect Polish space, it is the continuous image of  $\omega^\omega$  under an injective map  $f$ . We define a closed graph  $G^*$  on  $\omega^\omega$  as in the proof of Corollary 3. The finite support product of countably many copies of  $\mathbb{P}(G^*)$  is ccc by Lemma 7. A sufficiently generic filter of this finite support product adds countably many compact  $G^*$ -independent sets covering all of  $f^{-1}[Y]$ . The images of these sets are  $G$ -independent and cover  $Y$ .

b) Let  $E \subseteq [X]^2$  be the set of edges of  $G$ . Let  $\mathbb{P}$  be the forcing notion consisting of all finite  $G$ -independent subsets of  $X$  ordered by reverse inclusion. Clearly, forcing with  $\mathbb{P}$  adds a  $G$ -independent subset of  $X$ . We show that  $\mathbb{P}$  is ccc.

Suppose there is an uncountable antichain  $A \subseteq \mathbb{P}$ . We can assume that all elements of  $A$  are of the same size  $n$ . Fix a countable base  $\mathcal{B}$  for the topology of  $X$ . For each  $p \in A$  choose disjoint sets  $U_0^p, \dots, U_{n-1}^p \in \mathcal{B}$  such that for all  $i < n$ ,  $p \cap U_i^p$  has exactly one element and such that for all  $\{i, j\} \in [n]^2$ , if  $x \in U_i^p$  and  $y \in U_j^p$ , then  $\{x, y\} \notin E$ . This is possible since  $[X]^2 \setminus E$  is open.

Since  $\mathcal{B}$  is countable, there are only countably many possibilities for the sequence  $(U_i^p)_{i < n}$ . Hence we may assume that for all  $i < n$ ,  $U_i^p$  is the same set  $U_i$  for all  $p \in A$ .

For every  $p \in A$  and  $i < n$  let  $x_i^p$  be the unique element of  $U_i \cap p$ . For each  $i < n$  we define a coloring  $c_i : [A]^2 \rightarrow 3$  by letting

$$c_i(p, q) = \begin{cases} 0, & \text{if } x_i^p = x_i^q, \\ 1, & \text{if } x_i^p \neq x_i^q \text{ and } \{x_i^p, x_i^q\} \notin E \\ 2, & \text{if } x_i^p \neq x_i^q \text{ and } \{x_i^p, x_i^q\} \in E. \end{cases}$$

By the infinite Ramsey theorem there is an infinite set  $B \subseteq A$  that is homogeneous for all  $c_i$ . Note that for any two distinct elements  $p$  and  $q$  of  $A$  there is  $i < n$  such that  $c_i(p, q) = 2$  for otherwise,  $p$  and  $q$  would be compatible in  $\mathcal{P}$ . It follows that for some  $i < n$ ,  $c_i(p, q) = 2$  for all  $\{p, q\} \in [B]^2$ . But now  $\{x_i^p : p \in B\}$  is an infinite  $G$ -clique, contradicting the assumptions on  $G$ .

By using a  $\Delta$ -system argument first and then an argument almost identical to the one above but slightly more complicated as far as notation is concerned, we can also show that the finite support product  $\mathbb{Q}$  of countably many copies of  $\mathbb{P}$  is ccc. Since  $X$  is of size  $< 2^{\aleph_0}$ , MA implies the existence of a sufficiently generic filter of  $\mathbb{Q}$  that codes a countable family of  $G$ -independent sets that covers all of  $X$ .  $\square$

## 5. $F_\sigma$ -GRAPHS

There is an  $F_\sigma$ -graph that has an uncountable clique, but no perfect clique [14, Claim 3.2] (also see [11, Proposition 2.5]). For this  $F_\sigma$ -graph there is no  $\aleph_1$ -preserving forcing extension of the universe in which countably many independent sets cover the ground model vertices of the graph. This shows that the proof of Theorem 2 does not apply to  $F_\sigma$ -graphs.

Kubiś and Shelah proved that for every  $\alpha < \omega_1$  it is consistent that there is an  $F_\sigma$ -graph on  $2^\omega$  that has a clique of size  $\aleph_\alpha$  but no perfect clique [8]. This shows that Theorem 2 at least consistently fails for  $F_\sigma$ -graphs.

**Lemma 13.** *Let  $G = (X, E)$  be an  $F_\sigma$ -graph on a Polish space  $X$ . Let  $A \subseteq X$  be  $G$ -independent. Then there is a  $\sigma$ -centered forcing notion that adds countably many closed  $G$ -independent sets that cover the ground model set  $A$ .*

*Proof.* Since  $E$  is  $F_\sigma$ ,  $[X]^2 \setminus E$  is  $G_\delta$ . Let  $(O_n)_{n \in \omega}$  be a decreasing sequence of open subsets of  $[X]^2$  such that  $[X]^2 \setminus E = \bigcap_{n \in \omega} O_n$ . Fix a countable basis  $\mathcal{B}$  for the topology on  $X$ . Let  $\mathbb{P}$  consist of conditions of the form  $p = (n_p, \mathcal{U}_p, F_p)$  where

- (1)  $n_p \in \omega$ ,
- (2)  $F_p$  a finite subset of  $A$ ,
- (3)  $\mathcal{U}_p$  is a finite set of nonempty elements of  $\mathcal{B}$ ,
- (4)  $F_p \subseteq \bigcup \mathcal{U}_p$ ,
- (5) every  $U \in \mathcal{U}_p$  is of diameter at most  $2^{-n_p}$ ,
- (6) for all  $U, V \in \mathcal{U}_p$  with  $U \neq V$ ,  $\text{cl}(U) \cap \text{cl}(V) = \emptyset$ , and
- (7) for all  $U, V \in \mathcal{U}_p$  with  $U \neq V$ ,  $[U, V] \subseteq O_{n_p}$ .

Given  $p, q \in \mathbb{P}$ , we write  $p \leq q$  if

- (8)  $n_p \geq n_q$ ,
- (9) for all  $U \in \mathcal{U}_p$  either  $U \in \mathcal{U}_q$  or there is  $V \in \mathcal{U}_q$  such that  $\text{cl}(U) \subseteq V$ , and
- (10)  $F_p \supseteq F_q$ .

We first show that  $\mathbb{P}$  is  $\sigma$ -centered. Observe that there are only countably many possibilities for  $n_p$  and  $\mathcal{U}_p$ . Given  $n \in \omega$  and a finite set  $\mathcal{U} \subseteq \mathcal{B}$  consisting of nonempty sets, any two conditions  $p, q \in \mathbb{P}$  with  $n_p = n_q = n$  and  $\mathcal{U}_p = \mathcal{U}_q = \mathcal{U}$  are compatible, witnessed by the common extension  $(n, \mathcal{U}, F_p \cup F_q)$ . It follows that  $\mathbb{P}$  is  $\sigma$ -centered.

For  $n \in \omega$  let  $D_n = \{p \in \mathbb{P} : n_p \geq n\}$ . We show that each of the sets  $D_n$  is dense in  $\mathbb{P}$ . Let  $p \in \mathbb{P}$ . Since  $F_p \subseteq A$  and  $A$  is  $G$ -independent, for all  $\{x, y\} \in [F_p]^2$  we have  $\{x, y\} \in \bigcap_{m \in \omega} O_m$  and hence  $\{x, y\} \in O_n$ . Since  $F_p$  is finite, it follows that there are sets  $U_x \in \mathcal{B}$ ,  $x \in F_p$ , such that for distinct  $x, y \in F_p$ ,  $U_x \cap U_y = \emptyset$  and  $[U_x, U_y] \subseteq O_n$ . We may assume that the  $U_x$ ,  $x \in F_p$ , have diameter at most  $2^{-n}$  and that for each  $x \in F_p$  there is  $U \in \mathcal{U}_p$  such that  $\text{cl}(U_x) \subseteq U$ . Also, we

may assume that for distinct  $x, y \in F_p$ ,  $\text{cl}(U_x) \cap \text{cl}(U_y) = \emptyset$ . Now  $(\max(n, n_p), \{U_x : x \in F_p\}, F_p)$  is a condition in  $D_n$  that extends  $p$ .

Let  $\Gamma$  be a  $\mathbb{P}$ -generic filter over the ground model  $V$ . Let  $C = \bigcap_{p \in \Gamma} \text{cl}(\bigcup \mathcal{U}_p)$ . Clearly,  $C$  is a closed set. Observe that by the density of the sets  $D_n$ ,  $n \in \omega$ , and by the definition of the relation  $\leq$  we have  $C = \bigcap_{p \in \Gamma} (\bigcup \mathcal{U}_p)$ . Moreover, for every  $p \in \Gamma$ ,  $F_p \subseteq C$ . Again by the density of the sets  $D_n$ , for any two distinct points  $x, y \in C$  and all  $n \in \omega$ , there is a condition  $p \in \Gamma$  such that  $2^{-n_p} < \frac{1}{2} \cdot d(x, y)$  and  $n_p > n$ . Now if  $x \in U \in \mathcal{U}_p$  and  $y \in V \in \mathcal{U}_p$ ,  $U \cap V = \emptyset$  and  $[U, V] \subseteq O_n$ . It follows that  $\{x, y\} \in \bigcap_{n \in \omega} O_n = [X]^2 \setminus E$ . Therefore  $C$  is  $G$ -independent.

By the usual argument it now follows that the finite support product of countably many copies of  $\mathbb{P}$  adds countably many closed  $G$ -independent subsets of  $X$  that cover the ground model set  $A$ . It is well-known that a finite support product of countably many  $\sigma$ -centered forcing notions is again  $\sigma$ -centered.  $\square$

**Corollary 14.** *Assume Martin's Axiom for  $\sigma$ -centered partial orders. Then for every  $F_\sigma$ -graph  $G$  on a Polish space  $X$ , every  $G$ -independent set of size  $< 2^{\aleph_0}$  is contained in the union of countably many closed  $G$ -independent sets.*

*Proof.* Let  $A \subseteq X$  be  $G$ -independent and of size  $< 2^{\aleph_0}$ . We again use a finite support product  $\mathbb{F}$  of countably many copies of the forcing notion  $\mathbb{P}$  defined in the proof of Lemma 13. A sufficiently  $\mathbb{F}$ -generic filter yields countably many closed  $G$ -independent sets that cover  $A$ .  $\square$

**Theorem 15.** *Let  $G = (X, E)$  be an  $F_\sigma$ -graph and let  $\kappa$  be an uncountable cardinal. Suppose in every ccc forcing extension of the set-theoretic universe, the chromatic number of  $G$  is at most  $\kappa$ . Then there is a ccc forcing extension of the universe with an arbitrarily large size of the continuum in which  $X$  is covered by at most  $\kappa$  closed  $G$ -independent sets.*

*Proof.* First add Cohen reals in order to increase the size of the continuum to the desired value. Now cover  $X$  by  $\kappa$   $G$ -independent sets

and use Lemma 13 to cover each of these  $G$ -independent sets by countably many closed  $G$ -independent sets. This can be done simultaneously using a finite support product. Iterating this for  $\omega_1$  steps with finite supports yields a ccc forcing extension of the ground model with an arbitrarily large size of the continuum in which  $X$  is covered by  $\kappa$  closed  $G$ -independent sets.  $\square$

**Corollary 16.** *Let  $G = (X, E)$  be a locally countable  $F_\sigma$ -graph, i.e., an  $F_\sigma$ -graph in which every vertex has at most countably many neighbors. Then there is a ccc extension of the universe with an arbitrarily large size of the continuum in which the weak Borel chromatic number of  $G$  is at most  $\aleph_1$ .*

*Proof.* If  $G$  is locally countable, all the connected components of  $G$  are countable. Hence  $G$  is countably chromatic. The statement that  $G$  is locally countable is  $\Pi_2^1$  and thus absolute. Hence  $G$  is countably chromatic in every forcing extension of the universe. Now the corollary follows from Theorem 15.  $\square$

## 6. HYPERGRAPHS

Given a Hausdorff space  $X$  and  $n \in \omega$ , we topologize  $[X]^n$  in analogy to  $[X]^2$ . The notions *clique*, *independent set* and *weak Borel chromatic number* have natural generalizations to  $n$ -uniform hypergraphs  $G = (X, E)$ . Here  $(X, E)$  is  $n$ -uniform if  $E \subseteq [X]^n$ . We observe that the natural analogs of Corollary 3 and Theorem 12 fail even for clopen 3-uniform hypergraphs on  $2^\omega$ .

**Definition 17.** For all  $\{x, y\} \in [\omega^\omega]^2$  let  $\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}$ . Now let  $\{x, y, z\} \in [2^\omega]^3$  and assume that  $x, y, z$  are lexicographically increasing. Let

$$c_{\text{type}}^3(x, y, z) = \begin{cases} 1, & \Delta(x, y) < \Delta(y, z) \\ 0, & \Delta(y, z) < \Delta(x, y) \end{cases}.$$

Given  $\{x, y, z\} \in [2^\omega]^3$  with  $x, y, z$  lexicographically increasing, we say that  $\{x, y, z\}$  is of *type*  $c_{\text{type}}^3(x, y, z)$ . Let  $E_{\text{type}}^3 \subseteq [2^\omega]^3$  be the collection of all 3-element subsets of  $2^\omega$  of type 1.

Observe that there are no uncountable  $c_{\text{type}}^3$ -homogeneous sets. More precisely, if  $A \subseteq 2^\omega$  is uncountable, we may assume, after removing countably many points if necessary, that for every open set  $O \subseteq 2^\omega$ ,  $O \cap A$  is either empty or uncountable. Now the tree of finite initial segments of  $A$  is perfect. It follows that  $A$  has infinite subsets  $A_0$  and  $A_1$  such that for all  $i \in 2$ , all three-element subsets of  $A_i$  are of type  $i$ .

It follows that even though there are no perfect  $(2^\omega, E_{\text{type}}^3)$ -cliques,  $2^\omega$  is not the union of fewer than  $2^{\aleph_0}$  independent sets.

Galvin showed that for every continuous coloring  $c : [2^\omega]^3 \rightarrow 2$  there is a perfect set  $P \subseteq 2^\omega$  such that for all  $\{x, y, z\} \in [P]^3$ ,  $c(x, y, z)$  only depends on the type of  $\{x, y, z\}$  (see [2] for a proof). In other words,  $P$  is homogeneous for each of the two partial colorings obtained by restricting  $c$  to all three-element subsets of  $2^\omega$  of type 0, respectively of type 1.

Hence one might look for analogs of Corollary 3 and Theorem 12 that talk only about one type of three-element sets.

**Definition 18.** Let  $G = (2^\omega, E)$  be a 3-uniform hypergraph and  $i \in 2$ .  $C \subseteq 2^\omega$  is an  $i$ -clique of  $G$  if for all  $\{x, y, z\} \in [C]^3$  of type  $i$ ,  $\{x, y, z\} \in E$ . Likewise,  $I \subseteq 2^\omega$  is an  $i$ -independent set of  $G$  if for all  $\{x, y, z\} \in [I]^3$  of type  $i$ ,  $\{x, y, z\} \notin E$ .

**Definition 19.** For  $\{x, y\} \in [2^\omega]^2$  let  $c_{\min}(x, y) = \Delta(x, y) \pmod 2$ . Let

$$E_{\min}^3 = \{e \in [2^\omega]^3 : e \text{ is not } c_{\min}\text{-homogeneous}\}.$$

Since  $c_{\min}$  is continuous,  $G = (2^\omega, E_{\min}^3)$  a clopen graph on  $2^\omega$ . The following lemma shows that  $G$  is a counterexample to various weak 3-dimensional versions of Corollary 3 and Theorem 12.

**Lemma 20.** a)  $G$  has no cliques of size  $\geq 6$ .

b) For every  $i \in 2$ ,  $G$  has no uncountable  $i$ -cliques.

c) Every  $G$ -independent set is  $c_{\min}$ -homogeneous.

d) For every  $i \in 2$ , every  $i$ -independent set of  $G$  is the union of countably many  $c_{\min}$ -homogeneous sets.

e) If  $\kappa < 2^{\aleph_0}$  is an uncountable cardinal, then there is a set  $A \subseteq 2^\omega$  of size  $\kappa^+$  that cannot be covered by less than  $\kappa$   $i$ -independent subsets of  $G$ .

*Proof.* a) The Ramsey number for 3 is 6. It follows that every set  $A \subseteq 2^\omega$  of size at least 6 contains a  $c_{\min}$ -homogeneous set of size 3 and therefore fails to be a  $G$ -clique.

b) Let  $A \subseteq 2^\omega$  be uncountable. Then  $A$  has a subset  $F$  of size 6 such that all three-element subsets of  $F$  are of type  $i$ . As in the proof of a),  $F$  has a  $c_{\min}$ -homogeneous subset  $\{x, y, z\}$ . Now  $\{x, y, z\}$  is of type  $i$  and not in  $E_{\min}^3$ . It follows that  $A$  is not an  $i$ -clique of  $G$ .

c) This follows immediately from the definition of  $G$ .

d) Let  $A \subseteq 2^\omega$  be an  $i$ -independent subset of  $G$ . For reasons of symmetry we may assume that  $i = 1$ . If  $A$  is uncountable, after removing countably many points from  $A$ , we may again assume that the tree  $T(A) \subseteq 2^{<\omega}$  of finite initial segments of elements of  $A$  is perfect. Let  $t$  be a *splitting node* of this tree, i.e., an element with two immediate successors. Let  $j = |t| \bmod 2$ .

If  $a \in A$  and  $t \frown 1 \subseteq a$  we say that  $a$  is a *right extension* of  $t$ . If  $a \in A$  and  $t \frown 0 \subseteq a$ ,  $a$  is a *left extension* of  $t$ . Since  $t$  is a splitting node, it has both a left extension  $x$  and a right extension  $y$ . Now  $t$  is the maximal common initial segment of  $x$  and  $y$ .

If  $a, b \in A$  are distinct right extensions of  $t$ , then  $\{x, a, b\}$  is of type 1. Since  $A$  is  $i$ -independent,  $\{x, a, b\}$  is  $c_{\min}$ -homogeneous. It follows that for all distinct  $a, b \in A$  with  $t \frown 1 \subseteq a, b$ ,  $c_{\min}(a, b) = c_{\min}(x, a) = j$ . Hence the set of all right extensions of  $t$  is  $c_{\min}$ -homogeneous.

Except for possibly one element of  $A$ , every  $a \in A$  is a right extension of some splitting node in  $T(A)$ . Since  $T(A)$  is countable, this implies that  $A$  is the union of countably many  $c_{\min}$ -homogeneous sets.

e) The main argument of the proof of e) has already been used in [3]. For  $x, y \in 2^\omega$  let  $x \otimes y = (x(0), y(0), x(1), y(1), \dots)$ . The mapping  $\otimes$  is a homeomorphism between  $(2^\omega)^2$  and  $2^\omega$ . Fix  $a, b \in 2^\omega$ .

If  $H \subseteq 2^\omega$  is  $c_{\min}$ -homogeneous of color 0, then for every  $x \in 2^\omega$  there is at most one  $y \in 2^\omega$  with  $x \otimes y \in H$ . If  $H$  is maximal homogeneous, then there is some  $y$  with  $x \otimes y \in H$ . Thus, a maximal  $c_{\min}$ -homogeneous set  $H$  of color 0 gives rise to a function  $f_H : 2^\omega \rightarrow 2^\omega$  with  $H = \{x \otimes f(x) : x \in 2^\omega\}$ . We have  $a \otimes b \in H$  iff  $f_H(a) = b$ .

Similarly, every maximal  $c_{\min}$ -homogeneous set  $H$  of color 1 gives rise to a function  $f_H : 2^\omega \rightarrow 2^\omega$  with  $H = \{f(x) \otimes x : x \in 2^\omega\}$ . If  $a \otimes b \in H$ , then  $f_H(b) = a$ .

Now let  $B \subseteq 2^\omega$  be of size  $\kappa^+$ . Let  $A = \{a \otimes b : a, b \in B\}$ . Suppose  $\mathcal{H}$  is a family of size  $< \kappa$  of  $i$ -independent sets of  $G$  that covers  $A$ . We may assume that  $\mathcal{H}$  is infinite. By d) we may assume that  $\mathcal{H}$  consists of  $c_{\min}$ -homogeneous sets. By enlarging the sets in  $\mathcal{H}$ , we may assume that they are maximal  $c_{\min}$ -homogeneous sets. Using the connection between maximal  $c_{\min}$ -homogeneous sets and functions, we see that there is a family  $\mathcal{F}$  of size  $|\mathcal{H}| < \kappa$  of functions from  $2^\omega$  to  $2^\omega$  such that for all  $a, b \in B$  there is  $f \in \mathcal{F}$  such that  $f(a) = b$  or  $f(b) = a$ . But this contradicts a theorem of Kuratowski [9].

Kuratowski's theorem states that the minimal size of a family  $\mathcal{F}$  of functions from  $\kappa^+$  to  $\kappa^+$  such that for all  $a, b \in \kappa^+$  there is  $f \in \mathcal{F}$  with  $f(a) = b$  or  $f(b) = a$  is  $\kappa$ . For an English proof of this theorem see [1].  $\square$

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