INJECTIVE AND PROJECTIVE T-BOOLEAN ALGEBRAS

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Abstract. We introduce and analyze Boolean algebras acted on continuously (with respect to the discrete topology on the Boolean algebra) by a topological monoid T, so-called T-Boolean algebras. These are the duals of Boolean flows, and by analyzing injective T-Boolean algebras we are able to characterize projective Boolean flows. Moreover, we characterize the projective T-Boolean algebras in the case that T is a group. This characterization shows that the existence of nontrivial projective T-Boolean algebras depends on the properties of T.

1. Introduction

Boolean algebras with actions, herein termed T-Boolean algebras or T-algebras for short, is a subject of intrinsic interest and importance. Moreover, any systematic program of investigation of topological dynamics must place the topic of Boolean flows, i.e., Boolean spaces with actions, high on its list of priorities, and any investigation of Boolean flows leads directly to the subject of T-algebras via the Stone duality with actions outlined in Subsection 1.2.

In this article we take up the central issues of injective and projective T-Boolean algebras. This has already been for discrete monoids T, i.e., without the continuity assumption on the actions. (See Cornish's book [2] for a treatment and [6] for a general overview of categorical properties such as injectivity.) However, things become more involved once topology is added.

It is natural to consider continuous actions, since infinite monoids or groups often come with a topology on them, and many natural group or monoid actions are continuous in the sense that the group or monoid T acts on a set that has some topology and for every point in this set the evaluation map from T to the set is continuous.

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Here the right topology on the Boolean algebras is the discrete topology, since with this topology continuous actions on Boolean algebras are dual via Stone duality to continuous actions on the respective Stone spaces, i.e., to Boolean flows.

It develops that, although the usual categorical arguments establish the existence and uniqueness of the injective hull of any T-algebra A, the structure of this hull is quite complicated even when that of A is not. Among the more significant results of our research is a characterization of injective T-algebras in terms of systems of ideals; this can be found in Subsections 4.3, 4.4, and 4.5.

It also develops that nontrivial projective T-algebras exist only for rather special topological monoids T. We characterize those monoids within a class slightly broader than topological groups in Theorem 6.2.6. A fully satisfactory characterization of which topological monoids admit nontrivial projective T-algebras must await a deeper understanding than is currently available to the authors. Note, however, that nontrivial projectives always exist in the case of discrete monoids.

The neccessary background on Boolean algebras is provided by [1] and [7], for groups and monoids an introduction to algebra such as [5] is sufficient, for general topology and topological groups we refer the reader to [3] and especially the chapter on uniform spaces therein, and the concepts from category theory that we use can be found in [4].

1.1. **Actions.** We fix a topological monoid T throughout. We refer to the elements of T as *actions*, and denote them by the letters s, t, and r, often with subscripts. Let $\mathbf{C}$ be a category and C an object of $\mathbf{C}$. A *left action of T on C* (*right action of T on C*) is a monoid homomorphism (antimorphism) $\phi_C : T \to \mathrm{Hom}_{\mathbf{C}}(C, C)$, i.e., $\phi_C(1_T) = 1_C$ and for all $t_i \in T$, and

$$\phi_C(t_1)(\phi_C(t_2)) = \phi_C(t_1 t_2) \quad (\phi_C(t_1)(\phi_C(t_2)) = \phi_C(t_2 t_1)).$$

We suppress nearly all mention of ϕ_C, writing $\phi_C(t)(c)$ as $t_C c$ ($c t_C$) or simply tc (ct). In this simplified notation, the definition of a left (right) action is just that $1c = c$ and $(t_1 t_2) c = t_1 (t_2 c)$ ($c1 = c$ and $c (t_1 t_2) = (c t_1) t_2$) for all $c \in C$ and $t_i \in T$. We then have an enriched category $\mathbf{TC}$ ($\mathbf{CT}$) whose objects are the $\mathbf{C}$ objects C acted upon by T in such a way that the evaluation map $(t, c) \longmapsto tc$ ($(c, t) \longmapsto ct$) is continuous from $T \times C$ ($C \times T$) into C, where C either has a topological structure or is given the discrete topology, and $T \times C$ ($C \times T$) has the product topology. This is what we mean by the continuity of actions.

It is worth pointing out that for locally compact C there is a natural topology on the set $\mathrm{Hom}_{\mathbf{C}}(C, C)$ such that the evaluation map $(t, c) \longmapsto$

tc ($(c,t) \longmapsto ct$) from $T \times C$ ($C \times T$) into C is continuous if and only if $\phi_C : T \to \mathrm{Hom}_{\mathbf{C}}(C,C)$ is. By [3, Theorem 3.4.3] that topology is the *compact open topology* generated by the sets

$$M(A,B) = \{f \in \mathrm{Hom}_{\mathbf{C}}(C,C) : f(A) \subseteq B\}$$

where $A \subseteq C$ is compact and $B \subseteq C$ is open. If C is discrete, then the compact open topology on $\mathrm{Hom}_{\mathbf{C}}(C,C)$ is just the subspace topology inherited from the product topology on C^C.

The **TC** (**CT**) morphisms are the **C** morphisms $f : C \to B$ between **TC** (**CT**) objects C and B which commute with the actions, i.e., those which make this diagram commute. Maps commuting with the actions

$$\begin{tikzcd} C \arrow[r, "t_A"] \arrow[d, "f"'] & C \arrow[d, "f"] \\ B \arrow[r, "t_B"'] & B \end{tikzcd}$$

are also called T-*equivariant* maps.

We sometimes distinguish between the simpler objects of **C** and the more complicated objects of **TC** (**CT**) by referring to the former as *naked*. In the case $T = \{1\}$ the category **TC** (**CT**) is equivalent to the category **C** of naked objects. This is witnessed by the forgetful functor from **TC** (**CT**) to **C**. We refer to the case $T = \{1\}$ as the classical or no-action situation.

From the category **Ba** of Boolean algebras with Boolean homomorphisms we obtain the category **BaT** of T-*Boolean algebras* and T-*Boolean homomorphisms*. We consider no other types of algebras in this article, so we simplify the terminology by dropping the word Boolean, referring to the objects of **Ba** and of **BaT** as algebras and T-algebras, respectively, and to the corresponding morphisms as morphisms and T-morphisms. We reiterate that T-algebras carry the discrete topology, and that T acts on them on the right.

Likewise from the category **Sp** of spaces with continuous functions we obtain the category **TSp** of T-*spaces*, or T-*flows*, or simply *flows*, and likewise the category **SpT** of *antiflows*. The distinction between the two is that T acts on flows on the left and on antiflows on the right. T itself has two roles to play: it is a flow under the action of left multiplication and an antiflow under the action of right multiplication. Finally, from the full subcategory **BSp** of **Sp** consisting of the Boolean spaces, i.e., compact Hausdorff spaces with a clopen base, we obtain the category **TBSp** of Boolean flows.

1.2. **Stone duality with actions.** Stone duality extends to the action categories without a hitch. In fact, our interest in injective T-algebras arose from our desire to understand their duals, projective Boolean flows. However, we emphasize T-algebras in this article, pausing only occasionally to translate the results into terms of Boolean flows.

The main purpose of mentioning duality here is to point out that the definition of T-algebra is the right one. In particular, the continuity of evaluation on a Boolean flow is equivalent to the continuity of evaluation on its clopen algebra only when the latter carries the discrete topology. The following lemma characterizes continuity of actions both on discrete spaces and on Boolean spaces.

Lemma 1.2.1. *a) Let* $\mathbf{C}$ *be some category. Let* $C \in \mathbf{C}$ *be discrete and let* ϕ_C *be a right action of* T *on* C*. Then* ϕ *is continuous if and only if for all* $a, b \in C$ *the set* $\{t \in T : a = bt\}$ *is open in* T*.*

b) Let X *be a Boolean space and let* ϕ_X *be a left action of* T *on* X*. Then* ϕ_X *is continuous if and only if for all clopen sets* $a, b \subseteq X$ *the set* $\{t \in T : t(a) \subseteq b\}$ *is open in* T*.*

Proof. a) It is easily checked that for a discrete space C the compact open topology on $\mathrm{Hom}_{\mathbf{C}}(C, C)$ is generated by sets of the form $\{f \in \mathrm{Hom}_{\mathbf{C}}(C, C) : a = f(b)\}$, $a, b \in C\}$. It follows that ϕ_C is continuous if and only if for all $a, b \in C$, $\{t \in T : a = bt\}$ is open in T.

b) If ϕ_X is continuous, then $\{t \in T : t(a) \subseteq b\}$ is open since it is a preimage under ϕ_X of a generator of the compact open topology on $\mathrm{Hom}_{\mathbf{BSp}}(X, X)$.

Now suppose that for all clopen $a, b \subseteq X$ the set $\{t \in T : t(a) \subseteq b\}$ is open in T. In order to show the continuity of ϕ_X, it enough to show that preimages under ϕ_X of generators of the compact open topology on $\mathrm{Hom}_{\mathbf{BSp}}(X, X)$ are open in T.

Let $c \subseteq X$ be compact and let $U \subseteq X$ be open. We show that $\{t \in T : t(c) \subseteq U\}$ is open in T. Let t_0 be such that $t_0(c) \subseteq U$. Since $t_0 : X \to X$ is continuous, $t_0(c)$ is compact. Hence there exists a clopen set $b \subseteq U$ such that $t_0(c) \subseteq b$. Again by the continuity of t_0, $a = t_0^{-1}(b)$ is clopen. Now we have

$$t_0 \in \{t \in T : t(a) \subseteq b\} \subseteq \{t \in T : t(c) \subseteq U\}$$

and $\{t \in T : t(a) \subseteq b\}$ is open.

This shows that $\{t \in T : t(c) \subseteq U\}$ is open. $\square$

Theorem 1.2.2. *Let* X *be a Boolean space and* B *its algebra of clopen subsets. Then every left action* ϕ_X *on* X *gives rise to a right action* ϕ_B

on B by the rule

$$bt = \phi_B(t)(b) \equiv \phi_X(t)^{-1}(b) = t^{-1}(b).$$

Conversely, every right action ϕ_B on B gives rise to a left action ϕ_X on X by the rule

$$tx = \phi_X(t)(x) \equiv \bigcap_{x \in \phi_B(t)(b)} b = \bigcap_{x \in bt} b.$$

These two processes are inverses of one another. Furthermore, ϕ_X renders evaluation continuous on X if and only if ϕ_B renders evaluation continuous on B. Thus are the categories **BaT** *and* **TBSp** *equivalent.*

Proof. Let ϕ_X be a continuous left T-action on X. For every $t \in T$, $\phi_X(t)$ is a continuous map from X to X. Classical Stone duality tells us that $\phi_B(t) : B \to B : b \mapsto \phi_X(t)^{-1}(b)$ is well defined and a Boolean homomorphism. It is easily checked that $\phi_B : T \to \hom_{\mathbf{Ba}}(B, B)$ is a monoid homomorphism.

We now show that ϕ_B continuous with respect to the discrete topology on B. By Lemma 1.2.1 a) it is sufficient to prove that for all $a, b \in B$ the set $\{t \in T : a = bt\} = \{t \in T : t^{-1}(a) = b\}$ is open in T. But

$$\{t \in T : t^{-1}(a) = b\} = \{t \in T : t(b) \subseteq a\} \cap \{t \in T : t(X \setminus b) \subseteq X \setminus a\}$$

and by Lemma 1.2.1 it follows from the continuity of ϕ_X that the two sets on the right hand side of the equation are open in T.

Now assume that ϕ_B is a continuous right T-action on B with respect to the discrete topology on B. Again by Stone duality, for every $t \in T$, the Boolean homomorphism $\phi_B(t)$ dualizes to a continuous map $\phi_X(t) : X \to X$ as defined in the statement of the theorem. Again it is easily checked that $\phi_X : T \to \hom_{\mathbf{BSp}}(X, X)$ is a monoid homomorphism.

For the continuity of ϕ_X let $a, b \in B$. By Lemma 1.2.1 b) it is sufficient to show that $\{t \in T : t(a) \subseteq b\}$ is open in T. By the definition of ϕ_X, $t(a) \subseteq b$ if and only if $a \subseteq bt$. Let $t_0 \in T$ be such that $a \subseteq bt_0$. The set $\{t \in T : bt_0 = bt\}$ is open by the continuity of ϕ_B and Lemma 1.2.1 a). Moreover,

$$t_0 \in \{t \in T : bt_0 = bt\} \subseteq \{t \in T : t(a) \subseteq b\},$$

showing that $\{t \in T : t(a) \subseteq b\}$ is open in T. $\square$

Stone duality helps to clarify the nature of epimorphisms and monomorphisms in both **TBSp** and **BaT**.

Proposition 1.2.3. *In both* **TBSp** *and* **BaT**, *epimorphisms are surjective and monomorphisms are injective.*

Proof. Consider a **TBSp** morphism $f : X \to Y$. The product $P \equiv X \times X$ is a Boolean flow under componentwise actions, i.e., $t(x_1, x_2) \equiv (tx_1, tx_2)$ for all $t \in T$ and $(x_1, x_2) \in P$, and the projection maps $p_i : P \to X$, $i = 1, 2$, are flow surjections. Let

$$Z \equiv \{(x_1, x_2) \in P : f(x_1) = f(x_2)\},$$

a closed subflow of P and therefore itself a **TBSp** object. If f is not injective then there are $x_1 \neq x_2$ in X such that $f(x_1) = f(x_2)$. The corresponding point $p \equiv (x_1, x_2)$ lies in Z, and $p_1(p) = x_1 \neq x_2 = p_2(p)$. Since $fp_1 = fp_2$, f is not a monomorphism. This proves that monomorphisms in **TBSp** are injective, and it follows from Stone duality that epimorphisms in **BaT** are surjective. A similar argument shows that monomorphisms in **BaT** are injective, and dualizes to show that epimorphisms in **TBSp** are surjective. □

2. Pointed antiflows

Antiflows of a particular sort arise as a means of classifying the elements of a T-algebra according to the complexity of their orbits. This is because the orbit $aT \equiv \{at : t \in T\}$ of an element a in a T-algebra A is a discrete antiflow having source a.

2.1. **Pointed antiflows defined.**

Definition 2.1.1. A *pointed antiflow* is an object of the form (R, s), where R is a discrete antiflow and s is a *source* for R, i.e., for all $r \in R$ there is some $t \in T$ such that $st = r$. The pointed antiflow morphisms are the antiflow morphisms which take the designated source of the domain to the designated source of the codomain. We use **pSpT** to denote the category of pointed antiflows and pointed antiflow morphisms.

We remind the reader that T acts on any antiflow on the right, and does so in such a way that evaluation is continuous. A pointed antiflow is distinguished among general antiflows by two additional features: a pointed antiflow is discrete and has a source.

Observe that a pointed antiflow is really just a discrete antiflow quotient of T, with the image of the identity as source. In fact, for every pointed antiflow (R, s) there is a *unique* antiflow surjection $\rho_R : T \to R$ such that $\rho_R(1) = s$, and it is defined by the rule $\rho_R(t) = st$ for all $t \in T$. Consequently there are, up to antiflow isomorphism, only a set's worth of pointed antiflows.

Definition 2.1.2. We use $\{R_i : i \in I\}$ to designate the set of isomorphism types of pointed antiflows of T; more precisely, every pointed antiflow is **pSpT** isomorphic to exactly one R_i. And we use $\rho_i : T \to R_i$ to designate the corresponding antiflow morphism, i.e., $\rho_i(t) = s_i t$ for all $t \in T$. Moreover, any two pointed antiflows (R_1, s_1) and (R_2, s_2) admit at most one **pSpT** morphism $\rho_2^1 : R_1 \to R_2$, given by the rule $\rho_2^1 (s_1 t) = s_2 t$, and it satisfies $\rho_2^1 \rho_1 = \rho_2$ when it exists.

2.2. **Suitable relations.** The reason for the uniqueness of the morphisms ρ_i and ρ_j^i, $i, j \in I$, is that pointed antiflows are in one-to-one correspondence with certain equivalence relations on T. Given $i \in I$, define

$$t' \sim_i t'' \iff s_i t' = s_i t''.$$

Then $\sim_i$ is an equivalence relation on T which is *right invariant* in the sense that for all $t, t', t'' \in T$,

$$t' \sim_i t'' \Longrightarrow t' t \sim_i t'' t.$$

And $\sim_i$ has equivalence classes which are open, and hence clopen. We refer to an equivalence relation with these two properties as a *suitable relation*.

Given a suitable relation $\sim$ on T, let R designate the discrete space $T/{\sim}$, i.e.,

$$R \equiv \{[t] : t \in T\},$$

where $[t]$ denotes the equivalence class of t. Let $t \in T$ act on $[t'] \in R$ by the rule

$$[t']\, t = [t' t].$$

Then R is a pointed antiflow with source $s \equiv [1]$. Therefore there is a unique $i \in I$ for which (R, s) is antiflow isomorphic to (R_i, s_i), and $\sim$ is actually $\sim_i$.

Remark 2.2.1. The pointed antiflows are in one-to-one correspondence with the suitable relations.

1. $(T, 1)$ is a pointed antiflow if and only if T is discrete.
2. If T is connected then its only pointed antiflow contains a single point.
3. If T is compact then all its pointed antiflows are finite.

2.3. **The lattice of pointed antiflows.** The index set I inherits a partial order from the refinement ordering on suitable relations: we define $i \geq j$ in I if $\sim_i$ is finer than $\sim_j$, i.e., if $[t]_i \subseteq [t]_j$ for all $t \in T$, where $[t]_i$ designates the equivalence class of $t \in T$ with respect to $\sim_i$.

Proposition 2.3.1. *$i \geq j$ in I if and only if there is a* **pSpT** *morphism $\rho_j^i : R_i \to R_j$. The morphism is unique when it exists, and I is a lattice under this order.*

Proof. Consider $i, j \in I$. If $i \geq j$ then the map $[t]_i \longmapsto [t]_j$ is a **pSpT** morphism. Conversely if ρ^i_j exists then for all $t, t' \in T$ we would have

$$\begin{aligned} t \sim_i t' \Longleftrightarrow s_i t = s_i t' &\Longrightarrow \rho^i_j (s_i t) = \rho^i_j (s_i t') \Longrightarrow \rho^i_j (s_i) t = \rho^i_j (s_i) t' \\ &\Longrightarrow s_j t = s_j t' \Longleftrightarrow t \sim_j t'. \end{aligned}$$

That the order is a lattice ordering depends on three observations. First, the meet or join (in the lattice of equivalence relations on T) of two right-invariant relations is right invariant. Second, two equivalence relations on T with clopen classes have a join with the same feature. And third, any equivalence relation coarser than one whose classes are clopen also has clopen classes. □

Corollary 2.3.2. *$i \geq j$ in I if and only if*

$$s_i t = s_i t' \Longrightarrow s_j t = s_j t'$$

for all $t, t' \in T$.

At the expense of a little redundancy, we offer an exterior formulation of the fact that I is an upper semilattice. We use $i \vee j$ to denote the supremum of i and j in I.

Proposition 2.3.3. *For $i, j \in I$, $R_{i \vee j}$ is* **pSpT** *isomorphic to*

$$\{(r_i, r_j) \in R_i \times R_j : s_i t = r_i \text{ and } s_j t = r_j \text{ for some } t \in T\},$$

with coordinatewise actions and source (s_i, s_j). Furthermore, the projection maps are **pSpT** *morphisms.*

We close this subsection by pointing out that when T is a topological group, I is anti-isomorphic to its lattice of open subgroups.

Definition 2.3.4. For any element r in any pointed antiflow R_i we designate *the stabilizer of r in T* by

$$\operatorname{stab} r \equiv \{t \in T : rt = r\}.$$

Note that $\operatorname{stab} r$ is a clopen submonoid of T. When r is the source s_i of R_i, we call $\operatorname{stab} r$ a *source stabilizer.*

The terminology of Definition 2.3.4 applies to any element a in a T-algebra A, since the orbit $(aT, a) \equiv (\{at : t \in T\}, a)$ is a pointed antiflow. That is, $\operatorname{stab} a = \{t \in T : at = a\}$.

Proposition 2.3.5. *Suppose that T is a topological group. Then the lattice I of isomorphism types of pointed antiflows is anti-isomorphic to the lattice of open subgroups of T via the map*

$$i \mapsto \operatorname{stab} s_i.$$

The inverse of this map is

$$U \mapsto (T/U, U),$$

where T/U denotes the pointed antiflow of right cosets of U acted upon by right multiplication. Thus I is a lattice in this case.

Proof. For every $i \in I$, the $\sim_i$-class of 1 is just $\operatorname{stab} s_i$. Since the $\sim_i$-classes are clopen, so is $\operatorname{stab} s_i$. If T is a topological group, then for every $t \in T$ the $\sim_i$-class of t is simply the right coset $(\operatorname{stab} s_i)t$. In particular, $\sim_i$ is uniquely determined by $\operatorname{stab} s_i$ and R_i is isomorphic to $(T/\operatorname{stab} s_i, \operatorname{stab} s_i)$.

On the other hand, for every open subgroup U of T, $(T/U, U)$ is indeed a pointed antiflow whose source stabilzer is U. The continuity of the action of T on $(T/U, U)$ follows from the fact that for all $t_0, t_1 \in T$ the set $\{t : Ut_0t = Ut_1\}$ equals $t_0^{-1}Ut_1^{-1}$. The latter set is open since U is. □

It follows from Proposition 2.3.5 that for topological groups T, the suitable relations are in bijective order-reversing correspondence with the source stabilizers. The broader class of topological monoids with this feature will play a role in Section 6.

Definition 2.3.6. Let T be a topological monoid. We say that *the suitable relations on T correspond to the source stabilizers* if for $i, j \in I$,

$$i \geq j \iff \operatorname{stab} s_i \subseteq \operatorname{stab} s_j.$$

2.4. **Constructing $\hat{T}$.** For the purpose of analyzing T-algebras, the only pertinent feature of T is its actions on pointed antiflows. Thus we may exchange T for an associated topological monoid $\hat{T}$, formed by isolating this pertinent feature (Theorem 2.6.1).

We indulge in this development to point out that the action of a given topological monoid T on a T-algebra may, in effect, be other than what it first appears. For example, if T is connected then $\hat{T}$ has a single point and the action is, in fact, trivial; see Remark 2.2.1(2). On the other hand, there may be actions implicit in T which are not present in T but in $\hat{T}$, i.e., $\hat{T}$ may be larger than T; see Example 2.4.6. However, the reader who is only interested in injective and projective objects in **BaT**, the main content of this article, may choose to skip this development.

Let $i \in I$. We designate $\hom_{\mathbf{Sp}}(R_i, R_i) = R_i^{R_i}$ by V_i, and we make V_i into a topological monoid by using as basic neighborhoods of $v \in V_i$ sets of the form

$$\{v' \in V_i : sv' = sv \text{ for all } s \in S\},$$

for finite a subset $S \subseteq R_i$. In other words, V_i carries the product topology induced by the discrete topology on R_i. Note that the product topology on V_i coincides with the compact open topology since all subsets of R_i are open and the compact subsets of R_i are precisely the finite subsets.

Now V_i is a topological monoid acting continuously on R_i in the natural way. We designate the action of T on R_i by ϕ_i, i.e., $\phi_i : T \to V_i$ is the map such that $\phi_i(t)(r) = rt$ for every $r \in R_i$. We write T_i for $\phi_i(T)$. Note that T_i is the only part of T that is really visible from R_i's point of view. We will construct $\hat{T}$ from the T_i, $i \in I$.

Let V designate the product $\prod_I V_i$, regarded as a topological monoid with componentwise multiplication and product topology. Let $\hat{\phi}_i : V \to V_i$ designate the i^{th} projection map. Finally, define $\phi : T \to V$ by the rule

$$\phi\left(t\right)\left(i\right) \equiv \phi_i\left(t\right) = t_i.$$

For $i \geq j$ in I, the **pSpT** morphism $\rho^i_j : R_i \to R_j$ naturally induces a topological monoid homomorphism $\phi^i_j : T_i \to T_j$ as follows. For any $t_i \in T_i$ we define the action of $\phi^i_j\left(t_i\right)$ on an arbitrary $r_j \in R_j$ by the rule

$$\left(r_j\right)\phi^i_j\left(t_i\right) \equiv \rho^i_j\left(r_i t_i\right),$$

where $r_i \in R_i$ is chosen to satisfy $\rho^i_j\left(r_i\right) = r_j$. The definition is independent of the choice of r_i because ρ^i_j commutes with the actions. Writing r_i as $s_i t'_i$ for some $t' \in T$, so that $r_j = s_j t'_j$, gives the simpler formula

$$\left(s_j t'_j\right)\phi^i_j\left(t_i\right) = s_i t'_i t_i = s_i\left(t' t\right)_i.$$

Note that ϕ^i_j is the unique monoid homomorphism that makes the following diagram commutative:

$$\begin{tikzcd} T \arrow[d, "\phi_i"'] \arrow[dr, "\phi_j"] & \\ T_i \arrow[r, "\phi^i_j"'] & T_j \end{tikzcd}$$

The construct that emerges naturally here is the inverse limit $\bar{T}$ in the category of topological monoids, held together by the bonding maps ϕ^i_j for $i \geq j$ in I. In this case we can realize $\bar{T}$ concretely as

$$\bar{T} = \varprojlim_I T_i = \left\{\bar{t} \in \prod_I T_i : \forall i \geq j \ \left(\phi^i_j\left(\bar{t}\left(i\right)\right) = \bar{t}\left(j\right)\right)\right\}.$$

$\hat{T}$ is defined to be the closure of $\bar{T}$ in V. Of course, $\phi\left(T\right)$ is dense in $\bar{T}$, so $\hat{T}$ is also the closure of $\phi\left(T\right)$ in V. We abbreviate $\hat{\phi}_i\left(\hat{t}\right)$ to $\hat{t}_i$ for elements $\hat{t} \in \hat{T}$.

Remark 2.4.1. The following hold for any topological monoid T.

(1) All the maps ϕ, ϕ_i, and $\hat{\phi}_i$ are continuous monoid homomorphisms, and $\hat{\phi}_i\phi = \phi_i$ for each $i \in I$. In other words, the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "\phi"] \arrow[dr, "\phi_i"'] & \bar{T} \arrow[r, "\subseteq"] \arrow[d, "\hat{\phi}_i"] & \hat{T} \arrow[d, "\hat{\phi}_i"] \\ & T_i \arrow[r, "\subseteq"] & V_i \end{tikzcd}$$

Moreover, since $\hat{\phi}_i(\hat{T}) \subseteq V_i$, $\hat{T}$ acts continuously on R_i.

(2) If each pointed antiflow of T is finite then $\hat{T}$ is compact.

(3) ϕ is injective if and only if the common refinement of the suitable relations is the identity relation, i.e., each pair of distinct points of T is separated by a suitable relation.

(4) $\hat{T}$ is discrete if and only if $\phi(T)$ is discrete if and only if I has a greatest element.

Proof. (1) is straightforward.

For (2) observe that if every R_i is finite, then every V_i is finite. Thus, V is a product of finite spaces and hence compact. $\hat{T}$ is a closed subspace of V and therefore compact as well.

For (3) let $t, t' \in T$. Then $\phi(t) = \phi(t')$ if and only if for all $i \in I$, $\phi_i(t) = \phi_i(t')$. If for some $i \in I$, $t \not\sim_i t'$, then $s_i t \neq s_i t'$ and hence $\phi(t) \neq \phi(t')$. On the other hand, if $\phi_i(t) \neq \phi_i(t')$, then there is $r_i \in R_i$ such that $r_i t \neq r_i t'$. Now $(r_i T, r_i)$ is a pointed antiflow and hence it is isomorphic to R_j for some $j \in I$. Clearly, $\phi_j(t) \neq \phi_j(t')$.

For the proof of (4) notice that if $\hat{T}$ is discrete then so is $\phi(T)$, being a subspace of $\hat{T}$.

Now assume that $\phi(T)$ is discrete. T acts continuously on $\hat{T}$ by the rule

$$\hat{t}t \equiv \hat{t}\phi(t).$$

For every $t \in T$ we have $1_{\hat{T}} t = \phi(1_T)\phi(t) = \phi(t)$. Hence $\phi(T)$ is the orbit of $1_{\hat{T}}$ under the action of T on $\hat{T}$. It follows that $(\phi(T), 1_{\hat{T}})$ is a pointed antiflow of T. Let i be the corresponding element of I.

For all $t, t' \in T$ and all $j \in I$ we have

$$1_{\hat{T}} t = 1_{\hat{T}} t' \quad \Rightarrow \quad \phi(t) = \phi(t') \quad \Rightarrow \quad \phi_j(t) = \phi_j(t) \quad \Rightarrow \quad s_j t = s_j t'.$$

It follows that $t \sim_i t'$ implies $t \sim_j t'$. Hence i is the largest element of I.

Finally, assume that I has a greatest element i. Then $\bar{T} = \phi(T)$ and $\hat{\phi}_i : \phi(T) \to T_i$ is an isomorphism. Now the crucial observation is that an element t_i of T_i is already determined by $s_i t_i$.

To see this let $t, t' \in T$ be such that $s_i t = s_i t'$. We have to show that for every $r \in R_i$, $rt = rt'$. Let $r \in R_i$ and consider the pointed antiflow (rT, r). Since i is the greatest element of I, there is a **pSpT** morphism $\rho : (R_i, s_i) \to (rT, r)$. Now

$$rt' = \rho(s_i)t' = \rho(s_i t') = \rho(s_i t) = rt.$$

We are now ready to show that $\hat{T}$ is discrete.

Let $f \in \hat{T}$. The set

$$U \equiv \{f' \in \hat{T} : f'(i)(s_i) = f(i)(s_i)\}$$

an open neighborhood of f in $\hat{T}$. Since $\phi(T)$ is dense in $\hat{T}$, there is $t \in T$ with $\phi(t) \in U$. By the observation above, $\phi_i(t)$ is uniquely determined. Since i is the largest element of I, $\phi_i(t)$ detemines $\phi(t)$, i.e., the intersection of $\phi(T)$ and U is a singleton. Since f is in the closure of $\phi(T)$, we have $f = \phi(t)$. It follows that U is a singleton. i.e., f is an isolated point of $\hat{T}$. This shows that $\hat{T}$ is discrete. □

Proposition 2.4.2. *$\hat{T}$ has the same lattice I of pointed antiflows as T does.*

Proof. For each action ϕ_i of T on one of its pointed antiflows R_i we have the corresponding action $\hat{\phi}_i$ of $\hat{T}$ on R_i, and $\phi\hat{\phi}_i = \phi_i$ by construction. Conversely any action of $\hat{T}$ on a pointed antiflow, when followed by ϕ, gives an action of T on that flow. This shows that the pointed antiflows of T are the same as those for $\hat{T}$. Furthermore, the order on I imposed by T is the same as that imposed by $\hat{T}$. For if $i \geq j$ in I by virtue of the **pSpT** morphism $\rho^i_j : R_i \to R_j$ then this ρ^i_j is also a $\mathbf{pSp\hat{T}}$ morphism, i.e., it commutes with each $\hat{t} \in \hat{T}$. The reason is that for $r_i \in R_i$ and $r_j \equiv \rho^i_j(r_i)$ there is some $t \in T$ such that $r_i\hat{t}_i = r_i t_i$ and $r_j\hat{t}_j = r_j t_j$, hence

$$\rho^i_j\left(r_i\hat{t}_i\right) = \rho^i_j\left(r_i t_i\right) = \rho^i_j\left(r_i\right) t_i = r_j t_j = r_j\hat{t}_j = \rho^i_j\left(r_i\right)\hat{t}_j.$$

Thus $i \geq j$ in the order imposed on I by $\hat{T}$. □

Corollary 2.4.3. $\hat{\hat{T}} = \hat{T}$.

Remark 2.4.4. Let $i \in I$. We denote the equivalence relation on $\hat{T}$ that corresponds to R_i by $\sim_i$, just like the corresponding relation on T.

(1) For elements $t, t' \in T$, $t \sim_i t'$ if and only if $\phi(t) \sim_i \phi(t')$.
(2) Each $\sim_i$ class of $\hat{T}$ contains an element of $\phi(T)$.

Proof. The proof of (1) is straightforward. For (2) let $\hat{t} \in \hat{T}$. Then $\hat{\phi}_i(\hat{t}) \in T_i$. Hence for some $t \in T$, $\phi_i(t) = \hat{\phi}_i(\hat{t})$. But now $s_i\hat{t} = s_i\phi(t)$ and hence $\hat{t} \sim_i \phi(t)$. □

Let us have a closer look at the special case when T is a topological group.

Proposition 2.4.5. *Suppose that T is a topological group. Then $\bar{T}$ is a group, but $\hat{T}$ need not be a group. The identity element of $\bar{T}$ has a neighborhood base consisting of the open subgroups of $\bar{T}$.*

Proof. If T is a group, then each T_i, being a quotient of T by a monoid homomorphism, is a group as well. It follows that the inverse limit $\bar{T}$ is a group. Example 2.4.6 shows that $\hat{T}$ is not necessarily a group.

We show that $1_{\bar{T}}$ has a neighborhood base consisting of open subgroups of $\bar{T}$. Let O be an open neighborhood of $1_{\bar{T}}$. Since the topology on $\bar{T}$ is induced by the product topology on V, there are $n \in \mathbb{N}$, $i_0, \ldots, i_{n-1} \in I$ and finite sets $S_k \subseteq R_{i_k}$, $k < n$, such that

$$U \equiv \{\bar{t} \in \bar{T} : \forall k < n \forall r \in S_k(r\bar{t} = r)\} \subseteq O.$$

Clearly, U is an open subgroup of $\bar{T}$. □

Here is an example which illustrates the ideas in this subsection. It makes the point that T need not coincide with $\hat{T}$ even when $\phi : T \to \hat{T}$ is injective, and that $\hat{T}$ need not be a group even when T is.

Example 2.4.6. Let T be the group of permutations of the natural numbers N under composition, equipped with the topology inherited from the product topology on N^N. T is a topological group. For each $m \in N$ define the equivalence relation $\sim_m$ by declaring

$$t \sim_m t' \iff it = it' \quad \text{for all } i \leq m.$$

(We choose to write the permutation to the right of its input.) Then $\sim_m$ is a suitable relation, and every suitable relation is refined by one of these. Thus we may identify $[t]_m$ with the m-tuple $(1t, 2t, \ldots, mt)$, and identify $R_m \equiv T/{\sim_m}$ with the set of all m-tuples of distinct elements from N. The source of R_m is $s_m = (1, 2, \ldots, m)$, and the action of $t \in T$ on $(i_1, i_2, \ldots, i_m)$ is given by

$$(i_1, i_2, \ldots, i_m)\, t = (i_1 t, i_2 t, \ldots, i_m t).$$

The natural order on N coincides with the refinement ordering on the corresponding suitable relations, and if $m \geq n$ then the **pSpT** morphism $\rho^m_n : R_m \to R_n$ is simply restriction, i.e.,

$$\rho^m_n\,(i_1, i_2, \ldots, i_m) = (i_1, i_2, \ldots, i_n)\,.$$

Following the construction of $\hat{T}$ let $\phi_n : T \to V_n = R_n^{R_n}$ be the action of T on R_n and let $T_n \equiv \phi_n(T)$. The induced map $\phi : T \to V = \prod_N V_n$ is clearly injective. In fact,

$$s_n \phi\,(t)\,(n) = s_n = (1t, 2t, \dots, nt).$$

For every $f \in \prod_N V_n$, $f \in \hat{T} = \mathrm{cl}_V\,\phi\,(T)$ if and only if for all $n \in N$ there is a permutation $t_n \in T$ such that for all $m \leq n$, $(1, \dots, m)t_n = (1, \dots, m)f(m)$. However, the permutation t_n does not have to be the same for all n. It follows that $f \in \hat{T}$ implies that for all m and n with $m \leq n$, $(1, \dots, m)f(m)$ is the restriction of $(1, \dots, n)f(n)$ to the first m coordinates. Therefore every $f \in \hat{T} = \mathrm{cl}_V\,\phi\,(T)$ corresponds to a function $g : N \to N$ such that for every n,

$$(1, \dots, n)f(n) = (g(1), \dots, g(n)).$$

Every function g that arises in this way is one-to-one.

On the other hand, if $g : N \to N$ is one-to-one, for every n and every n-tuple $(i_1, \dots, i_n)$ of distinct natural numbers let

$$(i_1, \dots, i_n)f(n) \equiv (g(i_1), \dots, g(i_n)).$$

Now $f \in \hat{T}$ since for all n we can choose a permutation $t_n \in T$ such that

$$(1, \dots, n)t_n = (g(1), \dots, g(n)).$$

It follows that the elements of $\hat{T}$ correspond to one-to-one functions from N to N. This correspondence is in fact a monoid isomorphism. Hence $\hat{T}$ is not a group.

2.5. **The type of an element of a T-algebra.** Pointed antiflows arise as a means of classifying elements of a T-algebra A according to the complexity of their orbits. Let A be an algebra on which T acts, and for each $a \in A$ let $\sim_a$ designate the relation on T defined by the rule

$$t \sim_a t' \iff at_A = at'_A,$$

for $t, t' \in T$. (Here and in what follows we use t_A to abbreviate $\phi_A\,(t)$, where

$$\phi_A : T \to \hom A \equiv \hom_{\mathbf{Ba}}\,(A, A)$$

is the action of T on A.) Then $\sim_a$ is a right-invariant equivalence relation, and A is a T-algebra, i.e., evaluation is continuous, if and only if each $\sim_a$ is a suitable relation.

Definition 2.5.1. We say that *the type of an element a of a* T-algebra *A is* $i \in I$, and write $\operatorname{type} a = i$, provided that $\sim_a$ is $\sim_i$.

Remark 2.5.2. Let a be an element of a T-algebra A.

(1) To assert that a is of type at most i is to assert that for all $t, t' \in T$,

$$s_i t_i = s_i t'_i \Longrightarrow at_A = at'_A.$$

(2) If the suitable relations on T correspond to the source stabilizers, then to assert that a is of type at most i is to assert that

$$\operatorname{stab} s_i \subseteq \operatorname{stab} a.$$

Lemma 2.5.3. *Let a and b be elements of a T-algebra A. Then*

$$\operatorname{type}(a \vee b) \vee \operatorname{type}(a \wedge b) \leq \operatorname{type} a \vee \operatorname{type} b, \text{ and}$$
$$\operatorname{type} a = \operatorname{type} \overline{a},$$

where $\overline{a}$ denotes the complement of a. Therefore, for all $i \in I$,

$$\{a \in A : \operatorname{type} a \leq i\}$$

is a subalgebra of A, but need not be a T-subalgebra.

Proof. The map

$$\rho^a_{\overline{a}} : aT \to \overline{a}T; at \mapsto \overline{a}t$$

is a **pSpT** isomorphism. Hence $\operatorname{type}(a) = \operatorname{type}(\overline{a})$.

Now consider the pointed antiflow

$$R \equiv (\{(at, bt) : t \in T\}, (a, b)).$$

R is isomorphic to $R_{\operatorname{type} a \vee \operatorname{type} b}$. The map

$$\rho^{a,b}_{a \wedge b} : R \to ((a \wedge b)T, a \wedge b); (at, bt) \mapsto (a \wedge b)t$$

is a **pSpT** morphism showing that

$$\operatorname{type}(a \wedge b) \leq \operatorname{type} a \vee \operatorname{type} b.$$

Similarly,

$$\operatorname{type}(a \vee b) \leq \operatorname{type} a \vee \operatorname{type} b.$$

It follows that for every $i \in I$ the elements of A of type at most i form a subalgebra of A. Example 2.5.4 shows that this subalgebra does not have to be a T-subalgebra. □

Example 2.5.4. Let T be S_3, the group of all permutations of the set $\{1, 2, 3\}$. We consider the action of T on $\{1, 2, 3\}$ as a left-action since we consider $\{1, 2, 3\}$ as a Boolean space. Let A be the powerset algebra of $\{1, 2, 3\}$. The action of T on $\{1, 2, 3\}$ induces a right-action on A by letting $at = t^{-1}(a)$ for every $t \in T$.

Note that $\operatorname{stab}\{1\}$, $\operatorname{stab}\{2\}$ and $\operatorname{stab}\{3\}$ are pairwise incomparable (with respect to $\subseteq$) subgroups of T. It follows that the types of $\{1\}$, $\{2\}$ and $\{3\}$ are pairwise incomparable. Therefore

$$B \equiv \{a \in A : \operatorname{type} a \leq \operatorname{type}\{1\}\} = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}.$$

B is a subalgebra of A, but the T-subalgebra of A generated by B is already all of A.

A crucial observation is that morphisms reduce type.

Proposition 2.5.5. *If $f : A \to B$ is a T-morphism then for all $a \in A$,*

$$\operatorname{type} a \geq \operatorname{type} f(a).$$

T acts on I on the right, essentially by shifting each source s_i of R_i to $s_i t$. The key observation is that if $\sim_i$ is a suitable relation on T then so is the relation $\sim_{it}$ defined by the rule

$$t' \sim_{it} t'' \iff tt' \sim_i tt''$$

for $t', t'' \in T$.

Remark 2.5.6. The following hold for $i \in I$ and $t, t', t'' \in T$.

(1) $(it)\, t' = i\,(tt')$.
(2) $t' \sim_{it} t''$ if and only if $s_i t_i t'_i = s_i t_i t''_i$.
(3) A pointed antiflow corresponding to the suitable relation $\sim_{it}$ is $(s_i tT, s_i t)$, where $s_i tT \equiv \{s_i tt' : t' \in T\}$, and where the action is given by

$$(s_i tt')\, \phi_j\,(t'') = s_i \phi_i\,(tt't'')$$

for $t', t'' \in T$.

Proposition 2.5.7. *Suppose a is an element of a T-algebra A and $t \in T$. Then*

$$\operatorname{type}(at_A) = (\operatorname{type} a)\, t.$$

Proof. Let $\operatorname{type} a \equiv i$ and $\operatorname{type}(at_A) \equiv j$. Then for $t', t'' \in T$,

$$\begin{aligned} t' \sim_{it} t'' &\iff tt' \sim_i tt'' \iff s_i\,(tt')_i = s_i\,(tt'')_i \\ &\iff a\,(tt')_A = a\,(tt'')_A \iff (at_A)\, t'_A = (at_A)\, t''_A \\ &\iff t' \sim_j t''. \end{aligned}$$

Since the suitable relations $\sim_{it}$ and $\sim_j$ coincide, it follows that $it = j$ in I, which is the desired conclusion. ☐

One might idly conjecture that, in the notation of Lemma 2.5.7, $i \geq it$ by virtue of the map $r_i \longmapsto r_i t$, $r_i \in R_i$. But this is most assuredly not the case, since this map is generally not a **pSpT** morphism. This phenomenon occurs in Example 2.5.4: the types of $\{1\}$ and $\{2\}$ are incomparable, but if $t \in S_3$ is the transposition that exchanges 1 and 2, then $\{2\} = \{1\}t$, i.e., $(\operatorname{type}\{1\})t = \operatorname{type}\{2\}$.

Proposition 2.5.8. *Let A be a T-algebra and $i \in I$. Then for each $i \in I$,*

$$A_i \equiv \left\{ a \in A : \operatorname{type} a \le \bigvee_{T_0} it \text{ for some finite } T_0 \subseteq T \right\}$$

is a T-subalgebra of A, and

$$A = \lim_{\rightarrow} \left\{ A_i : i \in I \right\}.$$

Proof. The fact that A_i is a T-subalgebra follows from Proposition 2.5.7 and Lemma 2.5.3. And since each element $a \in A$ has a type $i \in I$ and is therefore contained in A_i, it follows that A is the direct limit of the A_i's. □

2.6. **Trading T for $\hat{T}$.**

Theorem 2.6.1. *For every action ϕ_A of T on a T-algebra A there is a unique corresponding action $\hat{\phi}_A$ of $\hat{T}$ on A such that $\hat{\phi}_A \phi = \phi_A$.*

$$\begin{tikzcd} T \arrow[r, "\phi"] \arrow[d, "\phi_A"'] & \hat{T} \arrow[dl, "\hat{\phi}_A"] \\ \hom A & \end{tikzcd}$$

Proof. Consider $\hat{t} \in \hat{T}$ and $a \in A$ such that $\operatorname{type} a = i \in I$, and find $t \in T$ such that $\phi(t) \sim_i \hat{t}$; see Remark 2.4.4(2). If an action $\hat{\phi}_A$ is to exist satisfying this theorem, it follows from Remarks 2.4.4(1) and 2.5.2(1) that

$$a\hat{\phi}_A\left(\hat{t}\right) \equiv a\hat{t}_A = at_A \equiv a\phi_A(t).$$

Therefore take this as the definition of $\hat{\phi}_A$. First observe that $a\hat{t}_A$ is well defined, for if t' is another element of T such that $\phi(t') \sim_i \hat{t}$ then $\phi(t) \sim_i \phi t'$, hence $s_i t_i = s_i t'_i$, with the result that $at_A = at'_A$ because $\operatorname{type} a = i$. Next we claim that $\hat{t}_A$ is a Boolean morphism. For

$$\overline{a\hat{t}_A} = \overline{at_A} = \overline{a}t_A = \overline{a}\hat{t}_A$$

because the same element $t \in T$ used to define $a\hat{t}_A$ can also be used to define $\overline{a}\hat{t}_A$ since $\operatorname{type} a = \operatorname{type} \overline{a}$ by Lemma 2.5.3. And for $a, b \in A$ we may take $k = \operatorname{type} a \vee \operatorname{type} b$ and find $t \in T$ such that $\phi(t) \sim_k \hat{t}$. Then because t_A is a Boolean morphism which agrees with $\hat{t}_A$ at a, b, $a \vee b$, and $a \wedge b$, we get

$$(a \vee b)\,\hat{t}_A = a\hat{t}_A \vee b\hat{t}_A \quad \text{and} \quad (a \wedge b)\,\hat{t}_A = a\hat{t}_A \wedge b\hat{t}_A.$$

To verify that $\hat{\phi}_A$ is a monoid morphism consider $\hat{t}', \hat{t} \in \hat{T}$ and $a \in A$, let $\operatorname{type} a = i$, and find $t \in T$ such that $\phi(t) \sim_i \hat{t}$. Then $a\hat{t}_A = at_A$, and

$$\operatorname{type}\left(a\hat{t}_A\right) = \operatorname{type}(at_A) = (\operatorname{type} a)\, t = it.$$

Next find $t' \in T$ such that $\phi(t') \sim_{it} \hat{t}'$, so that

$$\left(a\hat{t}_A\right)\hat{t}'_A = (at_A)\,\hat{t}'_A = (at_A)\,t'_A = a\,(tt')_A\,.$$

To show that $a\,(tt')_A = a\left(\hat{t}\hat{t}'\right)_A$ we must show that $\hat{t}\hat{t}' \sim_i \phi(tt')$. But this is easy. Because $\hat{t} \sim_i \phi(t)$ we know that $s_i\hat{t}_i = s_i t_i$, hence $s_i\hat{t}_i\hat{t}'_i = s_i t_i \hat{t}'_i$, and from the fact that $\hat{t}' \sim_{it} \phi(t')$ we know from Remark 2.5.6(2) that $s_i t_i \hat{t}'_i = s_i t_i t'_i$, hence $s_i\hat{t}_i\hat{t}'_i = s_i t_i t'_i$, from which the desired conclusion follows. This completes the verification that $\hat{\phi}_A$ is a monoid morphism. Finally, evaluation is continuous by construction. □

3. T-algebras

In this section we record the basic facts concerning T-algebras which will be necessary in what follows. We use $\perp$ and $\top$ to denote the smallest and the largest element of a given Boolean algebra.

3.1. T-Morphisms and T-ideals.

Definition 3.1.1. A *T-ideal* of a T-algebra A is an ideal I with the property that $at \in I$ for all $a \in I$ and all $t \in T$.

Such ideals determine the T-surjections.

Proposition 3.1.2. *For any T-morphism $f : A \to B$,*

$$I = \{a \in A : f(a) = \perp\}$$

is a T-ideal. Conversely, for a given T-ideal I there is one and only one way to have T act on the quotient A/I so as to make the actions commute with the quotient map g, namely by defining

$$g(a)\,t = g(at)$$

for each $t \in T$ and $a \in A$. In this case A/I is a T-algebra and g is a T-morphism.

Lemma 3.1.3. *Any ideal I of a T-algebra A has a largest T-ideal contained in it, namely*

$$I_T \equiv \{a \in A : at \in I \text{ for all } t \in T\}\,.$$

If $I \subseteq A$ is an ideal of the T-algebra A, then the quotient map $f : A \to A/I$ has a factorization $f = g\hat{f}$ where $\hat{f} : A \to A/I_T$ and $g : A/I_T \to A/I$ are the quotient maps. Note that $\hat{f}$ is a T-morphism. This observation gives rise to the following definition.

Definition 3.1.4. A factorization $g\hat{f} = f$ of a naked surjection f is a **Ba**-**BaT** *factorization* if $\hat{f}$ and g are surjections such that $g \in \mathbf{Ba}$ and $\hat{f} \in \mathbf{BaT}$. Such a factorization is *minimal* if it has the additional feature that any **Ba**-**BaT** factorization $kh = f$ has h as an initial factor of $\hat{f}$, i.e., $\hat{f} = lh$ for some $l \in \mathbf{BaT}$.

Proposition 3.1.5. *Every naked surjection out of a T-algebra has a minimal* **Ba**-**BaT** *factorization.*

Proof. If I is the kernel of f, then let $\hat{f} : A \to A/I_T$ and $g : A/I_T \to A/I$ be the canonical maps. We show that the factorization $f = g\hat{f}$ is minimal. Suppose $f = kh$ is another **Ba**-**BaT** factorization of f. Let J be the kernel of h. J is a T-ideal contained in I. Hence $J \subseteq I_T$. The image of h is canonically T-isomorphic to A/J and hence we may assume that h is just the quotient map from A onto A/J. Let $l : A/J \to A/I_T$ be the canonical map. Now clearly $\hat{f} = lh$, showing that $g\hat{f}$ is indeed minimal. □

Lemma 3.1.6. *Let A_0 be a subset of the T-algebra A, and let I be a T-ideal of A such that $I \cap A_0 \subseteq \{\bot\}$. Then there is a T-ideal J maximal with respect to $J \supseteq I$ and $J \cap A_0 \subseteq \{\bot\}$.*

Recall that in any category $\mathbf{C}$, a morphism $f : A \to B$ is called *essential* if it is injective, and every morphism out of B whose composition with f is injective must itself be injective.

Proposition 3.1.7. *The following are equivalent for a T-injection $f : A \to B$.*

1. *f is essential.*
2. *Every nontrivial T-ideal of B meets $f(A)$ nontrivially.*
3. *For every $\bot < b \in B$ there is some $\bot < a \in A$ and finite $T' \subseteq T$ such that $f(a) \le \bigvee_{T'} bt'$.*

Proof. By Proposition 3.1.2, the non-essentiality of f is equivalent to the existence of a T-ideal $I \subseteq B$ such that $I \cap f(A) = \{\bot\}$. An element $b > \bot$ of such an ideal would violate the condition of this proposition, and any $b \in B$ which violated this same condition would generate a proper T-ideal corresponding to a T-surjection denying essentiality. □

Proposition 3.1.8. *For every T-injection f there is a T-surjection g such that gf is essential.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Proof. Let J be a T-ideal of B maximal with respect to $J \cap f(A) = \{\perp\}$, and let $g : B \to C \equiv B/J$ be the natural map. □

Definition 3.1.9. A T-algebra A is *simple* if it has no proper T-homomorphic images.

Proposition 3.1.10. *Every T-algebra has a simple quotient.*

Proof. Any T-algebra A has a maximal proper T-ideal by Lemma 3.1.6, with A_0 and I there taken to be $\{\top\}$ and $\{\perp\}$, respectively. And the corresponding quotient is simple by Proposition 3.1.2 □

We use **2** to denote the algebra containing only greatest element $\top$ and least element $\perp$. When regarded as a T-algebra, the action is presumed to be trivial, as indeed it must be.

Proposition 3.1.11. *The following are equivalent for a T-algebra A.*

(1) *A is simple.*
(2) *A has no proper T-ideals.*
(3) **2** *is essentially embedded in A.*
(4) *For all $\perp < a \in A$ there is some finite subset $T_0 \subseteq T$ such that $\bigvee_{T_0} at_0 = \top$.*

3.2. **The reduction to discrete T.** Let T_d denote the monoid T with discrete topology. We will construct cofree T-algebras over naked algebras by the strategy of first constructing cofree T_d-algebras and then passing to the corresponding T-algebra by restricting to a particular subalgebra. The first use of this strategy comes in the following subsection.

Suppose B is an algebra on which T acts, i.e., a T_d-algebra. Slightly abusing notation, for $i \in I$ and $b \in B$ we write $\operatorname{type} b \le i$ if there is a T-equivariant map from R_i to aT that maps s_i to b. Now B_i can be defined as in Proposition 2.5.8.

Lemma 3.2.1. *Let B be as above.*

(1) *Let $b \in B$. Then T acts continuously on bT iff for some $i \in I$,* $\operatorname{type} b \le i$.
(2) *The set*
$$B_T \equiv \{b \in B : \exists i \in I \ (\operatorname{type} b \le i)\}$$
is the largest subalgebra of B which forms a T-algebra under the relativised actions. B_T is the direct limit of the algebras B_i, $i \in I$.

Proof. For (1) assume $\rho : R_i \to bT$ is a T-equivariant map such that $\rho(s_i) = b$. To show the continuity of the action of T on bT, we have to show that for all $a \in bT$ and all $t \in T$ the set $\{t' \in T : at = at'\}$ is open in T. But

$$\{t' \in T : at = at'\} = \bigcup \{\{t'' \in T : rt' = rt''\} : r \in \rho^{-1}(a) \wedge at' = at\}$$

and the sets $\{t'' \in T : rt' = rt''\}$ are open by the continuity of the action of T on R_i.

On the other hand, if T acts continuously on bT, then for some $i \in I$, (bT, b) is actually isomorphic to R_i.

(2) easily follows from (1). □

Theorem 3.2.2. **BaT** *is a mono-coreflective subcategory of* $\mathbf{BaT}_d$ [4, 36.1]*, and the coreflective morphism for* $B \in \mathbf{BaT}_d$ *is the insertion of* B_T *in* B*.*

Proof. Let $f : A \to B$ be a T_d-morphism for which $A \in \mathbf{BaT}$. We

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] & B \\ & B_T \arrow[u, "i"'] \end{tikzcd}$$

claim that $f(A) \subseteq B_T$. For if $a \in A$ then for any $t \in T$ we have a neighborhood $U \subseteq T$ of t such that $at = at'$ for all $t' \in U$. But then

$$f(a)\,t = f(at) = f(at') = f(a)\,t'$$

for all $t' \in U$. This proves the claim that $f(a) \in B_T$. □

3.3. **Freely adding actions to a naked algebra.** In this subsection we show that, although there are many ways to endow a naked algebra with actions, adding such actions "as freely as possible" can be done in one and only one way. More precisely, for a given naked algebra B there exist a unique T-algebra A and naked morphism $p : A \to B$ such that for any other T-algebra C and naked morphism $f : C \to B$ there is a unique **Bat** morphism $g : C \to A$ such that $pg = f$. That is, p

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "g"'] & B \\ & A \arrow[u, "p"'] \end{tikzcd}$$

is an F-co-universal map for B [4, VII 26.1], where $F : \mathbf{BaT} \to \mathbf{Ba}$ is the functor which forgets the actions. (We are, in effect, showing that F has a right adjoint [4, VII 27.3].) We refer to this situation by saying that **BaT** is cofree over **Ba**. The T-algebra A together with the morphism $p : A \to B$ is the cofree T-algebra over B.

We fix a naked algebra B, and set

$$A \equiv \prod_{t \in T} B_t,$$

where B_t is a copy of B for each $t \in T$. We view each element of A as a map from T into B. Let $t \in T$ act on $a \in A$ according to the rule

$$(at)(t') \equiv a(tt')$$

for all $t' \in T$. It is easy to check that $A \in \mathbf{BaT}_d$, i.e., that T acts on A. Project A onto B by the **Ba** morphism p defined by $p(a) = a(1)$.

Proposition 3.3.1. $\mathbf{BaT}_d$ *is cofree over* **Ba**, *and the cofree* T_d*-algebra over a naked algebra* B *is* $p : A \to B$.

Proof. Given a **Ba** morphism f whose domain C is a T_d-algebra, a

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "g"'] & B \\ & A \arrow[u, "p"'] \end{tikzcd}$$

T_d-morphism g which makes the diagram commute must satisfy

$$g(c)(t) = g(c)(t1) = (g(c)t)(1) = (g(ct))(1) = pg(ct) = f(ct).$$

So if we take this requirement as a definition of g we clearly get a homomorphism which makes the diagram commute. To check that g commutes with the actions, observe that

$$(g(c)t)(t') = g(c)(tt') = f(c(tt')) = f((ct)(t')) = g(ct)(t')$$

for all $c \in C$ and $t, t' \in T$. $\square$

Proposition 3.3.2. **BaT** *is cofree over* **Ba**, *and the cofree* T*-algebra over a naked algebra* B *is* $p : A_T \to B$.

Proof. Given a **Ba** morphism $f : C \to B$ whose domain is a **BaT** object, the morphism g of Proposition 3.3.1 factors through A_T by Theorem 3.2.2. $\square$

For $i \in I$ we say that an element a of the T_d-algebra A is *constant on every* $\sim_i$ *class* if for all $t, t' \in T$ with $t \sim_i t'$, $at = at'$.

Lemma 3.3.3. *An element of A has type at most i if and only if it is constant on each $\sim_i$ class. Therefore*

$$A_T = \{a \in A : \exists i \in I \ (a \text{ is constant on each } \sim_i \text{ class})\}.$$

Furthermore, for any $i \in I$ the elements of A_i are those $a \in A$ which are constant on all $\sim_j$ classes, where $j = \bigvee_{T_0} it$ for some finite $T_0 \subseteq T$.

Proof. If type $a \le i$ then for all $t, t' \in T$ we have

$$t \sim_i t' \iff s_i t = s_i t' \implies at = at'$$

by Remark 2.5.2(1). On the other hand, if a is constant on $\sim_i$ classes and $t \sim_i t'$ then by the right invariance of $\sim_i$ we have $tt'' \sim_i t't''$ for all $t'' \in T$. This implies that $a\,(tt'') = a\,(t't'')$, i.e., $(at)\,(t'') = (at')\,(t'')$ for all $t'' \in T$, which is to say that $at = at'$. This shows that type $a \le i$ by Remark 2.5.2(1). □

We summarize the results of this subsection.

Theorem 3.3.4. *The cofree T-algebra over B is the subalgebra of A consisting of those elements which are constant on the classes of some suitable relation on T.*

3.4. **Free T-algebras over pointed antiflows.** For $i \in I$ let F_i designate the free algebra over the generating set R_i. Recall that for $t \in T$, t_i denotes the action of t on R_i, i.e., the map $\phi_i(t) : R_i \to R_i$. Each t_i lifts uniquely to a morphism on F_i, and the actions thus defined make F_i a T-algebra.

Consider now a T-algebra A with element a of type at most i. The map $s_i t \mapsto at$ is a **pSpT** morphism from (R_i, s_i) onto the orbit (aT, a) of a. We show that this map lifts to a unique T-morphism f from F_i into A. It is in this sense that F_i serves as the free T-algebra over R_i.

$$\begin{tikzcd}
(R_i, s_i) \arrow[r, "\subseteq"] \arrow[d] & F_i \arrow[d, "f"] \\
(aT, a) \arrow[r, "\subseteq"] & A
\end{tikzcd}$$

Proposition 3.4.1. *Let a be an element of a T-algebra A of type at most $i \in I$. Then there is a unique T-morphism $f : F_i \to A$ such that $f\,(s_i) = a$.*

Proof. Since type $a \le i$, there is a unique T-equivariant map $\rho : R_i \to aT$ with $\rho(s_i) = a$. In order for f to respect the actions we must have $f \restriction R_i = \rho$ for each $t \in T$. The fact that F_i is the free algebra over R_i then provides a unique extension of ρ to a morphism $f : F_i \to A$.

We have to show that f commutes with the actions on F_i. Let $v, w \in F_i$ and suppose that we already know that for all $t \in T$, $f(vt) = (f(v))t$ and $f(wt) = (f(w))t$. Then

$$\begin{aligned} f((v \wedge w)t) &= f(vt \wedge wt) = f(vt) \wedge f(wt) \\ &= (f(v))t \wedge (f(w))t = (f(v) \wedge f(w))t = (f(v \wedge w))t \end{aligned}$$

and

$$f\left(\overline{vt}\right) = f(\overline{v}t) = (f(\overline{v}))t = \left(\overline{f(v)}\right)t = \overline{(f(v))t} = \overline{f(vt)}.$$

Since the elements of F_i are Boolean combinations of members of R_i and since the restriction of f to R_i is T-equivariant, it follows that f is T-equivariant. $\square$

Since the free T-algebra F_i over R_i plays a prominent role in what follows, particularly in Section 6, we allow ourselves a closer look at its elements. The context of this discussion is a little more general than that of Theorem 3.4.1. Suppose that F is a naked algebra with subset $R \subseteq F$. We use $\langle R \rangle_{\mathbf{Ba}}$ to designate the subalgebra generated by R. R generates F if $\langle R \rangle_{\mathbf{Ba}} = F$, and R freely generates F as a naked algebra if every set map from R into another algebra A lifts to a unique morphism from F into A. This is equivalent to the condition that

$$\langle S_1 \rangle_{\mathbf{Ba}} \cap \langle S_2 \rangle_{\mathbf{Ba}} = \langle S_1 \cap S_2 \rangle_{\mathbf{Ba}}$$

for finite $S_1, S_2 \subseteq R$. (See [1, V.3] for another equivalent formulation.) Therefore every $w \in F$ has a smallest subset $S \subseteq R$ for which $w \in \langle S \rangle_{\mathbf{Ba}}$; we refer to S as the *support of* w, and write $S = \operatorname{supp} w$. Note that $\operatorname{supp} w$ is a finite set and that $\operatorname{supp} w = \emptyset$ if and only if w is $\perp$ or $\top$.

We can understand the concept of support more concretely. Let w be an element of $\langle S \rangle_{\mathbf{Ba}} \setminus \{\perp, \top\}$. Then there is some finite set $S' \subseteq S$ such that $w \in \langle S' \rangle_{\mathbf{Ba}}$. So assume that S is finite. The laws of Boolean algebra allow w to be expressed as

$$w = \bigvee_{\Theta} \bigwedge_{S} s^{\theta(s)}$$

for some $\Theta \subseteq \{\pm 1\}^S$. (Here $\{\pm 1\}^S$ designates the set of all maps from S into $\{\pm 1\}$, and s^1 and s^{-1} designate s and the complement $\overline{s}$ of s, respectively.) This representation may be redundant because the laws of Boolean algebra may make it possible to omit an element s from S, restrict the functions of Θ to $S \setminus \{s\}$, and still have a representation of w. The criterion for being unable to omit s from S is exactly that there be a function $\theta \in \Theta$ such that changing its value only at s results in another function not in Θ. The support of w is precisely the subset of S consisting of those elements which cannot be omitted from S in this sense. Thus

$$w = \bigvee_{\Lambda} \bigwedge_{\operatorname{supp} w} s^{\theta(s)}$$

for some $\Lambda \subseteq \{\pm 1\}^{\operatorname{supp} w}$, and this representation is a normal form, i.e., it is unique to w.

Proposition 3.4.3 will find use in Subsection 6.2.

Lemma 3.4.2. *Suppose F is a T-algebra which is freely generated as a naked algebra by a subset $R \subseteq F$. Then*

$$\operatorname{supp}(wt) \subseteq (\operatorname{supp} w)\, t$$

for any $w \in F \setminus \{\bot, \top\}$ and any $t \in T$.

Proof. Abbreviate $\operatorname{supp} w$ to S. If we write w in normal form and act on it by t, we get

$$wt = \left(\bigvee_{\Theta} \bigwedge_{S} s^{\theta(s)} \right) t = \bigvee_{\Theta} \bigwedge_{S} (st)^{\theta(s)} = \bigvee_{\Lambda} \bigwedge_{St} s^{\lambda(s)},$$

where in the rightmost expression s ranges over St and Λ is some subset of $\{\pm 1\}^{St}$. Now this expression for wt may be redundant, but in that case it can be reduced to the normal form for wt by removing extraneous elements from St. That is, $\operatorname{supp}(wt) \subseteq St = (\operatorname{supp} w)\, t$. □

Proposition 3.4.3. *Suppose F is a T-algebra which is freely generated as a naked algebra by a subset $R \subseteq F$. Then*

$$\operatorname{supp}(wt) = (\operatorname{supp} w)\, t$$

for any $w \in F \setminus \{\bot, \top\}$ and any $t \in \operatorname{stab} w$.

Proof. Abbreviate $\operatorname{supp} w$ to S. Then we have

$$S = \operatorname{supp} w = \operatorname{supp}(wt) \subseteq (\operatorname{supp} w)\, t = St.$$

But these sets are finite, and the cardinality of St does not exceed that of S. Therefore $St = S$ □

For $w \in F_i$ and $t \in T$, Theorem 3.4.1 allows for a very useful notational device. Since an element $w \in F_i \setminus \{\bot, \top\}$ has the normal form

$$w = \bigvee_{\Theta} \bigwedge_{S} s^{\theta(s)}$$

for $S \equiv \operatorname{supp} w$ and $\Theta \subseteq \{\pm 1\}^S$, and since each $s \in S$ is a translate of the source, say $s = s_i t_s$, we may write

$$w = \bigvee_{\Theta} \bigwedge_{S} (s_i t_s)^{\theta(s)} .$$

Note that the latter form is no longer unique to w because of the multiplicity of choices of t_s for s.

By viewing s_i as an indeterminate, we can think of w as a Boolean word in translates of this free variable. So if A is a T-algebra, $a \in A$ is of type $\leq i$ and f is the unique **BaT** morphism from F_i to A that maps s_i to a as in Theorem 3.4.1, then we often write the image of w under f as

$$f(w) = w\,(a) = \bigvee_{\Theta} \bigwedge_{S} f(s_i t_s)^{\theta(s)} \bigvee_{\Theta} \bigwedge_{S} (a t_s)^{\theta(s)} = \bigvee_{\Theta} \bigwedge_{S} a^{\theta(s)} t_s .$$

This notation is unambiguous precisely because a has type at most i, i.e., meaning that for $t, t' \in T$, $at = at'$ whenever $s_i t = s_i t'$. Thus all references to $w\,(a)$ for words $w \in F_i \setminus \{\pm 1\}$ are implicitly references to Theorem 3.4.1. In particular, the notation $w\,(a)$ makes no sense unless a is of type at most i.

3.5. **Free products in BaT.** We leave the routine verification of the following lemma to the reader.

Lemma 3.5.1. *Let $\{A_j : j \in J\}$ be a family of T-algebras and let $e_j : A_j \to B$ be their coproduct in* **Ba**. *For $t \in T$ and $j \in J$ let $t_j : A_j \to A_j$ denote the actual action of t on A_j. Then for each $j \in J$, the map $e_j t_j : A_j \to B$ is a* **Ba** *morphism. By the coproduct property, there is a* **Ba** *morphism $t_B : B \to B$ such that for all $j \in J$, $e_j t_j = t_B e_j$. Let t act on B by the* **Ba** *morphism t_B. Then $e_j : A_j \to B$ is also the*

$$\begin{tikzcd}
A_1 \arrow[r, "e_1"] \arrow[d, "t_1"'] & B \arrow[d, "t_B"] & A_2 \arrow[l, "e_2"'] \arrow[d, "t_2"] \\
A_1 \arrow[r, "e_1"'] & B & A_2 \arrow[l, "e_2"]
\end{tikzcd}$$

coproduct of the A_j's in **BaT**.

The coproduct of algebras, respectively T-algebras, is also called their *free product*. The following corollary easily follows from Theorem 3.4.1.

Corollary 3.5.2. *Every T-algebra is an image under an epimorphism of a coproduct of T-algebras of the form F_i, $i \in I$.*

3.6. **Free T-algebras over sets.** A T-algebra F is free over the set X provided that there is an injective set map $e : X \to F$ such that for every T-algebra A and every set map $f : X \to A$ there is a unique T-morphism $g : F \to A$ such that $ge = f$. We say that *the set map f lifts uniquely to the T-morphism g.* Any two T-algebras free over the same set X are isomorphic over X, and so are determined up to isomorphism solely by the cardinality of X.

The trivial free T-algebra, namely the free T-algebra over $\emptyset$, always exists; it is the two-element algebra **2**. Now **BaT** is closed under free products by Lemma 3.5.1, and the free product of free T-algebras is free. Therefore free T-algebras exist over all sets if they exist over singletons.

The question is to determine when free T-algebras exist over all sets. We describe this situation succinctly by saying that nontrivial free T-algebras exist.

Recall from Section 2.4 the definition of $\bar{T}$ and $\hat{T}$. $\bar{T}$ is the inverse limit of the T_i's, where $T_i \subseteq R_i^{R_i}$ is the image $\phi_i(T)$ of T under the action on R_i, and $\hat{T}$ is the closure of $\bar{T}$ in the product of the topological monoids $R_i^{R_i}$.

Theorem 3.6.1. *The following are equivalent for a topological monoid T.*

(1) *Nontrivial free T-algebras exist.*
(2) *The lattice I of types of T has a greatest element.*
(3) *$\hat{T}$ is discrete.*

If T is a topological group, these conditions are equivalent to the following.

(4) *T possesses a smallest open subgoup.*
(5) *$\hat{T}$ is a discrete topological group.*

Proof. We have already remarked on the equivalence of (2) and (3) in Remark 2.4.1(4). To show that (1) implies (2), suppose that F is the free algebra over a singleton set $X \equiv \{x\}$, and identify x with its image $e(x) \in F$. Let $i \equiv \operatorname{type} x$ and fix $j \in I$. Let F_j be the T-algebra of Theorem 3.4.1, and let $g : F \to F_j$ be the T-morphism which results from lifting the set map $x \mapsto s_j$. Then by Proposition 2.5.5 we get

$$i = \operatorname{type} x \geq \operatorname{type} g(x) = \operatorname{type} s_j = j.$$

This shows that i is the largest element of I.

Now suppose that I has a greatest element i. We claim that the T-algebra F_i of Theorem 3.4.1 is the free T-algebra over the singleton set $\{s_i\}$. That is because any set map f from $\{s_j\}$ into a T-algebra

A takes s_j to an element of type $j \leq i$, and hence lifts to a unique T-morphism $g : F_j \to A$ by Theorem 3.4.1.

Now let T be a topological group. The equivalence of (2) and (4) follows from Proposition 2.3.5. By Theorem 2.4.5, $\bar{T}$ is a topological group. If $\hat{T}$ is discrete, then $\hat{T} = \bar{T}$ and hence $\hat{T}$ is a topological group. This shows the equivalence of (3) and (5). $\square$

3.7. **Extending mappings to morphisms.** For a subset B of a T-algebra C we let

$$BT \equiv \{bt : t \in T, b \in B\}.$$

We use $\langle B\rangle_{\mathbf{Ba}}$ and $\langle B\rangle_{\mathbf{BaT}}$ to denote the subalgebra generated in the category of the subscript. Note that $\langle B\rangle_{\mathbf{BaT}} = \langle BT\rangle_{\mathbf{Ba}}$.

Lemma 3.7.1. *Suppose that A and C are T-algebras and that B is a subset of C. Then a mapping $f : B \to A$ can be extended to a T-morphism $\hat{f} : \langle B\rangle_{\mathbf{BaT}} \to A$ if and only if it satisfies the following pair of conditions.*

(1) *$bt = b't' \Longrightarrow f(b)\,t = f(b')\,t'$ for all $t, t' \in T$ and $b, b' \in B$.*
(2) *For all finite subsets $B', B'' \subseteq B$ and $T', T'' \subseteq T$ we have*

$$\bigwedge_{T',B'} b't' \wedge \bigwedge_{T'',B''} \overline{b''}t'' = \bot \Longrightarrow \bigwedge_{T',B'} f(b')\,t' \wedge \bigwedge_{T'',B''} \overline{f(b'')}t'' = \bot$$

Proof. The existence of $\hat{f}$ certainly implies the conditions. Assuming them, first extend f to BT by declaring $\hat{f}(bt) = f(b)\,t$, an extension which is well-defined by the first condition. Then the second condition is a well-known criterion for the extension of $\hat{f}$ to all of $\langle BT\rangle_{\mathbf{Ba}}$; see [1, V.2]. Since $\langle BT\rangle_{\mathbf{Ba}} = \langle B\rangle_{\mathbf{BaT}}$, and since it is easy to verify that $\hat{f}$ commutes with the actions, the result follows. $\square$

Proposition 3.7.2. *Suppose that we have T-algebras C and A with elements c and a, respectively, a subalgebra $B \leq C$, and a morphism $f : B \to A$. Then f can be extended to a morphism $\hat{f} : \langle B, c\rangle \to A$ such that $\hat{f}(c) = a$ if and only if the following conditions are satisfied.*

(1a) *The type of a is at most the type of c, i.e., $ct = ct' \Longrightarrow at = at'$ for all $t, t' \in T$.*
(1b) *For all $t \in T$ and $b \in B$, $ct = b \Longrightarrow at = f(b)$.*
(2) *For all $b \in B$ and all finite subsets $T', T'' \subseteq T$ we have*

$$b \wedge \bigwedge_{T'} ct' \wedge \bigwedge_{T''} \bar{c}t'' = \bot \Longrightarrow f(b) \wedge \bigwedge_{T'} at' \wedge \bigwedge_{T''} \overline{a}t'' = \bot$$

Proof. All these conditions are clearly necessary. To prove their sufficiency apply Lemma 3.7.1 to $B \cup \{c\}$. Conditions (1a) and (1b) above together imply condition (1) of the lemma, while condition (2) here implies its counterpart in the lemma. $\square$

4. Injectives

4.1. **Injectivity defined.**

Definition 4.1.1. A T-algebra A is *injective* provided that for all morphisms $f : B \to A$ and all monomorphisms $g : B \to C$ there is a morphism $h : C \to A$ such that $hg = f$.

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[dr, "f"'] & C \arrow[d, "h"] \\ & A \end{tikzcd}$$

An *injective hull* of A is an essential embedding of A into an injective object.

We first show the existence and uniqueness of injective hulls of T-algebras in Subsection 4.2. Although this follows from simple categorical principles, we outline the construction because we need to understand the structure of these hulls as concretely as possible. We then characterize injectivity in terms of systems of ideals in Subsections 4.3, 4.4, and 4.5.

4.2. **The existence and uniqueness of the injective hull.**

We begin by showing that injective T-algebras exist. Recall that the cofree T-algebra of Theorem 3.3.4 over a naked algebra B is the subalgebra $(B^T)_T$ of B^T consisting of the elements that are constant on the classes of some suitable relation on T.

Lemma 4.2.1. *Let $p : A \to B$ be the cofree T-algebra over the naked complete algebra B. Then A is injective in* **BaT**.

Proof. Given the injection e and T-morphism f, let g be any morphism induced by the injectivity of B in **Ba** such that $ge = pf$. Then let h be

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[dr, "h"] & B \\ \mathrm{C} \arrow[u, "e"] \arrow[r, "f"'] & A \arrow[u, "p"'] \end{tikzcd}$$

the T-morphism induced by the cofree property of A such that $ph = g$.

To show that $he = f$ simply observe that for all $t \in T$,

$$\begin{aligned}(he(c))(t) &= (he(c))(t1) = (he(c)t)(1) = phe(ct) = ge(ct) = pf(ct)\\ &= f(ct)(1) = (f(c)t)(1) = f(c)(t1) = f(c)(t).\end{aligned}$$

This proves the lemma. □

We continue by showing that **BaT** has enough injectives.

Lemma 4.2.2. *Every T-algebra can be embedded in an injective object.*

Proof. Let e be the insertion of C in its injective hull B in **Ba**. (B

$$\begin{tikzcd} C \arrow[r, "e"] \arrow[dr, "f"'] & B \\ & A \arrow[u, "p"'] \end{tikzcd}$$

is just the completion of C.) Let $p : A \to B$ be the cofree T-algebra over B, and let f be the induced T-morphism. Then A is injective by Lemma 4.2.1, and f is injective because e is. □

For given morphisms $e_j : C \to E_j$, $j = 1, 2$, we say that a morphism $k : E_1 \to E_2$ is *over* C if $ke_1 = e_2$.

Proposition 4.2.3. *Every T-algebra C has a maximal essential extension $g : C \to E$. That is, g is essential, and every other essential extension of C embeds in E over C.*

Proof. Let f be the injection of Lemma 4.2.2, and let q be the quotient of Lemma 3.1.8. The composition $g \equiv qf$ is essential by construction.

$$\begin{tikzcd} D \arrow[d, "k"'] \arrow[dr, "h"] & C \arrow[l, "e"'] \arrow[d, "f"] \\ \mathrm{E} & A \arrow[l, "q"] \end{tikzcd}$$

Given an essential extension e, let h be a T-morphism produced by the injectivity of A. Then $k \equiv qh$ is injective because $ke = qhe = qf$ is injective and e is essential. □

For a proof of the next result, see [1, I.20].

Proposition 4.2.4. *The following are equivalent for a T-algebra E.*

(1) *E is injective.*
(2) *E is a retract of each of its extensions.*
(3) *E has no proper essential extensions.*

We summarize the development of this subsection.

Theorem 4.2.5. *Every T-algebra has an injective hull which is unique up to isomorphism over it.*

4.3. **Systems of ideals.** Having proved the existence and uniqueness of injective hulls of T-algebras in Theorem 4.2.5, we turn to the question of characterizing injective T-algebras. This is the content of Theorem 4.3.10, which requires the notion of an i-system of ideals.

Example 4.3.1. Let A be a naked algebra and let $L_0, L_1 \subseteq A$ be ideals such that for all $a \in L_0$ and $b \in L_1$, $a \wedge b = \perp$. Then there are an extension B of A and $b \in B$ such that $L_0 = \{a \in A : a \leq b\}$ and $L_1 = \left\{a \in A : a \leq \overline{b}\right\}$. (Recall that $\overline{b}$ stands for the complement of b in B.)

Now let F be the free Boolean algebra over a single generator x. Let $f : F \to B$ be the unique **Ba** morphism that maps x to b. For $w \in F$ let $L(w) \equiv \{a \in A : a \leq f(w)\}$. Clearly, $L(\perp) = \{\perp\}$, $L(\top) = A$, $L(x) = L_0$ and $L(\overline{x}) = L_1$. The *system of ideals* $\{L(w) : w \in F\}$ describes the position relative to A of an element of some extension of B.

With actions, the situation gets more complicated. The following definition gives a generalization of the systems of ideals of Example 4.3.1 to T-algebras.

Definition 4.3.2. Let A be a T-algebra and $i \in I$. Let F_i be the free T-algebra over (R_i, s_i) (see Subsection 3.4). Then an *i-system of ideals of A*, or simply an *i-system*, is a family

$$S = \{L\,(w) : w \in F_i\}$$

of ideals of A with the following properties.

(1) $L\,(\perp) = \{\perp\}$ and $L\,(\top) = A$.
(2) $\bigcap_K L\,(w_k) \subseteq L\,(w)$ for all finite subsets $\{w_k : k \in K\}$ and elements w of F_i such that $\bigwedge_K w_k \leq w$ in F_i.
(3) $L\,(w)\,t \equiv \{bt : b \in L\,(w)\} \subseteq L\,(wt)$ for all $t \in T$ and $w \in F_i$.

Example 4.3.3. Let A be a T-algebra and $a \in A$. Let $i \in I$ be such that $\operatorname{type} a \leq i$. Recall that $w \mapsto w(a)$ is the unique BaT morphism from F_i to A that maps s_i to a. For $w \in F_i$ let $L(w)$ be the principal ideal of A generated by $w(a)$. Then $\{L(w) : w \in F_i\}$ is an i-system of ideals.

The following example parallels Example 4.3.1.

Example 4.3.4. Let A, a, i and $\{L(w) : w \in F_i\}$ be as in Example 4.3.3. Suppose that B is is a T-subalgebra of A. Then

$$\{L(w) \cap B : w \in F_i\}$$

is an i-system of ideals in B. This will follow from Proposition 4.3.6(b). Theorem 4.3.8 tells us that every i-system of ideal arises in just this fashion.

Example 4.3.4 suggests the following definition.

Definition 4.3.5. We say that an element c in an extension $C \geq A$ *realizes* an i-system $S = \{L(w) : w \in F_i\}$ if c is of type at most i and

$$L(w) \subseteq \{a \in A : a \leq w(c)\}$$

for each $w \in F_i$. We say that c *exactly realizes* S if c realizes S and the containment is an equality for each $w \in F_i$.

Proposition 4.3.6. *(a) T-morphisms preserve i-systems. That is, if $f : A \to B$ is a T-morphism and if $\{L(w) : w \in F_i\}$ is an i-system of ideals of A, then $\{L_f^{\rightarrow}(w) : w \in F_i\}$ is an i-system of ideals of B where*

$$L_f^{\rightarrow}(w) \equiv \{b \in B : \exists a \in A \ (a \in L(w) \ \text{ and } f(a) \geq b)\}.$$

(b) If $\{L(w)\}$ is an i-system of ideals of B and $f : A \to B$ is a one-to-one T-morphism, then $\{L_f^{\leftarrow}(w) : w \in F_i\}$ is an i-system of ideals of A where

$$L_f^{\leftarrow}(w) \equiv \{a \in A : f(a) \in L(w)\} = f^{-1}(L(w)).$$

Proof. (a) Clearly, each $L_f^{\rightarrow}(w)$ is an ideal of B. Condition (1) of Definition 4.3.2 is easily verified. For condition (2) let $\{w_k : k \in K\}$ be a finite subset of F_i and $w \in F_i$ such that $\bigwedge_K w_k \leq w$. Let $b \in \bigcap_K L_f^{\rightarrow}(w_k)$. For each $k \in K$ let $a_k \in L(w_k)$ be such that $b \leq f(a_k)$. Then $b \leq f\left(\bigwedge_K a_k\right)$. But $\bigwedge_K a_k \in \bigcap_K L(w_k) \subseteq L(w)$. Hence $b \in L_f^{\rightarrow}(w)$.

For condition (3) let $w \in F_i$ and $t \in T$. If $b \in L_f^{\rightarrow}(w)$, then for some $a \in L(w)$, $b \leq f(a)$. Since f is a T-morphism, $bt \leq f(at)$. But $at \in L(w)t \subseteq L(wt)$. Hence $b \in L_f^{\rightarrow}(wt)$.

The proof of (b) is similar to the proof of (a) but even more straightforward. □

We characterize i-systems in Theorem 4.3.8, for which we need a simple lemma about naked algebras. In this lemma we consider the partitions of a given finite set B into two parts, B_1 and B_2. The symbol $\uplus$ stands for disjoint union, so that we refer to the partition by writing $B_1 \uplus B_2 = B$.

Lemma 4.3.7. *Let b be an element and B_0 a finite subset of a naked algebra B. Then b lies in the ideal generated by the set*

$$\left\{\bigwedge B_2 : \exists B_1 \left(B_1 \biguplus B_2 = B_0 \wedge b \not\leq \bigvee B_1\right)\right\}.$$

Proof. If not, then by Zorn's Lemma there is a prime ideal J containing the displayed set and omitting b. Put

$$B_1 \equiv \{b_1 \in B_0 : b_1 \in J\}, \quad B_2 \equiv \{b_2 \in B_0 : b_2 \notin J\}.$$

Then the primeness of J implies that $\bigwedge B_2 \notin J$, hence $b \le \bigvee B_1 \in J$, contrary to hypothesis. $\square$

Theorem 4.3.8. *A collection $S = \{L(w) : w \in F_i\}$ of ideals of A is an i-system if and only if there is some element c in some extension $C \ge A$ which exactly realizes S.*

Proof. Suppose that $C \ge A$ is an extension having an element c of type at most i in C. For every $w \in F_i$ let

$$L(w) = \{b \in C : a \le w(c)\}.$$

Then $\{L(w) : w \in F_i\}$ is an i-system of ideals of C as in Example 4.3.3. Let f denote the inclusion from A into C. By Proposition 4.3.6(b),

$$S \equiv \{L_f^{\leftarrow}(w) : w \in F_i\}$$

is an i-system of ideals of A. Note that for every $w \in F_i$,

$$L_f^{\leftarrow}(w) = \{a \in A : a \le w(c)\}.$$

In other words, c exactly realizes S.

Conversely assume that an i-system $\{L(w) : w \in F_i\}$ is given. Let B denote the coproduct of A with F_i, with insertion maps j_A and j_i. We want to define a quotient C of B with quotient map $g : B \to C$ such that gj_A is one-one and $g(j_i(s_i))$ exactly realizes $\{L_{j_A}^{\rightarrow}(w) : w \in F_i\}$. Regarding A as a subalgebra of C via gj_A, $L_{j_A}^{\rightarrow}(w)$ is just $L(w)$, i.e., $c \equiv g(j_i(s_i))$ is an element of an extension of A that exactly realizes $\{L(w) : w \in F_i\}$.

We have to define g in such a way that for all $w \in F_i$ and all $a \in L(w)$, $g(j_A(a)) \le g(j_i(w))$. In other words, for a and w as before we want that

$$g(a) \wedge \overline{g(j_i(w))} = g(j_A(a) \wedge j_i(\overline{w})) = \perp.$$

So let J be the ideal of B generated by all elements of the form $j_A(a) \wedge j_i(\overline{w})$ for $w \in F_i$ and $a \in L(w)$. Let $C \equiv B/J$ and let g be the quotient map. J is closed under the actions by the third defining property of an i-system of ideals, so that both C and g lie in **BaT** by Proposition 3.1.2. Finally set $f = gj_A$ and $c = gj_i(s_i)$. Observe that, since s_i is of type i in F_i, c is of type at most i in B by Proposition 2.5.5. And gj_i must be the function of Theorem 3.4.1 by virtue of its uniqueness, so that $gj_i(w) = w(c)$ for all $w \in F_i$.

We now show that $L(w) = \{a \in A : f(a) \leq w(c)\}$ for each $w \in F_i$. Since for $a \in L(w)$ we have

$$\perp = g(j_A(a) \wedge j_i(\overline{w})) = f(a) \wedge gj_i(\overline{w}) = f(a) \wedge \overline{w}(c) = f(a) \wedge \overline{w(c)},$$

it follows that $f(a) \leq w(c)$. Conversely, if $f(a) \leq w(c)$ then

$$\perp = f(a) \wedge \overline{w(c)} = f(a) \wedge \overline{w}(c) = g(j_A(a) \wedge j_i(\overline{w})),$$

i.e., $j_A(a) \wedge j_i(\overline{w}) \in J$. That means that there are finite subsets $\{a_k : k \in K\} \subseteq A$ and $\{w_k : k \in K\} \subseteq F_i$ such that $a_k \in L(w_k)$ for all $k \in K$, and such that

$$\begin{aligned} j_A(a) \wedge j_i(\overline{w}) &\leq \bigvee_K (j_A(a_k) \wedge j_i(\overline{w_k})) \\ &= \bigwedge_{K_1 \biguplus K_2 = K} \left(j_A\left(\bigvee_{K_1} a_k\right) \vee j_i\left(\bigvee_{K_2} \overline{w_k}\right)\right). \end{aligned}$$

Now for each partition $K_1 \biguplus K_2 = K$ it follows from properties of the coproduct ([1, VII 1(ii)]) that either $a \leq \bigvee_{K_1} a_k$ or $\overline{w} \leq \bigvee_{K_2} \overline{w_k}$. In the latter case we get $w \geq \bigwedge_{K_2} w_k$, from which the second defining property of i-systems implies that

$$\bigwedge_{K_2} a_k \in \bigcap_{K_2} L(w_k) \subseteq L(w).$$

Therefore $a \in L(w)$ by Lemma 4.3.7.

Finally, identify each element of A with its image under f. This identification makes C an extension of A because f must be one-to-one. The reason that f must be one-to-one is that by taking $w = \perp$ we get

$$f(a) = \perp = w(c) \Longrightarrow a \in L(w) = L(\perp) = \{\perp\}$$

for any $a \in A$ and hence $f^{-1}(\perp) = \{\perp\}$. □

The particular morphism f and element c constructed in the proof of Theorem 4.3.8 are universal with respect to their properties.

Theorem 4.3.9. *Suppose $S = \{L(w) : w \in F_i\}$ is an i-system of ideals of A for some $i \in I$. Let $C \geq A$ be the extension and c the element constructed in the proof of Theorem 4.3.8. Then for any other extension $D \geq A$ having an element d realizing S there is a unique morphism $h : C \to D$ over A taking c to d.*

Proof. Let l be the morphism of Proposition 3.4.1 from F_i into D taking s_i to d. The coproduct property of B applied to this map, together with the insertion of A in D, produces a unique morphism m making the top part of the diagram commute. We claim that m factors through

$$\begin{tikzcd}
A \arrow[d, "j_A"'] \arrow[drr] & & F \arrow[dll, "j_i"'] \arrow[d, "l"] \\
B \arrow[rr, "m"] \arrow[dr, "g"'] & & D \\
& C \arrow[ur, "h"'] &
\end{tikzcd}$$

g, i.e., that there is a morphism h such that $hg = m$. This is because for any $b \in B$ such that $g(b) = \perp$, i.e., $b \in J$, there are finite subsets $\{a_k : k \in K\} \subseteq A$ and $\{w_k : k \in K\} \subseteq F_i$ such that $a_k \in L(w_k)$ for all $k \in K$ and

$$b \leq \bigvee_K \left(j_A(a_k) \wedge j_i(\overline{w_k})\right).$$

But $a_k \in L(w_k)$ implies $a_k \leq w_k(d) = l(w_k)$, hence $a_k \wedge l(\overline{w_k}) = \perp$, with the result that $mj_A(a_k) \wedge mj_i(\overline{w_k}) = \perp$. Therefore

$$m(b) \leq \bigvee_K m\left(j_A(a_k) \wedge j_i(\overline{w_k})\right) = \perp.$$

Finally, the uniqueness of h is a consequence of the uniqueness of m and the fact that $\langle f(A) \cup \{c\}\rangle_{\mathbf{BaT}} = C$, where $f = gj_A$. $\square$

We have finally assembled the tools we need to characterize injective T-algebras.

Theorem 4.3.10. *A T-algebra A is injective if and only if for every $i \in I$ and for every i-system S there is an element of A realizing S.*

Proof. Suppose A is injective, $i \in I$, and S is an i-system. Let $c \in C \geq A$ be the items constructed in the proof of Theorem 4.3.8. By the injectivity of A there is some morphism $j : C \to A$ such that j is the identity map on A. Set $a_0 \equiv j(c)$, and observe that a_0 is of type at most i because morphisms preserve type by Proposition 2.5.5. And a_0 realizes S because

$$a \in L(w) \Longrightarrow a \leq w(c) \Longrightarrow a = j(a) \leq jw(c) = w(j(c)) = w(a_0).$$

Now suppose that for every $i \in I$ and for every i-system S there is an element of A realizing S. To test the injectivity of A consider a morphism $f : B \to A$ and superalgebra $C \geq B$ having element $c \in C$. It is sufficient to extend f to a morphism $\hat{f} : \langle B, c\rangle_{\mathbf{BaT}} \to A$, since a continuation of this process by transfinite induction results in an extension of f to all of C. We use Proposition 3.7.2 to achieve the extension by one element as follows. First let i be the type of c, and for each $w \in F_i$ let

$$L(w) = \{a_1 \in A : \exists b \in B \ (b \leq w(c) \text{ and } f(b) \geq a_1)\}.$$

Then $S \equiv \{L(w) : w \in F_i\}$ is an i-system by Proposition 4.3.6, and is therefore realized by some $a \in A$. We claim that this setup satisfies the hypotheses of Proposition 3.7.2. Condition (1a) is satisfied by virtue of the fact that a is of type at most i. To establish condition (1b) suppose that $ct = b$ for some $t \in T$ and $b \in B$, and let $s_i t \equiv w \in F_i$. Then

$$b \leq w(c) \Longrightarrow f(b) \in L(w) \Longrightarrow f(b) \leq w(a) = at,$$

and $\overline{c}t = \overline{b}$ implies $f\left(\overline{b}\right) \leq \overline{a}t$ in similar fashion, with the result that $f(b) = at$. To verify condition (2) consider $b \in B$ and finite subsets $T', T'' \subseteq T$, and let $w = \bigwedge_{T'} s_i t' \wedge \bigwedge_{T''} \overline{s_i} t''$. Then

$$\begin{aligned} b \wedge w(c) = \bot \Longrightarrow b \leq \overline{w}(c) \Longrightarrow f(b) \in L(\overline{w}) \\ \Longrightarrow f(b) \leq \overline{w}(a) \Longrightarrow f(b) \wedge w(a) = \bot. \end{aligned}$$

This completes the proof. □

4.4. **Maximal systems of ideals.** We need to consider the partitions of a given finite set $T_0 \subseteq T$ into two parts, T_1 and T_2. As before, we use the symbol $\biguplus$ for disjoint union, and refer to the partition by writing $T_1 \biguplus T_2 = T_0$.

Lemma 4.4.1. *Suppose c is an element of type at most i in some extension $C \geq A$, and fix $w \in F_i$ and $a \in A$. If there is a finite subset $T_0 \subseteq T$ and an element $\bot < a_0 \in A$ such that for all partitions $T_1 \biguplus T_2 = T_0$ we have*

$$a_0 \wedge \overline{\bigvee_{T_1} at_1} \leq \bigvee_{T_2} \overline{w}(c)\, t_2,$$

then $a \nleq w(c)$. Conversely, if $a \nleq w(c)$ then such a subset T_0 and element a_0 exist, provided that C is an essential extension of A.

Proof. If $a \nleq w(c)$ then $a \wedge \overline{w(c)} = a \wedge \overline{w}(c) > \bot$. If C is an essential extension of A, then by Proposition 3.1.7 there must be a finite subset $T_0 \subseteq T$ and element $\bot < a_0 \in A$ such that

$$a_0 \leq \bigvee_{T_0} (a \wedge \overline{w}(c))\, t_0 = \bigwedge_{T_1 \biguplus T_2 = T_0} \left(\bigvee_{T_1} at_1 \vee \bigvee_{T_2} \overline{w}(c)\, t_2 \right),$$

where the equality holds by the distributive law. This is to say that $a_0 \wedge \overline{\bigvee_{T_1} at_1} \leq \bigvee_{T_2} \overline{w}(c)\, t_2$ for all partitions $T_1 \biguplus T_2 = T_0$.

On the other hand suppose that $T_0 \subseteq T$ is a finite subset and $\bot < a_0 \in A$ an element for which every partition of T_0 satisfies the inequality displayed in the lemma. Then a reversal of the preceding argument leads to the conclusion that $a \wedge \overline{w(c)} > \perp$, i.e., $a \nleq w(c)$. $\square$

Observe that i-systems are ordered by containment, i.e.,

$$\{L(w) : w \in F_i\} \leq \{M(w) : w \in F_i\}$$

if and only if $L(w) \subseteq M(w)$ for all $w \in F_i$. Observe that in this case, any element (in any extension) which realizes $\{M(w) : w \in F_i\}$ also realizes $\{L(w) : w \in F_i\}$. Finally, observe also that the union of a tower of i-systems is an i-system, so that every i-system is contained in a maximal i-system by Zorn's Lemma. We characterize maximal i-systems in Theorem 4.4.2.

Theorem 4.4.2. *The following conditions are equivalent for an i-system $S = \{L(w) : w \in F_i\}$ of ideals of A.*

(1) *For every $a \in A$ and $w \in F_i$ with $a \notin L(w)$ there is some finite subset $T_0 \subseteq T$ and element $\perp < a_0 \in A$ such that*
$$a_0 \wedge \overline{\bigvee_{T_1} at_1} \in L\left(\bigvee_{T_2} \overline{w}t_2\right)$$
for all partitions $T_1 \biguplus T_2 = T_0$.

(2) *Every element which realizes S in an extension of A does so exactly.*

(3) *Every element which realizes S in an essential extension of A does so exactly.*

(4) *Every element which realizes S in the injective hull of A does so exactly.*

(5) *S is maximal among i-systems of ideals of A.*

Proof. Suppose (1) holds, let c be an element realizing S in some extension $C \geq A$, and consider $a \in A$ such that $a \leq w(c)$. Then by the first part of Lemma 4.4.1 there can be no finite $T_0 \subseteq T$ and $\perp < a_0 \in A$ such that

$$a_0 \wedge \overline{\bigvee_{T_1} at_1} \leq \bigvee_{T_2} \overline{w}(c)\, t_2$$

for all partitions $T_1 \biguplus T_2 = T_0$. Hence there can be no finite $T_0 \subseteq T$ and $\perp < a_0 \in A$ such that

$$a_0 \wedge \overline{\bigvee_{T_1} at_1} \in L\left(\bigvee_{T_2} \overline{w}t_2\right),$$

for all partitions $T_1 \biguplus T_2 = T_0$. It follows from (1) that $a \in L(w)$, i.e.,

$$L(w) = \{a \in A : a \leq w(c)\},$$

meaning c exactly realizes S.

The implications from (2) to (3) and (3) to (4) are obvious. Assume (4), and to prove (1) consider $w \in F_i$ and $a \notin L(w)$. Let $C \geq A$ be the injective hull of A. Then

$$S_C = \{\{c \in C : c \leq a_1 \text{ for some } a_1 \in L(w)\} : w \in F_i\}$$

is an i-system of ideals of C by Proposition 4.3.6, and thus is realized by an element $c_0 \in C$ by Theorem 4.3.10. Now c_0 clearly also realizes S, and does so exactly by (4), hence $a \not\leq w(c_0)$. From Lemma 4.4.1 we then get a finite subset $T_0 \subseteq T$ and an element $\perp < a_0 \in A$ which, in light of the exactness of the realization of S by c_0, satisfy (1).

If S were properly smaller than another i-system S' then by Theorem 4.3.8 we could find an extension $C \geq A$ with an element c realizing S' exactly. But then c would realize S inexactly. This proves that (5) follows from (2). On the other hand, if $C \geq A$ is any extension having an element c which realizes S then

$$S \leq S' \equiv \{L(w(c)) : w \in F_i\}.$$

Hence if S is maximal it follows that $S = S'$, which is to say that c realizes S exactly. This shows that (2) follows from (5), and completes the proof. □

Corollary 4.4.3. *Every i-system is contained in an i-system satisfying Theorem 4.4.2.*

Corollary 4.4.4. *A T-algebra A is injective if and only if for every $i \in I$ and for every maximal i-system S of ideals of A there is an element of A which realizes S exactly.*

It is tempting to speculate that the injective hull of a T-algebra A could be constructed as the algebra of maximal i-systems. Unfortunately, such a construction cannot be straightforward, since i-systems are not in one-to-one correspondence with the elements of the injective hull. Indeed, many different elements of the injective hull can give rise to the same maximal i-system.

Example 4.4.5. Let T be a (discrete) finite group. T acts trivially on the trivial Boolean algebra $\mathbf{2}$. The T-algebra $\mathbf{2}^T$ is the injective hull of $\mathbf{2}$ in **BaT**:

From Lemma 4.2.1 it follows that $\mathbf{2}^T$ is injective. Moreover, the embedding $f : \mathbf{2} \to \mathbf{2}^T$ is essential by Proposition 3.1.7 since $\mathbf{2}^T$ is the only T-ideal of $\mathbf{2}^T$ apart from $\{\perp\}$.

However, if $i \in I$ corresponds to the largest pointed antiflow of T, then the atoms of $\mathbf{2}^T$ all generate the same i-system of ideals of 2, namely $\{L(w) : w \in F_i\}$ where $L(w) = \{\perp\}$ for all $w \in F_i \setminus \{\top\}$ and $L(\top) = \mathbf{2}$.

4.5. **Checking only large types.** In this subsection we point out that, in order to verify the condition for injectivity of Theorem 4.3.10, we need not check all i-systems, but can confine the verification to i-systems for the larger (finer) i's in I.

Let us fix notation to be used throughout the rest of this subsection. Suppose we are given $i \geq j$ in I, and let $\rho^i_j : R_i \to R_j$ be the canonical **pSpT** surjection of Proposition 2.3.1, i.e., $\rho^i_j(s_i t) = s_j t$ for all $t \in T$. Let $p^i_j : F_i \to F_j$ be the T-morphism induced by ρ^i_j, where F_i and F_j are the free T-algebras over R_i and R_j, respectively.

It may be helpful to describe the action of p^i_j more concretely. As we mentioned in Subsection 3.4, an element $w \in F_i$ may be expressed in the form

$$w = \bigvee_{\Theta} \bigwedge_{S} (s_i t_s)^{\theta(s)},$$

for $S \equiv \operatorname{supp} w$ and $\Theta \subseteq \{\pm 1\}^S$. Therefore

$$p^i_j(w) = \bigvee_{\Theta} \bigwedge_{S} \left(p^i_j(s_i)\, t_s\right)^{\theta(s)} = \bigvee_{\Theta} \bigwedge_{S} \left(\rho^i_j(s_i)\, t_s\right)^{\theta(s)} = \bigvee_{\Theta} \bigwedge_{S} (s_j t_s)^{\theta(s)}.$$

Finally, consider a j-system $S_j = \{L_j(w) : w \in F_j\}$ of ideals on a given T-algebra A, and let $S_i = \{L_i(w) : w \in F_i\}$ be defined by the rule $L_i(w) \equiv L_j\left(p^i_j(w)\right)$ for all $w \in F_i$. Then it is easy to check that S_i is an i-system of ideals of A.

Proposition 4.5.1. *Let A, S_i and S_j be as above. If S_j is maximal among j-systems, then S_i is maximal among i-systems.*

Proof. Consider $a \in A$ and $w \in F_i$ such that $a \notin L_i(w) = L_j\left(p^i_j(w)\right)$. Then the maximality of S_j implies the existence of a finite subset $T_0 \subseteq T$ and an element $\perp < a_0 \in A$ such that

$$a_0 \wedge \overline{\bigvee_{T_1} a t_1} \in L_j\left(\bigvee_{T_2} \overline{p^i_j(w)} t_2\right)$$

for all partitions $T_1 \biguplus T_2 = T_0$. But $\bigvee_{T_2} \overline{p^i_j(w)} t_2 = p^i_j\left(\bigvee_{T_2} \overline{w} t_2\right)$ since p^i_j commutes with the Boolean operations and the actions, hence

$$a_0 \wedge \overline{\bigvee_{T_1} t_1 a} \in L_i\left(\bigvee_{T_2} \overline{w} t_2\right)$$

for all partitions $T_1 \biguplus T_2 = T_0$. That is, S_i is maximal as well. □

Corollary 4.5.2. *If S_j is maximal among j-systems, then any element of A which realizes S_i also realizes S_j, and does both exactly.*

Proof. Suppose $a_0 \in A$ realizes S_i. To show that a_0 has type at most j, consider actions $t', t'' \in T$ such that $s_j t' = s_j t''$. Since S_i is maximal by Proposition 4.5.1, a_0 realizes it exactly by Theorem 4.4.2. Hence

$$\begin{aligned}\{a \in A : a \leq a_0 t'\} &= L_i(s_i t') = L_j\left(p_j^i(s_i t')\right) \\ &= L_j\left(\rho_j^i(s_i t')\right) = L_j(s_j t') = L_j(s_j t'') = L_j\left(\rho_j^i(s_i t'')\right) \\ &= L_j\left(p_j^i(s_i t'')\right) = L_i(s_i t'') = \{a \in A : a \leq a_0 t''\},\end{aligned}$$

from which it follows that $a_0 t' = a_0 t''$. That is, a_0 has type at most j.

To demonstrate that a_0 realizes S_j, consider $w_j \in F_j$, and write w_j in the form

$$w_j = \bigvee_{\Theta} \bigwedge_{S} (s_j t_s)^{\theta(s)},$$

where $S = \operatorname{supp} w_j$ and $\Theta \subseteq \{\pm 1\}^S$. Put

$$w_i \equiv \bigvee_{\Theta} \bigwedge_{S} (s_i t_s)^{\theta(s)} \in F_i.$$

We are not claiming that w_i is uniquely determined by w_j, as indeed it may vary with the choice of t_s for each $s \in S$. Nevertheless it is still true that $p_j^i(w_i) = w_j$, and that

$$w_i(a_0) = \bigvee_{\Theta} \bigwedge_{S} (a_0 t_s)^{\theta(s)} = w_j(a_0).$$

Therefore

$$\begin{aligned}L_j(w_j) &= L_j\left(p_j^i(w_i)\right) = L_i(w_i) = \{a \in A : a \leq w_i(a_0)\} \\ &= \{a \in A : a \leq w_j(a_0)\}.\end{aligned}$$

That is, a_0 exactly realizes S_j. □

We summarize our characterizations of injective T-algebras.

Theorem 4.5.3. *The following are equivalent for a T-algebra A.*

1. *A is injective.*
2. *For every $i \in I$, every i-system is realized in A.*
3. *For every $i \in I$, every maximal i-system is realized in A.*
4. *For every $j \in I$ there is some $i \in I$ with $i \geq j$ such that every maximal i-system is realized in A.*

In particular, if T admits a finest pointed antiflow R_i, i.e., if I contains a largest element i, then a T-algebra A is injective if and only if every maximal i-system is realized in A.

Recall that an ideal J of a Boolean algebra A is *regular* if

$$J = \{a \in A : \forall b \in A(\forall c \in J(b \wedge c = \perp) \Rightarrow a \wedge b = \perp)\}.$$

The regular ideals of A correspond to regular open subsets of the Stone space of A.

Corollary 4.5.4. *If T is connected then a T-algebra A is injective if and only if A is complete.*

Proof. If T is connected, then all its continuous images are connected and hence all its pointed antiflows are singletons. It follows that T acts trivially on every T-algebra. Hence the characterization in the corollary is just the well known characterization of injectives in **Ba**. However, let us derive the proof from Theorem 4.5.3.

Let i be the single element of I. F_i is the free Boolean algebra generated by s_i. An i-system $\{L(w) : w \in F_i\}$ of ideals of A simply consists of $L(\top) = A$, $L(\perp) = \{\perp\}$ and two ideals $J_0 = L(s_i)$ and $J_1 = L(\overline{s}_i)$ with the property that for all $a \in J_0$ and all $b \in J_1$, $a \wedge b = \perp$.

Let us call a pair (J_0, J_1) of ideals of A maximal if it is maximal with respect to the property

$$\forall a \in J_0 \; \forall b \in J_1 \; (a \wedge b = \perp).$$

It now follows from Theorem 4.5.3 that A is injective if and only if for every maximal pair (J_0, J_1) of ideals both J_0 and J_1 are principal.

If A is complete, then clearly every maximal pair of ideals consist of principal ideals, the generators being the suprema of the ideals.

On the other hand, if every maximal pair of ideals consists of principal ideals, then A must be complete. This can be seen as follows: Let $D \subseteq A$. By Zorn's Lemma there is an ideal J_1 that is maximal with the property that for all $a \in D$ and all $b \in J_1$, $a \wedge b = \perp$. Let $J_0 = \{a \in A : \forall b \in J_1 \; (a \wedge b = \perp)\}$. Then (J_0, J_1) is a maximal pair of ideals. Hence, J_0 is generated by a single element a. It is easily checked that $a = \bigvee D$. It follows that A is complete. □

5. When all pointed antiflows are finite

Throughout this section we assume that all pointed antiflows R_i, $i \in I$, are finite. This means, in particular, that orbits of elements of T-algebras are finite. The assumption of finite pointed antiflows is equivalent to the assumption that $\hat{T}$ is compact (Remarks 2.2.1(3) and 2.4.1(2)).

5.1. **When $\hat{T}$ is a compact group.** In the presence of our running hypothesis that all antiflows are finite, either the surjectivity or the injectivity of all actions on all antiflows implies that $\hat{T}$ is a compact group (Lemma 5.1.1). In this case things are particularly simple: every T-algebra essentially extends its stationary subalgebra A_s (Proposition 5.1.3); every T-algebra A satisfies

$$A_s \leq A \leq E,$$

where E is the injective hull of A_s and the embeddings are essential (Proposition 5.1.5); the injective objects have a particularly simple structure (Theorem 5.1.6).

Lemma 5.1.1. *The following are equivalent for a T-algebra A.*

(1) *For each $i \in I$ and $t \in T$, the action of t_i on R_i is one-to-one.*
(2) *For each $i \in I$ and $t \in T$, t_i maps R_i onto itself.*
(3) *For each $i \in I$, T_i is a group.*
(4) *$\bar{T}$ is a compact group.*
(5) *$\hat{T}$ is a compact group.*

Proof. (1) and (2) are equivalent since each t_i maps the finite set R_i to itself and hence is onto if and only if it is one-one. In particular, if either (1) or (2) holds, then each t_i is a permutation of R_i.

Now fix $i \in I$ and $t \in T$ and assume that t_i is a permutation of R_i. Since R_i is finite, so is its permutation group. In particular, for some $n > 1$, t_i^n is the identity on R_i. Now t_i^{n-1} is the inverse of t_i and $t_i^{n-1} \in T_i$ since T_i is closed under composition. It follows that T_i is a group.

If (3) holds then $\prod_I T_i$ is a compact group because each T_i is, and so the closed submonoid

$$\bar{T} = \left\{ \bar{t} \in \prod_I T_i : \forall i \geq j \ \left(\phi_j^i \left(\bar{t}\,(i) \right) = \bar{t}\,(j) \right) \right\}$$

is clearly a compact group as well. And this implies in turn that $\hat{T} = \bar{T}$. The implication from (5) to (1) is Remark 2.2.1(3). □

Definition 5.1.2. For a T-algebra A, the *stationary subalgebra of* A is

$$A_s \equiv \{a \in A : at = a \text{ for all } t \in T\}.$$

We regard A_s to be a T-subalgebra of A acted upon trivially by T.

Proposition 5.1.3. *If $\hat{T}$ is a topological group then any T-algebra is an essential extension of its stationary subalgebra.*

Proof. Consider a T-algebra A with element $a > \perp$ and let $a_0 = \bigvee aT$. Note that $\bigvee aT$ exists since aT is finite. Each action must map aT into itself, and again because of the finiteness of aT and since the action is one-to-one, it must map aT onto itself. Therefore each action fixes a_0, i.e., $a_0 \in A_s$. This makes A an essential extension of A_s by Proposition 3.1.7. □

Here is an example to show that the Proposition 5.1.3 need not hold, even when both monoid and algebra are finite.

Example 5.1.4. Consider the flow $X = \{x_1, x_2, x_3, x_4\}$ with actions $\{1, f_1, f_2\}$ defined as follows.

fx	x_1	x_2	x_3	x_4
1	x_1	x_2	x_3	x_4
f_1	x_4	x_2	x_2	x_4
f_2	x_2	x_2	x_4	x_4

Let A be the Stone space $\mathbf{2}^X$ of X, and let T be $\{1, t_1, t_2\}$, where t_i is the dual of f_i, i.e., $at_i = f_i^{-1}\{a\}$ for subsets $a \subseteq X$. Let $a_1 = \{x_1, x_2\}$ and $a_2 \equiv \{x_2, x_3\}$, so that

$$\bigvee a_1 T = a_1 1 \vee a_1 t_1 \vee a_1 t_2 = a_1 \vee a_2 \vee a_1 = \{x_1, x_2, x_3\}.$$

However, $A_s = \{\emptyset, X\}$, and so A is not an essential extension of A_s.

Proposition 5.1.5. *If $\hat{T}$ is a topological group then any T-algebra A satisfies*

$$A_s \leq A \leq E,$$

where the embedding $A_s \leq E$ is the injective hull of A_s and the embedding $A \leq E$ is the injective hull of A.

Proof. Let $A_s \leq E$ be the injective hull of A_s (Theorem 4.2.5). By Proposition 5.1.3, $A_s \leq A$ is essential. By Proposition 4.2.4, $A_s \leq E$ is the maximal essential extension of Proposition 4.2.3, and so there is an embedding of A into E over A_s. It is in this sense that A is intermediate between A_s and E. It follows that $A \leq E$ is the injective hull of A. □

Proposition 5.1.6. *Suppose that $\hat{T}$ is a topological group. Then for any complete algebra B,*

$$E \equiv \left\{a \in B^T : a \text{ is constant on } \sim_i \text{ classes for some } i \in I\right\}$$

is an injective T-algebra, and every injective T-algebra has this form.

Proof. Given the complete algebra B, we see by Theorem 3.3.4 that E is the cofree T-algebra over B. Thus E is injective by Lemma 4.2.1. On the other hand, suppose we are given an arbitrary injective object A. We first claim that A_s is a complete algebra. For if we regard both A_s and its completion (without actions) B to be T-algebras with trivial action, then the embedding $A_s \leq B$ is a T-injection. Since A_s is dense in B, the extension $A_s \leq B$ is essential by Proposition 3.1.7. If follows that B embeds into A over A_s. Consider B as a subalgebra of A. Since the action on B is trivial, $B \subseteq A_s$ and hence i.e., $A_s = B$, i.e., A_s is complete.

Now A is the injective hull of $B = A_s$ by Proposition 5.1.3, so to finish the proof we argue that

$$E \equiv \left\{a \in B^T : a \text{ is constant on } \sim_i \text{ classes for some } i \in I\right\},$$

is also the injective hull of B. Since E is the cofree T-algebra over B of Theorem 3.3.4, it is injective by Lemma 4.2.1. Furthermore, by regarding B once again as a T-algebra with trivial action, the injection that maps every $b \in B$ to the function in E that is constantly b is in fact a T-injection since T acts trivially on the constant functions in E. We identify each element of B with the corresponding constant function, so that we have $B \leq E$. All that remains is to show that this extension is essential.

Consider an arbitrary $\perp < a \in E$. Then a is constant on $\sim_i$ classes for some $i \in I$, and in particular there is at least one such class $[t]_i$ on which $b \equiv a(t') > \perp$ for all $t' \in [t]_i$. But since $T/\sim_i$ is finite, so is T_i. Let $T' \subseteq T$ be finite such that for all $t \in T$ there is $t' \in T'$ with $t'_i = t_i$. Since T_i acts transitively on $T/\sim_i$, we have $\left(\bigvee_{T'} at'\right)(t) \geq b$ for all $t \in T$, i.e., $\bigvee_{T'} at'$ dominates the function that is constantly b. It follows from Proposition 3.1.7 that E is an essential extension of B. ☐

Example 5.1.7. Let $Z_2 = Z/2Z$ denote the group with two elements, 0 and 1. Z_2 carries the discrete topology. Let $T \equiv Z_2^N$ be the product of countably many copies of Z_2 equipped with the product topology. T is a compact zero dimensional group acting continuously on itself. The open subgroups of the form

$$\{t \in T : \forall n \leq m \ (t(n) = 0)\}$$

for $m \in N$ form a neighborhood base of 1_T in T. Hence $\bar{T} \cong T$. Since T is compact, so is $\bar{T}$ and we have $\hat{T} = \bar{T} \cong T$.

Let A be the algebra of clopen subsets of T. Then T acts continuously on A. A is a free Boolean algebra over countably many generators and in particular, A is countable.

Since the action of T on itself is transitive, the only clopen subsets of T that are fixed by every $t \in T$ are $\emptyset$ and T. It follows that the stationary subalgebra A_s of A is $\mathbf{2}$. We compute the injective hull of A. Since $\mathbf{2}$ is complete, the injective hull of A_s is the subalgebra $(\mathbf{2}^T)_T$ of $\mathbf{2}^T$ consisting of all elements that are constant on the classes of suitable relation on T (Lemma 4.2.1). Since $A_s \leq A$ is an essential extension (Proposition 5.1.3), A embeds into $(\mathbf{2}^T)_T$ over A_s. Hence $(\mathbf{2}^T)_T$ is the injective hull of A in $\mathbf{BaT}$.

Every open subgroup U of T is in fact clopen since the complement of U is a union of cosets of U and every coset of U is open. Since the antiflows of T correspond to the open subgroups of T and since T only has countably many clopen subsets, I is countable.

For every $i \in I$ there are only finitely many $\sim_i$ classes since R_i is finite. It follows that for each $i \in I$ there are only finitely many elements of $\mathbf{2}^T$ that are constant on the $\sim_i$ classes. It follows that $(\mathbf{2}^T)_T$ is countable.

So, the injective hull of A in $\mathbf{BaT}$ is countable. The injective hull of A in $\mathbf{Ba}$ (if we forget the action on A) is the completion of A, which is of size $2^{\aleph_0}$.

It is tempting to conjecture that the injective hull of A in $\mathbf{BaT}_d$ is $\mathbf{2}^T$. However, the extension $A \leq \mathbf{2}^T$ is not essential.

This can be seen as follows: The extension $\mathbf{2} \leq A$ is essential in $\mathbf{BaT}_d$ since essentiality of extensions does not depend on the topology on T. Now consider the function $f : T \to \mathbf{2}$ that is $\top$ on 1_T and $\bot$ everywhere else. For no finite set $S \subseteq T$ do we have $\bigvee_S ft = \top$. It follows from Proposition 3.1.11 that $\mathbf{2} \leq \mathbf{2}^T$ is not essential. Hence $A \leq \mathbf{2}^T$ is not essential.

6. Projectives

6.1. **Projectives in general.** We now turn to projectivity in $\mathbf{BaT}$.
Let us recall the definition.

Definition 6.1.1. A T-algebra A is *projective* if and only if for each morphism $f : A \to B$ and each epimorphism $g : C \to B$ there is a morphism $h : A \to C$ such that $gh = f$.

Recall that the T-epimorphisms are precisely the surjective morphisms by Proposition 1.2.3. Also note that we always have the trivial projective $\mathbf{2} \equiv \{\bot, \top\}$.

We can already state the first characterization of projectivity in $\mathbf{BaT}$. Recall that a retraction is an epimorphism $g : C \to A$ with a right inverse, i.e., there exists a morphism $h : A \to C$ such that $gh = 1_A$.

Lemma 6.1.2. *The following are equivalent for a T-algebra A.*

(1) *A is projective.*
(2) *Every epimorphism onto A is a retraction.*
(3) *Every epimorphism onto A out of a coproduct of T-algebras of the form F_i, $i \in I$, is a retraction.*

Proof. It is clear that (1) implies (2) and that (2) implies (3). Assume (3), and in order to prove (1) consider a given epimorphism $g : C \to B$ and a given homomorphism $f : A \to B$. For each $a \in A$ choose $c_a \in C$ such that $g(c_a) = f(a)$. Let $i_a \equiv \mathrm{type}\, a \vee \mathrm{type}\, c_a$, let (R_a, s_a) be a copy of (R_{i_a}, s_{i_a}), let F_a be a copy of F_{i_a}, the free T-algebra over (R_a, s_a), and let $l_a : F_a \to A$ and $k_a : F_a \to C$ be the unique morphisms given by Theorem 3.4.1 such that $l_a(s_a) = a$ and $k_a(s_a) = c_a$. Let F be the coproduct of the family $\{F_a : a \in A\}$, and let $l : F \to A$ and $k : F \to C$ be the unique maps induced by the l_a's and k_a's respectively. That is, $l(s_a) = a$ and $k(s_a) = c_a$ for all $a \in A$. Since gk and fl agree on $\{s_a : a \in A\}$, and since this set generates F, it follows that $gk = fl$. Now apply (3) to l to get a morphism $m : A \to F$ such that $lm = 1_A$. Set $h \equiv km$. Then we have

$$gh = gkm = flm = f,$$

as desired. □

6.2. **When do projectives exist?** We propose to do now for projective T-algebras what we did for free T-algebras in Subsection 3.6. We need a little notation in addition to that of Subsection 3.4. For $i \geq j$ in I let $p^i_j : F_i \to F_j$ be the unique T-morphism given by Theorem 3.4.1 such that $p^i_j(s_i) = s_j$. Note that the restriction of p^i_j to the generating set $R_i \subseteq F_i$ is just the **pSpT** surjection ρ^i_j of Definition 2.1.2. We use 1_j to designate the identity morphism on F_j.

Theorem 6.2.1. *For $j \in I$, the first five conditions are equivalent and imply the sixth. The first six conditions are equivalent if the suitable relations on T correspond to the source stabilizers. All seven conditions are equivalent if T is a topological group.*

(1) *For every $i \geq j$ in I there is some $k \geq i$ and some $w \in F_k$ of type at most j such that $p^k_j(w) = s_j$.*
(2) *For every $i \geq j$ in I there is some $w \in F_i$ of type at most j such that $p^i_j(w) = s_j$.*
(3) *For every $i \geq j$ in I there is a T-morphism $h : F_j \to F_i$ such that $p^i_j h = 1_j$.*
(4) *F_j is projective.*

(5) *For every T-epimorphism $f : A \to B$ and every $b \in B$ of type at most j there is some $a \in A$ of type at most j such that $f(a) = b$.*
(6) *For every $i \geq j$ in I there is a nonempty finite subset $R \subseteq \left(\rho^i_j\right)^{-1}(s_j)$ such that $Rt = R$ for all $t \in \operatorname{stab} s_j$.*
(7) *For every $i \geq j$ in I, $\left(\rho^i_j\right)^{-1}(s_j)$ is finite.*

Proof. To show that (2) follows from (1), consider $i \geq j$ in I and find $k \geq i$ and $w \in F_k$ for which $p^k_j(w) = s_j$. Then $w' \equiv p^k_i(w)$, which is of type at most j by Proposition 2.5.5 and which lies in F_i, satisfies (2) because

$$p^i_j(w') = p^i_j p^k_i(w) = p^k_j(w) = s_j.$$

If (2) holds then Theorem 3.4.1 provides a unique T-morphism $h : F_j \to F_i$ such that $h(s_j) = w$. Since $p^i_j h$ agrees with 1_j on s_j, the two T-morphisms must be the same by the uniqueness clause of Theorem 3.4.1. That is, (3) holds. To prove that (3) implies (4), consider a T-morphism f and a T-epimorphism g, choose $c \in C$ such that $f(s_j) = g(c)$, and set $i \equiv j \vee \operatorname{type} c$. Let k be the T-morphism given

$$\begin{tikzcd} F_i \arrow[r, "k"] \arrow[d, shift left, "p^i_j"] & C \arrow[d, "g"] \\ F_j \arrow[u, shift left, "h"] \arrow[r, "f"'] & B \end{tikzcd}$$

by Theorem 3.4.1 such that $k(s_i) = c$, and let h be the T-morphism whose existence is asserted in (3). Now gk and fp^i_j agree on s_i, and because T-morphisms out of F_i are determined by their values at s_i, we conclude that $gk = fp^i_j$. Therefore

$$gkh(s_j) = fp^i_j h(s_j) = f(s_j),$$

and we likewise conclude that $gkh = f$. This shows that F_j is projective.

To show that (4) implies (5), consider a given T-epimorphism $f : A \to B$ and element $b \in B$ of type at most j, choose $a_0 \in A$ such that $f(a_0) = b$, and set $i \equiv j \vee \operatorname{type} a_0$. Let $k : F_j \to B$ and $g : F_i \to A$ be the T-morphisms given by Theorem 3.4.1 such that $k(s_j) = b$ and $g(s_i) = a_0$. Let $h : F_j \to F_i$ be a T-morphism produced by the projectivity of F_j such that $p^i_j h = 1_j$. Now kp^i_j and fg agree at s_i and are therefore identical, with the consequence that

$$fgh(s_j) = kp^i_j h(s_j) = k(s_j) = b.$$

The desired element is $a \equiv gh(s_j)$. This works because $\operatorname{type} a \leq \operatorname{type} s_j = j$ by Proposition 2.5.5. Finally, to deduce (1) from (5) simply apply (5) to the T-epimorphism p^i_j and the element $s_j \in F_j$. We have proven the first five conditions equivalent.

To show that (2) implies (6) fix $i \geq j$ in I and use (2) to get $w \in F_i$ of type at most j such that $p^i_j(w) = s_j$. Put

$$R \equiv \operatorname{supp} w \cap \left(\rho^i_j\right)^{-1}(s_j).$$

Now $\operatorname{stab} s_j \subseteq \operatorname{stab} w$ because $\operatorname{type} w \leq j$, so any $t \in \operatorname{stab} s_j$ actually permutes the elements of $\operatorname{supp} w$ by Proposition 3.4.3. Since t maps $\left(\rho^i_j\right)^{-1}(s_j)$ into itself, it follows that t also permutes R, i.e., $Rt = R$.

Now assume that the suitable relations on T correspond to the source stabilizers. To show that (6) implies (2), consider $i \geq j$ in I, let R be the finite subset of $\left(\rho^i_j\right)^{-1}(s_j)$ such that $Rt = R$ for all $t \in \operatorname{stab} s_j$, and put $w \equiv \bigvee R \in F_i$. Then for any $t \in \operatorname{stab} s_j$ we have

$$wt = \bigvee Rt = \bigvee R = w,$$

so that $\operatorname{type} w \leq j$ by Remark 2.5.2(2). This is where we use the assumption that the suitable relations correspond to the source stabilizers. Clearly $p^i_j(w) = s_j$. This completes the proof that (6) implies (2).

Finally, assume that T is a group. Then T acts transitively on R_i. The actions $t \in T$ that take some element of $(\rho^i_j)^{-1}(s_j)$ to another element of $(\rho^i_j)^{-1}(s_j)$ are precisely the actions in $\operatorname{stab}(s_j)$. On the other hand, every action in $\operatorname{stab}(s_j)$ maps $(\rho^i_j)^{-1}(s_j)$ into itself. It follows that $\operatorname{stab}(s_j)$ acts transitively on $(\rho^i_j)^{-1}(s_j)$. Hence every nonempty set $R \subseteq (\rho^i_j)^{-1}(s_j)$ such that $Rt = R$ for every $t \in \operatorname{stab}(s_j)$ actually equals $(\rho^i_j)^{-1}(s_j)$. This shows the equivalence of (6) and (7). □

Definition 6.2.2. We say that an element $j \in I$ is *almost maximal* if it satisfies the first three conditions of Theorem 6.2.1.

Remark 6.2.3. Let i be a maximal element of I.

1. Then i is a maximum element because I is a lattice. By Remark 2.4.1(4), this happens if and only if $\hat{T}$ is discrete.
2. i is almost maximal.
3. Any element $j \in I$ is almost maximal if and only if there is some $w \in F_i$ such that $p^i_j(w) = s_j$ and $\operatorname{type} w \leq j$.

Proposition 6.2.4. *Suppose that the suitable relations on T correspond to the source stabilizers.*

1. *Any element of I above an almost maximal element is itself almost maximal.*

(2) *If I contains an almost maximal element, then every element is dominated by an almost maximal element.*

Proof. Since the suitable relations correspond to the source stabilizers, almost maximality is characterized by (6) in Theorem 6.2.1. If $i \geq j \geq k$ in I and k is almost maximal then there must be some nonempty finite subset R of $(\rho_k^i)^{-1}(s_k)$ such that $Rt = R$ for all $t \in \operatorname{stab} s_k$. But then $S \equiv \rho_j^i(R)$ is a nonempty finite subset of $\left(\rho_k^j\right)^{-1}(s_k)$ that satisfies $St = S$ for all $t \in \operatorname{stab} s_k$. This shows (1).

(2) follows from (1) and the fact that I is a lattice. $\square$

It turns out that the type of a nontrivial element of a projective T-algebra is almost maximal, provided that the suitable relations on T correspond to the source stabilizers.

Theorem 6.2.5. *Suppose the suitable relations on T correspond to the source stabilizers, and that A is a projective T-algebra with element $a \neq \bot, \top$. Then* $\operatorname{type} a$ *is almost maximal in I.*

Proof. We verify (6) in Theorem 6.2.1 for $j \equiv \operatorname{type} a$. Consider $i \geq j$ in I. For each $b \in A$ let (R_b, s_b) be a copy of (R_k, s_k), where k is $i \vee \operatorname{type} b$, and let F_b be a copy of F_k. Let C be the coproduct of the family $\{F_b : b \in A\}$, and let $p : C \to A$ be the unique T-morphism such that $p(s_b) = b$ for all $b \in A$. Since A is projective, there is a T-morphism $h : A \to C$ such that $ph = 1_A$. Put $c \equiv h(a)$. Note that $\operatorname{type} c = j$ by Proposition 2.5.5. Now C is freely generated as a naked algebra by $\bigcup_A R_b$, and $c \in C \setminus \{\bot, \top\}$, so c has nonempty support $S \subseteq \bigcup_A R_b$. Note that $S = St$ for any $t \in \operatorname{stab} s_j$ by Proposition 3.4.3. Let $b \in A$ be such that $S_b \equiv S \cap R_b \neq \emptyset$, and let $k \equiv i \vee \operatorname{type} b$. Since for any $t \in T$ it is true that $R_b t \subseteq R_b$, it follows that $S_b t = S_b$ for all $t \in \operatorname{stab} s_j$. Finally, put

$$R \equiv \{s_i t : s_b t \in S_b\} \subseteq S_i.$$

It follows from the fact that $k \geq i$ that R is finite, for S_b is finite and for all $t, t' \in T$ we have

$$s_b t = s_b t' \Longleftrightarrow s_k t = s_k t' \Longrightarrow s_i t = s_i t'.$$

Now consider $t \in \operatorname{stab} s_i$ and $r \in R$, say $r = s_i t_r$ for some $t_r \in T$ such that $s_b t_r \in S_b$. Then $rt = s_i t_r t$ lies in Rt; because $s_b t_r t$ lies in $S_b t \subseteq S_b$, this shows that $s_i t_r t \in R$ and therefore $Rt \subseteq R$. On the other hand, since $S_b t \supseteq S_b$ there is some $s_r \in S_b$ for which $s_r t = s_b t_r$, say $s_r = s_b t'$ for some $t' \in T$. We have

$$s_b t' t = s_r t = s_b t_r \Longleftrightarrow s_k t' t = s_k t_r \Longrightarrow s_i t' t = s_i t_r = r.$$

Now $s_i t' \in R$ because $s_b t' = s_r \in S_b$, and this shows that $r \in Rt$, i.e., that $Rt \supseteq R$. This completes the proof of the theorem. □

We summarize our results.

Theorem 6.2.6. *Suppose that the suitable relations on T correspond to the source stabilizers. Then nontrivial projective T-algebras exist if and only if I contains an almost maximal element. Furthermore, the projective objects are precisely the retracts of coproducts of T-algebras of the form F_i for i almost maximal in I.*

Proof. If T contains the almost maximal element i then F_i is a nontrivial projective T-algebra by Theorem 6.2.1. And if nontrivial projective T-algebras exist then I contains an almost maximal element by Theorem 6.2.5. Now on general principles, any retract of a coproduct of projectives is projective. And if A is any projective T-algebra then the first few sentences of the proof of Theorem 6.2.5 show that A is a retract of a coproduct of T-algebras of the form F_i for i almost maximal in I. □

For the readers who are familiar with topological groups, from Theorem 6.2.6 we derive a nice characterization of those topological groups T for which nontrivial projective T-algebras exist. Recall that a topological group H is *totally bounded* if for any nonempty open subset U of H there is a finite set $F \subseteq H$ such that $UF = H$.

Theorem 6.2.7. *Let T be a topological group. Nontrivial projective T-algebras exist if and only if T has an open subgroup H such that all open subgroups of H have finite index in H. If the identity element of T has a neighborhood base consisting of open subgroups, this is the same as to say that T has an open subgroup H which is totally bounded.*

Proof. Just note that $i \in I$ is almost maximal if and only if every open subgroup of $\operatorname{stab} s_i$ has finite index in $\operatorname{stab} s_i$. □

Here is an example which shows that the hypothesis that the suitable relations on T correspond to the source stabilizers cannot be omitted from Theorems 6.2.5 or 6.2.6, or from Proposition 6.2.4(1). This example also violates the implication from (7) to (2) in Theorem 6.2.1.

Example 6.2.8. Let T be the five element monoid whose multiplication table is below.

row×column	1	t_1	t_2	t_3	t_4
1	1	t_1	t_2	t_3	t_4
t_1	t_1	t_1	t_2	t_3	t_4
t_2	t_2	t_1	t_2	t_3	t_4
t_3	t_3	t_4	t_3	t_2	t_1
t_4	t_4	t_4	t_3	t_2	t_1

Here are four suitable relations on T whose types are almost maximal. In the right column are elements $w \in F_i$ which witness the almost maximality of each type as in Remark 6.2.3(3). (Here i designates the top element of I, the type corresponding to the identity suitable relation. Therefore F_i is the free algebra on the generating set T.)

suitable relation	witness
$\{\{1\}, \{t_1\}, \{t_2\}, \{t_3\}, \{t_4\}\}$	1
$\{\{1\}, \{t_1, t_4\}, \{t_2, t_3\}\}$	$1 \wedge t_1 \wedge t_4$
$\{\{1, t_1, t_4\}, \{t_2, t_3\}\}$	$t_1 \wedge t_4$
$\{\{t_1, t_4\}, \{1, t_2, t_3\}\}$	$t_2 \wedge t_3$

For example, if $w = t_1 \wedge t_4$ then

$$w1 = wt_1 = wt_4 = w, \quad wt_2 = wt_3 = t_2 \wedge t_3,$$

so type w corresponds to the third suitable relation in the table. Of the many other types, the authors believe only those in the table are almost maximal. In particular, the element $t_1 \wedge t_2^{-1}$ of the projective T-algebra F_i has type corresponding to the suitable relation $\{\{1\}, \{t_1, t_2, t_3, t_4\}\}$, and this type is not almost maximal.

Example 6.2.8 raises the question of whether every nontrivial projective T-algebra contains a nontrivial element of almost maximal type, i.e., whether Theorem 6.2.5 holds without the hypothesis that the suitable relations correspond to the source stabilizers. The authors are willing to conjecture that the answer to this question is positive.

We finish with an application to an almost finite case. Suppose that T is a topological group with only finite pointed antiflows. Then the group action can be replaced by the action of a compact group, namely $\bar{T}$, and all elements of I are almost maximal. Hence all T-algebras F_i, $i \in I$, are projective. In this case there is a very easy characterization of the finite projective T-algebras.

Theorem 6.2.9. *Let T be a topological group which has only finite pointed antiflows. Then any finite T-algebra A is projective if and only if it has an atom that is fixed by every element of the group, i.e., if and* *only if the Stone space of A has a fixed point with respect to the induced group action.*

Proof. Let $i \in I$. As before, we consider R_i as a subset of F_i. Let $a_0^i \equiv \bigwedge_{R_i} r$ and $a_1^i \equiv \bigwedge_{R_i} \overline{r}$. Here the infima exist since R_i is finite. Since T is a group, it acts transitively on R_i. Hence $R_i t = R_i$ for every $t \in T$. It follows that

$$a_0^i t = \bigwedge_{R_i} rt = \bigwedge_{R_i t} r = \bigwedge_{R_i} r = a_0^i$$

for all $t \in T$. Similarly, $a_1^i t = a_1^i$ for all $t \in T$. It is easily checked that a_0^i and a_1^i are atoms of F_i.

Now if B is a coproduct of any finite family $\{F_i : i \in I_0\}$, then B has at least two atoms that are fixed by the action, namely namely $\bigwedge_{I_0} a_0^i$ and $\bigwedge_{I_0} a_1^i$. This means that the Stone space Y of B, which can be identified with the set of atoms of B and which we regard as a Boolean flow as in Theorem 1.2.2, also has at least two fixed points. Now any finite projective T-algebra A is a retract of such a coproduct B; say $f : B \to A$ is a T-surjection and $h : A \to B$ a T-injection such that $fh = 1_A$. Let X be the Stone space of A and $f' : X \to Y$ and $h' : Y \to X$ be the flow maps dual to f and h, so that f' is injective and h' is surjective and $h'f' = 1_X$. Then any fixed point of Y is taken to a fixed point of X by h', so we conclude that X has at least one fixed point.

Now suppose that A is a finite T-algebra with Stone space X having fixed point x. Let B be the coproduct of the family $\{F_a : a \in A\}$ where F_a is a copy of $F_{\mathrm{type}\,a}$ for each $a \in A$, and let f be the epimorphism which takes the source s_a of F_a to a for each $a \in A$. B is finite and projective. Let Y be the Stone space of B and let $f' : X \to Y$ be the injective flow map dual to f. We will be done if we can show that f has a right inverse $h : A \to B$, or equivalently that there is a flow map $h' : Y \to X$ such that $h'f' = 1_X$. But since Y is finite, the continuity of h' is automatic. Define h' as follows: for each $y \in f'(X)$ let $h'(y)$ be the preimage of y under f'. For every $y \notin f'(X)$ let $h'(y) = x$. Since f' is a flow map, so is h'. Clearly h' is as required. □

Here is an example which shows that Corollary 6.2.9 is false without the group hypothesis.

Example 6.2.10. Let T be the monoid of Example 6.2.8, acting on the Boolean flows X and Y as follows.

tx	x_0	x_1	x_2
1	x_0	x_1	x_2
t_1	x_0	x_1	x_2
t_2	x_0	x_1	x_2
t_3	x_0	x_2	x_1
t_4	x_0	x_2	x_1

ty	y_0	y_1	y_2	y_3
1	y_0	y_1	y_2	y_3
t_1	y_0	y_1	y_2	y_1
t_2	y_0	y_1	y_2	y_2
t_3	y_0	y_2	y_1	y_1
t_4	y_0	y_2	y_1	y_2

Both X and Y have fixed points x_0 and y_0, respectively. However, the clopen algebra of X is a T-algebra which is not projective, for the flow map $f' : X \to Y$ which takes x_k to y_k, $k = 0, 1, 2$, has no left inverse.

If T is a topological group with only finite pointed antiflows we can get a sufficient condition for a (possibly infinite) T-algebra being projective which is internal, i.e., which only involves the structure of the T-algebra. A slight weakening of this sufficient condition turns out to be necessary. Unfortunately those conditions are very technical, and we do not have a complete characterization yet. That is why those conditions are not treated here.

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