An invitation to SYZ and GS

Calabi-Yau varieties: complex affine or projective variety s.th. $K_X = 0$ $\Updownarrow$ $\Omega^n = \mathcal{O}_X$ (canonical bdl)

(local rings have finite cyl. resolution)

Recs.:
- smooth complex variety $\Rightarrow$ Kähler manifold
- maybe non-cpt, allow mild (Gorenstein) singularities
- $\Omega^n \cong \mathcal{O}_X$ nowhere vanishing hol. volume form ~~[illegible]~~

$\Rightarrow c_1(X) = 0 \Leftrightarrow \exists$ Ricci-flat metric $\qquad h^{n,0} = 1$

$\Rightarrow b_1 = 0 \Rightarrow h^{1,0} = h^{0,1} = 0$ ($b_1 = h^{1,0} + h^{0,1}$)

- Serre duality & cplx conj. $\Rightarrow h^{p,q} = h^{q,p} = h^{n-p,n-q}$ ~~[illegible]~~

$h^{p,q} = \dim H^q(X, \Omega^p)$ $\Rightarrow$ Hodge diamond $(n=3)$ $\quad$ also: $h^{p,0} = h^{3-p,0}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & a & & 0 & \\ 1 & & b & & b & & 1 \\ & 0 & & a & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

($1 \leftarrow \Omega^n \cong \mathcal{O}_X$; $0 \leftarrow c_1 = 0$; $\updownarrow$ Serre & cplx conj; MS)

Expl.: - degree $d = n+1$ hypersurface in $\mathbb{P}^n$
- elliptic curves $\begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{smallmatrix}$
- K3 surfaces $\begin{smallmatrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{smallmatrix}$ $\quad \chi := \sum (-1)^r b^r = 24$
- quintic threefold $\begin{smallmatrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{smallmatrix}$ $\overset{[COGP]'91}{\rightsquigarrow}$ $\begin{smallmatrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{smallmatrix}$ $\longleftrightarrow$ cplx. conj.

A-model $\leftrightarrow$ B-model
- hypersurfaces in $W\mathbb{P}^4$ [CLS]'90 $\longrightarrow$ plot χ against $h^{11} + h^{21}$
dually "transverse" polys.

Mirror symmetry

- symmetry in Hodge numbers: $h^{p,q}(X) = h^{n-p,q}(\check{X}) \overset{\dim=3}{\Rightarrow} \chi(X) = -\chi(\check{X})$
- A-model $\longleftrightarrow$ B-model (top. string thy)

A-model	B-model
sympl. structure	complex structure
GW invariants = # hol. curves	period integrals
A-branes:	B-branes
special Lagr. submfd with flat U(1)-bdl	hol. submfd with hol. line bdl

[COGP] $\rightarrow$ # rational curves on quintic threefolds

Lagr.: $\begin{cases} \dim L = \frac{1}{2}\dim X \\ \omega|_L = 0 \end{cases}$ $\qquad$ $\operatorname{Im} \Omega|_L = 0$

Kähler form (sympl. form compatible with cplx str.) $\qquad$ hol. vol. form

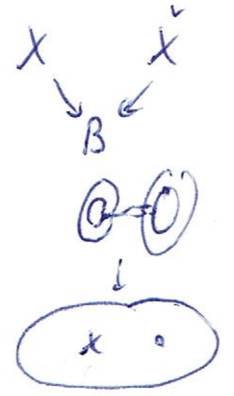
- HMS (Kontsevich '94) $\quad$ derived Fukaya

$D^b(\mathrm{Coh}\, X) \cong \mathrm{Fuk}(\check{X})$ as triangulated cat.

- SYZ conj. (SYZ '96)
$\rightsquigarrow$ geom. interpretation

SYZ conjecture: $X, \check{X}$ admit SL torus fibrations (w/ sing. fibres)

s.th. tori are duals: $X_b = H^1(\check{X}_b, \mathbb{R}/\mathbb{Z})$, $\check{X}_b = H^1(X_b, \mathbb{R}/\mathbb{Z})$



Rem: • no sing. fibres $\Rightarrow \chi = 0$ (boring, e.g. no K3)

(1) { • two affine structures on B (symplectic / c.s.)
• MS via "Legendre transform"

$\rightsquigarrow$ allow sing. along discriminant locus $\Gamma \subset B$

Q: construct $\check{X}$ from X

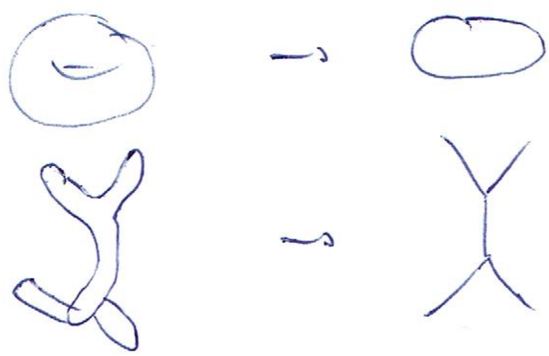
recipe: • dualize torus fibres $\rightsquigarrow \check{X}' \to B \setminus \Gamma$ (2)

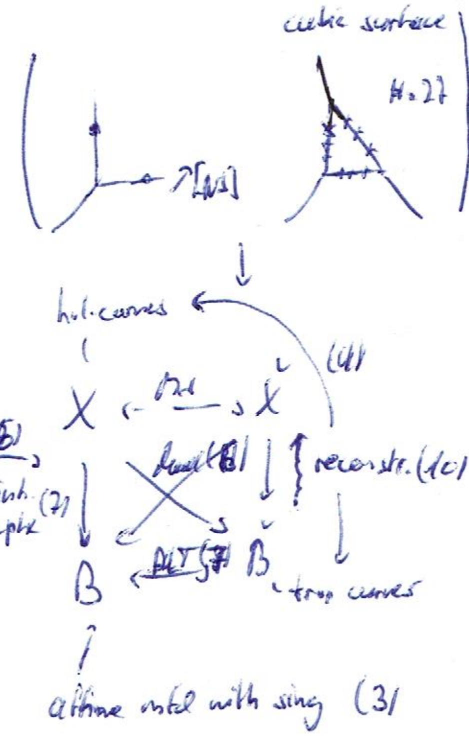
• extend across $\Gamma \rightsquigarrow \check{X} \to B$

Joyce et al.: smooth SL fibration has Γ of (Hausdorff) codim ≥ 2

$\rightsquigarrow$ take Gromov-Hausdorff (LCSL) limit (5)

$\check{X}_\varepsilon = \mathbb{R}^n / \varepsilon\Lambda \xrightarrow{\varepsilon \to 0}$

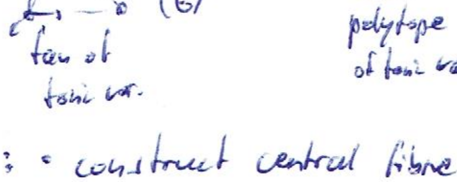




GS program: algebro geometric version of SYZ
MS is a feature of toric degenerations

X_0 = union of toric var. glued along toric divisors

toric var.: fan Σ, polytope Δ (4)
dual int. cplx: int. cplx.



reconstruction: • construct central fibre $\check{X}_0$ via fan (9) / cone (10) picture

• use logarithmic geometry (8)
and tropical geometry (4) to deform to a log smooth family

• involves scattering diagrams (11)
$\leftrightarrow$ hol. curves (12)