

THE TOPOLOGY AND GEOMETRY OF HYPERKÄHLER QUOTIENTS

by

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Abstract

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In this thesis we study the topology and geometry of hyperkähler quotients, as well as some related non-compact Kähler quotients, from the point of view of Hamiltonian group actions. The main technical tool we employ is Morse theory with moment maps. We prove a Lojasiewicz inequality which permits the use of Morse theory in the non-compact setting. We use this to deduce Kirwan surjectivity for an interesting class of non-compact quotients, and obtain a new proof of hyperkähler Kirwan surjectivity for hypertoric varieties. We then turn our attention to quiver varieties, obtaining an explicit inductive procedure to compute the Betti numbers of the fixed-point sets of the natural S^1 -action on these varieties. To study the kernel of the Kirwan map, we adapt the Jeffrey-Kirwan residue formula to our setting. The residue formula may be used to compute intersection pairings on certain compact subvarieties, and in good cases these provide a complete description of the kernel of the hyperkähler Kirwan map. We illustrate this technique with several detailed examples. Finally, we investigate the Poisson geometry of a certain family of Nakajima varieties. We construct an explicit Lagrangian fibration on these varieties by embedding them into Hitchin systems. This construction provides an interesting class of toy models of Hitchin systems for which the hyperkähler metric may be computed explicitly.

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Chapter 1

Introduction

1.1 Hyperkähler Quotients

A hyperkähler manifold is a Riemannian manifold (M, g) together with a triple (I, J, K) of parallel skew endomorphisms of TM which satisfy the quaternion relations

$$I^2 = J^2 = K^2 = IJK = -1. \quad (1.1)$$

From these we obtain a triple $(\omega_I, \omega_J, \omega_K)$ of Kähler forms, with $\omega_I(u, v) := g(Iu, v)$, etc. If a Lie group G acts on M preserving this structure, we call it Hamiltonian if it is Hamiltonian with respect to each of these symplectic forms, i.e. if there is a triple of moment maps $\mu_{\text{HK}} = (\mu_I, \mu_J, \mu_K) : M \rightarrow \mathfrak{g}^* \otimes \mathbf{R}^3$. One may then define the hyperkähler quotient $M //_{\alpha} G$ to be

$$M //_{\alpha} G := \mu_{\text{HK}}^{-1}(\alpha) / G. \quad (1.2)$$

It is straightforward to check that the complex 2-form $\omega_{\mathbf{C}} := \omega_J + \sqrt{-1}\omega_K$, is a holomorphic symplectic form, i.e. a closed non-degenerate $(2,0)$ -form. Hence, a hyperkähler quotient may alternatively be thought of as a *holomorphic symplectic quotient*. This leads to the natural question: which techniques from Hamiltonian group actions generalize to the setting of *holomorphic* Hamiltonian group actions? There are several fundamental obstacles which prevent an easy answer to this question: (i) there is no hyperkähler Darboux theorem, since the Riemann tensor is a local invariant; (ii) most interesting examples are non-compact; and (iii) the complex moment maps that we consider are never proper.

In this thesis, we focus on the special case $M = T^*\mathbf{C}^n$ with its Euclidean metric. This

may be identified with the flat quaternionic vector space \mathbf{H}^n via the complex structures

$$I(x, y) = (ix, iy), \quad J(x, y) = (-\bar{y}, \bar{x}), \quad K(x, y) = (-i\bar{y}, i\bar{x}). \quad (1.3)$$

For any $G \subseteq Sp(n, \mathbf{H})$ the natural linear G -action on M is Hamiltonian, and we may take the hyperkähler quotient $\mathfrak{M}_\alpha := M //_\alpha G$. Although $T^*\mathbf{C}^n$ is not interesting topologically, the hyperkähler varieties \mathfrak{M}_α constructed in this way turn out to be very interesting, both topologically and geometrically. This class of hyperkähler quotients includes hypertoric varieties and Nakajima quiver varieties. In all the examples that we will consider, we will in fact take $G \subset U(n) \subset Sp(n, \mathbf{H})$. In this case, there is a natural inclusion from the symplectic quotient $\mathfrak{X}_\alpha := \mathbf{C}^n //_\alpha G$ into the hyperkähler quotient $\mathfrak{M}_{(\alpha, 0)}$. Following Proudfoot [Pro04], we call $\mathfrak{M}_{(\alpha, 0)}$ the *hyperkähler analogue* of \mathfrak{X}_α .

There is a canonical map $H^*(BG) \rightarrow H^*(\mathfrak{M}_\alpha)$, called the *hyperkähler Kirwan map*. There is an analogous map for symplectic quotients which is well-known to be surjective in many cases. Three fundamental problems which serve as motivation for the work in this thesis are the following: (i) to determine whether the hyperkähler Kirwan map is surjective; (ii) to determine an effective method to compute the image of the hyperkähler Kirwan map; and (iii) to determine an effective method to compute the Betti numbers of \mathfrak{M}_α . We will address all three problems. Unfortunately, we do not obtain a complete solution to the first problem. However, solutions to the second and third problems provide an indirect way to prove or disprove Kirwan surjectivity in particular cases. The techniques we develop in this thesis provide a toolkit for producing many new examples satisfying hyperkähler Kirwan surjectivity. In addition, several techniques we develop may be applied to a larger class of non-compact quotients, not only to those arising from the hyperkähler quotient construction.

1.2 Motivation from Quantum Field Theory

Many of the constructions and examples considered in this thesis were motivated either directly from or by analogy with quantum field theory. The hyperkähler quotient construction itself has its origins in quantum field theory, having been first discovered in the context of gauged supersymmetric sigma model [HKLR87].

In Chapter 2, we study Morse theory with moment maps. Morse theory with $|\mu|^2$ was developed by Kirwan [Kir84] by direct analogy with Morse theory with the Yang-Mills functional in two dimensions [AB82]. In two dimensions, one may also study the Yang-Mills-Higgs functional [Wil08], which is analogous to $|\mu_{\text{HK}}|^2$ on a hyperkähler manifold.

In fact, it was in this context that we first encountered Lojasiewicz estimates, which provided the key idea of the main theorem of Chapter 2.

In four dimensions, one may also consider Yang-Mills instantons, which are solutions to the Euclidean equations of motion with finite action. These give important non-perturbative corrections to the quantum path integral and are the subject of intense study in theoretical physics. The self-duality equations defining instantons take the form of a hyperkähler moment map, and the moduli space of instantons is in fact an *infinite-dimensional* hyperkähler quotient. In [AHD78], the moduli space of instantons on \mathbf{R}^4 was famously constructed as a finite dimensional hyperkähler quotient of Euclidean space. This construction was later generalized to any ALE space [KN90], which are themselves hyperkähler quotients [Kro89]. In [DM96], a physical interpretation of these constructions was given in terms of gauge theories arising from certain D -brane configurations in string theory. The ALE and instanton moduli spaces above are all examples of Nakajima quiver varieties [Nak94], and for this reason we focus on moduli spaces of quiver representations in Chapter 3.

In Chapter 4, we revisit the residue formula in the context of hyperkähler quotients. The residue formula [JK95] was itself developed by analogy with Witten's quantum field theory arguments [Wit92], and to a certain extent made many of Witten's calculations rigorous. In Witten's recent work [GW09, Wit11a, Wit11b], he takes the fruitful viewpoint that symplectic manifolds should be thought of as middle-dimensional integration cycles in an ambient almost hyperkähler manifold. This served as motivation to re-examine the residue formula in the context of hyperkähler quotients.

In Chapter 5, we construct algebraic integrable systems from quiver representations. We prove integrability by embedding these spaces into certain Hitchin systems [Hit87b, Hit87a]. Hitchin systems are themselves moduli spaces of solutions to the Hitchin equations, which are a dimensional reduction of the self-duality equations. Using the quantum field theoretic techniques of BPS state counting and wall-crossing, Gaiotto, Moore, and Neitzke have given a conjectural construction of the hyperkähler metric on Hitchin systems [GMN13]. Although we will not have much to say about this, the primary motivation of Chapter 5 was to construct an interesting class of toy models to which the techniques of Gaiotto-Moore-Neitzke may be applied. We plan to investigate this in future work.

1.3 Summary of Results

In Chapter 2, we review the basic theory of Morse theory with moment maps. We develop some basic techniques to organize Morse theory calculations, and especially to streamline index calculations. The main new result that we prove is the Łojasiewicz estimate of Theorem 2.3.11. This gives good analytic control on the gradient flow of $|\mu|^2$ on \mathbf{C}^n . In the case that the group is abelian, this theorem also applies to $|\mu_c|^2$ and $|\mu_{\text{HK}}|^2$. This yields a new proof of hyperkähler Kirwan surjectivity for hypertoric varieties, and Morse theory provides a very short calculation of their cohomology rings (both equivariant and ordinary). Much of the material in Sections 2.3 and 2.4 appeared in an earlier article [Fis13].

In Chapter 3, we turn our attention to quiver varieties. In Theorem 3.2.13, we give a general procedure to compute the Poincaré polynomials of subvarieties of quiver varieties defined by relations in the path algebra. Nakajima quiver varieties are special cases, but our procedure applies to a larger class of non-compact varieties. In §3.3 we give a detailed example of this procedure.

In Chapter 4, we re-examine the Jeffrey-Kirwan residue formula in the context of hyperkähler quotients. We show that the residue formula can be interpreted as a procedure for producing cogenerators of cohomology rings of hyperkähler quotients. Theorem 4.2.8 provides a specialization of the residue formula applicable to quotients of \mathbf{C}^n by linear group actions. As a corollary, this gives an algorithmic procedure to compute the intersection pairings and cohomology ring of *any* compact symplectic quotient of the form $\mathbf{C}^n//G$. Using the residue formula, we define a natural ring associated to any compact symplectic quotient, which we call the *VGIT ring*. Conjecture 4.3.6 states that (under appropriate conditions), the VGIT ring is equal to the image of the hyperkähler Kirwan map. We give some explicit examples in §4.4. Theorem 4.3.11 provides an alternative procedure to compute a complete set of cogenerators for any Nakajima quiver variety, which is valid even if Conjecture 4.3.6 fails.

Finally, in Chapter 5 we turn our attention away from topology and focus on the symplectic geometry of hyperkähler quotients. In Theorem 5.1.9, we identify a certain class of hyperkähler quiver varieties with moduli spaces of parabolic Higgs bundles on \mathbf{P}^1 . Using this identification, together with standard arguments about Hitchin systems, we prove in Theorem 5.2.10 that these quiver varieties are algebraic completely integrable systems. This gives an interesting family of toy models of Hitchin systems for which the hyperkähler metric may be computed explicitly.

Chapter 2

Morse Theory with Moment Maps

2.1 Hamiltonian Group Actions

2.1.1 Symplectic Reduction

We begin by reviewing basic notions in equivariant symplectic geometry. A classic reference is [GS84]. A *symplectic manifold* is a pair (X, ω) where X is a smooth manifold and ω is a symplectic form on X . Let G be a compact Lie group with Lie algebra \mathfrak{g} , and suppose that G acts on a symplectic manifold X by symplectomorphisms. For each $\xi \in \mathfrak{g}$, we have the *fundamental vector field* v_ξ defined by

$$v_\xi(x) := \left. \frac{d}{dt} \right|_{t=0} e^{t\xi} \cdot x. \quad (2.1)$$

The action is said to be *Hamiltonian* if there exists a G -equivariant map $\mu : X \rightarrow \mathfrak{g}^*$ satisfying

$$d\langle \mu, \xi \rangle = i_{v_\xi} \omega, \quad (2.2)$$

for all $\xi \in \mathfrak{g}$. The map μ is called the *moment map*, and is unique (if it exists) up to the addition of a central constant. If α is a central value of μ , then the *symplectic reduction* $X//G$ is defined to be

$$X//G := \mu^{-1}(\alpha)/G. \quad (2.3)$$

When the group G is understood, we will often denote $X//G$ by \mathfrak{X}_α .

Theorem 2.1.1. *If α is a regular central value of μ , then \mathfrak{X}_α is a symplectic orbifold, and a symplectic manifold if G acts freely on $\mu^{-1}(\alpha)$. Now suppose that (g, I, ω) is a Kähler triple on X , and that the G -action preserves this Kähler structure. Then this*

Kähler structure descends to the symplectic reduction $X//G$, making it into a complex orbifold.

Remark 2.1.2. Suppose that $N \subset G$ is a normal subgroup, and let $H = G/N$. Then for appropriately chosen moment map levels, there is a natural identification

$$X//G \cong (X//N)//H. \quad (2.4)$$

This is called *reduction in stages*.

2.1.2 Hyperkähler Reduction

A *hyperkähler manifold* is a tuple (M, g, I, J, K) such that g is a Riemannian metric on M and I, J, K are skew-adjoint parallel endomorphisms of TM satisfying the quaternion relations. This endows M with a triple $(\omega_I, \omega_J, \omega_K)$ of symplectic forms, defined by

$$\omega_I(u, v) = g(Iu, v), \quad \omega_J(u, v) = g(Ju, v), \quad \omega_K(u, v) = g(Ku, v). \quad (2.5)$$

The action of a compact Lie group G on M is called *hyperhamiltonian* if it is Hamiltonian with respect to each of these symplectic forms. If we denote the corresponding moment maps by μ_I, μ_J, μ_K then we may package these together into a *hyperkähler moment map* $\mu_{\text{HK}} : M \rightarrow \mathfrak{g} \otimes \mathfrak{sp}_1$ given by

$$\mu_{\text{HK}}(x) = \mu_I(x) \otimes i + \mu_J(x) \otimes j + \mu_K(x) \otimes k \quad (2.6)$$

where $\{i, j, k\}$ is the standard basis of $\mathfrak{sp}_1 \cong \mathfrak{su}_2 \cong \mathbf{R}^3$.

Given a hyperkähler manifold, it is convenient to think of it as having a pair $(\omega_{\mathbf{R}}, \omega_{\mathbf{C}})$ of symplectic forms, where $\omega_{\mathbf{R}} := \omega_I$ is the *real symplectic form* and $\omega_{\mathbf{C}} := \omega_J + i\omega_K$ is the *complex symplectic form*. We denote the corresponding moment maps by μ_r and μ_c , respectively. The *hyperkähler quotient* of M by G , denoted $M//G$, is

$$M//G := \mu_{\text{HK}}^{-1}(\alpha)/G. \quad (2.7)$$

As in the symplectic case, when no confusion should arise, we will often denote $M//G$ by \mathfrak{M}_α .

Theorem 2.1.3 ([HKLR87]). *If α is a regular central value, then $M//G$ is a hyperkähler orbifold, and a hyperkähler manifold provided that the action of G on $\mu_{\text{HK}}^{-1}(\alpha)$ is free. The*

metric on $M//_{\alpha}G$ is complete provided that the metric on M is.

Denote the real and complex parts of α by $\alpha_{\mathbf{R}}$ and $\alpha_{\mathbf{C}}$, respectively. If $\alpha_{\mathbf{C}}$ is regular, then $\mu_c^{-1}(\alpha_{\mathbf{C}})$ is a complex submanifold of M , and hence is Kähler. Then we have

$$M//_{\alpha}G = \mu_r^{-1}(\alpha_{\mathbf{R}}) \cap \mu_c^{-1}(\alpha_{\mathbf{C}})/G = \mu_c^{-1}(\alpha_{\mathbf{C}})//_{\alpha_{\mathbf{R}}}G. \quad (2.8)$$

Since the Kähler quotient $\mu_c^{-1}(\alpha_{\mathbf{C}})//_{\alpha_{\mathbf{R}}}G$ may be identified with the geometric invariant theory quotient $\mu_c^{-1}(\alpha_{\mathbf{C}})//G_{\mathbf{C}}$, we may think of the hyperkähler quotient as a *holomorphic symplectic reduction* with respect to the action of $G_{\mathbf{C}}$.

2.1.3 Cuts and Modification

We review two basic constructions which we will be useful for our purposes. Let X be a symplectic manifold with Hamiltonian circle action.

Definition 2.1.4. For any $\sigma \in \mathbf{R}$ we define the *symplectic cut* [Ler95] X_{σ} to be

$$X_{\sigma} := (X \times \mathbf{C})//_{\sigma}S^1, \quad (2.9)$$

where the S^1 action is given by $s \cdot (x, z) = (s \cdot x, sz)$. Similarly, if M is hyperkähler with a hyperhamiltonian circle action, we define its *hyperkähler modification* [DS06] M_{σ} to be

$$M_{\sigma} := (M \times T^*\mathbf{C})//_{\sigma}S^1. \quad (2.10)$$

Note that if M is a hyperkähler analogue of X , then there is a natural inclusion $X_{\sigma} \hookrightarrow M_{\sigma}$.

We now consider the special case of $X = \mathbf{C}^n$, with the standard circle action given by $s \cdot z = sz$. For $\sigma > 0$, $\mathbf{C}_{\sigma}^n \cong \mathbf{P}^n$ as a variety, but its Kähler metric and symplectic form depend on σ . If $G \subset U(n)$, then G acts on \mathbf{C}^n as well as on \mathbf{C}_{σ}^n . We will need the following lemma.

Lemma 2.1.5. *Suppose that G acts linearly on $X = \mathbf{C}^N$ with proper moment map. Then for a sufficiently large cut parameter σ , we have a natural identification*

$$X//_{\alpha}G \cong X_{\sigma}//_{\alpha}G \quad (2.11)$$

as Kähler manifolds.

Proof. This is easy to see by reduction in stages. By the hypothesis that the G -moment map is proper, we can find an embedding $\chi : S^1 \hookrightarrow G$ so that the S^1 -moment map is proper. We define an action of $G \times S^1$ on $X \times \mathbf{C}$ by

$$(k, s) \cdot (x, z) = (\chi(s)kx, sz) \quad (2.12)$$

Note that $(X \times \mathbf{C}) //_{\alpha} G \cong (X //_{\alpha} G) \times \mathbf{C}$ and the residual S^1 action acts only on the second factor, so we have

$$(X \times \mathbf{C}) //_{(\alpha, \sigma)} (G \times S^1) \cong (X //_{\alpha} G) \times (\mathbf{C} //_{\sigma - \alpha'} S^1) \cong (X //_{\alpha} G) \times \{\text{pt}\} \cong X //_{\alpha} G \quad (2.13)$$

provided that $\sigma - \alpha' > 0$, where α' is the projection of α to the image of $d\chi$. On the other hand, we can perform the reductions in the opposite order. Again, assuming $\sigma > \alpha'$, we have

$$(X \times \mathbf{C}) //_{(\alpha, \sigma)} (G \times S^1) \cong ((X \times \mathbf{C}) //_{\sigma} S^1) //_{\alpha} G \cong X //_{\alpha} G \quad (2.14)$$

□

2.1.4 Equivariant Cohomology and the Kirwan Map

We will use equivariant cohomology throughout this thesis, so we review the basic constructions. A basic reference for equivariant cohomology and localization is [GS99].

First we recall the Borel model of equivariant cohomology. Let M be a G -space, and let EG be the classifying space of G . The equivariant cohomology¹ $H_G^*(M)$ is defined to be

$$H_G^*(M) := H^*(EG \times_G M) \quad (2.15)$$

where EG is the classifying space, and $EG \times_G M = (EG \times M)/G$ is the quotient by the diagonal G -action. The functor $H_G^*(\cdot)$ is an extraordinary cohomology theory on the category of G -spaces.

When G is a Lie group acting smoothly on a manifold M , there is an alternative construction of equivariant cohomology called the Cartan model. One simply defines $H_G^*(M)$ to be the cohomology of the complex

$$\Omega_G^*(M) := (\Omega^*(M) \otimes \text{Sym } \mathfrak{g}^*)^G, \quad (2.16)$$

¹We will always take cohomology with real coefficients.

where the equivariant differential D is defined to be

$$D = d \otimes 1 - \sum_a i_a \otimes \phi_a \quad (2.17)$$

where $\{\phi_a\}$ is a basis of \mathfrak{g}^* and i_a denotes the interior product with the fundamental vector field dual to the basis element ϕ_a . In all of the cases that we will consider, the Borel and Cartan models of equivariant cohomology are naturally isomorphic.

Definition 2.1.6. Let M be a symplectic manifold with Hamiltonian G -action. The *Kirwan map* is the natural map

$$\kappa : H_G^*(M) \rightarrow H^*(M//G) \quad (2.18)$$

induced by the inclusion $\mu^{-1}(0) \hookrightarrow M$. Similarly, if M is hyperkähler with a hyperhamiltonian G -action, the *hyperkähler Kirwan map* is the natural map

$$\kappa_{\text{HK}} : H_G^*(M) \rightarrow H^*(M///G). \quad (2.19)$$

It is well-known that the Kirwan map is surjective for a large class of symplectic quotients [Kir84]. However, the hyperkähler Kirwan map is not so well-understood.

Conjecture 2.1.7. *The hyperkähler Kirwan map is surjective.*

Remark 2.1.8. This conjecture is definitely false if one considers certain infinite-dimensional quotients. The moduli space of rank 2, fixed-determinant Higgs bundles provides an explicit counterexample [DWWW11].

Remark 2.1.9. While preparing this thesis, we became aware of the work of McGerty and Nevins [MN14]. By studying the Kirwan-Ness decomposition on the quotient stack $\mathcal{Y} = X/G$, they obtain a modified version of Kirwan surjectivity for the algebraic symplectic quotient $T^*X///G$.

2.2 Morse Theory

2.2.1 The Moment Map as a Morse-Bott Function

We begin by recalling (without proof) basic facts about Morse theory with moment maps. Standard references include Atiyah-Bott [AB82, AB84], Atiyah [Ati82], and Kirwan [Kir84]. Let G act on (X, ω) with moment map μ , and let $T \subseteq G$ be a maximal

torus. For $\beta \in \mathfrak{t}$, we define μ^β to be the pairing $\langle \mu, \beta \rangle$. Denote by T_β the closure in T of $\{e^{t\beta} \mid t \in \mathbf{R}\}$. Throughout, we will assume that we have fixed some compatible metric and almost complex structure.

Proposition 2.2.1. *For any $\beta \in \mathfrak{t}$, the function μ^β is Morse-Bott. The critical set of μ^β is equal to the fixed-point set X^{T_β} .*

Let C be a connected component of X^{T_β} . Since any $x \in C$ is fixed by T_β , we obtain an isotropy representation on $T_x X$. If we fix a compatible metric and almost complex structure, then we may decompose $T_x X$ into a direct sum of irreducibles

$$T_x X \cong \bigoplus_{i \in I} \mathbf{C}(\nu_i), \quad (2.20)$$

where I is some indexing set and ν_i are the weights of the isotropy representation, counted with multiplicity. We call a weight ν *positive* if $\langle \beta, \nu \rangle > 0$ and *negative* if $\langle \beta, \nu \rangle < 0$. Let $N^+ C$ and $N^- C$ be the subbundles of $TX|_C$ consisting of the positive and negative weight spaces of the isotropy representation, respectively.

Proposition 2.2.2. *The bundles $N^+ C$ and $N^- C$ are respectively the positive and negative eigenbundles of the Hessian of μ^β on C . In particular, the Morse index of μ^β on C is equal to twice the number of negative weights, counted with multiplicity.*

For the remainder of this section, we will assume that μ^β is proper and bounded below, and that X^{T_β} is compact. Let $C \subseteq X^{T_\beta}$ be a connected component of the fixed-point set. Denote by X_C^\pm the sublevel sets

$$X_C^\pm = \{x \in X \mid \mu^\beta(x) \leq \mu^\beta(C) \pm \epsilon\} \quad (2.21)$$

where ϵ is some positive number chosen small enough so that the only critical value occurring in $[\mu^\beta(C) - \epsilon, \mu^\beta(C) + \epsilon]$ is $\mu^\beta(C)$ itself.

Proposition 2.2.3. *The equivariant Thom-Gysin sequence for the Morse stratification with respect to μ^β splits into short exact sequences*

$$0 \rightarrow H_T^{*- \lambda}(C) \rightarrow H_T^*(X_C^+) \rightarrow H_T^*(X_C^-) \rightarrow 0. \quad (2.22)$$

The composition of $H_T^{*- \lambda}(C) \rightarrow H_T^*(X_C^+)$ with the restriction $H_T^*(X_C^+) \rightarrow H_T^*(C)$ is given by multiplication by the equivariant Euler class of the negative normal bundle to C . In particular, the function μ^β is equivariantly perfect.

Theorem 2.2.4. *Suppose that X^T is compact and there exists some component β such that μ^β is proper and bounded below. Then X is equivariantly formal, i.e. $H_T^*(X) \cong H^*(BT) \otimes H^*(X)$ as a $H^*(BT)$ -module. As a consequence, the function μ^β is perfect, and the Poincaré polynomial of X is given by*

$$P_t(X) = \sum_{C \subseteq X^{T\beta}} t^\lambda P_t(C), \quad (2.23)$$

where the sum is over connected components C of the fixed-point set.

Theorem 2.2.5. *Under the same hypotheses as above, the restriction $H_T^*(X) \rightarrow H_T^*(X^T)$ is an injection.*

2.2.2 Computing Isotropy Representations

We saw in the previous subsection that Morse theory with a moment map amounts to computing the fixed-point set as well as the isotropy representations. Not only does this allow us to compute the Poincaré polynomial of our manifold, but in addition we may use Proposition 2.2.3 to inductively construct the cohomology ring.

The most important special case that we consider is the following. Suppose that X is a symplectic manifold that we understand well—in most cases, a vector space, flag variety, or product thereof. Suppose that a compact group H acts on X , and $G \subset H$ is a normal subgroup. Then $K := H/G$ has a residual action on the symplectic reduction $\mathfrak{X} = X//G$. The goal of this section is to give a description of Morse theory on \mathfrak{X} with respect to the K action in terms of the H -action on X . We assume throughout this section that G acts freely on $\mu_G^{-1}(0)$, so that \mathfrak{X} is smooth.

Proposition 2.2.6. *$x \in \mathfrak{X}^K$ if and only if there is a lift $\tilde{x} \in X$ and a unique map $\phi : H \rightarrow G$ (depending on the lift \tilde{x}) such that for all $h \in H$, $h \cdot \tilde{x} = \phi(h) \cdot \tilde{x}$. Furthermore, the map ϕ satisfies the properties $\phi|_G = 1_G$ and $\phi(gh) = g\phi(h)g^{-1}\phi(g)$ for all $g, h \in H$.*

Proof. Suppose that $x \in \mathfrak{X}^K$. Then for any representative \tilde{x} , $h \cdot \tilde{x}$ must represent the same point in \mathfrak{X} , and hence differs from \tilde{x} by an element of G . Since the action of G on $\mu_G^{-1}(0)$ is free, they differ by a *unique* element of G , which we will denote by $\phi(h)$. This defines the map $\phi : H \rightarrow G$. Now consider two elements $g, h \in H$. We compute

$$\phi(gh) \cdot \tilde{x} = gh \cdot \tilde{x} = g\phi(h)\tilde{x} = g\phi(h)g^{-1}g\tilde{x} = g\phi(h)g^{-1}\phi(g)\tilde{x}, \quad (2.24)$$

and hence $\phi(gh) = g\phi(h)g^{-1}\phi(g)$. □

Proposition 2.2.7. *Let ϕ be as in the preceding proposition, for some lift $\tilde{x} \in X$ of a fixed point $x \in \mathfrak{X}^K$. Then the set $\ker \phi \subseteq H$ is a subgroup of H and is naturally isomorphic to K . Consequently, we have $H \cong K \times G$, where the semidirect product structure is given by $(k_1, g_1)(k_2, g_2) = (k_1 k_2, k_1 g_2 k_1^{-1} g_1)$*

Proof. A short calculation using the property $\phi(gh) = g\phi(h)g^{-1}\phi(g)$ verifies that $\ker \phi$ is indeed a subgroup of H . We define a map $\psi : K \rightarrow \ker \phi$ by $\psi(k) = \phi(\tilde{k})^{-1}\tilde{k}$, for any representative $\tilde{k} \in H$ of k . This is well-defined, since any other representative of k is of the form $\tilde{k}g$ with $g \in G$, and a short calculation shows that $\phi(\tilde{k}g)^{-1}\tilde{k}g = \phi(\tilde{k})^{-1}\tilde{k}$. It is easy to check that this is a group homomorphism, and is inverse to the quotient map $\ker \phi \rightarrow K$. The isomorphism $H \xrightarrow{\cong} K \times G$ is given by the map $h \mapsto ([h], \phi(h))$. \square

Now suppose that $x \in \mathfrak{X}^K$ is some particular fixed point with lift $\tilde{x} \in X$, and let $\phi : H \rightarrow G$ be the induced map as in the preceding propositions. Using ϕ , we define a new K -action on X , denoted $*$, by

$$k * y := \phi(k)^{-1}k \cdot y. \quad (2.25)$$

The property $\phi(k_1 k_2) = k_1 \phi(k_2) k_1^{-1} \phi(k_1)$ ensures that $k_1 * (k_2 * y) = (k_1 k_2) * y$, so this is indeed an action. This new action has the property that \tilde{x} will be fixed by all of K . Hence we have an isotropy representation of K on $T_{\tilde{x}}X$. Moreover, this new K -action on X induces the original K -action on \mathfrak{X} , since for any $y \in X$, $k \cdot y$ and $\phi^{-1}(k)k \cdot y$ lie on the same G -orbit.

Proposition 2.2.8. *Let $x \in \mathfrak{X}^K$, $\tilde{x} \in X$ a representative, and ϕ as above. We have the following complex of K -representations*

$$0 \rightarrow \mathfrak{g} \xrightarrow{d\rho} T_{\tilde{x}}X \xrightarrow{d\mu_G} \mathfrak{g}^* \rightarrow 0, \quad (2.26)$$

where ρ is the action map and K acts on $\mathfrak{g} \cong \mathfrak{g}^*$ by conjugation. Consequently, $T_{\tilde{x}}\mathfrak{X}$ is isomorphic as a K -representation to the cohomology of this complex.

Proof. By the preceding proposition, we identify K with $\ker \phi \subset H$. Then the twisted K -action is nothing more than the restriction of the H -action to K under this identification. Since G is a normal subgroup of H , then \mathfrak{g} is invariant under the adjoint action of H , so we may restrict this to a K -action on \mathfrak{g} . Since the action map and moment map are H -equivariant, this gives us an equivariant complex

$$0 \rightarrow \mathfrak{g} \xrightarrow{d\rho} T_{\tilde{x}}X \xrightarrow{d\mu_G} \mathfrak{g}^* \rightarrow 0 \quad (2.27)$$

as claimed. At the level of vector spaces, we have $T_x\mathfrak{X} \cong T_{\tilde{x}}\mu^{-1}(0)/T_{\tilde{x}}\mathcal{O}_{\tilde{x}}$, where $\mathcal{O}_{\tilde{x}}$ is the G -orbit through \tilde{x} . Since $T_{\tilde{x}}\mu^{-1}(0) = \ker d\mu_G(\tilde{x})$ and $T_{\tilde{x}}\mathcal{O}_{\tilde{x}} = \text{im}\rho$, we find that the cohomology of this complex is isomorphic to $T_x\mathfrak{X}$. \square

Since knowledge of the isotropy representation allows us to immediately compute the Morse index, the above proposition gives an effective method to compute Morse indices for a large class of Hamiltonian group actions. However, it will be convenient to have a slightly more general statement for invariant subvarieties. Suppose that E is a linear H -representation and that $f : X \rightarrow E$ is an H -equivariant map. Then $f^{-1}(0)$ is H -invariant and induces a subset² $V(f) \subset \mathfrak{X}$.

Proposition 2.2.9. *Let $f : X \rightarrow E$ an H -equivariant map to a linear representation of H , $x \in V(f)^K$, $\tilde{x} \in X$ a representative, and ϕ as above. We have the following complex of K -representations*

$$0 \rightarrow \mathfrak{g} \rightarrow T_{\tilde{x}}X \xrightarrow{d\mu_G \oplus df} \mathfrak{g} \oplus E \rightarrow 0, \quad (2.28)$$

and $T_xV(f)$ is isomorphic as a K -representation to the cohomology of this complex.

Proof. The argument is identical to that of the previous proposition, with $d\mu_G \oplus df$ in place of $d\mu_G$. \square

Our basic toolkit is nearly complete. However, since we only care about the weights and multiplicities of isotropy representations of maximal tori, it is most convenient to work at the level of the representation ring³ of K . We summarize the preceding arguments in the following theorem.

Theorem 2.2.10. *In the representation ring of K , we have the following equality*

$$[T_x\mathfrak{X}] = [T_{\tilde{x}}X] - [\mathfrak{g}_{\mathbf{C}}]. \quad (2.29)$$

If $f : X \rightarrow E$ is an H -equivariant map to a linear representation of H , then we also have

$$[T_xV(f)] = [T_{\tilde{x}}X] - [\mathfrak{g}_{\mathbf{C}}] - [E] \quad (2.30)$$

²In all cases that we consider, \mathfrak{X} will be a quasiprojective variety and $V(f)$ will in fact be an algebraic subvariety.

³Recall that the representation ring of a group is the ring consisting of formal \mathbf{Z} -linear combinations of isomorphism classes of representations, with addition and multiplication given by direct sum and tensor product, respectively. For a torus T of rank r , we have $\text{Rep}(T) \cong \mathbf{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$.

2.2.3 Equivariant Morse Theory

We now recall some of the basic results of equivariant Morse theory as developed by Atiyah-Bott [AB82] and Kirwan [Kir84]. Let M be a symplectic manifold with a Hamiltonian action of a compact group G , and assume that 0 is a regular value of the moment map.⁴ Let $T \subset G$ be a maximal torus, and let μ_T denote the restriction of the moment map. For $\beta \in \mathfrak{t}^*$, we define

$$T_\beta := \overline{\{\exp(t\beta) \mid t \in \mathbf{R}\}}, \quad (2.31)$$

$$Z_\beta := \mu^{-1}(\beta), \quad (2.32)$$

$$C_\beta := G \cdot (Z_\beta \cap M^{T_\beta}), \quad (2.33)$$

$$G_\beta := \text{Stab}_G(\beta). \quad (2.34)$$

Let \mathfrak{t}_+ be a positive Weyl chamber and let \mathcal{B} be the collection of $\beta \in \mathfrak{t}_+$ such that C_β is non-empty. Kirwan calls these *minimal combinations of weights*.

Proposition 2.2.11. *The critical set of $|\mu|^2$ is equal to the union $\bigcup_{\beta \in \mathcal{B}} C_\beta$.*

Theorem 2.2.12. *The function $|\mu|^2$ is Morse-Bott-Kirwan.*

Proposition 2.2.13. *Let $x \in Z_\beta \cap M^{T_\beta}$. Then the negative normal space to $Z_\beta \cap M^{T_\beta}$ at x is isomorphic to $N_x^-(M^{T_\beta})/T_x(G \cdot x)$. In particular, the Morse index at x is equal to*

$$\lambda_\beta = \lambda(\mu^\beta; C_\beta) - \dim G + \dim G_\beta. \quad (2.35)$$

We would like to do Morse theory with $|\mu|^2$, and to avoid analytic subtleties one usually assumes that M is compact or that μ is proper. However, we will need to work in a more general setting. As remarked in [Kir84, §9], it is enough to assume that the path of steepest descent through any point of M is contained in a compact subset. This notion is extremely useful, so we adopt the following definition.

Definition 2.2.14. A function $f : M \rightarrow \mathbf{R}$ is said to be *flow-closed* if every positive time trajectory of $-\nabla f$ is contained in a compact set.

For each component C_β of the critical set, denote by M_\pm^β sub-level sets

$$M_\pm^\beta := \{x \in M \mid |\mu|^2 \leq |\beta|^2 \pm \epsilon\} \quad (2.36)$$

⁴This assumes that we wish to study the symplectic reduction at moment map level 0. We can always assume that this is the case, by making the replacement $\mu \mapsto \mu - \alpha$.

where ϵ is taken sufficiently small so that the interval $[|\beta|^2 - \epsilon, |\beta|^2 + \epsilon]$ contains no critical values other than $|\beta|^2$.

Theorem 2.2.15. *If $|\mu|^2$ is flow-closed, then the stable manifolds S_β of its gradient flow form a smooth stratification of M . The Thom-Gysin sequence splits into short exact sequences*

$$0 \rightarrow H_G^{*-\lambda_\beta}(C_\beta) \rightarrow H_G^*(M_+^\beta) \rightarrow H_G^*(M_-^\beta) \rightarrow 0 \quad (2.37)$$

for each $\beta \in \mathcal{B}$. For any $\beta \in \mathcal{B}$, the restriction $H_G^*(M) \rightarrow H_G^*(M_\pm^\beta)$ is surjective. In particular, the Kirwan map is surjective, and its kernel is generated by those elements of $f \in H_G^*(M)$ such that for some β , the restriction of f to C_β is in the ideal generated by the equivariant Euler class of the negative normal bundle to C_β .

Corollary 2.2.16. *We have the following equality of Poincaré series*

$$P_t^G(M) = \sum_{\beta} t^{\lambda_\beta} P_t^G(C_\beta), \quad (2.38)$$

or equivalently,

$$P_t(M//G) = P_t^G(M) - \sum_{\beta \neq 0} t^{\lambda_\beta} P_t^G(C_\beta). \quad (2.39)$$

By the above theorems, we can understand the cohomology of the quotient in terms of the G -equivariant cohomology of the connected components of the critical set. Luckily, we have the following.

Proposition 2.2.17. *Let $\beta \in \mathcal{B}$. Then we have $H_G^*(C_\beta) \cong H_{G_\beta}^*(Z_\beta \cap M^{T_\beta})$.*

This proposition yields an inductive procedure as follows. By standard results in Hamiltonian group actions, the connected components of M^{T_β} are symplectic submanifolds. The group $G_\beta = \text{Stab}(\beta)$ acts on M^{T_β} with moment map given by the restriction of μ . Let $H_\beta \subseteq G_\beta$ be the kernel of the action map $G_\beta \rightarrow \text{Diff}(M^{T_\beta})$, and let $K_\beta = G_\beta/H_\beta$.

Proposition 2.2.18. *Suppose that K_β acts locally freely on $Z_\beta \cap M^{T_\beta}$. Then we have*

$$H_{G_\beta}^*(Z_\beta \cap M^{T_\beta}) \cong H^*(BH_\beta) \otimes H^*(M^{T_\beta}/K_\beta). \quad (2.40)$$

Proof. Simply compute:

$$H_{G_\beta}^*(Z_\beta \cap M^{T_\beta}) = H^*(EG_\beta \times_{G_\beta} (Z_\beta \cap M^{T_\beta})) \quad (2.41)$$

$$= H^*((EG_\beta/H_\beta) \times (Z_\beta \cap M^{T_\beta})/K_\beta) \quad (2.42)$$

$$= H^*((EG_\beta/H_\beta) \times ((Z_\beta \cap M^{T_\beta})/K_\beta)) \quad (2.43)$$

$$\cong H^*(BH_\beta) \otimes H^*((Z_\beta \cap M^{T_\beta})/K_\beta) \quad (2.44)$$

$$= H^*(BH_\beta) \otimes H^*(M^{T_\beta}/K_\beta). \quad (2.45)$$

□

Remark 2.2.19. There is a well-known equivalence between symplectic reduction on Kähler varieties and geometric invariant theory [MFK94, Chapter 8]. The fundamental result is that the Morse stratification with respect to $|\mu|^2$ agrees with the algebraic Kirwan-Ness stratification. In particular, the stable manifold of $\mu^{-1}(0)$ consists precisely of the semistable points, as defined by GIT. We will tend to take the symplectic point of view, but we will occasionally take advantage of the algebro-geometric description when it is convenient to do so.

2.2.4 Examples

Below we present some standard examples to illustrate Morse theory with moment maps, and in particular to elucidate our method of computing Morse indices. In Section 3.3 we present a very involved calculation which assumes familiarity with these techniques.

Example 2.2.20. Complex projective space \mathbf{P}^n may be identified with the symplectic quotient \mathbf{C}^{n+1}/S^1 . We use homogeneous coordinates $[z_0 : \cdots : z_n]$ on \mathbf{P}^n . Define an S^1 action on \mathbf{C}^{n+1} by

$$s \cdot (z_0, \dots, z_n) = (z_0, sz_1, s^2z_2, \dots, s^nz_n). \quad (2.46)$$

This induces an S^1 action on \mathbf{P}^n . Suppose that $[z_0 : \cdots : z_n]$ is fixed. Then according to Proposition 2.2.6, there is a map $\phi : S^1 \rightarrow S^1$ such that $s * z = \phi(s) \cdot z$ for all $s \in S^1$. Since S^1 is abelian, ϕ must be a genuine homomorphism, and hence is of the form $s \mapsto s^k$ for some k . The fixed-point set is non-empty only if $k \in \{0, \dots, n\}$. Given such a ϕ , the twisted action is given by

$$s * (z_0, \dots, z_n) = (s^{-k}z_0, s^{1-k}z_1, \dots, s^{n-k}z_n). \quad (2.47)$$

Hence the fixed-points are isolated and of the form $[0 : \cdots : 0 : 1 : 0 : \cdots : 0]$. According to Theorem 2.2.10, the isotropy representation then is given by

$$[T_x \mathbf{P}^n] = \sum_{i=0}^n s^{i-k} - 1, \quad (2.48)$$

hence the Morse index is $\lambda_k = 2\#\{i \mid i - k < 0\} = k$. Hence

$$P_t(\mathbf{P}^n) = \sum_{k=0}^n t^{2k} = 1 + t^2 + \cdots + t^{2n}. \quad (2.49)$$

Example 2.2.21. The hyperkähler ALE spaces may be constructed as hyperkähler quotients [Kro89]. We consider the type A_n ALE space \mathfrak{M} , which is the hyperkähler quotient of $T^*\mathbf{C}^{n+1}$ by $(S^1)^n$, where the action of $(S^1)^n$ on \mathbf{C}^{n+1} is given by

$$(s_1, \dots, s_n) \cdot (x_1, \dots, x_n, x_{n+1}) = (s_1 x_1, \dots, s_n x_n, (s_1 \cdots s_n)^{-1} x_{n+1}). \quad (2.50)$$

The moment map equations are

$$|x_j|^2 + |y_{n+1}|^2 = \alpha_j + |y_j|^2 + |x_{n+1}|^2, \quad (2.51)$$

$$x_j y_j - x_{n+1} y_{n+1} = 0, \quad (2.52)$$

for $j = 1, \dots, n$. Let us take $0 < \alpha_1 < \cdots < \alpha_n$. The S^1 -action on \mathfrak{M} is given by $s \cdot (x, y) = (sx, sy)$. As in the previous example, a point (x, y) represented by (\tilde{x}, \tilde{y}) is fixed if and only if there is a homomorphism $\phi : S^1 \rightarrow (S^1)^n$ such that

$$\phi(s)^{-1} s \cdot (\tilde{x}, \tilde{y}) = (\tilde{x}, \tilde{y}) \quad (2.53)$$

for all $s \in S^1$. Such a homomorphism is necessarily of the form $\phi(s) = (s^{w_1}, \dots, s^{w_n})$, and this must solve $(s\tilde{x}, s\tilde{y}) = \phi(s) \cdot (\tilde{x}, \tilde{y})$ which gives the system of equations

$$sx_j = s^{w_j} x_j, \quad j = 0, \dots, n \quad (2.54)$$

$$sy_j = s^{-w_j} y_j, \quad j = 0, \dots, n \quad (2.55)$$

$$sx_{n+1} = s^{-w_1 - \cdots - w_n} x_{n+1}, \quad (2.56)$$

$$sy_{n+1} = s^{w_1 + \cdots + w_n} y_{n+1}. \quad (2.57)$$

There are two possible cases: (i) $y_{n+1} = 0$ and (ii) $y_{n+1} \neq 0$. In the first case, the moment map equations force that $x_j \neq 0$ for $j = 1, \dots, n$, so that $w_1 = \dots = w_n = 1$. Then we must have $y_j = 0$ for $j = 1, \dots, n$ as well as $x_{n+1} = 0$. Hence this corresponds to a unique fixed point. The isotropy representation is given by $s^{n-1} + s^{1-n}$, and hence the Morse index is equal to 2 if $n > 1$ and 0 if $n = 1$.

In the second case, since $y_{n+1} \neq 0$ we have $\sum w_i = 1$, and hence $x_{n+1} = 0$. For each j , the complex moment map equations force at least one of x_j, y_j to be zero. Subtracting

the real moment map equation for index k from that of index j gives

$$|x_j|^2 + |y_k|^2 = \alpha_j - \alpha_k + |y_j|^2 + |x_k|^2. \quad (2.58)$$

For $j > k$, at least one of x_j or y_k must be non-zero. Let ℓ be the smallest index such that $y_\ell = 0$. Then $x_{\ell+1} \neq 0$, and therefore $y_{\ell+1} = 0$, hence $x_{\ell+2} \neq 0$, and so on. Hence we find $x_j = 0, y_j \neq 0, j < \ell, x_j \neq 0, y_j = 0, j \geq \ell, y_\ell = 0$. Now in order to have $\sum w_i = 1$, we have to choose $w_\ell = 2\ell - n$. Hence the fixed-points are parameterized by the integer $\ell \in \{1, \dots, n\}$, and the isotropy representation is $s^{1+n-2\ell} + s^{1-n+2\ell}$. Hence the Morse index is 0 if $\ell \in [(n-1)/2, (n+1)/2]$ and 2 otherwise. Combining this with the previous result, we find that the Poincaré polynomial of the A_n ALE space is

$$P_t(\mathfrak{M}) = 1 + nt^2. \quad (2.59)$$

Example 2.2.22. In this case we use equivariant rather than ordinary Morse theory. The Grassmannian $\text{Gr}(2, n)$ may be constructed as a symplectic quotient of the space of $2 \times n$ matrices by the left multiplication by $U(2)$, i.e. $\text{Gr}(2, n) \cong \mathbf{C}^{2n} // U(2)$, where the reduction is taken at the matrix $\text{diag}(\alpha, \alpha)$. Up to the action of the α Weyl group, the possible values β that can occur as minimal combinations of weights are $\beta = \text{diag}(0, 0), \text{diag}(0, -\alpha), \text{diag}(-\alpha, -\alpha)$. The critical set corresponding to $\beta = \text{diag}(0, 0)$ is $\mu^{-1}(0)$, i.e. the set whose $(U(2)$ -equivariant) cohomology we would like to compute, so there is nothing to be done. The critical set corresponding to $\beta = \text{diag}(-\alpha, -\alpha)$ is simply the origin in \mathbf{C}^{2n} . The negative normal bundle is equal to all of \mathbf{C}^n . In particular, the Morse index is equal to $4n$.

The non-trivial critical set is the one corresponding to $\beta = \text{diag}(0, -\alpha)$. We have $G_\beta \cong U(1) \times U(1)$, with $T_\beta \cong \{1\} \times U(1)$. $(\mathbf{C}^{2n})^{T_\beta}$ consists of matrices of the form

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad (2.60)$$

and Z_β consists of those matrices of this form such that the norm-square of the top row is equal to α . Hence $Z_\beta/U(1) \times \{1\} \cong \mathbf{P}^{n-1}$. The index of μ^β is equal to $2n$, and subtracting the dimension of $U(2)/\text{Stab}(\beta)$ we find that the Morse index is equal to $2n - 2$. Hence the Poincaré polynomial of $\text{Gr}(2, n)$ is equal to

$$P_t(\text{Gr}(2, n)) = \frac{1 - t^{4n}}{(1 - t^2)(1 - t^4)} - \frac{t^{2n-2}(1 + t^{2n})}{(1 - t^2)^2}. \quad (2.61)$$

2.3 Noncompact Quotients

2.3.1 Complex and Hyperkähler Moment Maps

One might expect to be able to develop an analogous Morse theory for $|\mu_{\text{HK}}|^2$. However, this appears not to be the case, except when G is a torus [Kir85]. The problem is that in the nonabelian situation, the norm of the gradient of $|\mu_{\text{HK}}|^2$ contains a term proportional to the structure constants of the group that is difficult to understand (see Remark 2.3.21). In [JKK09], it was found instead that the function $|\mu_c|^2$ is better behaved, owing to the fact that μ_c is I -holomorphic.

Theorem 2.3.1 (Jeffrey-Kiem-Kirwan [JKK09], Kirwan [Kir85]). *The function $f = |\mu_c|^2$ is minimally degenerate. If f is flow-closed, then the stable manifolds S_C form a K -invariant stratification of M and the equivariant Thom-Gysin sequence splits into short exact sequences. Consequently, the restriction $H_G^*(M) \rightarrow H_G^*(\mu_c^{-1}(0))$ is surjective. If G is a torus, the same conclusions hold for $|\mu_{\text{HK}}|^2$ provided that it is flow-closed.*

Note that $\mu_c^{-1}(0)$ is a G -invariant complex submanifold of M (provided 0 is regular), so that $M//G \cong \mu_c^{-1}(0)//G$. Hence surjectivity of map $H_G^*(\mu_c^{-1}(0)) \rightarrow H^*(M//G)$ can be studied by the usual methods (but see Corollary 2.3.17). In principle, this reduces the question of Kirwan surjectivity for hyperkähler quotients to the following conjecture.

Conjecture 2.3.2 ([JKK09]). *If μ_c is a complex moment map associated to a linear action by a compact group G on a vector space, then $|\mu_c|^2$ is flow-closed.*

This was proved for the special case of S^1 actions in [JKK09], but the method of proof does not admit any obvious generalization. We will prove the following.

Theorem 2.3.3. *Conjecture 2.3.2 is true when G is a torus.*

This is an immediate consequence of Proposition 2.3.7 and Theorem 2.3.11. To appreciate why this result requires some effort, let us contrast it with the analogous statement for $|\mu|^2$, where μ is an ordinary moment map associated to a linear action on a Hermitian vector space V . It is immediate from the definitions that $\nabla|\mu|^2 = 2Iv_\mu$, where v_μ denotes the fundamental vector field of μ . Hence, the gradient trajectory through a point x always remains in the $G_{\mathbb{C}}$ -orbit through x . Thus it suffices to restrict attention to the $G_{\mathbb{C}}$ -orbits. Flow-closedness is then a consequence of the following proposition, which follows easily from the KAK decomposition of reductive Lie groups [Kna02].

Proposition 2.3.4 (Sjamaar [Sja98, Lemma 4.10]). *Suppose $G_{\mathbb{C}}$ acts linearly on V , let $\{g_n\}$ be a sequence of points in $G_{\mathbb{C}}$, and let $\{x_n\}$ be a bounded sequence in V . Then the sequence $\{g_n x_n\}$ is bounded if $\{\mu(g_n x_n)\}$ is bounded.*

Now consider the gradient flow of $|\mu_c|^2$ on T^*V . Since $|\mu_c|^2 = |\mu_J|^2 + |\mu_K|^2$, we see that $\nabla|\mu_c|^2 = 2Jv_{\mu_J} + 2Kv_{\mu_K}$. The problem is immediate: due to the simultaneous appearance of J and K , the gradient trajectories appear to lie on the orbits of a “quaternification” of G , but in general no suitable quaternification of G exists. More precisely, if we let \mathcal{D} denote the distribution on V generated by the G action (i.e. the integrable distribution whose leaves are the G -orbits), then $\mathcal{D}_{\mathbf{C}} := \mathcal{D} + I\mathcal{D}$ is integrable, whereas the distribution $\mathcal{D}_{\mathbf{H}} := \mathcal{D} + I\mathcal{D} + J\mathcal{D} + K\mathcal{D}$ on T^*V is not integrable in general.

In a few specific examples, a detailed study of $\mathcal{D}_{\mathbf{H}}$ leads to definite conclusions about the gradient flow, but at present we cannot prove a general result in this direction. In any case, we will not pursue this approach in the present work (but see Remarks 2.3.21 and 2.3.22, which are related to this problem).

With these considerations in mind, it would be illuminating to have a proof of the flow-closedness of $|\mu|^2$ that relies neither on arguments involving $G_{\mathbf{C}}$ nor on properness, as such a proof might generalize to the hyperkähler setting. This is exactly the content the main result of this chapter, Theorem 2.3.11. The key idea is to relax the assumption of properness to the weaker condition of satisfying a certain gradient inequality.

In the case of ADHM spaces [AHD78], a gradient-like flow was studied by Boyer and Mann [BM93], following an idea of Taubes [Tau84] to perturb $\nabla|\mu_{\mathbf{HK}}|^2$ in such a way that its flow is better behaved analytically. However, this approach does not seem to be applicable to reduction at non-zero values of the moment map. The idea of using gradient estimates to study the flow of $\nabla|\mu_{\mathbf{HK}}|^2$ directly was inspired in part by [Wil08].

2.3.2 Łojasiewicz Inequalities

We begin by giving a precise definition of the type of inequality we wish to consider, as well as its most important consequence.

Definition 2.3.5. Let (M, g) be a complete Riemannian manifold. A smooth real-valued function f on M is said to satisfy a *global Łojasiewicz inequality* if for any real number f_c in the closure of the image of f , there exist constants $\epsilon > 0$, $k > 0$, and $0 < \alpha < 1$, such that

$$|\nabla f(x)| \geq k|f(x) - f_c|^\alpha, \quad (2.62)$$

for all $x \in M$ such that $|f(x) - f_c| < \epsilon$.

Remark 2.3.6. The term global Łojasiewicz inequality is borrowed from [JKS92]; however, we use it in a different way, as we are concerned specifically with bounding the gradient of f .

Proposition 2.3.7. *Suppose f satisfies a global Lojasiewicz inequality and is bounded below. Then f is flow-closed.*

Proof. Let $x(t)$ be a trajectory of $-\nabla f$. Since $f(x(t))$ is decreasing and bounded below, $\lim_{t \rightarrow \infty} f(x(t))$ exists. Call this limit f_c . Let ϵ, k , and α be the constants appearing in the global Lojasiewicz inequality for f with limit f_c . For large enough T , we have $|f(x(t)) - f_c| < \epsilon$ whenever $t > T$. Consider $t_2 > t_1 > T$, and let $f_1 = f(x(t_1))$ and $f_2 = f(x(t_2))$. Since $\dot{x} = -\nabla f$, we have that

$$d(x(t_1), x(t_2)) \leq \int_{t_1}^{t_2} |\nabla f(x(t))| dt, \quad (2.63)$$

where $d(x, y)$ denotes the Riemannian distance. By the change of variables $t \mapsto f(x(t))$, we obtain

$$d(x(t_1), x(t_2)) \leq \int_{f_2}^{f_1} |\nabla f|^{-1} df \quad (2.64)$$

$$\leq k^{-1} \int_{f_2}^{f_1} |f - f_c|^{-\alpha} df \quad (2.65)$$

$$= k^{-1} (1 - \alpha)^{-1} (|f_1 - f_c|^{1-\alpha} - |f_2 - f_c|^{1-\alpha}) \quad (2.66)$$

$$< k^{-1} (1 - \alpha)^{-1} |f(x(T)) - f_c|^{1-\alpha}. \quad (2.67)$$

Since $\alpha < 1$, the last expression can be made arbitrarily small by taking T sufficiently large, so we see that $\lim_{t \rightarrow \infty} x(t)$ exists, and in particular the gradient trajectory is contained in a compact set. \square

This argument establishes flow-closedness directly from the Lojasiewicz inequality, without appealing to compactness. Thus it would be sufficient to show that $|\mu|^2$ satisfies such an inequality. To motivate why we might expect this to be case, we recall the classical Lojasiewicz inequality.

Theorem 2.3.8 (Lojasiewicz Inequality [BM88, Loj64]). *Let f be a real analytic function on a domain in \mathbf{R}^N , and let c be a critical point of f . Then there is an open neighbourhood U of c such that for any compact subset $K \subset U$, there are constants $k > 0$ and $0 < \alpha < 1$ such that the inequality*

$$|\nabla f(x)| \geq k |f(x) - f(c)|^\alpha \quad (2.68)$$

holds for all $x \in K$.

If f is a *proper* real analytic function, then this immediately implies that f satisfies

a global Lojasiewicz inequality as defined above. Since our primary concern is Morse theory, we can relax the assumption of analyticity as follows.

Proposition 2.3.9. *Suppose f is a proper Morse function. Then f satisfies a global Lojasiewicz inequality.*

Proof. By the Morse lemma, near each critical point we can choose coordinates in which f is real analytic. Hence f satisfies the classical Lojasiewicz inequality near each critical point, and since f is proper this can be extended to a global inequality. \square

For a moment map μ , the function $|\mu|^2$ is in general neither Morse nor Morse-Bott, but minimally degenerate (in the sense of [Kir84]). Nonetheless, we can still obtain a global Lojasiewicz inequality whenever μ is proper.

Proposition 2.3.10. *Suppose μ is a moment map associated to an action of a compact Lie group, and suppose furthermore that μ is proper. Then $|\mu|^2$ satisfies a global Lojasiewicz inequality.*

Proof. Lerman [Ler05] proves that the moment map is locally real analytic with respect to the coordinates induced by the local normal form, and uses this fact to show that $|\mu|^2$ satisfies the classical Lojasiewicz inequality. Since μ is proper, this can be extended to a global inequality. \square

Since we would like to drop the assumption of properness, it is natural to ask whether there are examples of moment maps which are *not* proper but nevertheless satisfy a global Lojasiewicz inequality. The answer to this question is in the affirmative, at least when the action is linear. The following is the main theorem of this chapter, which we prove in Section 2.3.3.

Theorem 2.3.11. *Let μ be a moment map associated to a unitary representation of a compact group G on a Hermitian vector space V . Then $f = |\mu|^2$ satisfies a global Lojasiewicz inequality. In detail, for every $f_c \geq 0$, there exist constants $k > 0$ and $\epsilon > 0$ such that*

$$|\nabla f(x)| \geq k|f(x) - f_c|^{\frac{3}{4}} \tag{2.69}$$

*whenever $|f(x) - f_c| < \epsilon$. If G is a torus, then the same holds for the functions $|\mu_c|^2$ and $|\mu_{\text{HK}}|^2$ associated to the action of G on T^*V .*

Remark 2.3.12. This theorem is a generalization of [Nee85, Theorem A.1]. However, in [Nee85], it is assumed that the constant term in the moment map is chosen so that f is

homogeneous (see equation (2.85)), leading to an inequality of the form

$$|\nabla f| \geq kf^{\frac{3}{4}}, \quad (2.70)$$

which holds on all of V . Both sides are homogeneous of the same degree, so that it suffices to prove the inequality on the unit sphere, allowing the use of compactness arguments. In Theorem 2.3.11, we make no such assumption, and this complicates several steps of the proof.

Remark 2.3.13. If $X \subset V$ is a G -invariant subvariety, then it is easy to see that Theorem 2.3.11 implies that the restriction of f to X satisfies a global Lojasiewicz inequality. Similarly, if $X \subset T^*V$ is a G -invariant hyperkähler subvariety, we deduce the inequality for the restrictions of $|\mu_c|^2$ and $|\mu_{\text{HK}}|^2$.

Remark 2.3.14. In [DG05] and [Kur98] it is shown that certain classes of functions satisfy similar global Lojasiewicz inequalities; indeed this was the motivation to consider such inequalities. However, these general theorems cannot rule out the possibility $\alpha \geq 1$, which is not sharp enough to prove the boundedness of all gradient trajectories. In this sense, the real content of Theorem 2.3.11 is the bound on the exponent.

In what follows, let G be a compact group acting unitarily on a Hermitian vector space V , with moment map μ . Note that since the action is linear, we have $H_G^*(V) = H_G^*(\text{point}) =: H_G^*$. For $\alpha_{\mathbf{R}} \in \mathfrak{g}^*$ and $\alpha_{\mathbf{C}} \in \mathfrak{g}_{\mathbf{C}}^*$, we denote

$$\mathfrak{X}(\alpha_{\mathbf{R}}) := V //_{\alpha_{\mathbf{R}}} G, \quad (2.71)$$

$$\mathfrak{M}(\alpha_{\mathbf{R}}, \alpha_{\mathbf{C}}) := T^*V //_{(\alpha_{\mathbf{R}}, \alpha_{\mathbf{C}})} G. \quad (2.72)$$

Corollary 2.3.15. *Let $f = |\mu|^2$. Then for each component C of the critical set of f , the gradient flow defines a G -equivariant homotopy equivalence from the stable manifold S_C to the critical set C . If G is abelian, then the same holds for the functions $|\mu_c|^2$ and $|\mu_{\text{HK}}|^2$.*

Remark 2.3.16. If μ is assumed to be proper then this is a special case of the main theorem of [Ler05]. However, the essential ingredient of the proof is the Lojasiewicz inequality. We give the proof below for completeness, but the details do not differ significantly from [Ler05].

Proof. We define a continuous map $F : S_C \times [0, \infty) \rightarrow S_C$ by $(x, t) \mapsto x(t)$, where $x(t)$ is the trajectory of $-\nabla f$ beginning at x , evaluated at time t . By Proposition 2.3.7, we

can extend this to a map $F : S_C \times [0, \infty] \rightarrow S_C$ by $(x, \infty) \mapsto \lim_{t \rightarrow \infty} x(t)$. This map is the identity when restricted to C , and maps $S_C \times \{\infty\}$ to C . We must verify that this extended map is continuous.

We must show that for any $x_0 \in S_C$ and any sequence $\{(x_n, t_n)\}$ of points in $S_C \times [0, \infty]$ satisfying $\lim_{n \rightarrow \infty} x_n = x_0 \in S_C$ and $\lim_{n \rightarrow \infty} t_n = \infty$ that $\lim_{n \rightarrow \infty} x_n(t_n)$ exists. We will show that it is equal to $x_c := \lim_{t \rightarrow \infty} x_0(t)$. Let f_c be the value of f on C and let ϵ, k be the constants appearing in Theorem 2.3.11. Given such a sequence, let $\eta > 0$ and assume $\eta < \epsilon$ but is otherwise arbitrary; let $T > 0$ be chosen large enough so that $|f(x_0(t)) - f_c| < \eta$ for $t > T$; and let $N > 0$ be chosen so that $t_n > T$ for $n > N$. The map $S_C \rightarrow S_C$ given by $x \mapsto x(T)$ is continuous, so we can find $\delta > 0$ such that

$$|x - x_0| < \delta \implies |x(T) - x_0(T)| < \eta. \quad (2.73)$$

Since $x \mapsto f(x(T))$ is continuous, we can shrink δ if necessary so that

$$|x - x_0| < \delta \implies |f(x(T)) - f(x_0(T))| < \eta. \quad (2.74)$$

Choose N larger if necessary such that $|x_n - x_0| < \delta$ for $n > N$. We would like to show that $|x_n(t_n) - x_c| \rightarrow 0$ as $n \rightarrow \infty$. For $n > N$ we have

$$|x_n(t_n) - x_c| = |x_n(t_n) - x_n(T) + x_n(T) - x_0(T) + x_0(T) - x_c| \quad (2.75)$$

$$\leq |x_n(t_n) - x_n(T)| + |x_n(T) - x_0(T)| + |x_0(T) - x_c|. \quad (2.76)$$

By our choice of N , the second term is bounded by η , and we may apply the argument in the proof of Proposition 2.3.7 to show that the third term is bounded by $k^{-1}|f(x_0(T)) - f_c|^{\frac{1}{4}} < 4k^{-1}\eta^{\frac{1}{4}}$. Finally, to bound the first term we again apply the argument of Proposition 2.3.7 to obtain the bound

$$4k^{-1} \left(|f(x_n(T)) - f_c|^{\frac{1}{4}} - |f(x_n(t_n)) - f_c|^{\frac{1}{4}} \right) < 4k^{-1}|f(x_n(T)) - f_c|^{\frac{1}{4}}. \quad (2.77)$$

Since $|x_n - x_0| < \delta$, we have

$$|f(x_n(T)) - f_c| = |f(x_n(T)) - f(x_0(T)) + f(x_0(T)) - f_c| \quad (2.78)$$

$$\leq |f(x_n(T)) - f(x_0(T))| + |f(x_0(T)) - f_c| \quad (2.79)$$

$$< 2\eta. \quad (2.80)$$

Hence the first term is bounded by $4k^{-1}(2\eta)^{\frac{1}{4}} < 8k^{-1}\eta^{\frac{1}{4}}$, and we obtain

$$|x_n(t_n) - x_c| \leq 12k^{-1}\eta^{\frac{1}{4}} + \eta. \quad (2.81)$$

Since we may take η arbitrarily small, we see that $|x_n(t_n) - x_c| \rightarrow 0$. \square

Corollary 2.3.17. *If $\alpha_{\mathbf{C}} \in \mathfrak{g}_{\mathbf{C}}^*$ is regular central, then for any central $\alpha_{\mathbf{R}} \in \mathfrak{g}^*$, the set $\mu_r^{-1}(\alpha_{\mathbf{R}}) \cap \mu_c^{-1}(\alpha_{\mathbf{C}})$ is a G -equivariant deformation retract of $\mu_c^{-1}(\alpha_{\mathbf{C}})$, and in particular*

$$H_G^*(\mu_c^{-1}(\alpha_{\mathbf{C}})) \cong H^*(\mathfrak{M}(\alpha_{\mathbf{R}}, \alpha_{\mathbf{C}})). \quad (2.82)$$

Proof. Since $\alpha_{\mathbf{C}}$ is regular central, $\mu_c^{-1}(\alpha_{\mathbf{C}})$ is a G -invariant complex submanifold of T^*V , and furthermore G acts on $\mu_c^{-1}(\alpha_{\mathbf{C}})$ with at most discrete stabilizers. Hence the only component of the critical set of $|\mu_r - \alpha_{\mathbf{R}}|^2$ that intersects $\mu_c^{-1}(\alpha_{\mathbf{C}})$ is the absolute minimum, which occurs on $\mu_r^{-1}(\alpha_{\mathbf{R}})$. By Corollary 2.3.15, the gradient flow of $-|\mu_r - \alpha_{\mathbf{R}}|^2$ gives the desired G -equivariant deformation retract from $\mu_c^{-1}(\alpha_{\mathbf{C}})$ to $\mu_r^{-1}(\alpha_{\mathbf{R}}) \cap \mu_c^{-1}(\alpha_{\mathbf{C}})$. Thus

$$H_G^*(\mu_c^{-1}(\alpha_{\mathbf{C}})) \cong H_G^*(\mu_r^{-1}(\alpha_{\mathbf{R}}) \cap \mu_c^{-1}(\alpha_{\mathbf{C}})) \cong H^*(\mathfrak{M}(\alpha_{\mathbf{R}}, \alpha_{\mathbf{C}})). \quad (2.83)$$

\square

Corollary 2.3.18. *If α is a regular central value of μ , then the Kirwan map $H_G^* \rightarrow H^*(\mathfrak{X}(\alpha))$ is surjective. If G is a torus and $(\alpha_{\mathbf{R}}, \alpha_{\mathbf{C}})$ is a regular value of the hyperkähler moment map, then the hyperkähler Kirwan map $H_G^* \rightarrow H^*(\mathfrak{M}(\alpha_{\mathbf{R}}, \alpha_{\mathbf{C}}))$ is surjective.*

Proof. Theorem 2.3.11 and Proposition 2.3.7 show that $|\mu - \alpha|^2$ is flow-closed, so we obtain surjectivity of $H_G^* \rightarrow H^*(\mathfrak{X}(\alpha))$ by Theorem 2.2.15.

In the hyperkähler case, if G is a torus then $|\mu_c - \alpha_{\mathbf{C}}|^2$ is flow-closed, so by Theorem 2.3.1 we obtain surjectivity of $H_G^* \rightarrow H_G^*(\mu_c^{-1}(\alpha_{\mathbf{C}}))$. By Corollary 2.3.17, the map $H_G^*(\mu_c^{-1}(\alpha_{\mathbf{C}})) \rightarrow H^*(\mathfrak{M}(\alpha_{\mathbf{R}}, \alpha_{\mathbf{C}}))$ is an isomorphism, so we have surjectivity of $H_G^* \rightarrow H^*(\mathfrak{M}(\alpha_{\mathbf{R}}, \alpha_{\mathbf{C}}))$. \square

Remark 2.3.19. Konno proved surjectivity of the map $H_G^* \rightarrow H^*(\mathfrak{M}(\alpha_{\mathbf{R}}, \alpha_{\mathbf{C}}))$ when G is a torus using rather different arguments [Kon00a]. Konno also computed the kernel of the Kirwan map, giving an explicit description of the cohomology ring $H^*(\mathfrak{M}(\alpha_{\mathbf{R}}, \alpha_{\mathbf{C}}))$. We study this case in detail in Section 2.4, and we will see that Morse theory allows us to compute the kernel very easily.

2.3.3 Proof of the Łojasiewicz Inequality

Let G be a compact Lie group with Lie algebra \mathfrak{g} , and suppose G acts unitarily on a Hermitian vector space V . Without loss of generality we will regard G as a subgroup of $U(V)$ and identify \mathfrak{g} with a Lie subalgebra of $\mathfrak{u}(V)$, which we identify with the Lie algebra of skew-adjoint matrices. We will use the trace norm on $\mathfrak{u}(V)$ to induce an invariant inner product on \mathfrak{g} , and use this to identify \mathfrak{g} with its dual. For any $\xi \in \mathfrak{g}$ there is a fundamental vector field v_ξ which is given by $v_\xi(x) = \xi x$. We denote by $\text{stab}(x)$ the Lie algebra of the stabilizer of a point $x \in V$; i.e.

$$\text{stab}(x) = \{\xi \in \mathfrak{g} \mid v_\xi(x) = 0\}. \quad (2.84)$$

If we fix an orthonormal basis $\{e_a\}$ of \mathfrak{g} , then a moment map is given by

$$\mu(x) = \sum_a \frac{1}{2} \langle i e_a x, x \rangle - \alpha, \quad (2.85)$$

where $i = \sqrt{-1}$ is the complex structure on V and α is any central element of \mathfrak{g} . Then for $f = |\mu|^2$, we have

$$\nabla f(x) = 2i v_{\mu(x)}(x) = \sum_a 2i \mu^a(x) e_a x, \quad (2.86)$$

and since i is unitary, we have that

$$|\nabla f| = 2|v_\mu|. \quad (2.87)$$

Lemma 2.3.20. *Suppose G is abelian, and consider its action on T^*V . Let $f_i = |\mu_i|^2$ for $i = I, J, K$. Then $\langle \nabla f_i, \nabla f_j \rangle = 0$ for $i \neq j$.*

Proof. We compute:

$$\langle \nabla f_J, \nabla f_K \rangle = 4 \langle J v_{\mu_J}, K v_{\mu_K} \rangle \quad (2.88)$$

$$= 4 \langle I v_{\mu_J}, v_{\mu_K} \rangle \quad (2.89)$$

$$= 4 \omega_I(v_{\mu_J}, v_{\mu_K}) \quad (2.90)$$

$$= 4 \langle \mu_J, d\mu_I(v_{\mu_K}) \rangle \quad (2.91)$$

$$= 4 \langle \mu_J, [\mu_K, \mu_I] \rangle \quad (2.92)$$

$$= 0. \quad (2.93)$$

Similar computations show that the other two cross terms vanish. \square

Remark 2.3.21. The proof of this lemma makes it clear why the assumption that G is abelian is so useful in the hyperkähler setting. The cross term

$$\langle \nabla f_J, \nabla f_K \rangle = 4 \langle \mu_I, [\mu_J, \mu_K] \rangle \quad (2.94)$$

is exactly the obstruction to proving an estimate in the general nonabelian case. The function on the right hand side is very natural, and seems to be genuinely hyperkähler, having no analogue in symplectic geometry. Numerical experiments suggest that it is small in magnitude compared to $|\nabla f_J| + |\nabla f_K|$, but we do not know how to prove this. A theorem in this direction might be enough to prove flow-closedness (and hence Kirwan surjectivity) in general.

Remark 2.3.22. In light of the discussion following Proposition 2.3.4, the cross term $\langle \mu_I, [\mu_J, \mu_K] \rangle$ should have an interpretation in terms of the geometry of the non-integrable distribution $\mathcal{D}_{\mathbf{H}}$. Subriemannian geometry may have a key role to play in proving Kirwan surjectivity for hyperkähler quotients by nonabelian groups.

Proposition 2.3.23. *Let $T \subset G$ be a maximal torus of G , and let μ_G and μ_T be the corresponding moment maps. Let $f_G = |\mu_G|^2$ and $f_T = |\mu_T|^2$. Suppose that for any $f_c \geq 0$, f_T satisfies a global Lojasiewicz inequality. Then f_G satisfies a global Lojasiewicz inequality with the same constants and exponent.*

Proof. Since G is compact, for each $x \in V$ we can find some $g \in G$ so that $\text{Ad}_g \mu_G(x) \in \mathfrak{t}$. By equivariance of the moment map, we have $\text{Ad}_g \mu_G(x) = \mu_G(gx) \in \mathfrak{t}$. Hence $\mu_G(gx) = \mu_T(gx)$, so that $|v_{\mu_G(gx)}| = |v_{\mu_T(gx)}|$. Using equality (2.87), this tells us that $|\nabla f_G(gx)| = |\nabla f_T(gx)|$. Since $f_G(gx) = |\mu_G(gx)|^2 = |\mu_T(gx)|^2 = f_T(gx)$, we deduce the Lojasiewicz inequality for f_G from the inequality for f_T . \square

We assume for the remainder of this section that G is a torus.

Proposition 2.3.24. *Fix $f_c \geq 0$, and suppose that for each $\mu_c \in \mathfrak{g}$ satisfying $|\mu_c|^2 = f_c$, there exist constants $\epsilon' > 0$ and $c' > 0$ (depending on μ_c) such that*

$$|\nabla f(x)| \geq c' |f(x) - f_c|^{\frac{3}{4}} \quad (2.95)$$

whenever $|\mu(x) - \mu_c| < \epsilon'$. Then f satisfies a global Lojasiewicz inequality, i.e., there exist constants $\epsilon > 0$ and $c > 0$ so that the inequality (2.95) holds whenever $|f(x) - f_c| < \epsilon$.

Proof. Suppose that for each μ_c as above we can find constants $\epsilon(\mu_c)$ and $c(\mu_c)$ so that inequality (2.95) holds. Let $U(\mu_c)$ be the $\epsilon(\mu_c)$ -ball in \mathfrak{g} centered at μ_c . These open

sets cover the sphere S of radius $\sqrt{f_c}$ in \mathfrak{g} , and by compactness we can choose a finite subcover. Denote this finite subcover by $\{U_i\}_{i=1}^n$, with centers μ_i and constants c_i , and let $c = \min_i c_i$. The finite union $\cup_i U_i$ contains an ϵ -neighborhood of S for some sufficiently small ϵ . Since $f(x) = |\mu(x)|^2$, if we choose $\epsilon' > 0$ sufficiently small then $|f(x) - f_c| < \epsilon'$ implies that $||\mu(x)| - \sqrt{f_c}| < \epsilon$, so that $\mu(x) \in \cup_i U_i$. In particular, there is some j such that $\mu(x) \in U_j$, and by inequality (2.95) we have

$$|\nabla f(x)| \geq c_j |f(x) - f_c|^{\frac{3}{4}} \geq c |f(x) - f_c|^{\frac{3}{4}}, \quad (2.96)$$

as desired. \square

Before giving the proof of Theorem 2.3.11, we isolate some of the main steps in the following lemmas. Let us introduce the following notation. For $x \in V \setminus \{0\}$, let \hat{x} denote its projection to the unit sphere, i.e. $\hat{x} = x/|x|$ or equivalently $x = |x|\hat{x}$. Since the action is linear, we have that $v_\xi(x) = |x|v_\xi(\hat{x})$ and $\text{stab}(x) = \text{stab}(\hat{x})$.

Lemma 2.3.25. *Fix \hat{y} in the unit sphere in V . Let P be the orthogonal projection from \mathfrak{g} to $\text{stab}(\hat{y})^\perp$, and $Q = 1 - P$. Then there is a neighborhood U of \hat{y} such that for any $\xi \in \mathfrak{g}$, inequalities*

$$|v_\xi(x)| \geq c|x||P\xi| \quad (2.97)$$

$$|v_\xi(x)| \geq c'|v_{P\xi}(x)| \quad (2.98)$$

$$|v_\xi(x)| \geq c''(|v_{P\xi}(x)| + |v_{Q\xi}(x)|) \quad (2.99)$$

hold for all x such that $\hat{x} \in U$. The constants c, c', c'' are positive and depend only on \hat{y} and U but not on x or ξ .

Remark 2.3.26. A version of this lemma appears as part of the proof of [Nee85, Theorem A.1], though it is not stated exactly as above. We repeat the argument below so that our proof of Theorem 2.3.11 is self-contained.

Proof. Fix \hat{y} and let P and Q be as above. Let W be the smallest G -invariant subspace of V containing \hat{y} , and let $P_W : V \rightarrow W$ be the orthogonal projection. Note that W is generated by vectors of the form $\xi_1 \cdots \xi_l \hat{y}$, with $\xi_i \in \mathfrak{g}_{\mathbf{C}}$. Since P_W is a projection, $|v_\xi| \geq |P_W v_\xi|$, so to establish inequality (2.97) it suffices to show that $|P_W v_\xi| \geq c|P\xi|$. Note that P_W is equivariant, i.e. $\xi P_W = P_W \xi$ for all $\xi \in \mathfrak{g}$. Note also that since K is abelian, if $\xi \in \text{stab}(\hat{y})$ then $\xi \in \text{ann}(W)$, since $\xi \xi_1 \cdots \xi_l \hat{y} = \xi_1 \cdots \xi_l \xi \hat{y} = 0$. For any orthonormal basis $\{e_a\}_{i=1}^d$ of \mathfrak{g} chosen so that $\{e_a\}_{i=1}^n$ is an orthonormal basis of $\text{stab}(\hat{y})^\perp$

and $\{e_a\}_{i=n+1}^d$ is an orthonormal basis of $\text{stab}(\hat{y})$, we have $P\xi = \sum_{a=1}^d \xi^a P e_a = \sum_{a=1}^n \xi^a e_a$. Similarly, we find

$$P_W v_\xi(x) = \sum_{a=1}^d P_W \xi^a e_a x = \sum_{a=1}^d \xi^a e_a P_W x = \sum_{a=1}^n \xi^a e_a P_W x = P_W v_{P\xi}(x). \quad (2.100)$$

Taking norms, we see that

$$|P_W v_\xi(x)|^2 = \sum_{a=1}^n \sum_{b=1}^n \xi^a \xi^b \langle P_W e_a x, P_W e_b x \rangle = (P\xi)^T H(x) (P\xi), \quad (2.101)$$

where $H(x)$ is the matrix with entries $H_{ab}(x) = \langle P_W e_a x, P_W e_b x \rangle$ for $a, b = 1, \dots, n$. By construction, this matrix is positive definite at \hat{y} , so for a sufficiently small neighborhood U of \hat{y} , we obtain $|P_W v_\xi(\hat{x})|^2 \geq c |P\xi|^2$, with the positive constant c depending only on \hat{y} and the choice of neighborhood U . For any x with $\hat{x} \in U$, we obtain $|v_\xi(x)| = |x| |v_\xi(\hat{x})| \geq c |x| |P\xi|$, which is inequality (2.97).

We can deduce inequality (2.98) from inequality (2.97) as follows. We have

$$|v_{P\xi}(\hat{x})| = \left| \sum_{a=1}^n \xi^a e_a \hat{x} \right| \leq |P\xi| \sum_{a=1}^n |e_a \hat{x}|. \quad (2.102)$$

Shrinking U if necessary, we can assume that the functions $|e_a \hat{x}|$ are bounded on U , and so we obtain $|v_{P\xi}(\hat{x})| \leq c' |P\xi| \leq c^{-\frac{1}{2}} c' |v_\xi(\hat{x})|$. Hence for any x with $\hat{x} \in U$ we have

$$|v_{P\xi}(x)| = |x| |v_{P\xi}(\hat{x})| \leq c^{-\frac{1}{2}} c' |x| |v_\xi(\hat{x})| = c^{-\frac{1}{2}} c' |v_\xi(x)|, \quad (2.103)$$

which is inequality (2.98).

To establish inequality (2.99), first note the following consequence of the triangle inequality. If v, w are vectors in some normed vector space, and $|v + w| \geq a|v|$ with $a > 0$, then we have

$$|v| + |w| = |v| + |v + w - v| \leq 2|v| + |v + w| \leq \left(1 + \frac{2}{a}\right) |v + w|. \quad (2.104)$$

Since $\xi = P\xi + Q\xi$, we have $v_\xi(x) = v_{P\xi}(x) + v_{Q\xi}(x)$, so applying inequality (2.98) together with the inequality (2.104) above, we obtain

$$|v_\xi(x)| \geq c''' (|v_{P\xi}(x)| + |v_{Q\xi}(x)|), \quad (2.105)$$

with the constant c''' depending only on \hat{y} and the neighborhood U . \square

Lemma 2.3.27. *Let $f = |\mu|^2$ and fix $\mu_c \in \mathfrak{g}$. Then there exist constants $c > 0$ and $\epsilon > 0$ (depending on μ_c) such that whenever $|\mu(x) - \mu_c| < \epsilon$, we have*

$$|x|^2|f(x)| \geq c|f(x) - f_c|^{\frac{3}{2}}. \quad (2.106)$$

Proof. Fix some particular μ_c . Recall that μ is quadratic in the coordinate x with no linear terms. Thus μ is affine in the coordinates $v_{ij} = x_i \bar{x}_j$, and we may write $\mu(x) = \phi(v)$, where $\phi(v) = Av - \alpha$, for some linear transformation $A : V \otimes V \rightarrow \mathfrak{g}$. We have $|v_{ij}| = |x_i||x_j| \leq \frac{1}{2}(|x_i|^2 + |x_j|^2)$, so that $|v| \leq \frac{1}{2} \sum_{i,j} |x_i|^2 + |x_j|^2 = N|x|^2$, where $N = \dim V$. Thus $|x|^2|f(x)| = |x|^2|\phi(v)|^2 \geq N^{-1}|v||\phi(v)|^2$, so it suffices to show that

$$|v||\phi(v)|^2 \geq c|\phi(v)|^2 - f_c^{\frac{3}{2}}, \quad (2.107)$$

whenever $|\phi(v) - \mu_c| < \epsilon$. This follows immediately from Lemma 2.3.28, which we state and prove below. \square

Lemma 2.3.28. *Let V_1 and V_2 be inner product spaces, and consider an affine map $\phi : V_1 \rightarrow V_2$ given by $\phi(v) = Av - \alpha$ for some linear map $A : V_1 \rightarrow V_2$ and constant $\alpha \in V_2$. Then for any ϕ_c in the image of ϕ , there exist constants $c > 0$ and $\epsilon > 0$ such that*

$$|v||\phi(v)|^2 \geq c|\phi(v)|^2 - |\phi_c|^2^{\frac{3}{2}} \quad (2.108)$$

whenever $|\phi(v) - \phi_c| < \epsilon$.

Proof. To avoid unnecessary clutter, we use the shorthand $v^2 := |v|^2$ below. First note that if P is a projection such that $AP = A$, then we have $\phi(v) = \phi(Pv)$, and so

$$|v||\phi(v)|^2 = |v||\phi(Pv)|^2 \geq |Pv||\phi(Pv)|^2. \quad (2.109)$$

Taking P to be the orthogonal projection from V_1 to $(\ker A)^\perp$, without loss of generality we can assume that A is injective. Similarly, without loss of generality we may assume that A is surjective. Suppose that $\phi_c \in V_2$ is fixed. Pick $v_c \in V_1$ so that $\phi(v_c) = \phi_c$. There are four possible cases.

Case 1: $v_c = 0, \phi_c = 0$. In this case, $\alpha = 0$, and $\phi(v) = Av$, so that $|\phi(v)| \leq |A||v|$. Thus $|v||\phi(v)|^2 \geq |A|^{-1}|\phi(v)|^3$, as desired. We may take ϵ to be any positive number, and $c = |A|^{-1}$.

Case 2: $v_c = 0, \phi_c \neq 0$. Take $\epsilon \leq |\phi_c|/2$. Then

$$|\phi(v)^2 - \phi_c^2| = |\langle \phi(v) - \phi_c, \phi(v) + \phi_c \rangle| \quad (2.110)$$

$$\leq |\phi(v) - \phi_c| |\phi(v) + \phi_c| \quad (2.111)$$

$$\leq |\phi(v) - \phi_c| (2|\phi_c| + \epsilon) \quad (2.112)$$

$$\leq \frac{5}{2} |\phi(v) - \phi_c| |\phi_c|, \quad (2.113)$$

so we have $|\phi(v)^2 - \phi_c^2| \leq c_1 |\phi(v) - \phi_c| |\phi_c|$, where c_1 is a numerical constant independent of ϕ_c . Since $|\phi(v) - \phi_c| < |\phi_c|/2$, we also have $|\phi(v)^2 - \phi_c^2| \leq c_2 |\phi_c|^2$. Combining these two inequalities, we have $|\phi(v)^2 - \phi_c^2|^{\frac{3}{2}} \leq c_3 |\phi(v) - \phi_c| |\phi_c|^2$. Since $\phi(v) - \phi_c = A(v - v_c) = Av$, we have $|\phi(v) - \phi_c| \leq |A||v|$. Putting this back into the previous inequality, we obtain $|\phi(v)^2 - \phi_c^2|^{\frac{3}{2}} \leq |A||v| |\phi_c|^2$. On the other hand, with our choice of ϵ we have $|\phi| \geq \frac{1}{2} |\phi_c|$, so that $|\phi(v)^2 - \phi_c^2|^{\frac{3}{2}} \leq 4|A||v| |\phi|^2$, as desired.

Case 3: $v_c \neq 0, \phi_c = 0$. Take $\epsilon = |A^{-1}|^{-1} |A|^{-1} |\alpha|/2$. Since in this case $Av_c = \alpha$, we have $\epsilon \leq |A^{-1}|^{-1} |v_c|/2$, and

$$|v_c| = |v - (v - v_c)| \quad (2.114)$$

$$\leq |v| + |v - v_c| \quad (2.115)$$

$$= |v| + |A^{-1}\phi(v)| \quad (2.116)$$

$$\leq |v| + |A^{-1}|\epsilon \quad (2.117)$$

$$\leq |v| + \frac{|v_c|}{2}. \quad (2.118)$$

Thus $|v| \geq |v_c|/2 \geq |A|^{-1} |\alpha|/2$. Then

$$|\phi(v)|^3 \leq \epsilon |\phi(v)|^2 = \frac{1}{2} |A^{-1}|^{-1} |A|^{-1} |\alpha| |\phi(v)|^2 \leq |A^{-1}|^{-1} |v| |\phi(v)|^2, \quad (2.119)$$

which is the desired inequality.

Case 4: $v_c \neq 0, \phi_c \neq 0$. Let ϵ' be chosen as in case (3), and let $\epsilon = \min\{\epsilon', |\phi_c|/2\}$. As in the previous cases, this choice of ϵ guarantees that $|v| \geq |v_c|/2$, and that $|\phi(v)| \geq |\phi_c|/2$. As before,

$$|\phi(v)^2 - \phi_c^2| \leq |\phi(v) - \phi_c| |\phi_v + \phi_c| \quad (2.120)$$

$$\leq \epsilon (2|\phi_c| + \epsilon) \quad (2.121)$$

$$\leq \frac{5}{4} |\phi_c|^2. \quad (2.122)$$

Similarly, since $|v - v_c| \leq |v_c|/2$ and $|\phi(v) - \phi_c| \leq |A||v - v_c|$, we also have

$$|\phi(v)^2 - \phi_c^2| \leq (3/4) |A| |v_c| |\phi_c|. \quad (2.123)$$

Putting these together, we have that

$$|\phi(v)^2 - \phi_c^2|^{\frac{3}{2}} \leq c|v_c||\phi_c|^2 \leq c'|v||\phi(v)|^2, \quad (2.124)$$

where c and c' are numerical constants independent of ϕ_c . \square

Lemmas 2.3.25 and 2.3.27 allow us to prove the following local estimates, which are essential in the proof of Theorem 2.3.11.

Proposition 2.3.29. *Let $\mu_c \in \mathfrak{g}$ be fixed and y is some point in V with discrete stabilizer. Then there exists an open neighborhood U of \hat{y} and constants $c > 0$ and $\epsilon > 0$ such that*

$$|\nabla f(x)|^2 \geq k|f(x) - f_c|^{\frac{3}{2}} \quad (2.125)$$

for all $x \in V \setminus \{0\}$ such that $\hat{x} \in U$ and $|\mu(x) - \mu_c| < \epsilon$.

Proof. Suppose $y \in V$ and $\text{stab}(y) = 0$. Then by Lemma 2.3.25, there is a neighborhood U of \hat{y} so that

$$|v_\xi(x)|^2 \geq c|x|^2|\xi|^2 \quad (2.126)$$

for all x such that $\hat{x} \in U$. Take $\xi = \mu(x)$ and apply Lemma 2.3.27 to find c' and ϵ so that

$$|v_{\mu(x)}(x)|^2 \geq c' \left| |\mu(x)|^2 - f_c \right|^{\frac{3}{2}}, \quad (2.127)$$

whenever $|\mu(x) - \mu_c| < \epsilon$. Since $|\nabla f(x)| = 2|v_{\mu(x)}(x)|$, this gives the desired inequality. \square

Proposition 2.3.30. *Let $y \in V$ and suppose $\text{stab}(y)$ is a proper nontrivial subspace of \mathfrak{g} . Then there are proper subtori G_1, G_2 of G such that $G \cong G_1 \times G_2$, a neighborhood U of \hat{y} , and a constant $c > 0$ such that*

$$|\nabla f(x)| \geq c(|\nabla f_{G_1}(x)| + |\nabla f_{G_2}(x)|) \quad (2.128)$$

for all $x \in V \setminus \{0\}$ with $\hat{x} \in U$, where $f_{G_1} = |\mu_{G_1}|^2$ and $f_{G_2} = |\mu_{G_2}|^2$.

Proof. Let $\mathfrak{g}_1 = \text{stab}(y)$ and $\mathfrak{g}_2 = \mathfrak{g}_1^\perp$. Since \mathfrak{g} is abelian, both \mathfrak{g}_1 and \mathfrak{g}_2 are Lie subalgebras, and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then $\mu_G = \mu_{G_1} \oplus \mu_{G_2}$, and $|\mu_G|^2 = |\mu_{G_1}|^2 + |\mu_{G_2}|^2$, so the result follows immediately from Lemma 2.3.25 and inequality (2.99). \square

Proof of Theorem 2.3.11. By Proposition 2.3.23 we will assume that G is a torus. By Proposition 2.3.24, it suffices to show that for each $\mu_c \in \mathfrak{g}$, there is some $\epsilon > 0$ so that inequality (2.69) holds when $|\mu(x) - \mu_c| < \epsilon$. Additionally, since 0 is always a critical

point of f , it suffices to prove the estimate only on $V \setminus \{0\}$. Furthermore, it suffices to show that each point \hat{y} of the unit sphere has a neighborhood U such that the estimate holds for all x with $\hat{x} \in U$, since by compactness we can choose finitely many such neighborhoods to cover the unit sphere, and this yields the inequality on $V \setminus \{0\}$.

We will prove the estimate by induction on the dimension of G . First suppose $\dim G = 1$. Then we may assume without loss of generality that G acts locally freely on $V \setminus \{0\}$, since otherwise the fundamental vector field $v_\xi(x)$ vanishes on a nontrivial subspace and we can restrict our attention to its orthogonal complement. Then Proposition 2.3.29 yields the desired neighborhoods and estimates.

Now assume that $\dim G > n$ and we have proved the estimate for tori of dimension $\leq n$. Without loss of generality, we can assume that there is no nonzero vector in V which is fixed by all of G (since we can restrict to its orthogonal complement). Let \hat{y} be some point in the unit sphere, and let $\mathfrak{g}_1 = \text{stab}(\hat{y})$. If $\mathfrak{g}_1 = 0$, we may apply Proposition 2.3.29 to get a neighborhood $U_{\hat{y}}$ and a constant $k_{\hat{y}}$ such that the estimate holds on $U_{\hat{y}}$. Otherwise, let $\mathfrak{g}_2 = \mathfrak{g}_1^\perp$ so that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, corresponding to subtori G_1 and G_2 . Then we may apply Proposition 2.3.30 to find a neighborhood $U_{\hat{y}}$ so that

$$|\nabla f(x)| \geq k (|\nabla f_{G_1}(x)| + |\nabla f_{G_2}(x)|), \quad (2.129)$$

holds for all x with $\hat{x} \in U_{\hat{y}}$. Let $P_i : \mathfrak{g} \rightarrow \mathfrak{g}_i$, $i = 1, 2$ be the orthogonal projections, $\mu_{c,i} = P_i \mu_c$, and $f_{c,i} = |\mu_{c,i}|^2$. Since μ_{G_i} are the moment maps for the action of G_i , which are tori of dimension $\leq n$, we may apply the induction hypothesis to find a neighborhood U of \hat{y} and constants $\epsilon > 0$, $c > 0$ so that

$$|\nabla f_{G_1}(x)| \geq k' |f_{G_1}(x) - f_{c,1}|^{\frac{3}{4}}, \quad (2.130)$$

$$|\nabla f_{G_2}(x)| \geq k' |f_{G_2}(x) - f_{c,2}|^{\frac{3}{4}}, \quad (2.131)$$

for all x such that $\hat{x} \in U$ and $|\mu_{G_i}(x) - \mu_{c,i}| < \epsilon$. For any non-negative numbers a, b , we have $a^{\frac{3}{4}} + b^{\frac{3}{4}} \geq (a + b)^{\frac{3}{4}}$, so we obtain

$$|\nabla f(x)| \geq k'' (|f_{G_1}(x) - f_{c,1}| + |f_{G_2}(x) - f_{c,2}|)^{\frac{3}{4}} \geq k'' |f(x) - f_c|^{\frac{3}{4}}, \quad (2.132)$$

whenever $|\mu(x) - \mu_c| < \epsilon$, as desired.

To obtain the estimate for the functions $|\mu_c|^2$ and $|\mu_{\text{HK}}|^2$, we simply note that by Lemma 2.3.20, the norm of the gradient is bounded below by a sum of terms of the form $|\nabla |\mu_i|^2|$, and since we can bound each term individually we obtain a bound for the sum. \square

Remark 2.3.31. As pointed out in Remark 2.3.12, Theorem 2.3.11 is a generalization of [Nee85, Theorem A.1] and much of the argument is similar. The main new ingredients are Lemma 2.3.27 and Proposition 2.3.29, which are absolutely essential in handling the general case of a nonzero constant term in the moment map.

2.4 Application to Hypertoric Varieties

2.4.1 Toric Varieties

Let T be a subtorus of the standard N -torus $(S^1)^N$, with quotient $K := (S^1)^N/T$. We have a short exact sequence

$$1 \rightarrow T \xrightarrow{i} (S^1)^N \xrightarrow{\pi} K \rightarrow 1. \quad (2.133)$$

Taking Lie algebras, we have

$$0 \rightarrow \mathfrak{t} \xrightarrow{i} \mathbf{R}^N \xrightarrow{\pi} \mathfrak{k} \rightarrow 0, \quad (2.134)$$

$$0 \rightarrow \mathfrak{k}^* \xrightarrow{\pi^*} \mathbf{R}^N \xrightarrow{i^*} \mathfrak{t}^* \rightarrow 0. \quad (2.135)$$

Recall the standard Hamiltonian action of $(S^1)^N$ on \mathbf{C}^N . This restricts to a Hamiltonian action of T on \mathbf{C}^N , and hence induces an action on $T^*\mathbf{C}^N$. Given $\alpha \in \mathfrak{k}^*$, the *toric variety* associated to this data is the Kähler quotient

$$\mathfrak{X}_\alpha := \mathbf{C}^N //_{\alpha} T, \quad (2.136)$$

and the *hypertoric variety* associated to this data is the hyperkähler quotient

$$\mathfrak{M}_\alpha := T^*\mathbf{C}^N //_{(\alpha,0)} T. \quad (2.137)$$

Note that \mathfrak{X}_α has a residual Hamiltonian action of K , and that \mathfrak{M}_α has a residual hyperhamiltonian action of K .

It is convenient to organize the data determining \mathfrak{X}_α and \mathfrak{M}_α into a *hyperplane arrangement* as follows. Let $\{e_j\}$ be the standard basis of \mathbf{R}^N . Then we obtain a collection $\{u_j\}$ of *weights* defined by $u_j := i^*(e_j) \in \mathfrak{k}^*$, as well as a collection of *normals* $\{n_i\}$ defined by $n_i = \pi(e_i) \in \mathfrak{k}$. Note that we allow repetitions, i.e. u_i and u_j are considered to be distinct elements of A for $i \neq j$ even if $u_i = u_j$ as elements of \mathfrak{k}^* , and similarly for the normals. Pick some $d \in \mathbf{R}^N$ such that $i^*(d) = \alpha$. Then we can define affine hyperplanes

H_i by

$$H_i = \{x \in \mathfrak{k}^* \mid \langle n_i, x \rangle - d_i = 0\}, \quad (2.138)$$

as well as half-spaces

$$H_i^\pm = \{x \in \mathfrak{k}^* \mid \pm (\langle n_i, x \rangle - d_i) \geq 0\}. \quad (2.139)$$

This arrangement of hyperplanes will be denoted by \mathcal{A} . An example is pictured in Figure 2.1 below.

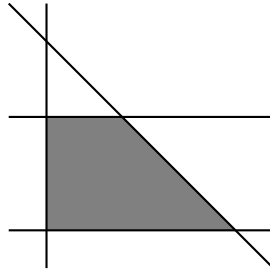


Figure 2.1: The hyperplane arrangement corresponding to the hyperkähler analogue of $\widetilde{\mathbf{P}^2}$, the blow-up of \mathbf{P}^2 at a point. The lines are the hyperplanes H_i and the shaded area is the common intersection $\cap H_i^+$, which is the moment polytope of $\widetilde{\mathbf{P}^2}$.

It is well-known that the geometry and topology of toric varieties is controlled by their moment polytopes, or dually by their normal fans [Ful93]. Hyperplane arrangements play an analogous role in hypertoric geometry. In fact, most of the geometry and topology can be recovered from the *matroid* underlying the hyperplane arrangement. We will show that Morse theory on hypertoric varieties is more naturally understood in terms of the *weight matroid* of the arrangement, which is dual to the matroid of the hyperplane arrangement.

Definition 2.4.1. Let $\mathcal{B} = \{u_1, \dots, u_N\}$ be the collection of weights of T acting on \mathbf{C}^N . For any subset $\mathcal{J} \subset \mathcal{B}$, the *rank* of \mathcal{J} , denoted $\text{rk}(\mathcal{J})$, is defined to be equal to the dimension of the span of \mathcal{J} . A subset \mathcal{J} is called a *flat* if for any $u_k \in \mathcal{B} \setminus \mathcal{J}$, $\text{rk}(\mathcal{J} \cup \{u_k\}) > \text{rk}(\mathcal{J})$. A flat is called a *hyperplane* if $\text{rk}(\mathcal{J}) = \dim(T) - 1$. The *weight matroid* is the set of all flats.

Remark 2.4.2. The relationship between hypertoric geometry and matroid theory was developed in [HS02]. The weight matroid of \mathcal{B} and the hyperplane arrangement \mathcal{A} are related by *Gale duality*.

Remark 2.4.3. In our notation, the collection of weights \mathcal{B} will be the set $\{u_1, \dots, u_N\}$, where $u_j = i^*(e_j)$ as above.

2.4.2 Analysis of the Critical Sets

We now consider a quotient of the form $\mathfrak{M}(\alpha_{\mathbf{R}}, \alpha_{\mathbf{C}})$ with $\alpha_{\mathbf{C}}$ a regular value of μ_c . Note that this includes quotients of the form $\mathfrak{M}(\alpha_{\mathbf{R}}, 0)$ as a special case, since we can always rotate the hyperkähler frame. We identify $T^*\mathbf{C}^N$ with $\mathbf{C}^N \times \mathbf{C}^N$ and use coordinates (x, y) . Shifting μ_c by $\alpha_{\mathbf{C}}$, we can take it to be

$$\mu_c(x, y) = \sum_{i=1}^N x_i y_i u_i - \alpha_{\mathbf{C}}, \quad (2.140)$$

and we will consider Morse theory with the function $f = |\mu_c|^2$. If $\mathcal{J} \subseteq \mathcal{B}$ is a flat, we define a subspace $V_{\mathcal{J}} \subset \mathbf{C}^N$ by

$$V_{\mathcal{J}} := \bigoplus_{u_j \in \mathcal{J}} V(u_j). \quad (2.141)$$

and a torus $T_{\mathcal{J}}$ by $T_{\mathcal{J}} = T/N_{\mathcal{J}}$, where $N_{\mathcal{J}}$ is the kernel of the action map on $V_{\mathcal{J}}$. We define $\mathfrak{M}_{\mathcal{J}}$ to be the hyperkähler quotient

$$\mathfrak{M}_{\mathcal{J}} := T^*V_{\mathcal{J}} //_{\alpha_{\mathcal{J}}} T_{\mathcal{J}}, \quad (2.142)$$

where $\alpha_{\mathcal{J}}$ is an induced moment map level which we define below.⁵

The inclusion $i : T \hookrightarrow (S^1)^N$ induces a surjective map of rings $i^* : H_{(S^1)^N}^* \rightarrow H_T^*$, so that $H_T^* \cong \mathbf{Q}[u_1, \dots, u_N] / \ker i^*$ as a graded ring. By abuse of notation we will write u_j to denote its image in H_T^* .⁶ If $\mathcal{J} \subseteq \mathcal{B}$ is a flat, then we define a class $u_{\mathcal{J}} \in H_T^*$ by

$$u_{\mathcal{J}} := \prod_{\beta \in \mathcal{J}^c} \beta, \quad (2.143)$$

and note that the product is taken over the *complement* of \mathcal{J} .

Proposition 2.4.4. *For a generic parameter β , the critical set of f is the disjoint union of sets $C_{\mathcal{J}}$, where the union runs over all flats \mathcal{J} of the weight matroid, where the sets $C_{\mathcal{J}}$ are defined by*

$$C_{\mathcal{J}} = \left(\bigcap_{\beta \in \mathcal{J}} \{\beta \cdot \mu_c = 0\} \right) \cap \left(\bigcap_{\beta \notin \mathcal{J}} \{(x_{\beta}, y_{\beta}) = 0\} \right). \quad (2.144)$$

⁵ α can always be chosen so that the moment map levels $\alpha_{\mathcal{J}}$ which appear for flats \mathcal{J} are all regular.

⁶Note that we also use the symbol u_j to denote the vectors $i^*(e_j)$, but no confusion should arise as it should be clear from context which of the two meanings is intended.

The Morse index of $C_{\mathcal{J}}$ is given by $\lambda_{\mathcal{J}} = 2(N - \#\mathcal{J})$. Up to a nonzero constant, the T -equivariant Euler class of the negative normal bundle to $C_{\mathcal{J}}$ is given by the restriction of the class $u_{\mathcal{J}}$ to $H_T^*(C_{\mathcal{J}})$, where $u_{\mathcal{J}}$ is defined by (2.143). The T -equivariant Poincaré series of $C_{\mathcal{J}}$ is equal to $(1 - t^2)^{-r} P_t(\mathfrak{M}_{\mathcal{J}})$, where r is the codimension of $T_{\mathcal{J}}$ in T and $\mathfrak{M}_{\mathcal{J}}$ is the quotient defined by (2.142).

Proof. Using equation (2.140), we see that

$$|\nabla f(x, y)|^2 = 4 \sum_{i=1}^N (|x_i|^2 + |y_i|^2) |u_i \cdot \mu_c(x, y)|^2. \quad (2.145)$$

Since this is a sum of non-negative terms, if $\nabla f(x, y) = 0$, each term in the sum must be 0. Thus for each weight u_i , we must have either that $(x_i, y_i) = 0$ or that $u_i \cdot \mu_c = 0$. Let us fix some particular critical point $(x_c, y_c) \in T^*\mathbf{C}^N$, and let \mathcal{J} be the set of weights for which $u_i \cdot \mu_c(x_c, y_c) = 0$. By construction \mathcal{J} is a flat and $(x_c, y_c) \in C_{\mathcal{J}}$. Hence every critical point is contained in $C_{\mathcal{J}}$ for some flat \mathcal{J} . Conversely, $\nabla f = 0$ on $C_{\mathcal{J}}$ by construction, so we see that $\text{Crit} f = \cup_{\mathcal{J}} C_{\mathcal{J}}$, where the union runs over the flats $\mathcal{J} \subseteq \mathcal{B}$.

To see that the union is disjoint, write $\mu_c = \mu_{\mathcal{J}} + \mu_{\mathcal{J}^c}$, where

$$\mu_{\mathcal{J}}(x, y) = \sum_{u_i \in \mathcal{J}} u_i x_i y_i - (\alpha_{\mathbf{C}})_{\mathcal{J}}, \quad (2.146)$$

$$\mu_{\mathcal{J}^c}(x, y) = \sum_{u_i \notin \mathcal{J}} u_i x_i y_i - (\alpha_{\mathbf{C}})_{\mathcal{J}}^{\perp}, \quad (2.147)$$

$(\alpha_{\mathbf{C}})_{\mathcal{J}}$ is the projection of $\alpha_{\mathbf{C}}$ to $\mathfrak{t}_{\mathcal{J}}^*$, and $(\alpha_{\mathbf{C}})_{\mathcal{J}}^{\perp} = \alpha_{\mathbf{C}} - (\alpha_{\mathbf{C}})_{\mathcal{J}}$. Then at (x_c, y_c) , we have $\mu_{\mathcal{J}}(x_c, y_c) = 0$ and $\mu_{\mathcal{J}^c}(x_c, y_c) = -(\alpha_{\mathbf{C}})_{\mathcal{J}}^{\perp}$. Thus on $C_{\mathcal{J}}$, μ_c takes the value $-(\alpha_{\mathbf{C}})_{\mathcal{J}}^{\perp}$. For generic $\alpha_{\mathbf{C}}$, we have $(\alpha_{\mathbf{C}})_{\mathcal{J}}^{\perp} \neq (\alpha_{\mathbf{C}})_{\mathcal{J}'}^{\perp}$, for $\mathcal{J} \neq \mathcal{J}'$, hence $C_{\mathcal{J}} \cap C_{\mathcal{J}'} = \emptyset$ for $\mathcal{J} \neq \mathcal{J}'$.

To determine the Morse index of $C_{\mathcal{J}}$, we compute

$$|\mu_c(x, y)|^2 = |\mu_{\mathcal{J}}(x, y)|^2 + |\mu_{\mathcal{J}^c}(x, y)|^2 + 2\text{Re} \langle \mu_{\mathcal{J}}(x, y), \mu_{\mathcal{J}^c}(x, y) \rangle. \quad (2.148)$$

The term $|\mu_{\mathcal{J}}(x, y)|^2$ has an absolute minimum at (x_c, y_c) , and so does not contribute to the Morse index. Since $(\alpha_{\mathbf{C}})_{\mathcal{J}}^{\perp}$ is orthogonal to $\mu_{\mathcal{J}}(x, y)$ for all (x, y) , the third term can be rewritten as

$$2\text{Re} \langle \mu_{\mathcal{J}}(x, y), \mu_{\mathcal{J}^c} + (\alpha_{\mathbf{C}})_{\mathcal{J}}^{\perp} \rangle. \quad (2.149)$$

Looking at the expressions (2.146) and (2.147) for $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{J}^c}$, we see that at (x_c, y_c) , $\mu_{\mathcal{J}}$ vanishes to first order, whereas $\mu_{\mathcal{J}^c} + (\alpha_{\mathbf{C}})_{\mathcal{J}}^{\perp}$ vanishes to second order. Hence the inner product of these terms vanishes to third order and does not affect the Morse index.

Thus the Morse index is determined solely by the second term, which is

$$|\mu_{\mathcal{J}^c}(x, y)|^2 = |(\alpha_{\mathbf{C}})_{\mathcal{J}}^\perp|^2 - 2\operatorname{Re} \sum_{i \in \mathcal{J}^c} \langle (\alpha_{\mathbf{C}})_{\mathcal{J}}^\perp, u_i \rangle x_i y_i + \text{fourth order}. \quad (2.150)$$

For generic $\alpha_{\mathbf{C}}$, we have $\langle (\alpha_{\mathbf{C}})_{\mathcal{J}}^\perp, u_i \rangle \neq 0$ for all $u_i \in \mathcal{J}^c$, and since each term in the sum is the real part of the holomorphic function $\langle (\alpha_{\mathbf{C}})_{\mathcal{J}}^\perp, u_i \rangle x_i y_i$ it must contribute 2 to the Morse index. Hence the Morse index is $\lambda_{\mathcal{J}} := 2\#\mathcal{J}^c = 2(N - \#\mathcal{J})$. Since the j th factor of $(S^1)^N$ acts on (x_j, y_j) with weight $(1, -1)$, this also shows that the equivariant Euler class is given by a nonzero multiple of $u_{\mathcal{J}}$ (defined by (2.143)), as claimed.

Finally, we compute the equivariant Poincaré series of $C_{\mathcal{J}}$. Let $V_{\mathcal{J}} \in \mathbf{C}^N$ be defined as above, and let $N_{\mathcal{J}} \subset T$ be the subtorus that acts trivially on $V_{\mathcal{J}}$. Then we have an isomorphism $T \cong T_{\mathcal{J}} \times N_{\mathcal{J}}$. Let r be the dimension of $N_{\mathcal{J}}$, which is the codimension of $T_{\mathcal{J}}$ in T . The moment map for the action of $T_{\mathcal{J}}$ on $T^*V_{\mathcal{J}}$ is given by the restriction of $\mu_{\mathcal{J}}$ (as defined by equation (2.146)) to $T^*V_{\mathcal{J}}$. Hence $C_{\mathcal{J}} = \mu_{\mathcal{J}}^{-1}(0) \cap T^*V_{\mathcal{J}}$, and

$$P_t^T(C_{\mathcal{J}}) = P_t^{T_{\mathcal{J}} \times N_{\mathcal{J}}}(C_{\mathcal{J}}) = (1 - t^2)^{-r} P_t^{T_{\mathcal{J}}}(C_{\mathcal{J}}) = (1 - t^2)^{-r} P_t(\mathfrak{M}_{\mathcal{J}}). \quad (2.151)$$

□

Remark 2.4.5. Note that the critical sets $C_{\mathcal{J}}$ are all nonempty, and that the Morse indices do not depend on $\alpha_{\mathbf{C}}$ (as long as it is generic). This is due to the fact that μ_c is holomorphic. In the real case, i.e. $|\mu_r - \alpha_{\mathbf{R}}|^2$, the critical sets and Morse indices have a much more sensitive dependence on the level $\alpha_{\mathbf{R}}$.

By Theorem 2.3.1 and Corollary 2.3.18, the hyperkähler Kirwan map $\kappa : H_T^* \rightarrow H^*(\mathfrak{M})$ is surjective, and its kernel is the ideal generated by the equivariant Euler classes of the negative normal bundles to the components of the critical set. Since we described these explicitly in Proposition 2.4.4, we immediately obtain the following description of $H^*(\mathfrak{M})$.

Theorem 2.4.6. *Assume $\alpha_{\mathbf{C}}$ is generic. Then the cohomology ring $H^*(\mathfrak{M})$ is isomorphic to $H^*(BT)/\ker \kappa$, where κ is the hyperkähler Kirwan map. Its kernel is the ideal generated by the classes $u_{\mathcal{J}}$, for every proper flat $\mathcal{J} \subset \mathcal{B}$.*

Remark 2.4.7. The cohomology ring $H^*(\mathfrak{M})$ was first computed by Konno [Kon00a, Theorem 3.1]. The relations defining $\ker \kappa$ obtained by Konno are not identical to those in Theorem 2.4.6, but it is not difficult to see that they are equivalent. It was pointed out to us by Proudfoot that this equivalence is a special case of Gale duality [BLVS⁺99].

Remark 2.4.8. Under the assumption that \mathfrak{M} is smooth (and not just an orbifold), the same result holds with \mathbf{Z} coefficients [Kon00a]. In principle, it should be possible to extend our Morse-theoretic arguments to obtain this stronger result using the ideas and results of [TW03, §7].

Quotients of the form $\mathfrak{M}(\alpha_{\mathbf{R}}, 0)$ inherit an additional S^1 -action induced by the S^1 action on $T^*\mathbf{C}^N$ given by $t \cdot (x, y) = (x, ty)$. This action preserves the Kähler structure and rotates the holomorphic symplectic form (i.e. $t^*\omega_{\mathbf{C}} = t\omega_{\mathbf{C}}$). Let us fix some particular $\alpha_{\mathbf{R}}$ and denote $\mathfrak{M} := \mathfrak{M}(\alpha_{\mathbf{R}}, 0)$. We would like to understand the S^1 -equivariant cohomology $H_{S^1}^*(\mathfrak{M})$ (which, unlike the ordinary cohomology of \mathfrak{M} , *does* depend on the choice of $\alpha_{\mathbf{R}}$). To compute the S^1 -equivariant cohomology, it is more convenient to work directly with $|\mu_{\text{HK}}|^2 = |\mu_r|^2 + |\mu_c|^2$, where

$$\mu_r = \sum_i u_i (|x_i|^2 - |y_i|^2) - \alpha_{\mathbf{R}}, \quad (2.152)$$

$$\mu_c = \sum_i u_i x_i y_i. \quad (2.153)$$

By Theorems 2.3.1 and 2.3.11 $|\mu_{\text{HK}}|^2$ is minimally degenerate and flow-closed, and since it is also S^1 -invariant we can consider the $T \times S^1$ -equivariant Thom-Gysin sequence. The usual arguments of the Kirwan method extend to the S^1 -equivariant setting, so we obtain surjectivity of map $\kappa_{S^1} : H_{T \times S^1}^* \rightarrow H_{S^1}^*(\mathfrak{M})$, and its kernel is generated by the $T \times S^1$ -equivariant Euler classes of the negative normal bundles to the components of the critical set.

To find the components of the critical set of $|\mu_{\text{HK}}|^2$ and to compute the equivariant Euler classes, we can repeat the arguments of Proposition 2.4.4 almost without modification. The components of the critical set are again indexed by flats \mathcal{J} . The only important difference is that since we now work with $T \times S^1$ -equivariant cohomology, we have to be more careful in computing the equivariant Euler classes. Let us make the identification $H_{T \times S^1}^* \cong H_T^*[u_0]$. When we expand $|\mu_{\text{HK}}|^2$ about a critical point as in the proof of Proposition 2.4.4, the relevant term is now (cf. equation (2.150))

$$-2 \sum_{u_i \in \mathcal{J}^c} \langle (\alpha_{\mathbf{R}})_{\mathcal{J}}^{\perp}, u_i \rangle (|x_i|^2 - |y_i|^2), \quad (2.154)$$

where $(\alpha_{\mathbf{R}})_{\mathcal{J}}^{\perp}$ is defined in a manner analogous to $(\alpha_{\mathbf{C}})_{\mathcal{J}}^{\perp}$ as in the proof of Proposition 2.4.4. We see that the x_i term appears with an overall negative sign if $\langle (\alpha_{\mathbf{R}})_{\mathcal{J}}^{\perp}, u_i \rangle > 0$; otherwise it is the y_i term that appears with a negative sign. Since S^1 acts on x with weight 0 and acts on y with weight 1, and since T acts on x and y with oppositely signed

weights, we find that the equivariant Euler class is given (up to an overall constant) by

$$\tilde{u}_{\mathcal{J}} := \left(\prod_{u_i \in \mathcal{J}^+} u_i \right) \left(\prod_{u_i \in \mathcal{J}^-} (u_0 - u_i) \right), \quad (2.155)$$

where

$$\mathcal{J}^{\pm} := \{u_i \in \mathcal{J}^c \mid \pm \langle (\alpha_{\mathbf{R}})_{\mathcal{J}}^{\perp}, u_i \rangle > 0\}. \quad (2.156)$$

We thus obtain the following.

Theorem 2.4.9. *The S^1 -equivariant cohomology $H_{S^1}^*(\mathfrak{M})$ is isomorphic to*

$$H_{T \times S^1}^* / \ker \kappa_{S^1}, \quad (2.157)$$

where κ_{S^1} is the S^1 -equivariant Kirwan map. Its kernel is the ideal generated by the classes $\tilde{u}_{\mathcal{J}}$, for every proper flat \mathcal{J} .

Remark 2.4.10. The S^1 -equivariant cohomology rings were first computed by Harada and Proudfoot [HP05a]. Note that unlike the ordinary cohomology ring, the S^1 -equivariant cohomology ring depends explicitly on the parameter α .

2.4.3 Hyperkähler Modifications

Recall from Section 2.1.3 that there is a natural operation called *hyperkähler modification*, which is the hyperkähler analogue of symplectic cutting. If $\mathfrak{M} = T^*\mathbf{C}^N // T$ is a hypertoric variety with a residual torus action by K , then for any choice of $S^1 \hookrightarrow K$ we can consider the modification

$$\tilde{\mathfrak{M}} := \mathfrak{M} \times T^*\mathbf{C} // S^1 \cong T^*\mathbf{C}^{N+1} // \tilde{T}, \quad (2.158)$$

where we have set $\tilde{T} = T \times S^1$. We also consider the quotient

$$\hat{\mathfrak{M}} := \mathfrak{M} // S^1 \cong T^*\mathbf{C}^N // \tilde{T}, \quad (2.159)$$

which are both hypertoric varieties. As we saw in the previous section, for any hypertoric variety, equivariant Morse theory with $|\mu_c|^2$ is controlled entirely by the underlying matroid. Hence if we can deduce a relation between the matroids of \mathfrak{M} , $\tilde{\mathfrak{M}}$, and $\hat{\mathfrak{M}}$, we will be able to determine a relation between their Poincaré polynomials and cohomology rings.

Let \mathcal{B} , $\tilde{\mathcal{B}}$, and $\hat{\mathcal{B}}$ denote the respective collections of weights defining the hypertoric varieties \mathfrak{M} , $\tilde{\mathfrak{M}}$, and $\hat{\mathfrak{M}}$, and let μ , $\tilde{\mu}$, and $\hat{\mu}$ denote the respective (complex) moment maps. We can relate collections of weights $\tilde{\mathcal{B}}$ and $\hat{\mathcal{B}}$, corresponding to the modification and quotient of M , to the weights \mathcal{B} as follows.

Lemma 2.4.11. *Let $\tilde{\mathcal{B}} = \{\tilde{u}_j\}_{j=1}^{N+1}$. Then $\hat{\mathcal{B}} = \{\tilde{u}_j\}_{j=1}^N$ and $\mathcal{B} = \{u_j\}_{j=1}^N$, where u_j is the image of \tilde{u}_j after quotienting by $\text{span}\{\tilde{u}_{N+1}\}$.*

Proof. The weights are determined by the embeddings $\mathfrak{t} \hookrightarrow \mathbf{R}^N$, $\tilde{\mathfrak{t}} \hookrightarrow \mathbf{R}^{N+1}$, and $\hat{\mathfrak{t}} \hookrightarrow \mathbf{R}^N$. Note that $\tilde{\mathfrak{t}} \cong \mathfrak{t} \oplus \mathbf{R}$. If we pick a basis of \mathfrak{t} (and use the standard basis of \mathbf{R}^N), we can represent the embedding $\mathfrak{t} \hookrightarrow \mathbf{R}^N$ by some matrix B . The S^1 action on M is determined by specifying its weights on \mathbf{C}^N ; this is equivalent to adjoining a column to B . This gives the matrix \hat{B} determining $\hat{\mathfrak{t}} \hookrightarrow \mathbf{R}^N$. Finally, to obtain the modification \tilde{M} , we let the S^1 act on an additional copy of \mathbf{C} with weight -1 . This amounts to adjoining the row $(0, \dots, 0, -1)$ to \hat{B} , to obtain \tilde{B} determining $\tilde{\mathfrak{t}} \hookrightarrow \mathbf{R}^{N+1}$. Since the weights u_j , \tilde{u}_j , and \hat{u}_j correspond to the rows of B , \tilde{B} , and \hat{B} , respectively, the result follows. \square

By the above lemma, we have $\mathcal{B}, \hat{\mathcal{B}} \cong \{1, \dots, N\}$ and $\tilde{\mathcal{B}} \cong \{1, \dots, N+1\}$ as sets. Hence any subset $\mathcal{J} \subset \mathcal{B}$ may be regarded as a subset of $\hat{\mathcal{B}}$ as well as of $\tilde{\mathcal{B}}$. We will use this identification for the remainder of this section.

Lemma 2.4.12. *Let $\mathcal{J} \subseteq \mathcal{B}$. Then*

1. \mathcal{J} is a flat of \mathcal{B} if and only if $\mathcal{J} \cup \{N+1\}$ is a flat of $\tilde{\mathcal{B}}$.
2. \mathcal{J} is a flat of $\hat{\mathcal{B}}$ if \mathcal{J} is a flat of \mathcal{B} .
3. \mathcal{J} is a flat of $\hat{\mathcal{B}}$ if and only if at least one of \mathcal{J} or $\mathcal{J} \cup \{N+1\}$ is a flat of $\tilde{\mathcal{B}}$.

Proof. Let $\tilde{\mathcal{B}} = \{\tilde{u}_j\}_{j=1}^{N+1}$. By the previous lemma, $\hat{\mathcal{B}} = \{\tilde{u}_j\}_{j=1}^N$ and $\mathcal{B} = \{u_j\}_{j=1}^N$, where u_j is the image of \tilde{u}_j after quotienting by $\text{span}\{\tilde{u}_{N+1}\}$.

To prove (1), first suppose that \mathcal{J} is a flat of \mathcal{B} . Suppose that there is some $\tilde{u}_i \in \tilde{\mathfrak{t}}_{\mathcal{J} \cup \{N+1\}}$. If $i = N+1$ then $i \in \mathcal{J} \cup \{N+1\}$ and there is nothing to check, so suppose that $i \neq N+1$. Applying the quotient map, we see that $u_i \in \mathfrak{t}_{\mathcal{J}}$ (since \tilde{u}_{N+1} goes to 0), and since \mathcal{J} was a flat of \mathcal{B} we see that $i \in \mathcal{J} \subset \mathcal{J} \cup \{N+1\}$. Hence $\mathcal{J} \cup \{N+1\}$ is a flat of $\tilde{\mathcal{B}}$.

Conversely, suppose that $\mathcal{J} \cup \{N+1\}$ is a flat of $\tilde{\mathcal{B}}$. Suppose that there is some $u_i \in \mathfrak{t}_{\mathcal{J}}$. Then if $u_i = \sum_{j \in \mathcal{J}} a_j u_j$, we see that $\tilde{u}_i - \sum_{j \in \mathcal{J}} a_j \tilde{u}_j$ is in the kernel of the projection, and so is some multiple of \tilde{u}_{N+1} . Hence $\tilde{u}_i \in \tilde{\mathfrak{t}}_{\mathcal{J} \cup \{N+1\}}$. Since $\mathcal{J} \cup \{N+1\}$ is

a flat of $\tilde{\mathcal{B}}$, we must have $i \in \mathcal{J} \cup \{N+1\}$. But $i \neq N+1$ by assumption, so we have $i \in \mathcal{J}$.

To prove (2), suppose that \mathcal{J} is a flat of \mathcal{B} . If $\tilde{u}_i \in \tilde{\mathfrak{t}}_{\mathcal{J}}$, then applying the quotient map we find $u_i \in \mathfrak{t}_{\mathcal{J}}$. Hence $i \in \mathcal{J}$.

To prove (3), first suppose that $\mathcal{J} \cup \{N+1\}$ is a flat of $\tilde{\mathcal{B}}$. Then by (1), \mathcal{J} is a flat of \mathcal{B} , and by (2) \mathcal{J} is a flat of for $\hat{\mathcal{B}}$. On the other hand, if \mathcal{J} is a flat of $\tilde{\mathcal{B}}$, then it is certainly a flat of $\hat{\mathcal{B}}$. This establishes one direction.

Conversely, suppose that \mathcal{J} is a flat of $\hat{\mathcal{B}}$. If $\tilde{u}_{N+1} \notin \tilde{\mathfrak{t}}_{\mathcal{J}}$ then \mathcal{J} is a flat of $\tilde{\mathcal{B}}$; otherwise $\tilde{u}_{N+1} \in \tilde{\mathfrak{t}}_{\mathcal{J}}$ and thus $\mathcal{J} \cup \{N+1\}$ is a flat of $\tilde{\mathcal{B}}$. \square

We rephrase the preceding lemma as the following trichotomy:

Lemma 2.4.13. *Let $\mathcal{J} \subseteq \{1, \dots, N\}$ be a flat of $\hat{\mathcal{B}}$. Then exactly one of the following cases occurs:*

1. \mathcal{J} is a flat of $\tilde{\mathcal{B}}$ and \mathcal{J} is not a flat of \mathcal{B} .
2. \mathcal{J} is a flat of \mathcal{B} and both \mathcal{J} and $\mathcal{J} \cup \{N+1\}$ are flats of $\tilde{\mathcal{B}}$.
3. \mathcal{J} is a flat of \mathcal{B} and $\mathcal{J} \cup \{N+1\}$ is a flat of $\tilde{\mathcal{B}}$, while \mathcal{J} is not.

Moreover, every critical subset for \mathcal{B} and $\tilde{\mathcal{B}}$ occurs as exactly one of the above.

Theorem 2.4.14. *If $\tilde{\mathfrak{M}}$ is a hyperkähler modification of \mathfrak{M} and $\hat{\mathfrak{M}}$ is the corresponding quotient, then*

$$P_t(\tilde{\mathfrak{M}}) = P_t(\mathfrak{M}) + t^2 P_t(\hat{\mathfrak{M}}). \quad (2.160)$$

Proof. We will prove this by induction on $N = \#\mathcal{B}$, the number of the number of weights (equivalently, the number of hyperplanes in \mathcal{A}). The base case can be verified easily, so we assume that the result is true for modifications $(\tilde{\mathfrak{M}}', \mathfrak{M}', \hat{\mathfrak{M}}')$, where \mathfrak{M}' is a quotient of $T^*\mathbf{C}^{N'}$, with $N' < N$.

By Theorems 2.3.1 and 2.3.11, the functions $f = |\mu_c|^2$, $\tilde{f} = |\tilde{\mu}_{\mathbf{C}}|^2$, and $\hat{f} = |\hat{\mu}_{\mathbf{C}}|^2$ are equivariantly perfect. \mathfrak{M} is a quotient of $T^*\mathbf{C}^N$ by a torus of rank d , while $\tilde{\mathfrak{M}}$ and $\hat{\mathfrak{M}}$ are quotients of $T^*\mathbf{C}^{N+1}$ and $T^*\mathbf{C}^N$, respectively, by a torus of rank $d+1$. Hence we have

$$\frac{1}{(1-t^2)^d} = \sum_C t^{\lambda_C} P_t^T(C), \quad (2.161)$$

$$\frac{1}{(1-t^2)^{d+1}} = \sum_{\tilde{C}} t^{\tilde{\lambda}_C} P_t^{\tilde{T}}(\tilde{C}), \quad (2.162)$$

$$\frac{1}{(1-t^2)^{d+1}} = \sum_{\hat{C}} t^{\hat{\lambda}_C} P_t^{\hat{T}}(\hat{C}). \quad (2.163)$$

Since 0 is the absolute minimum in each case, we obtain

$$\frac{1}{(1-t^2)^d} = P_t(\mathfrak{M}) + \sum_{C, f(C) > 0} t^{\lambda_C} P_t^T(C), \quad (2.164)$$

and similarly for $\tilde{\mathfrak{M}}$ and $\hat{\mathfrak{M}}$. Since

$$\frac{1}{(1-t^2)^{d+1}} = \frac{1}{(1-t^2)^d} + \frac{t^2}{(1-t^2)^{d+1}} = 0, \quad (2.165)$$

we just have to show that

$$\sum_{\tilde{C}, \tilde{f}(\tilde{C}) > 0} t^{\tilde{\lambda}_C} P_t^{\tilde{T}}(\tilde{C}) = \sum_{C, f(C) > 0} t^{\lambda_C} P_t^T(C) + \sum_{\hat{C}, \hat{f}(\hat{C}) > 0} t^{\hat{\lambda}_C + 2} P_t^{\hat{T}}(\hat{C}). \quad (2.166)$$

to obtain the desired recurrence relation among the Poincaré polynomials of \mathfrak{M} , $\tilde{\mathfrak{M}}$, and $\hat{\mathfrak{M}}$. By Lemma 2.4.13, there is a trichotomy relating the critical sets of \tilde{f} to the critical sets of f and \hat{f} . We will consider each case separately.

Case (1): We have $\tilde{C} = \tilde{C}_{\mathcal{J}}$ where \mathcal{J} is a flat of $\tilde{\mathcal{B}}$ and $\hat{\mathcal{B}}$ but not of \mathcal{B} . From Proposition 2.4.4, we have $\tilde{\lambda}_{\mathcal{J}} = \hat{\lambda}_{\mathcal{J}} + 2$ and $\tilde{C}_{\mathcal{J}} = \hat{C}_{\mathcal{J}} \times (0, 0)$. Hence $t^{\tilde{\lambda}_{\mathcal{J}}} P_t^{\tilde{T}}(\tilde{C}_{\mathcal{J}}) = t^{\hat{\lambda}_{\mathcal{J}} + 2} P_t^{\hat{T}}(\hat{C}_{\mathcal{J}})$.

Case (2): We have $\tilde{C} = \tilde{C}_{\mathcal{J}}$ where \mathcal{J} is a flat of $\hat{\mathcal{B}}$ and $\mathcal{J} \cup \{N+1\}$ is a flat of $\tilde{\mathcal{B}}$. Then $\tilde{\lambda}_{\mathcal{J}} = \hat{\lambda}_{\mathcal{J}} + 2$ and $\tilde{\lambda}_{\mathcal{J} \cup \{N+1\}} = \lambda_{\mathcal{J}}$. The terms involving $\tilde{C}_{\mathcal{J}}$ and $\hat{C}_{\mathcal{J}}$ are equal as in case (1). Since both \mathcal{J} and $\mathcal{J} \cup \{N+1\}$ are flats of $\tilde{\mathcal{B}}$, it must be that $\tilde{u}_{N+1} \notin \tilde{\mathfrak{t}}_{\mathcal{J}}$. Hence $\tilde{T}_{\mathcal{J} \cup \{N+1\}} \cong \tilde{T}_{\mathcal{J}} \times S^1$, where the last factor is generated by \tilde{u}_{N+1} , and we find that $\tilde{\mathfrak{M}}_{\mathcal{J} \cup \{N+1\}} \cong \mathfrak{M}_{\mathcal{J}}$. Hence $P_t^{\tilde{T}}(\tilde{C}_{\mathcal{J} \cup \{N+1\}}) = P_t^T(C_{\mathcal{J}})$.

Case (3): $\tilde{C} = \tilde{C}_{\mathcal{J} \cup \{N+1\}}$, where \mathcal{J} is critical for \mathcal{B} and $\hat{\mathcal{B}}$ but not for $\tilde{\mathcal{B}}$. By Proposition 2.4.4, we have $P_t^{\tilde{T}}(\tilde{C}_{\mathcal{J} \cup \{N+1\}}) = (1-t^2)^{-r} P_t(\tilde{\mathfrak{M}}_{\mathcal{J} \cup \{N+1\}})$, $P_t^T(C_{\mathcal{J}}) = (1-t^2)^{-r} P(\mathfrak{M}_{\mathcal{J}})$, and $P_t^{\hat{T}}(\hat{C}_{\mathcal{J}}) = (1-t^2)^{-r} P_t(\hat{\mathfrak{M}}_{\mathcal{J}})$, where r is the codimension of $T_{\mathcal{J}}$ in T . Thus we have to show that

$$P_t(\tilde{\mathfrak{M}}_{\mathcal{J} \cup \{N+1\}}) = P_t(\mathfrak{M}_{\mathcal{J}}) + t^2 P_t(\hat{\mathfrak{M}}_{\mathcal{J}}). \quad (2.167)$$

But $\tilde{\mathfrak{M}}_{\mathcal{J} \cup \{N+1\}}$ is a modification of $\mathfrak{M}_{\mathcal{J}}$, and $\hat{\mathfrak{M}}_{\mathcal{J}}$ is the corresponding quotient of $\mathfrak{M}_{\mathcal{J}}$. Since \mathcal{J} is a proper subset of $\{1, \dots, N\}$, the relation (2.167) is true by induction. \square

The relation (2.160) is equivalent to the following recurrence relation among the Betti numbers of \mathfrak{M} , $\tilde{\mathfrak{M}}$, and $\hat{\mathfrak{M}}$:

$$\tilde{b}_{2k} = b_{2k} + \hat{b}_{2k} + \hat{b}_{2k-2}. \quad (2.168)$$

If we let d_k denote the number of bounded k -dimensional facets of the polyhedral complex

generated by the half-spaces H_i^\pm (with \tilde{d}_k and \hat{d}_k defined similarly), it is easy to see that these satisfy the same relation:

$$\tilde{d}_k = d_k + \hat{d}_k + \hat{d}_{k-1}. \quad (2.169)$$

Since any toric hyperkähler orbifold can be constructed out of a finite sequence of modifications starting with $T^*\mathbf{C}^n$, an easy induction argument then yields the following explicit description of $P_t(\mathfrak{M})$.

Corollary 2.4.15 (Bielawski-Dancer [BD00, Theorem 7.6]). *The Poincaré polynomial of \mathfrak{M} is given by*

$$P_t(\mathfrak{M}) = \sum_k d_k (t^2 - 1)^k. \quad (2.170)$$

Chapter 3

Quivers with Relations

3.1 Moduli of Quiver Representations

3.1.1 Quiver Representations

In this chapter, we will study varieties associated to quivers. The basic results concerning theory of quiver representations can be found in [CB], and the theory of moduli spaces of representations from the algebro-geometric point of view was developed in [Kin94]. We will focus on hyperkähler varieties constructed from quivers, studying a mild generalization of the varieties introduced by Nakajima [Nak94]. For the most part, we will not rely on any deep results from these works, and will instead work directly with the tools developed in Chapter 2.

Definition 3.1.1. A *quiver* $\mathcal{Q} = (\mathcal{I}, \mathcal{E})$ is a finite directed graph with vertex set \mathcal{I} and edge set \mathcal{E} . We will assume that the vertices of \mathcal{Q} are labelled by $\{\circ, \square\}$ so that $\mathcal{I} = \mathcal{V} \sqcup \mathcal{W}$, where \mathcal{V} consists of the vertices labelled “ \circ ” and \mathcal{W} consists of the vertices labelled “ \square ”. A *dimension vector* $\mathbf{d} = (d_i)_{i \in \mathcal{I}}$ is an assignment of a non-negative integer d_i to each $i \in \mathcal{I}$. We will often write $\mathbf{d} = (\mathbf{v}, \mathbf{w})$ so that $d_i = v_i$ for $i \in \mathcal{V}$ and $d_i = w_i$ for $i \in \mathcal{W}$.

Given a quiver with dimension vector $(\mathcal{Q}, \mathbf{d} = (\mathbf{v}, \mathbf{w}))$ we define vector spaces

$$V = \bigoplus_{i \in \mathcal{V}} V_i, \quad (3.1)$$

$$W = \bigoplus_{i \in \mathcal{W}} W_i, \quad (3.2)$$

$$\text{Rep}(\mathcal{Q}, \mathbf{d}) = \bigoplus_{i \rightarrow j \in \mathcal{E}} \text{Hom}(D_i, D_j), \quad (3.3)$$

where $V_i = \mathbf{C}^{v_i}$, $W_i = \mathbf{C}^{w_i}$, and $D_i = V_i$ for $i \in \mathcal{V}$ and $D_i = W_i$ for $i \in \mathcal{W}$. The vector space $\text{Rep}(\mathcal{Q}, \mathbf{d})$ is the space of representations of \mathcal{Q} of dimension \mathbf{d} . We will often write this as $\text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w})$ to emphasize the decomposition $\mathbf{d} = (\mathbf{v}, \mathbf{w})$. We define groups

$$G_{\mathbf{v}} = \prod_{i \in \mathcal{V}} U(v_i), \quad (3.4)$$

$$G_{\mathbf{w}} = \prod_{i \in \mathcal{W}} U(w_i), \quad (3.5)$$

$$G_{\mathbf{d}} = G_{\mathbf{v}} \times G_{\mathbf{w}}, \quad (3.6)$$

which act naturally on $\text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w})$. We denote by $\mu_{\mathbf{v}}, \mu_{\mathbf{w}}, \mu_{\mathbf{d}}$ the corresponding moment maps. We also consider the *double* $\overline{\overline{\mathcal{Q}}}$ of a quiver, defined by $\overline{\overline{\mathcal{Q}}} = (\mathcal{V}, \mathcal{E} \sqcup \overline{\mathcal{E}})$ where $\overline{\mathcal{E}}$ denotes reversing the orientation of all the edges in \mathcal{E} .

For a fixed dimension vector $\mathbf{d} = (\mathbf{v}, \mathbf{w})$ we now define several moduli spaces of representations as follows:

$$\mathfrak{X}_{\alpha}(\mathbf{v}, \mathbf{w}) := \text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w}) //_{\alpha} G_{\mathbf{v}}, \quad (3.7)$$

$$\overline{\overline{\mathfrak{X}}}_{\alpha}(\mathbf{v}, \mathbf{w}) = \text{Rep}(\overline{\overline{\mathcal{Q}}}, \mathbf{v}, \mathbf{w}) //_{\alpha} G_{\mathbf{v}}, \quad (3.8)$$

$$\mathfrak{M}_{\alpha}(\mathbf{v}, \mathbf{w}) := T^* \text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w}) // //_{\alpha} G_{\mathbf{v}}. \quad (3.9)$$

We call $\mathfrak{X}_{\alpha}(\mathbf{v}, \mathbf{w})$ the *Kähler quiver variety*, $\overline{\overline{\mathfrak{X}}}_{\alpha}(\mathbf{v}, \mathbf{w})$ the *doubled quiver variety*, and $\mathfrak{M}_{\alpha}(\mathbf{v}, \mathbf{w})$ the *hyperkähler quiver variety*. Note that there are natural inclusions¹

$$T^* \mathfrak{X}_{\alpha}(\mathbf{v}, \mathbf{w}) \subset \mathfrak{M}_{\alpha}(\mathbf{v}, \mathbf{w}) \subset \overline{\overline{\mathfrak{X}}}_{\alpha}(\mathbf{v}, \mathbf{w}) \quad (3.10)$$

and that all three of these spaces have a residual Hamiltonian action of $G_{\mathbf{w}}$. Hence we may consider their quotients $\mathfrak{X}_{\alpha}(\mathbf{v}, \mathbf{w}) // G_{\mathbf{w}}$ and $\mathfrak{M}_{\alpha}(\mathbf{v}, \mathbf{w}) // // G_{\mathbf{w}}$. Of course, these quotients do not depend on the particular choice of decomposition $\mathcal{I} = \mathcal{V} \sqcup \mathcal{W}$ —this is just reduction in stages (cf. Remark 2.1.2). However, we will see in the following sections that we will be able to understand these quotients by studying the $G_{\mathbf{w}}$ -equivariant stratification on varieties of the form $\mathfrak{X}_{\alpha}(\mathbf{v}, \mathbf{w})$.

Example 3.1.2. The quiver pictured in Figure 3.1 is called the ADHM quiver. By the ADHM construction [AHDM78], the hyperkähler quiver variety $\mathfrak{M}_{\alpha}(k, n)$ may be identified with the moduli space of framed $SU(n)$ instantons on \mathbf{R}^4 of charge k . This construc-

¹It is easy to check that $\overline{\overline{\mathfrak{X}}}$ is holomorphic Poisson, $\mathfrak{M} \hookrightarrow \overline{\overline{\mathfrak{X}}}$ is a symplectic leaf, and $\mathfrak{X} \hookrightarrow \mathfrak{M}$ is a Lagrangian subvariety.

tion was later generalized by Kronheimer and Nakajima [KN90] to construct Yang-Mills instanton moduli spaces from any affine ADE quiver.

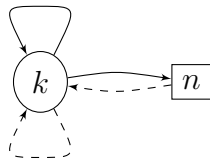


Figure 3.1: The ADHM quiver, with doubled edges indicated by dashed arrows.

Example 3.1.3. See the quiver pictured in Figure 3.2. We have $\mathbf{v} = (2)$ and $\mathbf{w} = (1, 1, 1, 1)$. Then $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w}) \cong \text{Gr}(2, 4)$, and $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})//G_{\mathbf{w}} \cong \text{Gr}(2, 4)//(S^1)^4$. If we interchange the labels $\square \leftrightarrow \circ$, then we have $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w}) \cong (\mathbf{P}^1)^4$, and $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})//G_{\mathbf{w}} \cong (\mathbf{P}^1)^4//SU(2)$. This is the most basic example of a polygon space [Kly94], and its hyperkähler analogue is a hyperpolygon space [Kon00b, HP05a].



Figure 3.2: Left: a star quiver with dimension vector; right: its double.

Remark 3.1.4. The definition of quiver variety that we use here is a slightly different than the usual the notion of a Nakajima quiver variety [Nak94]. To obtain a Nakajima variety from the construction above, first start with a quiver \mathcal{Q} all of whose vertices are labelled by \circ . Then construct a new quiver \mathcal{Q}' from \mathcal{Q} by adding a “shadow vertex” i' , labelled by \square , for each vertex i of \mathcal{Q} , as well as a single edge from i' to i . Then the hyperkähler quiver variety $\mathfrak{M}_\alpha(\mathbf{v}, \mathbf{w})$ associated to \mathcal{Q}' by the above construction coincides with the Nakajima variety associated to \mathcal{Q} . We illustrate this in Figure 3.3.

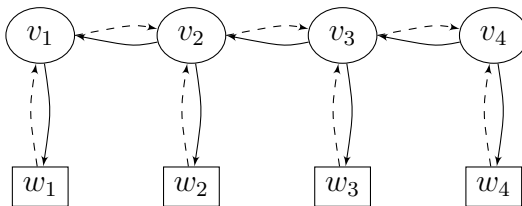


Figure 3.3: A Nakajima quiver constructed by adding “shadow” vertices.

3.1.2 Relations in the Path Algebra

We wish to consider certain natural subvarieties of quiver varieties, constructed in the following way. Suppose that E is a linear representation of $G_{\mathbf{d}}$, and $f : \text{Rep}(\mathcal{Q}, \mathbf{d}) \rightarrow E$ is an equivariant algebraic map. Then $f^{-1}(0) \subset \text{Rep}(\mathcal{Q}, \mathbf{d})$ is $G_{\mathbf{d}}$ -invariant, hence it defines a subvariety

$$V(f) \subset \mathfrak{X}_{\alpha}(\mathbf{v}, \mathbf{w}). \quad (3.11)$$

By equivariance, this subvariety is preserved by the $G_{\mathbf{w}}$ -action, so we also obtain a subvariety

$$V(f)//G_{\mathbf{w}} \subset \mathfrak{X}_{\alpha}(\mathbf{v}, \mathbf{w})//G_{\mathbf{w}}. \quad (3.12)$$

Understanding compact subvarieties of this form will be the primary focus of this chapter. To understand the relevance of these subvarieties, let us recall the path algebra of a quiver.

Definition 3.1.5. Let \mathcal{Q} be a quiver. The *path algebra* $\mathbf{C}[\mathcal{Q}]$ of \mathcal{Q} is the unital associative algebra generated by directed paths in \mathcal{Q} . The product of paths p_1 and p_2 is defined to be the composition of p_1 and p_2 if they are composable, and zero otherwise.

Construction 3.1.6. Let $f \in \mathbf{C}[\mathcal{Q}]$. Then for any dimension vector \mathbf{d} , f induces a $G_{\mathbf{d}}$ -equivariant map $\text{Rep}(\mathcal{Q}, \mathbf{d}) \rightarrow E_{\mathbf{d}}$, where $E_{\mathbf{d}}$ is a linear representation of $G_{\mathbf{d}}$. We can write f (uniquely) as

$$f = \sum_{(i,j) \in \mathcal{I} \times \mathcal{I}} f_{ij} \quad (3.13)$$

where each summand f_{ij} is a linear combination of paths from vertex i to vertex j . Hence f_{ij} induces a map $\text{Rep}(\mathcal{Q}, \mathbf{d}) \rightarrow \text{Hom}(D_i, D_j)$. Hence we get an induced map (which, by abuse of notation, we denote by f)

$$f : \text{Rep}(\mathcal{Q}, \mathbf{d}) \rightarrow \bigoplus_{(i,j):f_{ij} \neq 0} \text{Hom}(D_i, D_j). \quad (3.14)$$

It is easy to see that this map is $G_{\mathbf{d}}$ -equivariant.

Hence given any polynomial f in the edges, we obtain an equivariant map $f : \text{Rep}(\mathcal{Q}, \mathbf{d}) \rightarrow E$, and the subvariety $V(f) \subset \mathfrak{X}_{\alpha}(\mathcal{Q}, \mathbf{d})$ can be interpreted as the subvariety of representations which satisfy the relation $f = 0$, i.e. we can interpret $V(f)$ as a moduli space of representations of the quotient path algebra $\mathbf{C}[\mathcal{Q}]/\langle f \rangle$.

Remark 3.1.7. The complex moment map on $\text{Rep}(\overline{\mathcal{Q}}, \mathbf{d})$ is a polynomial in the edges, and the hyperkähler quiver variety is the subvariety of $\overline{\mathfrak{X}}_{\alpha}(\mathcal{Q}, \mathbf{d})$ defined by the corresponding relation.

Proposition 3.1.8. *Given an equivariant map $f : \text{Rep}(\mathcal{Q}, \mathbf{d}) \rightarrow E$ as above, there is a natural $G_{\mathbf{w}}$ -equivariant bundle $\mathcal{E} \rightarrow \mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ with section s such that $V(f) = s^{-1}(0)$. In particular, $V(f)$ is smooth if s is transverse to 0. Similarly, $V(f)//G_{\mathbf{w}}$ is the vanishing set of a section of a vector bundle over $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})//G_{\mathbf{w}}$.*

Proof. Let $P = \mu_{\mathbf{v}}^{-1}(\alpha)$, where $\mu_{\mathbf{v}}$ is the moment map for the $G_{\mathbf{v}}$ action. This is the total space of a principal $G_{\mathbf{v}}$ -bundle over $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$. Hence we may take the associated bundle $\mathcal{E} = P \times_{G_{\mathbf{v}}} E$. Since f is $G_{\mathbf{v}}$ -equivariant, it induces a section s of \mathcal{E} . \square

Remark 3.1.9. Given $V(f) \subset \mathfrak{X}$ we can always compactify via symplectic cuts to obtain $\overline{V(f)} \subset \overline{\mathfrak{X}}$, where $\overline{\mathfrak{X}}$ is projective. Moreover, $\overline{V(f)}$ may be identified with the vanishing set of a section of a bundle $E \rightarrow \overline{\mathfrak{X}}$. If this bundle happens to be positive, then Sommese's Theorem [Laz04, Chapter 7] implies that the restriction $H^*(\overline{\mathfrak{X}}) \rightarrow H^*(\overline{V(f)})$ is surjective in degrees less than $\dim \mathfrak{X} - \text{rk} E$. It may be possible to deduce hyperkähler Kirwan surjectivity by arguments along these lines. We plan to investigate this in future work.

3.2 Morse Theory on Quiver Varieties

3.2.1 Circle Actions

We define a *circle compact variety* to be a Kähler variety together with a Hamiltonian circle action such that the fixed-point set is compact, and the moment map is proper and bounded below. Now let $\mathcal{Q} = (\mathcal{I}, \mathcal{E})$ be a quiver. For each edge $e \in \mathcal{E}$, pick a non-negative integer w_e . Then we can define an S^1 -action on $\text{Rep}(\mathcal{Q}, \mathbf{d})$ by

$$s \cdot x_e = s^{w_e} x_e. \quad (3.15)$$

This commutes with the $G_{\mathbf{d}}$ -action, and hence descends to an S^1 -action on $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ (as well as $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})//G_{\mathbf{w}}$). The moment map for this action is given by

$$\mu_{S^1}(x) = \frac{1}{2} \sum_{e \in \mathcal{E}} w_e |x_e|^2 \quad (3.16)$$

which is non-negative. We have the following lemma.

Lemma 3.2.1. *Every quiver variety $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ admits an S^1 -action of the form $s \cdot [x_e] = [s^{w_e} x_e]$ such that the moment map is proper and bounded below.*

Proof. By the preceding discussion, we need only show that the weights w_e may always be chosen so that μ_{S^1} is proper on $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$. We can always arrange this by taking $w_e = 1$

for every $e \in \mathcal{E}$, since in this case μ_{S^1} is proper on $\text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w})$, before quotienting, and hence descends to a proper map on $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$. \square

Remark 3.2.2. Note that the action given by taking $w_e = 1$ for all $e \in \mathcal{E}$ is, in some sense, the worst case scenario. It is often the case that we can pick a different S^1 -action, which is more convenient for certain purposes. The most important special case is when \mathcal{Q} is an acyclic quiver. Then for its double we have $\text{Rep}(\overline{\mathcal{Q}}, \mathbf{v}, \mathbf{w}) \cong T^*\text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w})$ and it suffices to take the weight one action on the fibers, i.e. $t \cdot (x, y) = (x, ty)$. In worked examples, we will always make it clear what S^1 -action we have chosen.

The only remaining condition we must check for circle compactness of $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ is that the fixed-point set is compact. Since μ_{S^1} is proper, the fixed-point set is a union of compact subvarieties. What remains to be shown is that this is a finite union, i.e. that there are only finitely many connected components of the fixed-point set. We will do this by enumerating the connected components combinatorially.

Suppose that we have fixed an S^1 -action on $\text{Rep}(\mathcal{Q}, \mathbf{d})$ as above, and denote this action by $x_e \mapsto s \cdot x_e$. Now suppose that there is some S^1 action on $D = V \oplus W$. This induces a new S^1 -action on $\text{Rep}(\mathcal{Q}, \mathbf{d})$ which we denote by $x_e \mapsto s * x_e$.

Definition 3.2.3. Given two S^1 -actions on $\text{Rep}(\mathcal{Q}, \mathbf{d})$, denoted \cdot and $*$, a *compatible representation* is a representation $x \in \text{Rep}(\mathcal{Q}, \mathbf{d})$ such that for all $s \in S^1$, $s \cdot x = s * x$.

The set of compatible representations may be naturally identified with a vector subspace isomorphic to $\text{Rep}(\tilde{\mathcal{Q}}, \tilde{\mathbf{d}})$, for some auxiliary quiver $(\tilde{\mathcal{Q}}, \tilde{\mathbf{d}})$. We can take the vertex set \tilde{I} to consist of the direct summands $D_i(\lambda)$ in the weight space decomposition of D_i , for $i \in \mathcal{I}$, and the dimension of each vertex to be $\dim D_i(\lambda)$. For each edge $e \in \mathcal{E}$ mapping i to j , we add edges in $\tilde{\mathcal{Q}}$ from $D_i(\lambda)$ to $D_j(\lambda')$, for all λ, λ' such that $\lambda' - \lambda$ is equal to the weight of the edge e under the \cdot action.

Example 3.2.4. Consider the quiver with one vertex and a single edge taking this vertex to itself. Let the S^1 -action be given by weight 1 on this edge. Consider the case of dimension vector (3). In order to produce a compatible representation with at least one edge, the S^1 -action on \mathbf{C}^3 must have at least two distinct weights differing by 1. The inequivalent choices of weights are then $(0, 0, 1)$, $(0, 1, 1)$, $(0, 1, 2)$, and $(0, 1, a)$ with $a \geq 3$. These lead to the quivers in the Figure 3.4 below.

Remark 3.2.5. It is clear that there are only finitely many quivers produced by this construction for any finite quiver \mathcal{Q} with dimension vector \mathbf{d} .



Figure 3.4: Compatible representations of the quiver with one node and a single edge loop, with dimension vector $\mathbf{d} = (3)$.

Proposition 3.2.6. *Let \mathcal{Q} be a quiver and suppose that we have picked an S^1 -action on $\text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w})$ as above, inducing an S^1 -action on $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$. Then every connected component of the fixed-point set consists of equivalence classes of compatible representations, for some auxiliary S^1 -action on $V \oplus W$ which acts trivially on W . Consequently, the S^1 -fixed point set of $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ consists of finitely many connected components.*

Proof. By Proposition 2.2.6, on any connected component of $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})^{S^1}$ we can choose a homomorphism $\phi : S^1 \rightarrow G_{\mathbf{v}}$ such that for each edge x , we have

$$s \cdot x = s^{w_e} x = \phi(g) \cdot x = \phi(g)_j x \phi(g)_i^{-1}. \quad (3.17)$$

Moreover, since S^1 is abelian, we can assume (by conjugation by an element of $G_{\mathbf{v}}$ if necessary) that ϕ maps to the maximal torus of $G_{\mathbf{v}}$. Hence the vector space V_i associated to each vertex $i \in \mathcal{V}$ decomposes into weight spaces, and equation (3.17) implies that for each edge x mapping i to j , the component of x mapping weight space $V_i(\lambda)$ to $V_j(\nu)$ is identically zero unless $\nu - \lambda = w_e$. Hence every representation on this connected component is a compatible representation as described above. \square

Example 3.2.7. We consider the double of the polygon space of Example 3.1.3, with the S^1 -action given by the fiberwise action as in Remark 3.2.2. We assume that the moment map level is chosen so that the polygon space is non-empty. Figure 3.5 depicts all possible components of the fixed-point set. (Not all of these will be non-empty, as this depends on the particular choice of moment map level.)

3.2.2 Morse Theory with $\mu_{\mathbf{w}}$ and $|\mu_{\mathbf{w}}|^2$

In this section we would like to understand the $G_{\mathbf{w}}$ -equivariant Morse stratification on $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ with respect to the $G_{\mathbf{w}}$ -moment map. We will assume throughout that $\mu_{\mathbf{w}}$ is proper on $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$.

Before we proceed, let us note that there is a natural direct sum operation

$$\text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w}) \oplus \text{Rep}(\mathcal{Q}, \mathbf{v}', \mathbf{w}') \hookrightarrow \text{Rep}(\mathcal{Q}, \mathbf{v} + \mathbf{v}', \mathbf{w} + \mathbf{w}'), \quad (3.18)$$

given by sending a pair of representations (x, x') to the representation $x \oplus x'$. We define

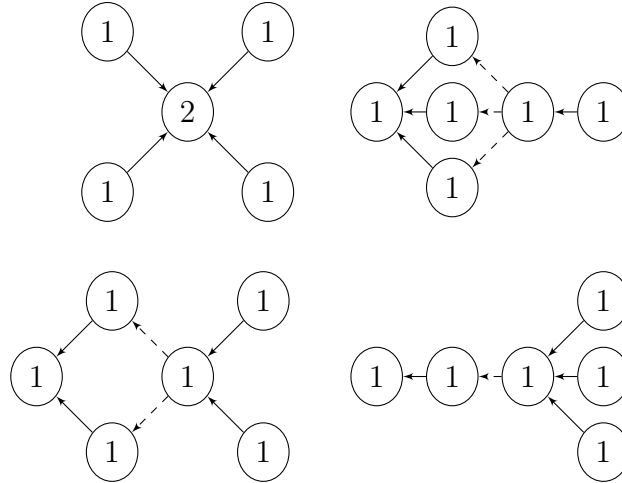


Figure 3.5: The four possible types of fixed-point set for the S^1 -action on the doubled polygon space.

a *partition* of (\mathbf{v}, \mathbf{w}) to be a finite sequence of dimension vectors $(\mathbf{v}_1, \mathbf{w}_1), \dots, (\mathbf{v}_l, \mathbf{w}_l)$ such that $(\mathbf{v}_1, \mathbf{w}_1) + \dots + (\mathbf{v}_l, \mathbf{w}_l) = (\mathbf{v}, \mathbf{w})$. Denote by $\mathcal{P}(\mathbf{v}, \mathbf{w})$ the set of all partitions of (\mathbf{v}, \mathbf{w}) . For each $\lambda \in \mathcal{P}(\mathbf{v}, \mathbf{w})$, there is a subspace of $\text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w})$ consisting of those representations which decompose as a direct sum according to the partition λ . Such representations can be enumerated diagrammatically, as shown by the following lemma, which follows directly from the definition of direct sum.

Lemma 3.2.8. *Let $\lambda \in \mathcal{P}(\mathbf{v}, \mathbf{w})$. Then the representations of \mathcal{Q} that decompose according to λ are naturally identified with representations of $|\lambda|$ disjoint copies of \mathcal{Q} , where $|\lambda|$ denotes the length of the partition.*

Example 3.2.9. Consider the star quiver as in Figure 3.2 with dimension vector $\mathbf{d} = (2, 1, 1, 1, 1)$, and consider the partition $\mathbf{d} = (1, 1, 0, 0, 0) + (1, 0, 1, 0, 0) + (0, 0, 0, 1, 1)$. Then we can represent the space of representations which decompose according to this partition by Figure 3.6.

Theorem 3.2.10. *Let $\beta \in t_{\mathbf{w}}$ be a topological generator of a torus $T_\beta \subseteq T_{\mathbf{w}}$. Then $[x] \in \mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ is fixed by T_β only if there is a representative x which decomposes as a direct sum for some partition $\lambda \in \mathcal{P}(\mathbf{v}, \mathbf{w})$. Conversely, if x decomposes as a non-trivial direct sum, then $[x]$ is fixed by some non-trivial sub-torus of $T_{\mathbf{w}}$.*

Proof. Since $[x]$ is fixed by T_β , for each $s \in T_\beta$, there is some $\phi(s) \in G_{\mathbf{v}}$ such that

$$s \cdot x = \phi(s) \cdot x, \tag{3.19}$$

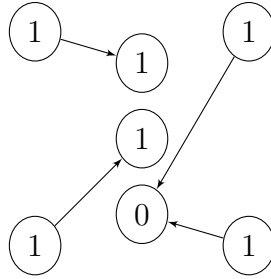


Figure 3.6: Diagrammatic depiction of the representations decomposing according to the partition described above.

and in particular we get a homomorphism $\phi : T_\beta \rightarrow G_{\mathbf{v}}$. Define subtorus $T \subset G_{\mathbf{d}}$ as the image of $s \mapsto s\phi(s)^{-1}$. Then we can decompose $V \oplus W$ into weight spaces under the action of T , and the above equation implies that for each edge e , the component x_e decomposes as a direct sum.

Conversely, if x decomposes according to a nontrivial partition λ , we can always choose a subtorus $T_\beta \subset T_{\mathbf{w}}$ and a map $\phi : T_\beta \rightarrow T_{\mathbf{v}}$ such that the weight space decomposition of $V \oplus W$ corresponds to λ . Hence $[x]$ will be fixed by T_β . \square

Remark 3.2.11. The above theorem gives an explicit enumeration of the connected components of the critical set of $|\mu_{\mathbf{w}}|^2$ in terms of simpler quivers, since every critical point of $|\mu_{\mathbf{w}}|^2$ is conjugate to a point fixed by some $T_\beta \subseteq T_{\mathbf{w}}$. By the results of Section 2.2.3, we may explicitly compute the Morse indices, and inductively compute the equivariant cohomology.

Remark 3.2.12. In the special case $\mathcal{W} = \mathcal{I}, \mathcal{V} = \emptyset$, all of the edges are even and so representations fixed by T_β decompose as direct sums in the ordinary sense. Additionally, in this special case we may drop the hypothesis that $\mu_{\mathbf{w}}$ is proper, since by Theorem 2.3.11 the gradient flow of $|\mu_{\mathbf{w}}|^2$ always converges. This special case was studied in detail in [HW11], where it was proved that the $G_{\mathbf{d}}$ -equivariant Morse stratification agrees with the algebraic stratification induced by the Harder-Narasimhan filtration.

3.2.3 Cohomology of Subvarieties

The results of Section 3.2.1 show that any quiver variety $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ is circle compact, and moreover that the circle action can be chosen so that the fixed-point set is a finite union of simple quiver varieties. The results of Chapter 2 and Section 3.2.2 then allow us to compute the Betti numbers of the connected components of the fixed-point set, and in addition we may compute the cohomology *rings* of these compact quiver varieties using the more sophisticated techniques of Chapter 4.

However, hyperkähler quiver varieties are defined as *subvarieties* cut out by the complex moment map equation, so there is still some work to be done. In this section we will describe a technique that allows us to piggyback the results of Chapter 2 and the preceding sections, so as to settle this question for a large class of subvarieties of quiver varieties. This technique is heavily computational, and we will give a detailed case study in Section 3.3 below.

Let us fix a quiver \mathcal{Q} with dimension vector $\mathbf{d} = (\mathbf{v}, \mathbf{w})$, and assume that we have fixed a circle action so that $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ is circle compact. Let $f : \text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w}) \rightarrow E$ be some $G_{\mathbf{d}} \times S^1$ -equivariant polynomial in the edges, and assume that $V(f) \subset \mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ is smooth. Note that

$$V(f)^{S^1} = V(f) \cap \mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})^{S^1} \tag{3.20}$$

and hence

$$\text{Crit} \left(|\mu_{\mathbf{w}}|^2|_{V(f)} \right) = V(f) \cap \text{Crit} |\mu_{\mathbf{w}}|^2, \tag{3.21}$$

so that the Morse stratifications with respect to the S^1 -action and to $|\mu_{\mathbf{w}}|^2$ on $V(f)$ can be understood by the combinatorial arguments of Sections 3.2.1 and 3.2.2. Putting everything together, we have the following theorem.

Theorem 3.2.13. *Let $(\mathcal{Q}, \mathbf{v}, \mathbf{w})$ be a quiver with dimension vector, and suppose that we have chosen a circle action on $\text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w})$ as in the preceding sections so that $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ is circle compact and such that $f : \text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w}) \rightarrow E$ is $S^1 \times G_{\mathbf{d}}$ -equivariant, and such that $V(f)$ is smooth. Then we may compute $P_t(V(f)//G_{\mathbf{w}})$ by the following procedure.*

1. *By Morse theory with the S^1 -action, express the Poincaré polynomial of $V(f)//G_{\mathbf{w}}$ in terms of the Poincaré polynomials of the connected components of $V(f)^{S^1}//G_{\mathbf{w}}$. Working one connected component at a time, we may now assume that $\mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ is compact.*
2. *By equivariant formality, $P_t^{G_{\mathbf{w}}}(V(f)) = P_t(BG_{\mathbf{w}})P_t(V(f))$.*
3. *Compute $P_t(V(f))$ by Morse theory with a generic component of $\mu_{\mathbf{w}}$.*
4. *Compute $P_t(V(f)//G_{\mathbf{w}})$ by Morse theory with $|\mu_{\mathbf{w}}|^2$. The equivariant cohomology of the higher critical sets of $|\mu_{\mathbf{w}}|^2$ may be computed from the ordinary cohomology of subvarieties of simple quiver varieties, which we compute by induction.*

Remark 3.2.14. The crucial hypothesis is that we have chosen a decomposition $\mathcal{I} = \mathcal{V} \sqcup \mathcal{W}$ so that $V(f) \subset \mathfrak{X}_\alpha(\mathbf{v}, \mathbf{w})$ is smooth, so that our reduction in stages procedure involves only smooth varieties. We do not claim that it is always possible to do this, but we

are not aware of any counterexamples. We suspect that it is always possible for acyclic quivers. We will see in the following section that it is always possible for star-shaped quivers.

3.3 Rank Three Star Quivers

3.3.1 Star Quivers

In this section we will illustrate the techniques developed in this chapter by giving a detailed calculation of the cohomology of the fixed-point sets of a particular family of hyperkähler quiver varieties. We will revisit this family in Chapters 4 and 5.

The family that we consider are the hyperkähler varieties associated to the quiver pictured in Figure 3.7. In the rank two case (where the sink has dimension two rather than three), the Kähler quiver varieties are polygon spaces [Kly94, HK98]. Their hyperkähler analogues are called hyperpolygon spaces, and have been well-studied [Kon00b, HP05a]. In particular, hyperkähler Kirwan surjectivity is known to hold and their cohomology rings have been explicitly computed.

In the rank three case, the calculations become considerably more involved. It will take the remainder of this chapter to compute their Betti numbers, and we will not show that hyperkähler Kirwan surjectivity holds until Chapter 4. Our Morse theory calculation closely parallels that of rank three parabolic Higgs bundles [GPGMn07]. This is no coincidence, as we shall see in Chapter 5.

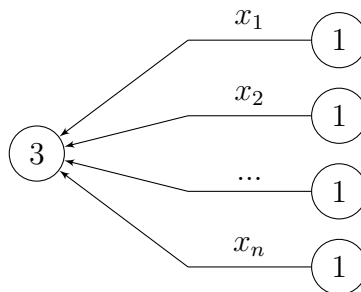


Figure 3.7: Rank three star quiver.

Let x_1, \dots, x_n denote the incoming edges, and let y_1, \dots, y_n denote the outgoing edges on the doubled quiver. We will explicitly write these as column and row vectors, respectively. Note that the overall diagonal S^1 subgroup of $G_{\mathbf{d}}$ acts trivially on $\text{Rep}(\mathcal{Q}, \mathbf{d})$, so strictly speaking we should quotient only by $PG_{\mathbf{d}} := G_{\mathbf{d}}/S^1$. We have $\mathfrak{pg}_{\mathbf{w}} \cong \mathfrak{su}_3 \oplus \mathbf{R}^n$ and hence we take (central) moment map levels $\alpha \in \mathbf{R}^n$. We always assume that the

components α_i of α are taken to be positive. We denote the Kähler and hyperkähler quotients by $\mathfrak{X}_\alpha(n)$ and $\mathfrak{M}_\alpha(n)$, respectively. In what follows, we adopt the following notation. We denote by $[n]$ the set $\{1, \dots, n\}$, and for a subset $S \subseteq [n]$, we denote by α_S the sum

$$\alpha_S = \sum_{i \in S} \alpha_i. \quad (3.22)$$

When it is clear from context, we will denote $\alpha_{[n]} = \sum_{i=1}^n \alpha_i$ simply by α .

Proposition 3.3.1. *The real and complex moment maps on $T^*\text{Rep}(\mathcal{Q}, \mathbf{d})$ are given by*

$$\mu_r = \frac{1}{2} \left((xx^* - y^*y)_0, |y_1|^2 - |x_1|^2, \dots, |y_n|^2 - |x_n|^2 \right) \in \mathfrak{su}_3 \oplus \mathbf{R}^n, \quad (3.23)$$

$$\mu_c = ((xy)_0, -y_1x_1, \dots, -y_nx_n) \in \mathfrak{sl}_3 \oplus \mathbf{C}^n, \quad (3.24)$$

where $(\cdot)_0$ denotes the traceless part of a matrix. For a regular value $(\alpha, 0)$ of μ_{HK} , both $\mathfrak{X}_\alpha(n)$ and $\mathfrak{M}_\alpha(n)$ are smooth.

Proof. The formulas for the moment maps follow by direct calculation. Smoothness follows from [Nak94, Theorem 2.8] (this is a general fact about quiver varieties). \square

3.3.2 Critical Sets for the Circle Action

The first order of business is to reduce the problem to computing the Betti numbers of compact subvarieties, by choosing an appropriate circle action. We define the action on $T^*\text{Rep}(\mathcal{Q}, \mathbf{d})$ to be

$$s \cdot (x, y) = (x, sy). \quad (3.25)$$

The moment map for this action is

$$\mu_{S^1}(x, y) = \frac{1}{2} \sum_i |y_i|^2. \quad (3.26)$$

Lemma 3.3.2. *This action makes $\mathfrak{M}_\alpha(n)$ circle compact.*

Proof. From the moment map equations,² we have

$$|x_i|^2 = \alpha_i + |y_i|^2, \quad (3.27)$$

²To avoid factors of $1/2$, we are taking the reduction at moment map level $\alpha/2$.

for each $i \in [n]$. Since the moment map for the S^1 -action is given by $\frac{1}{2}|y|^2$, we have that

$$|x|^2 + |y|^2 = \sum_i |x_i|^2 + |y|^2 = \sum_i \alpha_i + 2|y|^2 = \sum_i \alpha_i + 2\mu_{S^1}(x, y), \quad (3.28)$$

and hence μ_{S^1} is proper on $\mathfrak{M}_\alpha(n)$. \square

Now that we have a circle action making $\mathfrak{M}_\alpha(n)$ circle compact, we must enumerate the connected components of the fixed-point set. A point $[x, y] \in \mathfrak{M}_\alpha(n)$ will be fixed only if there is a homomorphism $\phi : S^1 \rightarrow U(3) \times (S^1)^n$ such that

$$x_i = g(s)x_i h_i^{-1}(s), \quad (3.29)$$

$$s y_i = h_i(s) y_i g(s)^{-1}, \quad (3.30)$$

for each $i \in [n]$, where we have denoted the components of ϕ by $g \in U(2)$ and $h_i \in S^1$. Since the moment map equations force $x_i \neq 0$ for each $i \in [n]$, we have that x_i is an eigenvector of $g(s)$ with eigenvalue $h_i(s)$. The second equation shows that y_i^* is an eigenvector of $g(s)$ with eigenvalue $s^{-1}h_i(s)$.

There are four possible cases depending on the multiplicities of the eigenvalues of $g(s)$. In the first case, suppose that $g(s)$ has a single eigenvalue of multiplicity three. Then this forces all y_i corresponding to $\mathfrak{X}_\alpha(n) \subset \mathfrak{M}_\alpha(n)$. We call this type (3).

Next suppose that $g(s)$ has two distinct eigenvalues. In order to avoid the previous case, at least one of the y_i must be non-zero. Hence the eigenvalues of $g(s)$ are of the form $(\lambda, \lambda s^{-1})$. There are two possible cases for their multiplicities: $(\lambda, \lambda, s^{-1}\lambda)$ and $(\lambda, s^{-1}\lambda, s^{-1}\lambda)$. We call these type (2, 1) and (1, 2), respectively. Each of these types corresponds to many connected components of the critical set, enumerated by the possible choices of $h_i(s)$. Since every x_i is non-zero, $h_i(s)$ can only take the value λ or $s^{-1}\lambda$. Hence each particular connected component is indexed by the set $S = \{i \in [n] \mid h_i(s) = \lambda\}$. These are pictured in Figure 3.8 below.

In the last case, we assume that $g(s)$ has three distinct eigenvalues. We can assume that they are of the form $(\lambda, s^{-1}\lambda, s^{-a}\lambda)$ and with $a \neq 1, 0, -1$. Now, if α is assumed to be generic so that $\mathfrak{M}_\alpha(n)$ is smooth, by Theorem 3.2.10 no representation $[x, y] \in \mathfrak{M}_\alpha(n)$ admits a non-trivial direct sum decomposition. Hence the quiver representing this connected component must be connected, and the only way to obtain this is to take $a = 2$. We call this a type (1, 1, 1) critical set. The type (1, 1, 1) critical sets are indexed by the choice of

$$S_1 = \{i \in [n] \mid h_i(s) = \lambda\}, \quad (3.31)$$

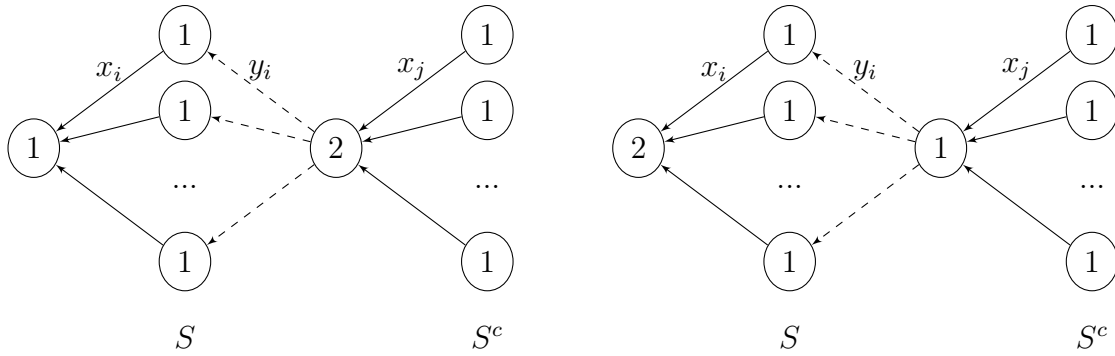


Figure 3.8: Left: type (1, 2); right: type (2, 1).

$$S_2 = \{i \in [n] \mid h_i(s) = s^{-1}\lambda\}, \tag{3.32}$$

$$S_3 = \{i \in [n] \mid h_i(s) = s^{-2}\lambda\}. \tag{3.33}$$

The diagrammatic representation of the type (1, 1, 1) critical set is pictured in Figure 3.9.

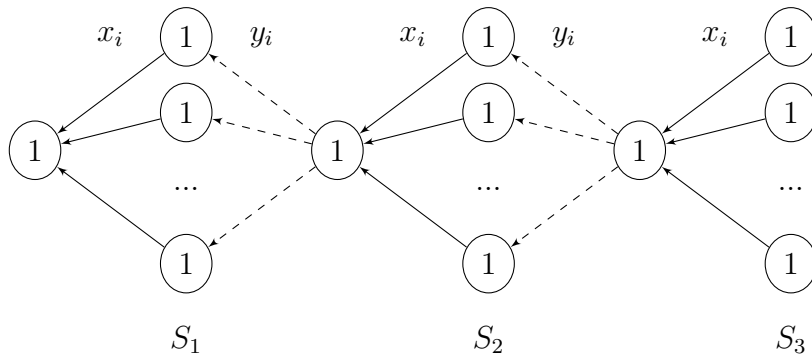


Figure 3.9: Type (1, 1, 1) critical set.

We summarize the above arguments in the following theorem.

Proposition 3.3.3. *Let \mathfrak{M}_α be a rank 3 star quiver variety, and assume that α is generic. Then the connected components of $\mathfrak{M}_\alpha^{S^1}$ can be classified into three types: (3), (2, 1), (1, 2), and (1, 1, 1). Type (3) consists of the single component $\mathfrak{X}_\alpha(n)$. The type (2, 1) and (1, 2) critical sets are indexed by a subset $S \subset [n]$, and the type (1, 1, 1) critical sets are indexed by a pair of disjoint subsets $S_1, S_2 \subset [n]$.*

In the remaining sections, we will show how to use Theorem 3.2.13 to compute the Betti numbers of these critical sets. The homotopy type of $\mathfrak{M}_n(\alpha)$ does not depend on the particular choice of α , as long as it is generic,³ so we can simplify the calculation by

³This is a well-known fact about hyperkähler quotients. Various special cases appear in the literature,

choosing α to be *extreme*—we can ensure that the type (3) and (1, 2) critical sets are always empty. This reduces the problem to calculating the Betti numbers of the type (2, 1) and (1, 1, 1) critical sets, and of these only type (2, 1) turns out to be non-trivial. However, it should be emphasized that the method of extreme stability is in no way whatsoever essential—it is just a trick to reduce the amount of calculation necessary.

3.3.3 The Type (3) Critical Set

As noted above, the type (3) critical set is just the Kähler quiver variety $\mathfrak{X}_\alpha \subset \mathfrak{M}_\alpha$, and thus has index 0, since it is the absolute minimum of μ_{S^1} . Its Poincaré polynomial can be computed straightforwardly using the equivariant Morse theory of Chapter 2. However, we can avoid this calculation entirely simply by choosing α to be extreme.

Proposition 3.3.4. *Suppose that α satisfies $\alpha_n > \sum_{i=1}^{n-1} \alpha_i$. Then the Kähler quiver variety $\mathfrak{X}_\alpha(n)$ is empty.*

Proof. The moment map equations are

$$\sum_i (x_i x_i^*)_0 = 0, \quad (3.34)$$

$$|x_i|^2 = \alpha_i. \quad (3.35)$$

Now consider the map sending x_i to $v_i := (x_i x_i^*)_0 \in \mathfrak{su}_3$. The first moment map equation demands that $\sum v_i = 0$, and hence

$$|v_n| = \left| \sum_{i=1}^{n-1} v_i \right| \leq \sum_{i=1}^{n-1} |v_i|. \quad (3.36)$$

On the other hand, we have

$$|v_i|^2 = \text{Tr}(v_i v_i^*) \quad (3.37)$$

$$= \text{Tr} \left(\left(x_i x_i^* - \frac{1}{3} |x_i|^2 \right) \left(x_i x_i^* - \frac{1}{3} |x_i|^2 \right) \right) \quad (3.38)$$

$$= |x_i|^4 - \frac{2}{3} |x_i|^4 + \frac{1}{3} |x_i|^4 \quad (3.39)$$

$$= \frac{2}{3} |x_i|^4 = \frac{2}{3} \alpha_i^2. \quad (3.40)$$

e.g. [Nak94, Corollary 4.2], and in our case it is easily deduced from Corollary 2.3.17.

Hence if we choose $\alpha_n > \sum_{i=1}^{n-1} \alpha_i$, then $|v_n|$ violates the above upper bound, and there can be no solution to the moment map equations. \square

3.3.4 The Type (1, 2) Critical Sets

Next we consider the type (1, 2) critical sets. Their Poincaré polynomials can be computed in a manner analogous to the technique described below for the type (2, 1). However, we can always eliminate this case by choosing α to be extreme.

Proposition 3.3.5. *The type (1, 2) critical set indexed by $S \subset [n]$ is empty unless the following conditions hold: (i) $\alpha_{S^c} > 2\alpha_S$, (ii) $\#S \geq \max(1, 6 - n)$, and (iii) $\#S^c \geq 2$. The isotropy representation at any point in this connected component of the critical set is given by*

$$n + \#S - 6 + (n + \#S - 6)t + (\#S^c - 2)t^2 + (\#S^c - 2)t^{-1} \quad (3.41)$$

In particular, the complex dimension of this connected component is $n + \#S - 6$ and its Morse index is $2(\#S^c - 2)$.

Proof. A type (1, 2) critical set labelled by $S \subset [n]$ corresponds to a map $\phi : S^1 \rightarrow U(3) \times (S^1)^n$ given by

$$g(s) = \text{diag}(1, s^{-1}, s^{-1}), \quad h_i(s) = 1 \text{ for } i \in S, \quad h_i(s) = s^{-1} \text{ for } i \in S^c. \quad (3.42)$$

For $i \in S$, we have $x = (*, 0, 0)^T$, $y_i = (0, *, *)$, and for $i \in S^c$ we have $x_i = (0, *, *)^T$, $y_i = 0$. Then the real moment map equation $\sum_i x_i x_i^* - y_i^* y_i = (\alpha/3)1_{3 \times 3}$ gives

$$\sum_{i \in S} |x_i|^2 = \alpha/3, \quad (3.43)$$

$$\sum_{i \in S^c} |x_i|^2 - \sum_{i \in S} |y_i|^2 = 2\alpha/3. \quad (3.44)$$

Using the real moment map equation $|x_i|^2 = |y_i|^2 + \alpha_i$, we find

$$\sum_i |y_i|^2 = \sum_{i \in S} |y_i|^2 = \sum_{i \in S} (|x_i|^2 - \alpha_i) = \alpha/3 - \alpha_S = \frac{\alpha_{S^c} - 2\alpha_S}{3}, \quad (3.45)$$

from which we deduce $\alpha_{S^c} > 2\alpha_S$. From equation 3.43 we see that S is nonempty, so $\#S \geq 1$. From the real moment map equation, we also see that $\sum_{i \in S^c} x_i x_i^*$ has rank 2,

so $\#S^c \geq 2$. Finally, the map ϕ determines representations

$$[T_x \text{Rep}(\mathcal{Q})] = 2n + \#S + (2n + \#S)t + (\#S^c)t^{-1} + (\#S^c)t^2 \quad (3.46)$$

$$[\mathfrak{g}] = n + 4 + 2t + 2t^{-1} \quad (3.47)$$

$$[E] = 2 + (n + 4)t + 2t^2. \quad (3.48)$$

Using Theorem 2.2.10 we obtain the isotropy representation as desired. \square

Now using condition (1) above, we can ensure that neither the type (3) nor the type (1, 2) critical sets contribute to $P_t(\mathfrak{M}_\alpha(n))$.

Proposition 3.3.6. *Suppose that α satisfies $\alpha_n > 2 \sum_{i=1}^{n-1} \alpha_i$. Then there are no type (1, 2) critical sets.*

Proof. We repeat essentially the same argument as in the type (3) case. From the proof of Proposition 3.3.5 above, we have that $|x_i|^2 = \alpha_i$ for $i \in S^c$, and that

$$\sum_{i \in S^c} x_i x_i^* = \frac{\alpha_{S^c}}{3} 1_{3 \times 3}. \quad (3.49)$$

Now, the condition $\alpha_{S^c} > 2\alpha_S$ implies that necessarily $n \in S^c$. As in the type (3) case, consider the vectors $v_i := (x_i x_i^*)_0$. The moment map equation implies $\sum v_i = 0$, so by the triangle inequality we find

$$|v_n| \leq \sum_{i=1}^{n-1} |v_i|, \quad (3.50)$$

but as in the type (3) case this contradicts $\alpha_n > 2 \sum_{i=1}^{n-1} \alpha_i > \sum_{i=1}^{n-1} \alpha_i$. \square

3.3.5 The Type (2, 1) Critical Sets

Finally, we encounter the first case which cannot be eliminated by extreme stability.

Proposition 3.3.7. *The type (2, 1) critical set is empty unless the following conditions hold: (i) $2\alpha_{S^c} > \alpha_S$, (ii) $\#S \geq 3$, and (iii) $\#S^c \geq 1$. The isotropy representation of any point is given by*

$$2\#S - 6 + (2\#S - 6)t + (2\#S^c - 2)t^2 + (2\#S^c - 2)t^{-1}. \quad (3.51)$$

In particular, if it is non-empty then its complex dimension is $2\#S - 6$ and its Morse index is $2(2\#S^c - 2)$.

Proof. A type $(2, 1)$ critical set labelled by $S \subseteq [n]$ corresponds to a homomorphism $\phi : S^1 \rightarrow U(3) \times (S^1)^n$ given by

$$g(s) = \text{diag}(1, 1, s^{-1}), \quad h_i(s) = 1 \text{ for } i \in S, \quad h_i(s) = s^{-1} \text{ for } i \in S^c. \quad (3.52)$$

For $i \in S$, we have $x = (*, *, 0)^T$, $y_i = (0, 0, *)$, and for $i \in S^c$ we have $x_i = (0, 0, *)^T$, $y_i = 0$. Then the real moment map equation $\sum_i x_i x_i^* - y_i^* y_i = (\alpha/3)1_{3 \times 3}$ gives

$$\sum_{i \in S} |x_i|^2 = 2\alpha/3, \quad (3.53)$$

$$\sum_{i \in S^c} |x_i|^2 - \sum_{i \in S} |y_i|^2 = \alpha/3. \quad (3.54)$$

Using the real moment map equation $|x_i|^2 = |y_i|^2 + \alpha_i$ and rearranging, we find

$$\sum_i |y_i|^2 = \frac{2\alpha_{S^c} - \alpha_S}{3}, \quad (3.55)$$

from which we deduce $2\alpha_{S^c} > \alpha_S$. From equation 3.54 we see that S^c is nonempty, so $\#S \geq 1$. From the real moment map equation, we also see that $\sum_{i \in S} x_i x_i^*$ has rank 2, so $\#S \geq 2$. Finally, the subgroup determines representations

$$[T_x \text{Rep}(\mathcal{Q})] = n + 2\#S + (n + 2\#S)t + (2\#S^c)t^2 + (2\#S^c)t^{-1} \quad (3.56)$$

$$[\mathfrak{g}] = n + 4 + 2t + 2t^{-1} \quad (3.57)$$

$$[E] = 2 + (n + 4)t + 2t^2 \quad (3.58)$$

where $E \cong \mathfrak{psl}_2 \oplus \mathbf{C}^n$ is the target space of the complex moment map. Hence by Theorem 2.2.10 we obtain the desired expression for the isotropy representation. \square

We will compute the Poincaré polynomials of the type $(2, 1)$ critical sets by Theorem 3.2.13. To do this, we have to choose a decomposition $\mathcal{I} = \mathcal{V} \sqcup \mathcal{W}$ in order to apply reduction in stages. To ensure that the subvarieties $V(f)$ (defined by the residual part of the complex moment map) are smooth, we make the following choice: $\mathcal{W} = S$. The decorated quiver corresponding to this decomposition is pictured in Figure 3.10.

To simplify things, let us note that up to isomorphism we can ignore the edges x_i for $i \in S^c$. This is pictured in Figure 3.11.

Lemma 3.3.8. *The type $(2, 1)$ critical set indexed by $S \subset [n]$ is isomorphic to the moduli space of representations of the quiver with the edges x_i deleted for $i \in S^c$, as in Figure*

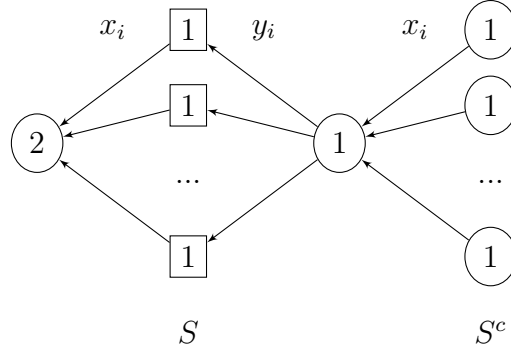


Figure 3.10: The reduction in stages used for the type $(2, 1)$ critical set.

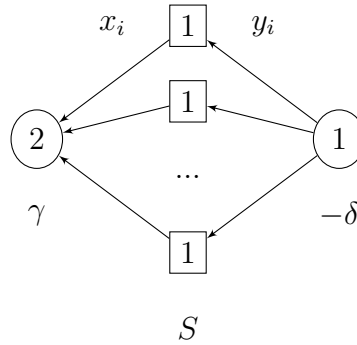


Figure 3.11: Simplification of the type $(2, 1)$ critical set, with the moment map level (γ, δ) as indicated.

3.11. Under this isomorphism, the moment map level at the leftmost vertex is given by $(\gamma/2)\mathbf{1}_{2 \times 2}$ where $\gamma = 2\alpha/3$, and at the rightmost vertex by $-\delta$, where $\delta = (2\alpha_{S^c} - \alpha_S)/3$.

Proof. For each $i \in S^c$, the moment map equation is given by $|x_i|^2 = \alpha_i$. From Proposition 3.3.7, the moment map equation at the vertex is

$$\sum_{i \in S^c} |x_i|^2 - \sum_{i \in S} |y_i|^2 = \frac{\alpha}{3} \tag{3.59}$$

Now, since for $i \in S^c$ x_i is a 1×1 matrix of norm α_i , up to the S^1 action we simply have $x_i = \sqrt{\alpha_i}$ and there are no additional moduli. Hence we can simply delete these edges. Rearranging the above equation, we have

$$-\sum_{i \in S} |y_i|^2 = \frac{\alpha}{3} - \sum_{i \in S^c} |x_i|^2 = \frac{2\alpha_{S^c} - \alpha_S}{3} = -\delta, \tag{3.60}$$

as claimed. The moment map equation

$$\sum_i x_i x_i^* = \frac{\alpha}{3} 1_{3 \times 3} + \sum_i y_i^* y_i, \quad (3.61)$$

also implies $\sum_{i \in S} x_i x_i^* = (\alpha/3) 1_{2 \times 2}$, so $\gamma = 2\alpha/3$ as claimed. \square

For the remainder of this section, we let $\mathcal{Q} = (\mathcal{V} \sqcup \mathcal{W}, \mathcal{E})$ denote the type $(2, 1)$ quiver pictured in Figure 3.11. The first step of Theorem 3.2.13 is to compute the Poincaré polynomial of $V(f) \subset \mathfrak{X}_\alpha(\mathcal{Q}, \mathbf{v}, \mathbf{w})$ where $f : \text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w}) \rightarrow E \cong \mathbf{C}^2$ is the residual part of the complex moment map, given by

$$f(x, y) = \sum_{i \in S} x_i y_i. \quad (3.62)$$

Proposition 3.3.9. *The Poincaré polynomial of $V(f) \subset \mathfrak{X}_\alpha(\mathcal{Q}, \mathbf{v}, \mathbf{w})$ is given by*

$$P_t(V(f)) = P_t(\text{Gr}(2, n)) P_t(\mathbf{P}^{n-3}) \quad (3.63)$$

Proof. Pick a generic component β of $\mathfrak{t}_{\mathbf{w}}$ to that $T_\beta = T_{\mathbf{w}}$. (We will make a particular choice of β once we need it to compute Morse indices.) Let $h = \langle \mu_{\mathbf{w}}, \beta \rangle$ so that h is a perfect Morse-Bott function and $\text{Crit}(h) = \mathfrak{X}_\alpha(\mathcal{Q}, \mathbf{v}, \mathbf{w})^{T_{\mathbf{w}}}$. To proceed, we have to compute the $T_{\mathbf{w}}$ -fixed point set of $V(f)$. As in the proof of Theorem 3.2.10, a point $[x, y] \in \mathfrak{X}_\alpha(\mathcal{Q}, \mathbf{v}, \mathbf{w})$ is fixed if there is a map $\phi : T_{\mathbf{w}} \rightarrow G_{\mathbf{v}}$ such that $t \cdot (x, y) = \phi(t) \cdot (x, y)$. If we denote $t = (t_1, \dots, t_n)$ then, writing this out in components, we have the equations

$$g_1(t) x_i = t_i x_i, \quad (3.64)$$

$$y_i g_2(t)^{-1} = t_i^{-1} y_i. \quad (3.65)$$

Now, by the moment map equations, at least two of the x_i must be non-zero and linearly independent, say x_a and x_b with $a < b$. Then the first of the above equations force $g_1(t) = \text{diag}(t_a, t_b)$. On the other hand, the moment map equations also force at least one of the y_i to be non-zero, say y_c . The second equation then forces $g_2(t) = t_c$, and this determines ϕ completely. Furthermore, the residual complex moment map equation is $\sum_{i \in S} x_i y_i = 0$, and this can only be satisfied if $c \neq a, b$. Hence the fixed-points are isolated and are indexed by tuples (a, b, c) with $a < b$ and $c \neq a, b$.

To compute Morse indices, we first compute the isotropy representations using The-

orem 3.2.10. We find

$$[T_x \text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w})] = \sum_l (t_a t_l^{-1} + t_b t_l^{-1} + t_l t_c^{-1}) \quad (3.66)$$

$$[\mathfrak{g}_{\mathbf{v}}] = t_a t_b^{-1} + t_b t_a^{-1} + 3 \quad (3.67)$$

$$[E] = t_a t_c^{-1} + t_b t_c^{-1}. \quad (3.68)$$

If we take the particular generic component $(1, \epsilon, \epsilon^2, \dots, \epsilon^{n-1})$ for ϵ small, then a weight of the form $t_i t_j^{-1}$ is negative if and only if $j < i$. Hence the Morse index is given by

$$\lambda(a, b, c) = 2n - 6 + 2a + 2b + 2c - 2d, \quad (3.69)$$

where $d = 0$ if $c < a$, 1 if $a < c < b$, and 0 if $c > b$. Hence

$$P_t(V) = \sum_{a=1}^{n-1} \sum_{b=a+1}^n \sum_{c \neq a, b} t^{\lambda(a, b, c)} \quad (3.70)$$

and with a bit of algebraic manipulation this yields the claimed formula for $P_t(V(f))$. \square

Remark 3.3.10. A quicker way to calculate $P_t(V)$ is to simply observe that the projection $[x, y] \mapsto [x]$ makes $V(f)$ the total space of a fiber bundle over $Gr(2, n)$ with fiber \mathbf{P}^{n-3} . However, Morse theory with the $T_{\mathbf{w}}$ -action provides a systematic way to compute $V(f)$ for a large class of subvarieties of quiver varieties without relying on such coincidences.

Now that we have calculated the Poincaré polynomial of $P_t(V(f))$, we must next enumerate the connected components of the critical set of $|\mu_{\mathbf{w}}|^2$. By Theorem 3.2.10, these are enumerated by the possible direct sum decompositions. At the left-most vertex, which has dimension 2, there are two possibilities: either it does not decompose, or it decomposes as $1 + 1$. We call these two possibilities type (2) and type (1, 1) respectively.

First let us consider type (2). Under a direct sum decomposition of this form, the only possibility is that some of the edges are identically zero. Define

$$S = \{i \in [n] \mid x_i \neq 0 \text{ generically}\}, \quad (3.71)$$

$$T = \{i \in [n] \mid y_i \neq 0 \text{ generically}\}. \quad (3.72)$$

The type (2) critical set corresponding to the choice $S, T \subset [n]$ is pictured in Figure 3.12.

Now, suppose that $S \cap T \neq \emptyset$, and let $i \in S \cap T$. Since $x_i \neq 0$ generically, this forces $g_1 = \text{diag}(t_i, t_i)$. Similarly, since $y_i \neq 0$ generically, we have $g_2 = t_i$. Hence the generic stabilizer is given by the subtorus (s_1, \dots, s_n) of $T_{\mathbf{w}}$, where $s_l = t_i$ for $l \in S \cup T$, and

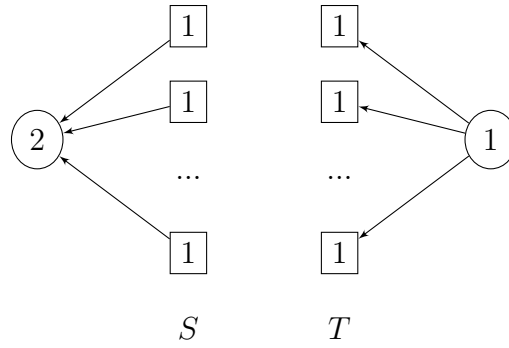


Figure 3.12: The type (2) stratum of the type (2, 1) critical set.

$s_l = t_l$ for $l \neq S \cup T$. However, the fixed-point set of this stabilizer subgroup consists of

$$\{x_i = 0 \mid i \notin (S \cup T)\} \cup \{y_i = 0 \mid i \notin (S \cup T)\}, \tag{3.73}$$

so in fact $S = T$. This divides the type (2) critical sets into two subcases, $S = T$ and $S \cap T = \emptyset$, which we call type *A* and type *B* respectively.

Next we consider the remaining case, type (1, 1). Now we define S and T as in the type (2) case, and the decomposition $2 = 1 + 1$ further decomposes S as $S = S_1 \sqcup S_2$, as pictured in Figure 3.13. Now, as in the type (2) case there are additional subcases. By arguments analogous to those above, either T is equal to one of the S_i (which we may assume is S_1 , simply by relabelling), or T is disjoint from both S_1 and S_2 . We denote these subcases by type *C* and type *D*, respectively.

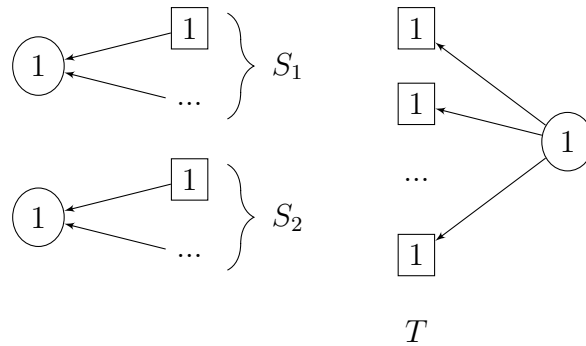


Figure 3.13: The type (1, 1) stratum on the type (2, 1) quiver.

Summarizing the above arguments, we have the following theorem.

Theorem 3.3.11. *The critical sets for $|\mu_{\mathbf{w}}|^2$ on the type (2, 1) are classified into the types A, B, C, and D described above. Hence The Poincaré polynomial of the type (2, 1)*

critical set on $\mathfrak{M}_\alpha(n)$ may be computed as

$$P_t(V(f)//G_{\mathbf{w}}) = P_t(BG_{\mathbf{w}})P_t(V(f)) - R_A - R_B - R_C - R_D, \quad (3.74)$$

where R_A, R_B, R_C, R_D are corrections due to the type A, B, C, D critical sets of $|\mu_{\mathbf{w}}|^2$, respectively. These may be computed by Propositions 3.3.14, 3.3.19, 3.3.22, and 3.3.25 below.

Type A Critical Sets

We have a subset $S \subset [n]$ satisfying $S^c \neq \emptyset$. This critical set is given by $x_i = 0, y_i = 0$ for $i \in S^c$.

Lemma 3.3.12. *The stabilizer of a generic point of the type A critical set is the subgroup $T_S \subseteq T_{\mathbf{w}}$ consisting of elements of the form $t_i = s$ for $i \in S$ with no constraint on the t_i for $i \in S^c$. The map $\phi : T_S \rightarrow T_{\mathbf{v}}$ is given by $g_1 = \text{diag}(s^{-1}, s^{-1}), g_2 = s^{-1}$. The isotropy representation is given by*

$$3(\#S) - 7 + \sum_{i \in S^c} (2st_i^{-1} + s^{-1}t_i). \quad (3.75)$$

Proof. The generic stabilizer follows from the definition of the type A critical set. To compute the isotropy representation, we simply compute

$$[\text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w})] = 3(\#S) + \sum_{i \in S^c} (2st_i^{-1} + s^{-1}t_i) \quad (3.76)$$

$$[\mathfrak{g}_{\mathbf{v}}] = 5 \quad (3.77)$$

$$[E] = 2 \quad (3.78)$$

and appeal to Theorem 2.2.10. □

Lemma 3.3.13. *The Morse index λ is given by*

$$\lambda = 2\#S^c + 2\#\{j \in S^c \mid \mu_S + \alpha_j > 0\}. \quad (3.79)$$

where $\mu_S = (\gamma - \delta - \alpha_S)/(\#S)$. The equivariant topology of the type A critical set is that of the type $(2, 1)$ critical set with stability parameters $(\gamma, \alpha', \delta)$, where $\alpha' = \{\alpha'_i\}_{i \in S}$ and $\alpha'_i = \alpha_i + \mu_S$. Hence its Poincaré polynomial may be computed inductively.

Proof. The condition of being critical forces $\mu(x)$ to lie in $\text{stab}(x)$, and hence $\mu_i = \mu_{i'}$ for

$i, i' \in S$. Denote this common value by μ_S . Using the moment map equations, we find

$$(\#S)\mu_S = \sum_{i \in S} (|x_i|^2 - |y_i|^2 - \alpha_i) = \gamma - \delta - \alpha_S. \quad (3.80)$$

Since $x_j, y_j = 0$ for $j \in S^c$, we have $\mu_j = -\alpha_j$ for $j \in S^c$. Appealing to the isotropy representation, we obtain the desired formula for the Morse index. \square

Proposition 3.3.14. *The correction R_A of a type A critical set labelled by $S = T \subset [n]$ is given by*

$$R_A(S) = \frac{t^\lambda}{(1-t^2)^{\#S^c}} P_t(C^{(2,1)}(\gamma, \alpha', \delta)) \quad (3.81)$$

where λ is the Morse index as computed above and $C^{(2,1)}(\gamma, \alpha', \delta)$ denotes a type (2, 1) critical set with moment map level $(\gamma, \alpha', \delta)$, as computed above.

Type B Critical Sets

We have disjoint subsets $S_1, S_2 \subset [n]$ satisfying $\#S_1 \geq 3$ and $\#S_2 \geq 1$. The coordinates (x_i, y_i) satisfy the conditions $x_i = 0$ for $i \in S_1^c$ and $y_i = 0$ for $i \in S_2^c$.

Lemma 3.3.15. *The stabilizer of a generic point of the type B critical set is the subgroup $T_S \subseteq T_{\mathbf{w}}$ consisting of elements of the form $t_i = s_1$ for $i \in S_1$, $t_i = s_2$ for $i \in S_2$ with no constraint on the t_i for $i \in (S_1 \cup S_2)^c$. The map $\phi : T_S \rightarrow T_{\mathbf{v}}$ is given by $g_1 = \text{diag}(s_1^{-1}, s_1^{-1}), g_2 = s_2^{-1}$. The isotropy representation is given by*

$$(2\#S_1 + \#S_2 - 5) + (\#S_1 + 2\#S_2 - 2)s_1s_2^{-1} + \sum_{i \notin S_1 \cup S_2} (2s_1t_i^{-1} + s_2^{-1}t_i) \quad (3.82)$$

Proof. The form of the stabilizer follows from the definition of the type B critical set, and to obtain the isotropy representation, we simply compute

$$[\text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w})] = \sum_{i \in S_1} (2 + s_1s_2^{-1}) + \sum_{i \in S_2} (2s_1s_2^{-1} + 1) + \sum_{i \notin S_1 \cup S_2} (2s_1t_i^{-1} + s_2^{-1}t_i) \quad (3.83)$$

$$[\mathfrak{g}_{\mathbf{v}}] = 5 \quad (3.84)$$

$$[E] = 2s_1s_2^{-1} \quad (3.85)$$

and appeal to Theorem 2.2.10. \square

Lemma 3.3.16. *The Morse index is given by $2\lambda_0 + 4\lambda_1 + 2\lambda_2$, where $\lambda_0 = \#S_1 + 2\#S_2 - 2$*

if $\mu_{S_1} > \mu_{S_2}$ and $\lambda_0 = 0$ otherwise, and

$$\lambda_1 = \#\{i \notin S_1 \cup S_2 \mid \mu_{S_1} + \alpha_i > 0\}, \quad (3.86)$$

$$\lambda_2 = \#\{i \notin S_1 \cup S_2 \mid \mu_{S_2} + \alpha_i < 0\}, \quad (3.87)$$

and where $\mu_{S_1} = (\gamma - \alpha_{S_1})/(\#S_1)$ and $\mu_{S_2} = -(\delta + \alpha_{S_2})/(\#S_2)$. The equivariant topology of the type B critical set is that of a rank 2 polygon space with moment map level $\alpha' = \{\alpha'_i\}_{i \in S}$ and $\alpha'_i = \alpha_i + \mu_{S_1}$.

Proof. The condition of being critical forces $\mu(x)$ to lie in $\text{stab}(x)$, and hence $\mu_i = \mu_{i'}$ for $i, i' \in S_1$ and similarly for $i, i' \in S_2$. Denote these common values by μ_{S_1}, μ_{S_2} , respectively. Using the moment map equations, we find

$$(\#S_1)\mu_{S_1} = \sum_{i \in S_1} (|x_i|^2 - \alpha_i) = \gamma - \alpha_{S_1}, \quad (3.88)$$

$$(\#S_2)\mu_{S_2} = \sum_{i \in S_2} (-|y_i|^2 - \alpha_i) = -\delta - \alpha_{S_2}, \quad (3.89)$$

Since $x_j, y_j = 0$ for $j \in S^c$, we have $\mu_j = -\alpha_j$ for $j \in S^c$. Appealing to the isotropy representation, we obtain the desired formula for the Morse index. \square

Lemma 3.3.17. *The Poincaré polynomial of the polygon space $\mathcal{P}(\alpha)$ may be computed inductively by equivariant Morse theory.*

Proof. The calculation is very similar to those carried out in the remainder of this chapter, so we will just give a sketch. Let $(\mathcal{Q}, \mathbf{d})$ be the star quiver with dimension vector $(2, 1, \dots, 1)$. We choose the decomposition $\mathbf{d} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v} = (2, 0, \dots, 0)$ and $\mathbf{w} = (0, 1, \dots, 1)$. Then $G_{\mathbf{v}} = U(2)$ and $G_{\mathbf{w}} = T_{\mathbf{w}} = (S^1)^n$. If a subtorus $T_\beta \subset T_{\mathbf{w}}$ stabilizes a point $[x]$, then we can choose some homomorphism $\phi : T_\beta \rightarrow G_{\mathbf{v}}$ so that for all $t \in T_\beta$ we have

$$t \cdot x = \phi(t) \cdot x, \quad (3.90)$$

for some representative x of $[x]$. We can assume that $\phi(t) = \text{diag}(t^{\nu_1}, t^{\nu_2})$ for characters ν_1, ν_2 of T_β . Then there are two possible cases: $\nu_1 = \nu_2$, and $\nu_1 \neq \nu_2$. In the first case, the stabilizer T_β must take the form $h_i(t) = t^\nu$, for all $i \in S$ for some subset $S \subset [n]$. The equivariant topology of the critical set is that of the polygon space $\mathcal{P}(\alpha|_S)$, where $\alpha|_S$ denotes the vector $\{\alpha_i + \mu_S\}_{i \in S}$, where $\mu_S = \alpha_{S^c}/(\#S)$. In the second case, the critical set corresponds to a choice of disjoint subsets $S_1, S_2 \subset [n]$ where $h_i(t) = t^{\nu_j}$ for $i \in S_j$. The Morse index is straightforward to compute, and one finds that the equivariant topology

of the critical set is that of a point. By Corollary 2.2.16, we have

$$P_t(\mathcal{P}(\alpha)) = \frac{P_t(\text{Gr}(2, n))}{(1 - t^2)^{n-1}} - \sum_C t^{\lambda_C} P_t^G(C), \quad (3.91)$$

where the sum is over the higher critical sets C of $|\mu|^2$, which we have just enumerated. Since the terms on the right hand side can be expressed in terms of simpler polygon spaces, we obtain a recursion relation for their Poincaré polynomials. \square

Remark 3.3.18. Klyachko [Kly94] obtained a similar recursion relation for the special moment map level $(1, \dots, 1)$. Later, Konno [Kon00b] obtained an explicit formula by studying the circle action on hyperpolygons. In this case, the fixed-point set consists of $\mathcal{P}(\alpha)$ together with a finite union of projective spaces. Hence the Poincaré polynomial of $\mathcal{P}(\alpha)$ can be expressed as a difference of the Poincaré polynomial of its hyperkähler analogue and a weight sum of Poincaré polynomials of projective spaces. By taking an extreme moment map level so that $\mathcal{P}(\alpha)$ is empty, Konno obtains an explicit formula for the Poincaré polynomial of the hyperpolygon space. Since the latter is independent of the moment map level, substituting it into the former yields an explicit expression for the Poincaré polynomial of $\mathcal{P}(\alpha)$.

Proposition 3.3.19. *The correction R_B of a type B critical set labelled by $S_1, S_2 \subset [n]$ is given by*

$$R = \frac{t^\lambda}{(1 - t^2)^{1 + \#S^c}} P_t(\mathcal{P}(\alpha')) \quad (3.92)$$

where λ is the Morse index as computed above and $\mathcal{P}(\alpha')$ is the polygon space with moment map level α' as computed above.

Type C Critical Sets

For the type C critical set, we have disjoint subsets $S_1, S_2 \subset [n]$ satisfying $\#S_1 \geq 2$, $\#S_2 \geq 1$.

Lemma 3.3.20. *The stabilizer of a generic point of the type C critical set is the subgroup $T_S \subseteq T_{\mathbf{w}}$ consisting of elements of the form $t_i = s_j$ for $i \in S_j$, with no constraint on t_i for $i \notin S_1 \cup S_2$. The map $\phi : T_S \rightarrow T_{\mathbf{v}}$ is given by $g_1 = \text{diag}(s_1^{-1}, s_2^{-1}), g_2 = s_1^{-1}$. The isotropy representation is given by*

$$\begin{aligned} 2(\#S_1) + \#S_2 - 4 + (\#S_1 + \#S_2 - 1)s_1^{-1}s_2 + (\#S_2 - 1)s_1s_2^{-1} \\ + \sum_{i \notin S_1 \cup S_2} s_1t_i^{-1} + s_2t_i^{-1} + s_1^{-1}t_i. \end{aligned} \quad (3.93)$$

Proof. The form of the stabilizer follows from the definition of the type C critical set, and to obtain the isotropy representation, we simply compute

$$\begin{aligned} [\text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w})] &= \sum_{i \in S_1} (2 + s_1^{-1} s_2) + \sum_{i \in S_2} (1 + s_1^{-1} s_2 + s_1 s_2^{-1}) \\ &\quad + \sum_{i \notin S_1 \cup S_2} (s_1 t_i^{-1} + s_2 t_i^{-1} + s_1^{-1} t_i) \end{aligned} \quad (3.94)$$

$$[\mathbf{g}_{\mathbf{v}}] = 3 + s_1 s_2^{-1} + s_1^{-1} s_2 \quad (3.95)$$

$$[E] = 1 + s_1^{-1} s_2 \quad (3.96)$$

and appeal to Theorem 2.2.10. \square

Lemma 3.3.21. *Let $\mu_{S_1} = (\gamma/2 - \delta - \alpha_{S_1})/(\#S_1)$ and $\mu_{S_2} = (\gamma/2 - \alpha_{S_2})/(\#S_2)$. Then the Morse index is given by $2\lambda_0 + 2\lambda_1 + 2\#(S_1 \cup S_2)^c$, where*

$$\lambda_0 = \begin{cases} \#S_2 - 1, & \mu_{S_1} > \mu_{S_2} \\ \#S_1 + \#S_2 - 1, & \mu_{S_1} < \mu_{S_2} \end{cases} \quad (3.97)$$

$$\lambda_1 = \#\{i \notin S_1 \cup S_2 \mid \mu_{S_2} + \alpha_i > 0\}, \quad (3.98)$$

The equivariant topology of the type C critical set is that of the projective space $\mathbf{P}^{\#S_1-2}$.

Proof. The condition of being critical forces $\mu(x)$ to lie in $\text{stab}(x)$, and hence $\mu_i = \mu_{i'}$ for $i, i' \in S_1$ and similarly for $i, i' \in S_2$. Denote these common values by μ_{S_1}, μ_{S_2} , respectively. Using the moment map equations, we find

$$(\#S_1)\mu_{S_1} = \sum_{i \in S_1} (|x_i|^2 - |y_i|^2 - \alpha_i) = \frac{\gamma}{2} - \delta - \alpha_{S_1}, \quad (3.99)$$

$$(\#S_2)\mu_{S_2} = \sum_{i \in S_1} (|x_i|^2 - \alpha_i) = \frac{\gamma}{2} - \alpha_{S_2}, \quad (3.100)$$

$$(3.101)$$

Since $x_j, y_j = 0$ for $j \in (S_1 \cup S_2)^c$, we have $\mu_j = -\alpha_j$ for $j \in S^c$. Appealing to the isotropy representation, we obtain the desired formula for the Morse index. \square

Proposition 3.3.22. *The correction R_C of a type C critical set labelled by $S, T \subset [n]$ is given by*

$$R_C = \frac{t^\lambda}{(1-t^2)^{1+\#(S_1 \cup S_2)^c}} P_t(\mathbf{P}^{\#S_1-2}) \quad (3.102)$$

where λ is the Morse index as computed above.

Type D Critical Sets

We have disjoint subsets $S_1, S_2, S_3 \subset [n]$ satisfying $\#S_1 \geq 1, \#S_2 \geq 1, \#S_3 \geq 1$.

Lemma 3.3.23. *The stabilizer of a generic point of the type D critical set is the subgroup $T_S \subseteq T_{\mathbf{w}}$ consisting of elements of the form $t_i = s_j$ for $i \in S_j$, with no constraint on t_i for $i \notin S_1 \cup S_2 \cup S_3$. The map $\phi : T_S \rightarrow T_{\mathbf{v}}$ is given by $g_1 = \text{diag}(s_1^{-1}, s_2^{-1}), g_2 = s_3^{-1}$. The isotropy representation is given by*

$$\begin{aligned} & (\#S_1 + \#S_2 + \#S_3 - 3) + (\#S_1 - 1)s_1^{-1}s_2 + (\#S_2 - 1)s_1s_2^{-1} \\ & + (\#S_1 + \#S_3 - 1)s_1s_3^{-1} + (\#S_2 + \#S_3 - 1)s_2s_3^{-1} \\ & + \sum_{i \notin S_1 \cup S_2 \cup S_3} (s_1t_i^{-1} + s_2t_i^{-1} + s_3^{-1}t_i) \end{aligned} \quad (3.103)$$

Proof. The form of the stabilizer follows from the definition of the type C critical set, and to obtain the isotropy representation, we simply compute

$$\begin{aligned} [\text{Rep}(\mathcal{Q}, \mathbf{v}, \mathbf{w})] &= \sum_{i \in S^1} (1 + s_2s_1^{-1} + s_1s_3^{-1}) + \sum_{i \in S_2} (1 + s_1s_2^{-1} + s_2s_3^{-1}) \\ &+ \sum_{i \in S_3} (1 + s_1s_3^{-1} + s_2s_3^{-1}) + \sum_{i \notin S_1 \cup S_2} (s_1t_i^{-1} + s_2t_i^{-1} + s_3^{-1}t_i) \end{aligned} \quad (3.104)$$

$$[\mathfrak{g}_{\mathbf{v}}] = 3 + s_1s_2^{-1} + s_1^{-1}s_2 \quad (3.105)$$

$$[E] = s_1s_3^{-1} + s_2s_3^{-1} \quad (3.106)$$

and appeal to Theorem 2.2.10. \square

Lemma 3.3.24. *Let $\mu_{S_1} = (\gamma/2 - \alpha_{S_1})/(\#S_1)$, $\mu_{S_2} = (\gamma/2 - \alpha_{S_2})/(\#S_2)$, and $\mu_{S_3} = -(\delta + \alpha_{S_3})/(\#S_3)$, and assume that $\mu_{S_1} > \mu_{S_2}$. Then the Morse index is given by $2(\#S_2 - 1) + 2\lambda_0 + 2\lambda_1 + 2\lambda_2 + 2\lambda_3$, where*

$$\lambda_0 = (\#S_1 + \#S_3 - 1)\theta(\mu_{S_1} - \mu_{S_3}) + (\#S_2 + \#S_3 - 1)\theta(\mu_{S_2} - \mu_{S_3}), \quad (3.107)$$

$$\lambda_1 = \#\{i \notin S_1 \cup S_2 \cup S_3 \mid \mu_{S_1} + \alpha_i > 0\}, \quad (3.108)$$

$$\lambda_2 = \#\{i \notin S_1 \cup S_2 \cup S_3 \mid \mu_{S_2} + \alpha_i > 0\}, \quad (3.109)$$

$$\lambda_3 = \#\{i \notin S_1 \cup S_2 \cup S_3 \mid \mu_{S_3} + \alpha_i < 0\}, \quad (3.110)$$

where θ denotes the Heaviside step function. The equivariant topology of the type D critical set is that of a point.

Proof. The condition of being critical forces $\mu(x)$ to lie in $\text{stab}(x)$, and hence $\mu_i = \mu_{i'}$ for $i, i' \in S_1$ and similarly for $i, i' \in S_2$. Denote these common values by μ_{S_1}, μ_{S_2} ,

respectively. Using the moment map equations, we find

$$(\#S_1)\mu_{S_1} = \sum_{i \in S_1} (|x_i|^2 - \alpha_i) = \frac{\gamma}{2} - \alpha_{S_1}, \quad (3.111)$$

$$(\#S_2)\mu_{S_2} = \sum_{i \in S_1} (|x_i|^2 - \alpha_i) = \frac{\gamma}{2} - \alpha_{S_2}, \quad (3.112)$$

$$(\#S_3)\mu_{S_3} = \sum_{i \in S_1} (-|y_i|^2 - \alpha_i) = -\delta - \alpha_{S_3}, \quad (3.113)$$

Appealing to the isotropy representation, we obtain the desired formula for the Morse index. \square

Proposition 3.3.25. *The correction R_D of a type D critical set labelled by subsets $S_1, S_2, S_3 \subset [n]$ is equal to*

$$R_D = \frac{t^\lambda}{(1 - t^2)^{2 + \#(S_1 \cup S_2 \cup S_3)^c}} \quad (3.114)$$

where λ is the Morse index as computed above.

3.3.6 The Type $(1, 1, 1)$ Critical Sets

Proposition 3.3.26. *Let S_1, S_2, S_3 be a decomposition of $[n]$ indexing a critical set of type $(1, 1, 1)$. Then these subsets must satisfy (i) $\alpha_{S_2 \cup S_3} > 2\alpha_{S_1}$, (ii) $2\alpha_{S_3} > \alpha_{S_1 \cup S_2}$, (iii) $\#S_1 \geq 2$, (iv) $\#S_2 \geq 2$, and (v) $\#S_3 \geq 1$. The isotropy representation is given by*

$$(\#S_1 + \#S_2 - 4)(1 + t) + (n - 3)(t^{-1} + t^2) + (\#S_3 - 1)(t^{-2} + t^3). \quad (3.115)$$

The dimension of the critical set is $\#S_1 + \#S_2 - 4$ and its Morse index is $2(n + \#S_3 - 4)$. It is isomorphic to $\mathbf{P}^{\#S_1 - 2} \times \mathbf{P}^{\#S_2 - 2}$

Proof. A fixed-point of this type corresponds to the map $\phi : S^1 \rightarrow G_{\mathbf{v}} \cong U(3) \times (S^1)^n$ given by $g(t) = \text{diag}(1, t^{-1}, t^{-2})$ and

$$h_i = \begin{cases} 1, & i \in S_1 \\ t^{-1}, & i \in S_2 \\ t^{-2}, & i \in S_3 \end{cases} \quad (3.116)$$

To compute the isotropy representation, we compute

$$[T_x \text{Rep}(Q)] = n + (\#S_1 + \#S_2)t + (\#S_1)t^2 + (\#S_2 + \#S_3)t^{-1} + (\#S_3)t^{-2} \quad (3.117)$$

$$[\mathfrak{pg}_v] = n + 2 + 2t + 2t^{-1} + t^2 + t^{-2} \quad (3.118)$$

$$[E] = 2 + (n + 2)t + 2t^2 + t^3 + t^{-1} \quad (3.119)$$

and appeal to Theorem 2.2.10. The isomorphism with the product of projective spaces is given as follows. First, we consider the map to $\mathbf{P}^{\#S_1-1} \times \mathbf{P}^{\#S_2-1}$ given by

$$[x_i, y_i] \mapsto [x_i y_i : i \in S_1] \times [x_j y_j : j \in S_2]. \quad (3.120)$$

This is easily seen to be a well-defined embedding (at least for generic α). However, the complex moment map equation gives the relations $\sum_{i \in S_1} x_i y_i = 0$ and $\sum_{i \in S_2} x_i y_i = 0$, so the image is the subvariety defined by these equations, which is isomorphic to $\mathbf{P}^{\#S_1-2} \times \mathbf{P}^{\#S_2-2}$. \square

3.3.7 The Poincaré Polynomial

The preceding sections described the computation of the Morse indices of the S^1 -fixed point sets, as well as inductive methods to compute the Poincaré polynomials of types $(2, 1)$ and $(1, 1, 1)$. Since we can ensure that only these types appear by choosing the moment map level α to be extreme, this gives an explicit inductive procedure to compute $P_t(\mathfrak{M}_\alpha(n))$ by Morse theory.⁴ We have implemented this procedure in the computer software Sage. For small values of n , we summarize the results in the following theorem.

Theorem 3.3.27. *For small values of n , the Poincaré polynomials of the rank 3 star quiver varieties $\mathfrak{M}_\alpha(n)$ are the following:*

$$P_t(\mathfrak{M}_\alpha(4)) = 1$$

$$P_t(\mathfrak{M}_\alpha(5)) = 1 + 5t^2 + 11t^4$$

$$P_t(\mathfrak{M}_\alpha(6)) = 1 + 6t^2 + 22t^4 + 51t^6 + 66t^8$$

$$P_t(\mathfrak{M}_\alpha(7)) = 1 + 7t^2 + 29t^4 + 85t^6 + 190t^8 + 308t^{10} + 302t^{12}$$

$$P_t(\mathfrak{M}_\alpha(8)) = 1 + 8t^2 + 37t^4 + 121t^6 + 311t^8 + 653t^{10} + 1115t^{12} + 1450t^{14} + 1191t^{16}$$

Remark 3.3.28. Using the Weil conjectures, Hausel produced an explicit generating function for the Poincaré polynomials of Nakajima quiver varieties [Hau06]. As a consistency check, we have verified that our calculation is consistent with Hausel's generating function for small values of n . Note that unlike the method of this chapter, Hausel's method does not give any direct information about the S^1 -fixed point set.

⁴Recall that the Betti numbers of $\mathfrak{M}_\alpha(n)$ do not depend on α , as long as α is generic.

Chapter 4

The Residue Formula Revisited

4.1 Generating Functions for Intersection Pairings

4.1.1 Cogenerators of Rings

Let k be a field of characteristic 0,¹ V a graded k -vector space, and V^* its dual. Let $R = k[V] \cong \text{Sym } V^*$ and $R^\vee = k[V^*] \cong \text{Sym } V$. Note that R has a natural grading $R = \bigoplus_{d \geq 0} R^d$, where $R^d := \text{Sym}^d V^*$ consists of homogeneous polynomials of degree d . The natural pairing between V and V^* gives R^\vee the structure of an R -module, with $\text{Sym}^d V^*$ acting on R^\vee as homogeneous linear differential operators of degree d .

Lemma 4.1.1. *Suppose that $I \subseteq R$ is a homogeneous ideal such that $S = R/I$ is finite-dimensional over k . Then there exists a finitely-generated R -submodule $S^\vee \subset R^\vee$ such that $I = \text{ann}_R(S^\vee)$. Consequently, the action of R on S^\vee descends to an action of S on S^\vee .*

Proof. We define $S^\vee := \text{Hom}_R(R/I, R^\vee)$. Since an element $\phi \in \text{Hom}_R(R/I, R^\vee)$ is uniquely determined by its image $\phi(1)$, S^\vee is naturally a sub-module of R^\vee . An element $r \in R^\vee$ is the image of 1 under $\phi \in \text{Hom}_R(S, R^\vee)$ if and only if $I \cdot r = 0$. Hence

$$S^\vee \cong \{r \in R^\vee \mid I \cdot r = 0\}. \quad (4.1)$$

By construction, $I \subseteq \text{ann}_R(S^\vee)$. To see that this is in fact an equality, simply note that for any degree d the vector spaces R^d and $(R^\vee)^d$ are naturally dual. Since I is homogeneous, we have that $(S^\vee)^d$ is the annihilator of I^d under the dual pairing, and conversely. \square

¹For our purposes we will only need to consider \mathbf{Q} , \mathbf{R} , and \mathbf{C} .

Remark 4.1.2. The conclusion of this lemma is definitely not true if we drop the hypothesis that I is homogeneous. For example, if $R = k[u]$ and $I = \langle u - 1 \rangle$ then $S = R/I \cong k$, but $S^\vee = 0$, since the differential equation

$$\left(\frac{\partial}{\partial u} - 1\right) f(u) = 0 \tag{4.2}$$

has no non-trivial polynomial solutions. Hence $\text{ann}(S^\vee) = R \neq I$.

Definition 4.1.3. Let $S = R/I$ be as above. We call $S^\vee = \text{Hom}_R(S, R^\vee)$ the *Fourier dual* of S . A *cogenerator* of S is an element $f \in S^\vee$. A set $\{f_1, \dots, f_n\}$ of cogenerators is said to be *complete* if this set generates S^\vee as an R -module.

Definition 4.1.4. A finite-dimensional graded k -algebra is called *level* if every non-zero element divides a non-zero element of top degree. It is said to satisfy *Poincaré duality* if it is both level and one-dimensional in top degree.

Remark 4.1.5. A connected compact oriented manifold M satisfies Poincaré duality, which implies that the ring $H^*(M)$ satisfies Poincaré duality as defined above. In the non-compact setting, levelness can be a suitable replacement for Poincaré duality in many arguments. This property is known to hold for hypertoric varieties, and is conjectured to hold for a large class of holomorphic symplectic quotients [Pro11].

Proposition 4.1.6. *Let $I \subseteq R$ be as above. Then the quotient ring $S = R/I$ is level if and only if the dual R -module S^\vee is generated in top degree, and S satisfies Poincaré duality if and only if S^\vee is generated by a single polynomial of top degree.*

Proof. Let $S_0^\vee \subseteq S^\vee$ be the R -submodule of S^\vee generated by $(S^\vee)^n$, where n is the top non-vanishing degree of S . Then S^\vee is generated in top degree if and only if $S^\vee = S_0^\vee$. By Lemma 4.1.1 this is true if and only if $\text{ann}(S^\vee) = \text{ann}(S_0^\vee)$. Note that we always have the containment $\text{ann}(S^\vee) \subseteq \text{ann}(S_0^\vee)$, so S^\vee is not generated in top degree if and only if there exists $f \in R, f \notin I$ such that $f \cdot S_0^\vee = 0$. If there is such an f , then for all g of complementary degree we have

$$(fg) \cdot (S^\vee)^n = (fg) \cdot (S_0^\vee) = g \cdot (f \cdot (S_0^\vee)) = 0. \tag{4.3}$$

By the lemma, the image of f in S does not divide any non-zero element of top-degree, hence S is not level. □

In what follows, R will be the torus-equivariant cohomology of a point and I is the kernel of the Kirwan map. The surprising result is that while the cohomology ring

S might have a very complicated presentation in terms of generators and relations, in many cases we can compute generators for the Fourier dual module rather easily. Hence, Fourier duality gives a convenient way to encode the entire cohomology ring as a single polynomial.

Example 4.1.7. Let $R = \mathbf{Q}[u]$ and $I = \langle u^{n+1} \rangle$ with $\deg u = 2$. Then $R^\vee = \mathbf{Q}[\lambda]$ where λ is dual to u and u acts on R^\vee as $d/d\lambda$. The dual module S^\vee consists of all polynomials in λ of total degree less than or equal to $2n$, where $\deg \lambda = 2$. Hence S^\vee is generated as an R -module by λ^n . It is well-known that S is isomorphic to the cohomology ring of \mathbf{P}^n , and we will see that its Jeffrey-Kirwan residue is (up to an overall constant) given by λ^n .

We will give many more examples in Section §4.4.

Let us record two easy lemmas, which we will need later in the chapter.

Lemma 4.1.8. *Let I be a homogeneous ideal of $R[u]$ such that $S = R[u]/I$ is finite-dimensional. Then the Fourier dual of $S/\langle u \rangle$ is given by*

$$\{f \in S^\vee \mid \frac{\partial f}{\partial u} = 0\}. \quad (4.4)$$

Proof. We have that $S/\langle u \rangle = R[u]/\langle I, u \rangle$, so its Fourier dual consists of those elements of $R[u]$ which are simultaneously annihilated by I and u , i.e. those elements of S^\vee which do not depend on u . \square

Lemma 4.1.9. *Let $\varpi \in R$ be homogeneous, let I be a homogeneous ideal of R such that $S = R/I$ is finite dimensional, and suppose that $\{f_i\}_{i \in I}$ is a complete set of generators of S^\vee . Then a complete set of generators of $(S/\text{ann}_S(\varpi))^\vee$ is given by $\{\varpi \cdot f_i\}_{i \in I}$.*

Proof. If $a \in \text{ann}_S(\varpi)$, then $a \cdot \varpi \in I$, so that

$$a \cdot (\varpi \cdot f_i) = (a\varpi) \cdot f_i = 0, \quad (4.5)$$

and hence $\text{ann}_R(D_\varpi f_i) \subset \langle I, \text{ann}_S(\varpi) \rangle$. Conversely, suppose that $r \in R$ such that $[r] \in S/\text{ann}_S(\varpi)$ is non-zero. Then $r\varpi \notin I$, and hence $(r\varpi) \cdot f_i \neq 0$ for some $i \in I$. Hence $\bigcap_{i \in I} \text{ann}_R(\varpi \cdot f_i) = \langle I, \text{ann}_S(\varpi) \rangle$. \square

4.1.2 Generating Functions

Let G be a compact Lie group and $T \subset G$ a maximal torus. Denote the Weyl group $N(T)/T$ by W . Let $R = H^*(BT) \cong \mathbf{Q}[u_1, \dots, u_r]$, with $\deg X_i = 2$ and r the rank of G . Then $H^*(BG) \cong R^W$ and we regard elements of $H^*(BG)$ as Weyl-invariant polynomials

in the u_i . Then $R^\vee = \mathbf{Q}[\lambda_1, \dots, \lambda_r]$ for dual variables λ_i , and for any polynomial $f = \sum a_I x^I$ we have the Fourier dual differential operator

$$D_f := \sum a_I \left(-i \frac{\partial}{\partial \lambda} \right)^I. \quad (4.6)$$

Suppose that G acts linearly on $X = \mathbf{C}^N$ and assume that X has a G -invariant Hermitian inner product. Then for any regular central $\alpha \in \mathfrak{g}^*$ the symplectic reduction $\mathfrak{X}_\alpha := X // G$ is a symplectic orbifold. We assume throughout this section that the moment map $\mu : X \rightarrow \mathfrak{g}^*$ is proper so that \mathfrak{X}_α is compact.

Definition 4.1.10. We define the *generating function* $Z : \mathfrak{t}^* \rightarrow \mathbf{Q}$ by

$$Z(\lambda) = \frac{1}{|W|} \int_{\mathfrak{X}_\alpha} \kappa \left(\sum_{s \in W} e^{i(\lambda - \alpha)(s \cdot u)} \right) e^\omega, \quad (4.7)$$

where $\kappa : H^*(BG) \rightarrow H^*(\mathfrak{X}_\alpha)$ is the Kirwan map.

Note that although Z is defined as a Weyl-invariant formal power series in λ , due to the finite-dimensionality of \mathfrak{X}_α it is actually a polynomial. It is a generating function in the following sense.

Lemma 4.1.11. *For any Weyl-invariant polynomial $f \in R^W$ we have*

$$D_f Z(\alpha) = \int_{\mathfrak{X}_\alpha} \kappa(f) e^\omega. \quad (4.8)$$

Hence the intersection pairings in \mathfrak{X}_α completely determine Z , and conversely.

Proof. Simply note that for any polynomial $f(u)$ we have

$$D_f e^{i(\lambda - \alpha)(u)} = f(u) e^{i(\lambda - \alpha)(u)}. \quad (4.9)$$

Now assume that f is Weyl-invariant. Average over the Weyl group, apply the Kirwan map, multiply both sides by e^ω , and integrate over \mathfrak{X}_α to obtain

$$D_f Z(\lambda) = \frac{1}{|W|} \int_{\mathfrak{X}_\alpha} \kappa \left(f(u) \sum_{s \in W} e^{i(\lambda - \alpha)(s \cdot u)} \right) e^\omega. \quad (4.10)$$

Now evaluate at $\lambda = \alpha$ to obtain the desired result. This shows that the intersection pairings are the Taylor coefficients of Z about $\lambda = \alpha$, and since Z is a polynomial, its Taylor coefficients determine it completely. \square

Theorem 4.1.12. $f \in \ker \kappa$ if and only if $D_f Z = 0$. Hence the R^W -module generated by Z has annihilator equal to $\ker \kappa$, and therefore it is the Fourier dual of $H^*(\mathfrak{X}_\alpha)$.

Proof. Without loss of generality assume that f is homogeneous. Then by Kirwan surjectivity and Poincaré duality, $f \in \ker \kappa$ if and only if for all $g \in R^W$,

$$\int_{\mathfrak{X}_\alpha} \kappa(fg)e^\omega = 0. \quad (4.11)$$

By the above lemma, $\int_{\mathfrak{X}_\alpha} \kappa(fg)e^\omega = 0$ if and only if $(D_f D_g Z)(\alpha) = 0$ for all g . Then the Taylor expansion of $D_f Z$ about $\lambda = \alpha$ is identically 0, hence $D_f Z = 0$ identically. \square

4.2 Jeffrey-Kirwan Residues

4.2.1 The Residue Operation

We desire an effective method to compute the generating function Z associated to any compact symplectic quotient \mathfrak{X}_α as in the preceding section. We begin by recalling the residue operation:

Proposition 4.2.1. [JK97, Proposition 3.2] *Let $\Lambda \subset \mathfrak{t}$ be a non-empty open cone and suppose that $\beta_1, \dots, \beta_N \in \mathfrak{t}^*$ all lie in the dual cone Λ^* . Suppose that $\lambda \in \mathfrak{t}^*$ does not lie in any cone of dimension at most $n-1$ spanned by a subset of $\{\beta_1, \dots, \beta_N\}$. Let $\{u_1, \dots, u_n\}$ be any system of coordinates on \mathfrak{t} and let $du = du_1 \wedge \dots \wedge du_n$ be the associated volume form. Then there exists a residue operation Res^Λ defined on meromorphic differential forms of the form*

$$h(u) = \frac{q(u)e^{i\lambda(u)}}{\prod_{j=1}^N \beta_j(u)} du \quad (4.12)$$

where $q(u)$ is a polynomial. The operation Res^Λ is linear in its argument and is characterized uniquely by the following properties:

(i) *If $\{\beta_1, \dots, \beta_N\}$ does not span \mathfrak{t} as a vector space then*

$$\text{Res}^\Lambda \left(\frac{u^J e^{i\lambda(u)}}{\prod_{j=1}^N \beta_j(u)} du \right) = 0 \quad (4.13)$$

(ii) *For any multi-index J , we have*

$$\text{Res}^\Lambda \left(\frac{u^J e^{i\lambda(u)}}{\prod_{j=1}^N \beta_j(u)} du \right) = \sum_{m \geq 0} \lim_{s \rightarrow 0^+} \text{Res}^\Lambda \left(\frac{u^J (i\lambda(u))^m e^{is\lambda(u)}}{m! \prod_{j=1}^N \beta_j(u)} du \right). \quad (4.14)$$

(iii) *The limit*

$$\lim_{s \rightarrow 0^+} \text{Res}^\Lambda \left(\frac{u^J e^{is\lambda(u)}}{\prod_{j=1}^N \beta_j(u)} du \right) \quad (4.15)$$

is zero unless $N - |J| = n$.

(iv) *If $N = n$ and $\{\beta_1, \dots, \beta_n\}$ spans \mathfrak{t}^* as a vector space then*

$$\text{Res}^\Lambda \left(\frac{u^J e^{i\lambda(u)}}{\prod_{j=1}^n \beta_j(u)} du \right) = 0 \quad (4.16)$$

unless λ is in the cone spanned by $\{\beta_1, \dots, \beta_n\}$. If λ is in this cone, then the residue is equal to $\bar{\beta}^{-1}$, where $\bar{\beta}$ is the determinant of an $n \times n$ matrix whose columns are the coordinates of β_1, \dots, β_n with respect to any orthonormal basis defining the same orientation as β_1, \dots, β_n .

Although the above properties determine the residue operation completely, they can be difficult to work with. However, in the one-dimensional case it is essentially the usual residue operation from complex analysis. Consider meromorphic functions of $z \in \mathbf{C}$ which take the form

$$f(z) = \sum_{j=1}^m g_j(z) e^{i\lambda_j z} \quad (4.17)$$

where $g_j(z)$ are rational functions and $\lambda_j \in \mathbf{R} \setminus \{0\}$. We introduce an operation Res_z^+ defined by

$$\text{Res}_z^+ f(z) dz = \sum_{\lambda_j \geq 0} \sum_{b \in \mathbf{C}} \text{Res} (g_j(z) e^{i\lambda_j z} dz; z = b). \quad (4.18)$$

The following proposition reduces the residue operation to a series of iterated one-dimensional residues.

Proposition 4.2.2. *[JK97, Proposition 3.4] Let*

$$h(u) = \frac{q(u) e^{i\lambda(u)}}{\prod_{j=1}^N \beta_j(u)} \quad (4.19)$$

where $q(u)$ is a polynomial, and λ is not in any proper subspace spanned by a subset of $\{\beta_1, \dots, \beta_N\}$. For a generic choice of coordinate system $\{u_1, \dots, u_n\}$ on \mathfrak{t} for which $(0, \dots, 0, 1) \in \Lambda$ we have

$$\text{Res}^\Lambda (h(u) du) = \Delta \text{Res}_{u_1}^+ \cdots \text{Res}_{u_n}^+ h(u) du_1 \wedge \cdots \wedge du_n \quad (4.20)$$

where the variables u_1, \dots, u_{n-1} are held constant while calculating $\text{Res}_{u_n}^+$, and Δ is the determinant of any $n \times n$ matrix whose columns are the coordinates of an orthonormal basis of \mathfrak{t} defining the same orientation as the chosen coordinate system.

Example 4.2.3. We compute the residue $e^{i\lambda u}/u^{n+1}$ with respect to the dual cone $\Lambda^* = \{\lambda | \lambda > 0\}$ using Proposition 4.2.2. In this case, it is zero if $\lambda < 0$ and otherwise it is equal to the usual residue from elementary complex analysis. Hence we have

$$\text{Res}^\Lambda \left(\frac{e^{i\lambda u}}{u^{n+1}} \right) = \theta(\lambda) \text{Res} \left(\frac{e^{i\lambda u}}{u^{n+1}}; u = 0 \right) = \theta(\lambda) \frac{(-i\lambda)^n}{n!}, \quad (4.21)$$

where $\theta(\lambda)$ is the Heaviside step function, defined to be equal to 1 for $\lambda > 0$ and 0 for $\lambda < 0$.

Example 4.2.4. We compute the residue

$$\text{Res}^\Lambda \left(\frac{e^{i\lambda_1 u_1 + i\lambda_2 u_2}}{u_1^2 u_2 (u_2 - u_1)} \right) \quad (4.22)$$

where Λ is the dual cone $\Lambda^* = \{(\lambda_1, \lambda_2) | 0 < \lambda_2, \lambda_1 < \lambda_2\}$. Since in this coordinate system $(0, 1) \in \Lambda$, we may evaluate it via Proposition 4.2.2. We first take the residue with respect to u_2 , keeping u_1 fixed. There are two poles: $u_2 = 0$ and $u_2 = u_1$. In the first case we obtain $-e^{i\lambda_1 u_1}/u_1^3$, and in the second $e^{i(\lambda_1 + \lambda_2)u_1}/u_1^3$. Taking the residue at $u_1 = 0$ in each case, we obtain

$$\text{Res}^\Lambda \left(\frac{e^{i\lambda_1 u_1 + i\lambda_2 u_2}}{u_1^2 u_2 (u_2 - u_1)} \right) = \theta(\lambda_1) \theta(\lambda_2) \frac{\lambda_1^2}{2} - \theta(\lambda_1 + \lambda_2) \theta(\lambda_2) \frac{(\lambda_1 + \lambda_2)^2}{2}, \quad (4.23)$$

where the Heaviside factors are due to the fact that in each iterated one-dimensional residue, only those poles for which $\lambda_j > 0$ contribute.

Remark 4.2.5. In the examples above, the chosen coordinate systems do not satisfy the genericity assumption of Proposition 4.2.2. However, this assumption is needed only to avoid ambiguous Heaviside factors of the form $\theta(0)$. As long as we assume that λ is sufficiently generic, we are free to work in non-generic coordinate systems. However, the assumption that $(0, \dots, 1) \in \Lambda$ is crucial.

We shall need the following lemma in the next section.

Lemma 4.2.6. *If λ is not in the dual cone Λ^* then*

$$\text{Res}^\Lambda \left(\frac{u^J e^{i\lambda(u)}}{\prod_{j=1}^N \beta_j(u)} \right) = 0 \quad (4.24)$$

Proof. If $\{\beta_1, \dots, \beta_N\}$ do not span \mathfrak{t}^* as a vector space, then by property (i) the residue is zero and there is nothing to show. Otherwise, by properties (ii) and (iii) it suffices to consider residues of the form

$$\text{Res}^\Lambda \left(\frac{u^J e^{i\lambda(u)}}{\prod_{j=1}^N \beta_j(u)} \right) \quad (4.25)$$

with $|J| = N - n$. Since $\{\beta_1, \dots, \beta_N\}$ span \mathfrak{t}^* as a vector space, we can write

$$u_k = \sum_{j=1}^N c_{kj} \beta_j(u) \quad (4.26)$$

for some constants c_{kj} , for each $k = 1, \dots, n$. Hence

$$\frac{u_k}{\prod_{j=1}^N \beta_j(x)} = \sum_j \frac{c_{kj}}{\prod_{j \neq k} \beta_j(u)}. \quad (4.27)$$

By induction on $|J|$, $u^J / \prod_{j=1}^N \beta_j(u)$ can be written as linear combination of terms of the form $\prod_{j \in I} \beta_j(u)^{-1}$ where $|I| = n$. By property (iv), the corresponding residues are 0 if λ is not in the dual cone Λ^* . \square

4.2.2 The Linear Residue Formula

We now recall the Jeffrey-Kirwan residue formula. Let G be a compact group acting on a compact symplectic manifold X with moment map μ . Let $\alpha \in \mathfrak{g}^*$ be regular central and let $\mathfrak{X}_\alpha = X // G$ be the symplectic reduction. Let $T \subset G$ be a maximal torus, and let X^T be the T -fixed point set. Let \mathcal{F} denote the collection of connected components of X^T . For each component $F \in \mathcal{F}$, we have the T -equivariant Euler class $e_F \in H_T^*(F)$ of the normal bundle. By the splitting principle, we can assume that each e_F factors as a product

$$e_F = \prod_{i=1}^{n_F} (c_i^F + \beta_i^F) \quad (4.28)$$

where the c_i^F are the Chern classes and the β_i^F the weights of the decomposition of the pullback of the normal bundle into line bundles. Let \mathcal{B} be the collection of all weights appearing in this decomposition, for all components $F \in \mathcal{F}$. Let $\Lambda \subset \mathfrak{t}$ be any cone complementary to the hyperplanes $\{\beta(u) = 0 \mid \beta \in \mathcal{B}\}$.

Theorem 4.2.7. [JK95, JK97] *Let G act in a Hamiltonian fashion on a compact symplectic manifold (X, ω) , and let $f \in H_G^*$. Then*

$$\int_{\mathfrak{X}_\alpha} \kappa(f)e^\omega = n_0 c_G \text{Res}^\Lambda \left(\varpi(u)^2 \sum_{F \in \mathcal{F}} r_F^f du \right), \quad (4.29)$$

where ϖ is the product of the positive roots of G , n_0 is the order of the stabilizer in G of a generic point of $\mu^{-1}(\alpha)$, and the constant c_G is defined by

$$c_G = \frac{i^n}{|W| \text{vol}(T)}; \quad (4.30)$$

u is a coordinate on $\mathfrak{t} \otimes \mathbf{C}$ and \mathcal{F} is the set of components of the fixed point set X^T of the action of T on M . If F is one of these components the meromorphic function r_F^f on $\mathfrak{t} \otimes \mathbf{C}$ is defined by

$$r_F^f(u) = e^{i(\mu_T(F) - \alpha)(u)} \int_F \frac{i_F^*(\kappa(f)(u)e^\omega)}{e_F(u)} \quad (4.31)$$

where $i_F : F \rightarrow M$ is the inclusion and e_F the T -equivariant Euler class of the normal bundle to F in M . The individual terms of the sum on the right hand side depend on the choice of Λ , but the overall sum does not.

We would like to apply the residue formula to quotients of the form $\mathfrak{X}_\alpha = X //_\alpha G$, where $X = \mathbf{C}^N$. Unfortunately, this does not satisfy the hypotheses of Theorem 4.2.7 as X is not compact. We will need the following.

Theorem 4.2.8. *Let G be a compact group with maximal torus T , and suppose G acts unitarily on $X = \mathbf{C}^N$. Let $\alpha \in \mathfrak{g}^*$ be regular central and let $\mathfrak{X}_\alpha = X //_\alpha G$. Let $\{\beta_1, \dots, \beta_N\}$ be the multiset of weights of the action of T on X (counted with multiplicity). Suppose further that there is some $\xi \in \mathfrak{t}$ such that the ξ -component of the moment map is proper and bounded below. Then for all $f \in H_G^*$, we have*

$$\int_{\mathfrak{X}_\alpha} \kappa(f)e^\omega = n_0 c_G \text{Res}^\Lambda \left(\frac{\varpi^2(u) f(u) e^{-i\alpha(u)}}{\prod_{j=1}^N \beta_j(u)} \right), \quad (4.32)$$

where n_0 and c_G are as in Theorem 4.2.7, and the cone Λ is defined by

$$\Lambda = \{u \in \mathfrak{t} \mid \beta_j(u) < 0, j = 1, \dots, N\}. \quad (4.33)$$

Proof. The moment map for the T action is given by

$$\mu_T(x) = \frac{1}{2} \sum_{j=1}^N \beta_j |x_j|^2, \quad (4.34)$$

hence the ξ -component is

$$\mu_T^\xi(x) = \frac{1}{2} \sum_{j=1}^N \beta(\xi) |x_j|^2. \quad (4.35)$$

Thus μ_T^ξ is proper and bounded below if and only if $\beta_j(\xi) > 0$ for all $j = 1, \dots, N$. This shows that $-\xi \in \Lambda$ hence Λ is non-empty. Note that the dual cone Λ^* contains $\{-\beta_1, \dots, -\beta_N\}$. To apply the residue formula, we compactify by taking a symplectic cut (see §2.1.3). Let $\sigma > 0$ to be determined. We set

$$X_\sigma := (X \oplus \mathbf{C}) //_\sigma S^1 \quad (4.36)$$

where the S^1 action on $X \oplus \mathbf{C}$ is given by

$$e^{i\theta} \cdot (x \oplus z) = (e^{i\theta} x) \oplus (e^{i\theta} z) \quad (4.37)$$

As a variety we have $X_\sigma \cong \mathbf{P}^m$, but the Kähler metric and hence symplectic form depend on the parameter σ . Any point $[x, z] \in X_\sigma$ can be represented by $x \oplus z \in X \oplus \mathbf{C}$ satisfying

$$\frac{1}{2}|x|^2 + \frac{1}{2}|z|^2 = \sigma. \quad (4.38)$$

Note that this implies that for any $[x, z] \in X_\sigma$ a representative $x \oplus z$ satisfies

$$\frac{1}{2}|x|^2 \leq \sigma. \quad (4.39)$$

Moreover, G acts naturally on X_σ and the moment map for the action is given by

$$\mu_G([x, z]) = \mu_G(x). \quad (4.40)$$

By Lemma 2.1.5, for $\sigma \gg \alpha$, we may identify $X // G \cong X_\sigma // G$ and hence it suffices to compute the intersection pairings of $X_\sigma // G$ for large σ . Since X_σ is compact we may apply the usual residue formula, Theorem 4.2.7. We have to identify the components of X_σ^T . In addition to the isolated fixed point $[0, \sqrt{2\sigma}] \in X_\sigma$ corresponding to the origin in X , the cutting procedure introduces additional fixed-points at infinity. These are all of

the form

$$F_\beta = \{[x, 0] \mid x \in X(\beta)\} \quad (4.41)$$

where $X(\beta) = \{x \in X \mid t \cdot x = \beta(t)x \forall t\}$. On a component F_β , we have

$$\mu_T(F_\beta) = \frac{1}{2}\beta|x_\beta|^2 = \frac{1}{2}\sigma \quad (4.42)$$

where we used equation (4.38) and the fact that the points of F_β have the form $[x_\beta, 0]$. Now consider

$$\langle \mu_T(F_\beta) - \alpha, \xi \rangle = \frac{1}{2}\sigma\beta(\xi) - \alpha(\xi). \quad (4.43)$$

If $\sigma > 2\alpha(\xi)/\beta(\xi)$ then the above quantity is strictly positive, and hence $\mu_T(F_\beta) - \alpha$ does not lie in the cone spanned by $\{-\beta_1, \dots, -\beta_N\}$. By Lemma 4.2.6, the residue of the term corresponding to F_β is necessarily zero. Hence none of the fixed-points at infinity contribute to the residue formula, and the only remaining term is the contribution from the origin, which is given exactly by equation (4.32). \square

As stated, Theorem 4.2.8 provides a method to calculate the intersection pairings directly, but does not immediately yield the generating function for the intersection pairings. To obtain this, we define a residue operation Res_α as follows. Given a meromorphic differential form $h(u)e^{i\lambda(u)}du$, the residue $\text{Res}^\Lambda(h(u)e^{i\lambda(u)}du)$ is a piecewise polynomial function on \mathfrak{t}^* . We define $\text{Res}_\alpha(h(u)e^{i\lambda(u)}du)$ to be the unique polynomial in λ which is equal to $\text{Res}^\Lambda(h(u)e^{i\lambda(u)})$ when λ is in the same chamber as α .

Corollary 4.2.9. *Suppose $\alpha \in \mathfrak{g}^*$ is regular central. Then the generating function $Z(\lambda)$ of \mathfrak{X}_α is given by*

$$Z(\lambda) = n_0 c_G \text{Res}_\alpha \left(\frac{\varpi^2(u)e^{-i\lambda(u)}}{\prod_{j=1}^N \beta_j(u)} \right) \quad (4.44)$$

Proof. By the formal properties of the residue operation, for any $f \in R^W$ we have

$$D_f \text{Res}_\alpha \left(\frac{\varpi^2(u)e^{-i\lambda(u)}}{\prod_{j=1}^N \beta_j(u)} \right) = \text{Res}_\alpha \left(\frac{\varpi^2(u)f(u)e^{-i\lambda(u)}}{\prod_{j=1}^N \beta_j(u)} \right). \quad (4.45)$$

By Theorem 4.2.8, the evaluation at $\lambda = \alpha$ gives the intersection pairing $\int_{\mathfrak{X}_\alpha} \kappa(f)e^\omega$, and this property determines the generating function $Z(\lambda)$ uniquely. \square

Remark 4.2.10. Generating functions for intersection pairings were also studied in [GS95].

Corollary 4.2.11. *Let \mathcal{Q} be any acyclic quiver with dimension vector \mathbf{d} , and let α be a generic moment map level. Then the Kirwan map $H^*(BG_{\mathbf{d}}) \rightarrow H^*(\mathfrak{X}_\alpha(\mathcal{Q}, \mathbf{d}))$ is surjective, and the intersection pairings of Kirwan classes may be computed algorithmically by*

the Theorem 4.2.8. In particular, the ring structure on $H^*(\mathfrak{X}_\alpha(\mathcal{Q}, \mathbf{d}))$ may be computed algorithmically.

4.3 Hyperkähler Residues

4.3.1 Hyperkähler Residues

Let $X = \mathbf{C}^N$ and $M = T^*\mathbf{C}^N$. As before, we assume that a compact Lie group G acts linearly on X , and as before we let $T \subset G$ be a maximal torus. Then $M = T^*X$ which is naturally a flat hyperkähler manifold. As before we let $R = H^*(BT)$ and $R^W \cong H^*(BG)$. We have the *hyperkähler Kirwan map*

$$\kappa_{\text{HK}} : R^W \rightarrow H^*(\mathfrak{M}_\alpha). \quad (4.46)$$

We would like to understand the kernel of the hyperkähler Kirwan map by computing cogenerators. Since \mathfrak{M}_α is non-compact, there is no natural way to integrate over it.² However, we can integrate over cycles. We define the *hyperkähler residue operation* to be

$$Z : H_*(\mathfrak{M}) \rightarrow (R^W)^\vee, \quad \gamma \mapsto Z_\gamma(\lambda) = \frac{1}{|W|} \int_\gamma \kappa \left(\sum_{s \in W} e^{i(\lambda - \alpha)(s \cdot u)} \right) e^\omega. \quad (4.47)$$

As before, the following is an immediate consequence of the definition.

Lemma 4.3.1. *For any $f \in R^W$, we have*

$$D_f Z_\gamma(\alpha) = \int_\gamma \kappa(f) e^\omega. \quad (4.48)$$

Theorem 4.3.2. *The residues $\{Z_\gamma \mid \gamma \in H_*(\mathfrak{M}_\alpha)\}$ are a complete set of cogenerators for the image of the hyperkähler Kirwan map.*

Proof. If $\kappa(f) \neq 0$, then there is some $\gamma \in H_*(\mathfrak{M}_\alpha)$ such that $\langle \gamma, \kappa(f) \rangle \neq 0$. Hence $D_f Z_\gamma \neq 0$, and we find that the annihilator of the $(R^W)^\vee$ -module generated by the residues is exactly equal to the kernel of the Kirwan map. \square

In the symplectic case, the residue formula gave an effective method to compute the residue, and hence the cohomology ring. In the hyperkähler case, we will develop an algorithm to compute a complete set of cogenerators.

²It is possible to define an equivariant integration theory [HP05b], but we wish to work in ordinary cohomology. An equivariant version of the residue formula was developed in [Mar08] as well as [Szi13].

Remark 4.3.3. It is possible to define formal S^1 -equivariant integration theory on hyperähler quotients [HP05b]. However, due to the complicated structure of the fixed-point set (as evidenced by the calculations in Chapter 3), these equivariant integrals can be difficult to compute in general. Instead, we choose to work with ordinary integration. The main feature of this approach is that much information about the cohomology of hyperkähler quotients can be deduced from the cohomology of simpler, compact symplectic quotients.

4.3.2 Kähler Quotients as Integration Cycles

In this section we will show that a large set of hyperkähler residues may be calculated directly from the residue formula. Suppose that $V \subset M$ is a G -invariant linear subspace such that $\mu_c|_V = 0$ and that $\mu_r|_V$ is proper. Assume furthermore that α is a regular value of $\mu_r|_V$. Then define

$$\mathfrak{Y}_\alpha := V //_\alpha G. \quad (4.49)$$

This is a natural compact subvariety of \mathfrak{M}_α , and hence defines a homology class $[\mathfrak{Y}_\alpha] \in H_*(\mathfrak{M}_\alpha)$. Now recall that $H^*(\mathfrak{M}_\alpha)$ is independent of α as long as α is generic.

Definition 4.3.4. A *VGIT residue* is a hyperkähler residue Z_γ associated to a class $\gamma = [\mathfrak{Y}_\alpha]$, for some choice of $V \subset M$ and α as above. The *VGIT ring* is the quotient R^W/I , where the ideal I is the annihilator of the $(R^W)^\vee$ -module generated by the VGIT residues.

In the above definition, VGIT is short for variation of GIT. The reason for this terminology is the following proposition, which follows immediately from the definition.

Proposition 4.3.5. *The VGIT ideal is equal to*

$$\bigcap_{V \subset M, \alpha} \ker(\kappa : R^W \rightarrow H^*(\mathfrak{Y}_\alpha)) \quad (4.50)$$

where the intersection runs over all $V \subset M$ as above and α generic. As a consequence, the ordinary Kirwan map $R^W \rightarrow H^*(\mathfrak{Y}_\alpha)$ factors through the VGIT ring, i.e.

$$R^W \twoheadrightarrow R^W/I \twoheadrightarrow H^*(\mathfrak{Y}_\alpha), \quad (4.51)$$

where I is the VGIT ideal.

The residues associated to the classes $[\mathfrak{Y}_\alpha]$ may be computed by the linear residue formula, Theorem 4.2.8. Hence the VGIT ideal and VGIT ring may be computed algo-

rithmically. Moreover, there is a natural inclusion $\ker \kappa_{\text{HK}} \subseteq I_{\text{VGIT}}$, and hence a natural surjection $\text{im}(\kappa_{\text{HK}}) \rightarrow R_{\text{VGIT}}$.

Conjecture 4.3.6. *Let G act linearly on $M = T^*X$ as above, and assume that the restriction of the real moment map to X is proper. Then the image of the hyperkähler Kirwan map is equal to the VGIT ring.*

Remark 4.3.7. The hypothesis that the restriction of the real moment map to X be proper is essential. One may check explicitly that the Nakajima varieties associated to the ADHM quiver (Figure 3.1) do not satisfy $\text{im}(\kappa_{\text{HK}}) = R_{\text{VGIT}}$. However, because this quiver has a vertex self-loop, it is impossible to find a Lagrangian subspace $X \subset M$ such that the restriction of μ_r is proper.

This conjecture is motivated by the following results.

Theorem 4.3.8 (Proudfoot [Pro11]). *Conjecture 4.3.6 is true for hypertoric varieties.*

Theorem 4.3.9 (Konno [Kon00b]). *Conjecture 4.3.6 is true for hyperpolygon spaces.*

In Theorem 4.4.5 we compute the VGIT rings of rank 3 star quivers (for small values of n) and find that they satisfy this conjecture.

4.3.3 Quiver Varieties

We expect that Conjecture 4.3.6 may be true for all acyclic quivers, but unfortunately we do not have a general result.³ Instead, we will use abelianization to give an explicit algorithm to compute a complete set of hyperkähler residues.

Let $\mathfrak{M}_\alpha = M //_{\alpha} G$ be a Nakajima quiver variety, $T \subset G$ a maximal torus, and suppose that we have chosen a circle action on M so that \mathfrak{M}_α is circle compact (see §3.2.1). Then we may consider the S^1 -equivariant cohomology $H_{S^1}^*(\mathfrak{M}_\alpha)$, as well as the S^1 -equivariant Kirwan map

$$\kappa_{S^1} : H^*(BG) \otimes H^*(BS^1) \rightarrow H_{S^1}^*(\mathfrak{M}_\alpha). \quad (4.52)$$

Now $H^*(BS^1) \cong \mathbf{Q}[u]$, and by Theorem 2.2.4 we have $H_{S^1}^*(\mathfrak{M}_\alpha) \cong H^*(\mathfrak{M}_\alpha)[u]$ as a $\mathbf{Q}[u]$ -module. Hence we have a natural ring map

$$q : H_{S^1}^*(\mathfrak{M}_\alpha) \twoheadrightarrow H^*(\mathfrak{M}_\alpha) \quad (4.53)$$

which is given by quotienting by the ideal generated by $u \cdot 1 \in H_{S^1}^*(\mathfrak{M}_\alpha)$. The main technical tool we need is the following.

³But see §4.4 for some special cases.

Theorem 4.3.10. [HP05b] *The image of the S^1 -equivariant Kirwan map is isomorphic to*

$$H_{S^1}^*(M///T)^\alpha / \text{ann}(e), \quad (4.54)$$

where e is the class given by

$$e = \prod_{\alpha \in \Delta} \alpha(u - \alpha). \quad (4.55)$$

The above theorem gives a complete solution to the problem of computing hyperkähler residues of Nakajima quiver varieties.

Theorem 4.3.11. *The following algorithm produces a complete list of hyperkähler residues for any Nakajima quiver variety.*

1. Compute $H_{S^1}^*(M///T)$ using the techniques of Chapter 2.
2. Compute a complete set of Weyl-invariant cogenerators of the above ring in dimensions less than or equal to $\dim_{\mathbf{R}} \mathfrak{M}_\alpha + \deg(e)$.
3. Apply the differential operator D_e to the above set of cogenerators.
4. Compute the u -independent linear combinations of the above cogenerators.

Proof. Lemmas 4.1.8 and 4.1.9 ensure that the above procedure produces a complete set of cogenerators for the ring $H_{S^1}^*(M///T)^\alpha / \langle \text{ann}(e), u \rangle$, and by Theorem 4.3.10 and formality this is equal to the image of the hyperkähler Kirwan map. \square

4.4 Examples

4.4.1 Hypertoric Varieties

Example 4.4.1. We consider the hyperplane arrangement pictured in figure 4.1. The normals are given by the column vectors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & -1 \end{pmatrix} \quad (4.56)$$

The kernel is generated by the vectors $(1, 0, -1, 1)^T$ and $(0, 1, 1, 0)^T$. Hence this arrangement corresponds to the $S^1 \times S^1$ representation

$$V = \mathbf{C}(1, 0) \oplus \mathbf{C}(0, 1) \oplus \mathbf{C}(-1, 1) \oplus \mathbf{C}(1, 0). \quad (4.57)$$

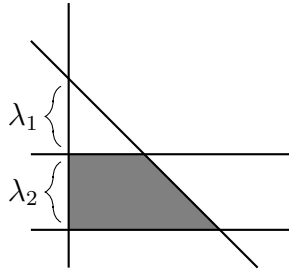


Figure 4.1: The hyperplane arrangement corresponding to the blow-up of \mathbf{P}^2 at a point.

The moment map is given by

$$\mu(x) = \frac{1}{2} \begin{pmatrix} |x_1|^2 - |x_3|^2 + |x_4|^2 \\ |x_2|^2 + |x_4|^2 \end{pmatrix} - \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad (4.58)$$

To compute the symplectic volume, we have to calculate

$$\begin{aligned} \operatorname{res} \left(\frac{e^{+i\lambda_1 u_1 + i\lambda_2 u_2}}{u_1^2 u_2 (u_2 - u_1)} \right) &= -\theta(\lambda_2) \operatorname{res} \left(\frac{e^{i\lambda_1 u_1}}{u_1^3} \right) + \theta(\lambda_2) \operatorname{res} \left(\frac{e^{i(\lambda_1 + \lambda_2) u_1}}{u_1^3} \right) \\ &= \theta(\lambda_1 | \lambda_2) \frac{\lambda_1^2}{2} - \theta(\lambda_1 + \lambda_2 | \lambda_2) \frac{(\lambda_1 + \lambda_2)^2}{2} \end{aligned}$$

Thus we find that the symplectic volume is (proportional to)

$$Z(\lambda) = \frac{1}{2} \begin{cases} (\lambda_1 + \lambda_2)^2 - \lambda_1^2, & \lambda_1 > 0, \lambda_2 > 0 \\ (\lambda_1 + \lambda_2)^2, & \lambda_1 < 0, \lambda_1 + \lambda_2 > 0, \\ 0, & \text{otherwise} \end{cases} \quad (4.59)$$

Hence there are two Jeffrey-Kirwan residues, Z_1 and Z_2 . The wall-crossing behavior of the residue formula is pictured in Figure 4.2. Taking derivatives, we find

	∂_1	∂_2	∂_1^2	∂_{12}	∂_2^2	
Z_1	λ_2	$\lambda_1 + \lambda_2$	0	1	1	(4.60)
Z_2	$\lambda_1 + \lambda_2$	$\lambda_1 + \lambda_2$	1	1	1	

Hence we find a single common relation in degree 4, $\partial_{12} = \partial_2^2$, and of course in degrees ≥ 6 every constant coefficient differential operator acts as 0. Hence the cohomology ring of the corresponding hypertoric variety is given by

$$H^*(\mathfrak{M}) \cong \frac{\mathbf{C}[u_1, u_2]}{\langle u_1 u_2 - u_2^2, u_1^3, u_1^2 u_2 \rangle} \quad (4.61)$$

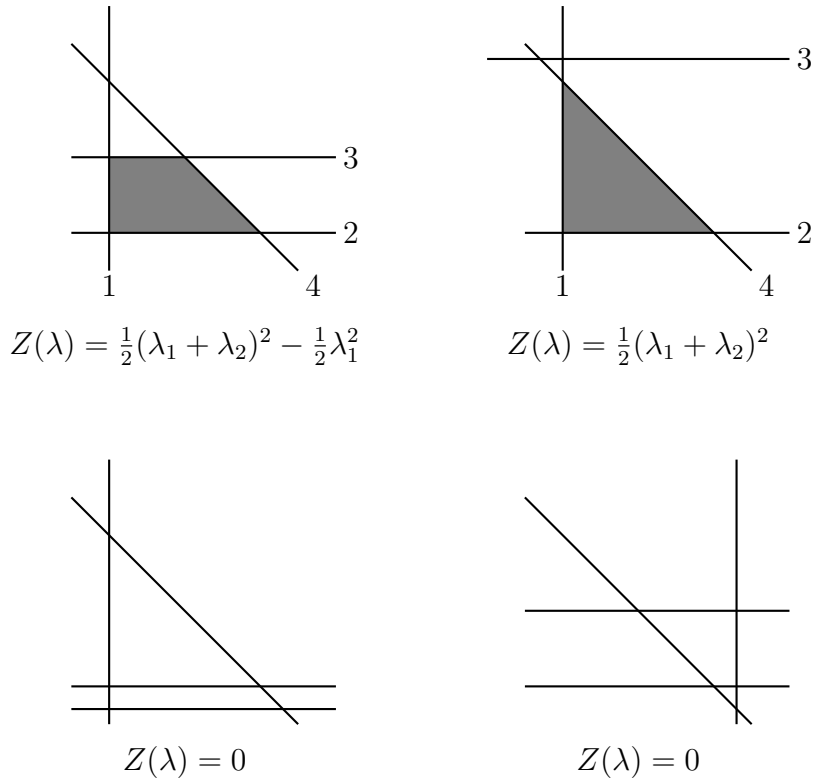


Figure 4.2: The three possible walls.

4.4.2 Polygon Spaces

We will show here that Theorem 4.2.8 gives a very straightforward calculation of the cohomology rings of polygon spaces. These correspond to a star quiver with n incoming arrows and dimension vector $\mathbf{d} = (2, 1, \dots, 1)$. Let x_1, \dots, x_n be coordinates on the vector space $X = \text{Rep}(\mathcal{Q}, \mathbf{d})$, where each x_i is a 2×1 column vector. The action of $G_{\mathbf{d}}$ on X is given by

$$x_i \mapsto gx_i h_i^{-1} \tag{4.62}$$

where $g \in SU(2)$ and $h_i \in S^1$. (We restrict to $g \in SU(2)$ because the overall diagonal in $G_{\mathbf{d}}$ acts trivially.) Let u_0, \dots, u_n be coordinates on the maximal torus of $SU(2) \times (S^1)^n$. Then the weights of the above action are $(u_i \pm u_0)$ for $i = 1, \dots, n$. Hence the product of all weights is

$$\prod_{\beta \in \mathcal{B}} \beta = \prod_{i=1}^n (u_i^2 - u_0^2). \tag{4.63}$$

Also, we have $\varpi^2 = -u_0^2$. Hence we have to compute

$$\text{Res} \left(\frac{u_0^2 e^{-i\lambda_0 u_0 - i \sum_j \lambda_j u_j}}{\prod_{j=1}^n (u_j^2 - u_0^2)} \right). \quad (4.64)$$

For each j , we have a pole at $u_j = \pm u_0$. Taking the residues for $j = n$, we have

$$\text{Res} \left(\frac{u_0^2 e^{-i\lambda_0 u_0 - i \sum_j \lambda_j u_j}}{\prod_{j=1}^n (u_j^2 - u_0^2)} \right) = \theta(\lambda_n) \text{Res} \left(\frac{u_0^2 e^{-i(\lambda_0 + \lambda_n)u_0 - i \sum_{j \neq n} \lambda_j u_j}}{2u_0 \prod_{j \neq n} (u_j^2 - u_0^2)} \right) \quad (4.65)$$

$$- \theta(\lambda_n) \text{Res} \left(\frac{u_0^2 e^{-i(\lambda_0 - \lambda_n)u_0 - i \sum_{j \neq n} \lambda_j u_j}}{2u_0 \prod_{j \neq n} (u_j^2 - u_0^2)} \right) \quad (4.66)$$

Next we take the residues at $u_{n-1} = \pm u_0$, and so on. Continuing inductively, we obtain

$$\sum_{\sigma \in \{+, -\}^n} (-1)^\sigma \text{Res} \left(\frac{u_0^2 e^{-i(\lambda_0 + \sigma \cdot \lambda)u_0}}{2^n u_0^n} \right), \quad (4.67)$$

where $\sigma \cdot \lambda = \sum_i \sigma_i \lambda_i$.

Proposition 4.4.2. *Let \mathfrak{X}_α be a polygon space. Then a cogenerator for its cohomology ring is $D_{\lambda_0}^2 \sum_{\sigma \in \mathcal{S}} (\lambda_0 + \sigma \cdot \lambda)^{n-1}$.*

Example 4.4.3. Take $\alpha = (1, 1, 1, 2, 2)$. Then according to the above proposition, the cogenerator for the cohomology ring of \mathfrak{X}_α is (up to an irrelevant overall constant)

$$\lambda_0^2 + \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 + \lambda_3^2 + \lambda_4^2 - 2\lambda_4\lambda_5 + \lambda_5^2 \quad (4.68)$$

Taking partial derivatives, we find the relations $\partial_4 + \partial_5 = 0$ and $\partial_0^2 - \partial_i^2 = 0$ for $i = 1, \dots, 5$. Hence the Poincaré polynomial is $1 + 4t^2 + t^4$ and the cohomology ring is

$$H^*(\mathfrak{X}_\alpha) = \frac{\mathbf{Q}[u_0^2, u_1, \dots, u_5]}{\langle u_0^2 - u_i^2, u_4 + u_5, \text{deg } 6 \rangle}. \quad (4.69)$$

4.4.3 Rank 3 Star Quivers

In this case, let u_1, \dots, u_n be generators of $(S^1)^n$ and let v_1, v_2 be generators of the maximal torus in $SU(3)$. Then $\varpi^2 = (v_1 - v_2)(2v_1 + v_2)(v_1 + 2v_2)$. The weights are given by $u_i - v_1, u_i - v_2, u_i + v_1 + v_2$ for each i . For each i , there are three possible residues with respect to u_i : $u_i \rightarrow v_1, u_i \rightarrow v_2$, and $u_i \rightarrow -v_1 - v_2$. The residue as $u_n \rightarrow v_1$ produces the term

$$\text{Res} \left(\frac{\varpi^2 e^{-i\mu \cdot v - i\lambda \cdot u - i\lambda_n v_1}}{(v_1 - v_2)(2v_1 + v_2)} \right). \quad (4.70)$$

The residue as $u_n \rightarrow v_2$ produces the term

$$\text{Res} \left(\frac{\varpi^2 e^{-i\mu \cdot v - i\lambda \cdot u - i\lambda_n v_2}}{(v_2 - v_1)(v_1 + 2v_2)} \right). \quad (4.71)$$

The residue as $u_n \rightarrow -v_1 - v_2$ produces the term

$$\text{Res} \left(\frac{\varpi^2 e^{-i\mu \cdot v - i\lambda \cdot u + i\lambda_n (v_1 + v_2)}}{(2v_1 + v_2)(v_1 + 2v_2)} \right). \quad (4.72)$$

Iterating as in the previous case, we see that the residue will be a sum over disjoint subsets $S_1, S_2, S_3 \subset [n]$, where $i \in S_1$ indicates that we take the residue $u_i \rightarrow v_1$, $i \in S_2$ denotes the residue $u_i \rightarrow v_2$, and $i \in S_3$ denotes the residue $u_i \rightarrow -v_1 - v_2$. The denominator of such a term is

$$(-1)^{|S_2|} (v_1 - v_2)^{|S_1| + |S_2|} (2v_1 + v_2)^{|S_1| + |S_3|} (v_1 + 2v_2)^{|S_2| + |S_3|}. \quad (4.73)$$

The argument of the exponential of such a term is

$$(S_1 \cdot \lambda - S_3 \cdot \lambda)v_1 + (S_2 \cdot \lambda - S_3 \cdot \lambda)v_2. \quad (4.74)$$

It remains to take the residues at $v_2 \rightarrow v_1$, $v_2 \rightarrow -2v_1$, and $v_2 \rightarrow -\frac{1}{2}v_1$, followed by the remaining residue at $v_1 \rightarrow 0$.

Proposition 4.4.4. *A cogenerator for the rank 3 star quiver is given by*

$$\sum_{S_1 \sqcup S_2 \sqcup S_3 = [n]} (R_1(S_1, S_2, S_3) + R_2(S_1, S_2, S_3) + R_3(S_1, S_2, S_3)) \quad (4.75)$$

where

$$\begin{aligned} R_1(S_1, S_2, S_3) &= \theta(S_1 \cdot \alpha + S_2 \cdot \alpha - 2S_3 \cdot \alpha) \theta(S_2 \cdot \alpha - S_3 \cdot \alpha) \\ &\quad \times \text{Res}_{v_1 \rightarrow 0} \text{Res}_{v_2 \rightarrow v_1} (f(S_1, S_2, S_3)) \end{aligned} \quad (4.76)$$

$$\begin{aligned} R_2(S_1, S_2, S_3) &= \theta(S_1 \cdot \alpha + S_3 \cdot \alpha - 2S_2 \cdot \alpha) \theta(S_2 \cdot \lambda - S_3 \cdot \lambda) \\ &\quad \times \text{Res}_{v_1 \rightarrow 0} \text{Res}_{v_2 \rightarrow -2v_1} (f(S_1, S_2, S_3)) \end{aligned} \quad (4.77)$$

$$\begin{aligned} R_3(S_1, S_2, S_3) &= \theta(2S_1 \cdot \alpha - S_2 \cdot \alpha - 2S_3 \cdot \alpha) \theta(S_2 \cdot \alpha - S_3 \cdot \alpha) \\ &\quad \times \text{Res}_{v_1 \rightarrow 0} \text{Res}_{v_2 \rightarrow -v_1/2} (f(S_1, S_2, S_3)) \end{aligned} \quad (4.78)$$

and $f(S_1, S_2, S_3)$ is the meromorphic function

$$f(S_1, S_2, S_3) = \frac{\exp(-i(\mu_1 + S_1 \cdot \lambda - S_3 \cdot \lambda)v_1 - i(\mu_2 + S_2 \cdot \lambda - S_3 \cdot \lambda)v_2)}{(-1)^{|S_2|}(v_1 - v_2)^{|S_1|+|S_2|-2}(2v_1 + v_2)^{|S_1|+|S_3|-2}(v_1 + 2v_2)^{|S_2|+|S_3|-2}} \quad (4.79)$$

The above expression, though explicit, is rather difficult to use directly by hand, but it is very easy to implement in computer algebra software. By varying the moment map level α across all the chambers, we produce cogenerators for the VGIT ring. By comparison with the Betti numbers from Theorem 3.3.27, we can then deduce Kirwan surjectivity. Using our computer implementation, we find the following.

Theorem 4.4.5. *For $n = 4, 5, 6$, the hyperkähler Kirwan map is surjective and its kernel is equal to the VGIT ideal. Moreover, the cohomology ring is level. For $n = 5$ the cohomology ring is*

$$H^*(\mathfrak{M}_\alpha) \cong \mathbf{Q}[v_1^2 + v_1^2 + (v_1 + v_2)^2, v_1^3 + v_2^3 - (v_1 + v_2)^3, u_1, \dots, u_5]/I \quad (4.80)$$

where I is the ideal generated by all degree 6 polynomials in the generators as well as the following degree 4 polynomials

$$4u_2u_3 + 4u_2u_4 + 4u_3u_4 + 4u_2u_5 + 4u_3u_5 + 4u_4u_5 + 3v_1^2 + 3v_2^2, \quad (4.81)$$

$$3u_1u_2 + u_2u_3 + u_2u_4 - 2u_3u_4 + u_2u_5 - 2u_3u_5 - 2u_4u_5, \quad (4.82)$$

$$3u_1u_3 + u_2u_3 - 2u_2u_4 + u_3u_4 - 2u_2u_5 + u_3u_5 - 2u_4u_5, \quad (4.83)$$

$$- 2u_2u_3 + 3u_1u_4 + u_2u_4 + u_3u_4 - 2u_2u_5 - 2u_3u_5 + u_4u_5, \quad (4.84)$$

$$- 2u_2u_3 - 2u_2u_4 - 2u_3u_4 + 3u_1u_5 + u_2u_5 + u_3u_5 + u_4u_5. \quad (4.85)$$

Remark 4.4.6. The computer calculation can easily be extended to higher values of n as well as higher ranks, but as the list of generators of the kernel of the Kirwan map grows very quickly we have decided only to state the results for $r = 3$ and $n = 5$.

Chapter 5

Star Quivers and Integrable Systems

5.1 Flags and Stars

5.1.1 The Springer Resolution

We begin by recalling the Springer resolution, which is of fundamental importance in geometric representation theory [CG97]. Let G be a simply connected complex semisimple group with Lie algebra \mathfrak{g} . The *nilpotent cone* $\mathcal{N} \subset \mathfrak{g}$ is a singular affine variety which carries a natural Poisson structure induced by the Lie-Poisson structure on \mathfrak{g} . Let \mathcal{B} be the set of all Borel sub-algebras of \mathfrak{g} . Then we define a variety $\tilde{\mathcal{N}}$ by

$$\tilde{\mathcal{N}} = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}. \quad (5.1)$$

Note that there is a natural projection map $\mu : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$.

Theorem 5.1.1. *The variety $\tilde{\mathcal{N}}$ is isomorphic to $T^*(G/B)$, where G/B is the flag variety of G (for B a fixed Borel subgroup). The map $\mu : T^*(G/B) \rightarrow \mathcal{N}$ is a resolution of singularities. Moreover, the map μ is the moment map for the canonical G -action on $T^*(G/B)$, and in particular, it is a Poisson map.*

Definition 5.1.2. The map $\mu : T^*(G/B) \rightarrow \mathcal{N}$ is called the *Springer resolution*.

In the case of type A (i.e., \mathfrak{sl}_n), the Springer resolution may be constructed using Nakajima quiver varieties.¹ Consider the type A quiver pictured in Figure 5.1, and let \mathfrak{X}_α be the Kähler quiver variety associated this quiver. There is a natural residual action of $U(r)$ given by the action on the left-most vertex. This action is Hamiltonian,

¹In fact, Nakajima varieties provide a large class of *conical symplectic resolutions*, of which the Springer resolution is a special case.

with moment map $\mu : \mathfrak{X}_\alpha \rightarrow \mathfrak{u}_r$. Similarly, the hyperkähler quotient \mathfrak{M}_α has a residual hyperhamiltonian action of $U(r)$, with real and complex moment maps given by

$$\mu_r([x, y, x_1, y_1, \dots, x_l, y_l]) = \frac{1}{2}xx^* - \frac{1}{2}y^*y \tag{5.2}$$

$$\mu_c([x, y, x_1, y_1, \dots, x_l, y_l]) = xy \tag{5.3}$$

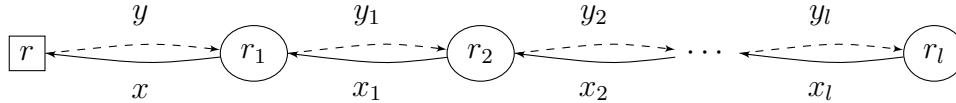


Figure 5.1: Type A quiver for the Springer resolution.

Proposition 5.1.3. *The moment map $\mu : \mathfrak{X}_\alpha \rightarrow \mathfrak{u}_r$ is an embedding and identifies \mathfrak{X}_α with a $U(r)$ -coadjoint orbit of flag type $(r_l, r_{l-1}, \dots, r_1, r)$. The hyperkähler quotient \mathfrak{M}_α is naturally identified with $T^*\mathfrak{X}_\alpha$. The complex moment map $\mu_c : \mathfrak{M}_\alpha \rightarrow \mathfrak{gl}_r$ takes values in the nilpotent cone, and in the case of a complete flag (i.e. $(1, 2, \dots, r)$), it is the Springer resolution for \mathfrak{sl}_r .*

Proof. This is a standard exercise. □

Remark 5.1.4. Note that since $\mu_c : \mathfrak{M}_\alpha \rightarrow \mathfrak{gl}_r$ is a moment map, it is in particular Poisson with respect to the Lie-Poisson structure on \mathfrak{gl}_r . That is,

$$\{\mu_{ij}, \mu_{kl}\} = \delta_{jk}\mu_{il} - \delta_{il}\mu_{kj}, \tag{5.4}$$

where μ_{ij} denotes the (i, j) th entry of the moment map μ , and $\{\cdot, \cdot\}$ is the Poisson bracket. This observation will be crucial when we try to understand the Poisson geometry of star quiver varieties in the sections that follow.

5.1.2 Star Quivers

Next we consider a more general class of star quivers than those studied in Chapter 3.

Definition 5.1.5. *A generalized star quiver is a quiver \mathcal{Q} consists of a single sink with n incoming paths, such that the dimensions of the vertices are strictly increasing as one moves toward the sink. We call the dimension of the sink the *rank*, and denote it by r . If the dimension vectors of all of the incoming paths are complete, i.e. $(r, r-1, r-2, \dots, 1)$ we call the star quiver *complete*, otherwise we call the star quiver *incomplete*.*

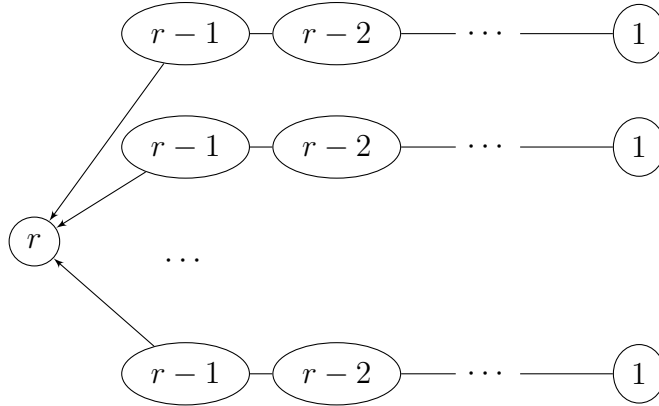


Figure 5.2: A complete star quiver.

It follows immediately from Proposition 5.1.3 that we have the following alternative description of \mathfrak{X}_α and \mathfrak{M}_α .

Proposition 5.1.6. *The Kähler quiver variety associated to a generalized star quiver may be identified, via reduction in stages, with the Kähler quotient of a product of $U(r)$ coadjoint orbits by the diagonal $SU(r)$ -action. Similarly, the hyperkähler quiver variety associated to a star quiver may be identified with the hyperkähler quotient of a product of cotangent bundles of flag varieties. In the case of a complete star quiver, \mathfrak{M}_α has complex dimension $(n - 2)r^2 - nr + 2$.*

Remark 5.1.7. This identification allows for an easy proof of hyperkähler Kirwan surjectivity for these spaces, see [HP05b, Remark 4.3].

5.1.3 Parabolic Bundles

Let X be a Riemann surface and $E \rightarrow X$ a rank r holomorphic vector bundle. Let $D = \sum_{i=1}^n a_i$ be a fixed divisor, and assume that the a_i are pairwise distinct. For each i , fix a flag type $n > n_1 > \dots > n_{l_i} > 0$.

Definition 5.1.8. A *parabolic structure* on E is a choice of flag

$$E_{a_i} = E_{a_i}^0 \supset E_{a_i}^1 \supset \dots \supset E_{a_i}^{l_i} \supset 0, \tag{5.5}$$

at each of the points a_i of the divisor. A *parabolic Higgs field* is a section

$$\phi \in H^0(X, \text{End}E \otimes \mathcal{O}_X(D) \otimes K_X) \tag{5.6}$$

such that at each point a_i of the divisor, we have

$$\phi_i(E_{a_i}^j) \subseteq E_{a_i}^j, \quad (5.7)$$

where ϕ_i is the residue of ϕ at the point $z = a_i$. We say that ϕ is *strictly parabolic* if $\phi_i(E_{a_i}^j) \subseteq E_{a_i}^{j+1}$ for all $i = 1, \dots, n, j = 0, \dots, l_i$.

Theorem 5.1.9. *Let \mathcal{Q} be a generalized star quiver of type (r, n) and let $D = \sum_{i=1}^n a_i$ be a divisor on \mathbf{P}^1 . Then the Kähler quiver variety \mathfrak{X}_α is naturally identified with a moduli space of parabolic structures on the trivial rank r vector bundle $E \rightarrow \mathbf{P}^1$, and the hyperkähler quiver variety \mathfrak{M}_α is naturally identified with a moduli space of strictly parabolic Higgs bundles on \mathbf{P}^1 , whose underlying bundle type is trivial.*

Proof. Let $E \rightarrow X$ be a trivial bundle. Then we may choose some trivialization such that $E \cong \mathbf{C}^r \times \mathbf{P}^1$. Hence under this trivialization, the set of possible parabolic structures is a product of flag varieties. The diagonal action of $GL(r)$ corresponds to a change of trivialization, and hence the space of parabolic structures is naturally identified with the GIT quotient of a product of flag varieties by the diagonal $GL(r)$ -action. By Proposition 5.1.6, this may be identified with a star quiver variety \mathfrak{X}_α (see Remark 2.2.19).

Now consider a product of cotangent bundles of flag varieties, $M = \prod_{i=1}^n T^*\mathfrak{F}_i$. Let μ_i denote the complex moment map on $T^*\mathfrak{F}_i$. Let z be an affine coordinate on \mathbf{P}^1 (assume that $z = 0$ is not a point in the divisor D). Then consider the \mathfrak{gl}_r -valued meromorphic form

$$\phi(z)dz = \sum_{i=1}^n \frac{\mu_i dz}{z - a_i}. \quad (5.8)$$

Then $\phi(z)dz$ is well-defined as a section of $\text{End}E \otimes K(D)$ on the vanishing set $V(\sum_i \mu_i) \subset M$, and hence defines a Higgs field.² Moreover, by Proposition 5.1.3, $\mu_i(x_i, y_i)$ is a nilpotent matrix preserving the flag defined by $x_i \in \mathfrak{F}_i$, and hence $\phi(z)$ is a strictly parabolic Higgs field. The residual action of $GL(r)$ corresponds to change of trivialization, so the GIT quotient $V(\sum_i \mu_i) // GL(r)$ is a moduli space of strictly parabolic Higgs bundles. But $\sum_i \mu_i$ is exactly the complex moment map for the diagonal $GL(r)$ -action, and hence this quotient may be identified with $M // U(r) \cong \mathfrak{M}_\alpha$. \square

Remark 5.1.10. This construction is a natural generalization of the rank 2 construction in [GM13], wherein it was also shown that the stability condition on quiver representations is equivalent to parabolic slope stability, for an appropriate choice of parabolic weights.

²The condition $\sum_i \mu_i = 0$ is easily seen to be equivalent to the condition that $\phi(z)dz$ is regular at $z = \infty$.

We suspect that this remains true in the higher rank case, but as we shall not need this result, we make no attempt to prove it.

Remark 5.1.11. The moduli space of strictly parabolic Higgs bundles is a symplectic leaf in the moduli space of parabolic Higgs bundles [LM10], and it is known to be completely integrable. However, we take the viewpoint that star quivers should be regarded as “toy models” of Hitchin systems, so we will instead prove integrability by direct calculation.

Remark 5.1.12. If we fix a line bundle L , then a *Hitchin pair* is a pair (E, ϕ) where E is a vector bundle over a curve and $\phi \in H^0(\text{End}E \otimes L)$.³ The moduli space of (stable) Hitchin pairs carries a natural family of Poisson structures which are all completely integrable with respect to the Hitchin map [Bot95, Mar94]. The construction of this section can be generalized to give a construction of Hitchin pairs (over \mathbf{P}^1) associated to the quiver pictured in Figure 5.3. However, it is not clear whether this construction preserves Poisson structures.

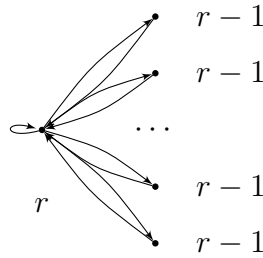


Figure 5.3: A quiver representing a space of Hitchin pairs.

5.2 Complete Integrability

5.2.1 The Hitchin Fibration

In this section, we describe the Hitchin map on \mathfrak{M}_α and show by direct calculation that it is a coisotropic fibration.

Definition 5.2.1. The *Hitchin base* is

$$B = \bigoplus_{i=1}^r H^0(\mathbf{P}^1, K(D)^i), \tag{5.9}$$

³Of particular interest is the case $L = -K_X$, which corresponds to *co-Higgs fields* [Ray11]. These may be identified with holomorphic bundles in the sense of generalized complex geometry [Gua11].

where K is the canonical sheaf of \mathbf{P}^1 . The *Hitchin map* $h : H^0(\mathbf{P}^1, \text{End}E \otimes K(D)) \rightarrow B$ is the map

$$\phi \mapsto (\text{Tr}(\phi), \text{Tr}(\phi^2), \dots, \text{Tr}(\phi^r)). \quad (5.10)$$

Note that this map is invariant under $\phi \mapsto g^{-1}\phi g$, and hence descends to a well defined map on \mathfrak{M}_α , which we also denote by h .

Lemma 5.2.2. *On \mathfrak{M}_α , the Hitchin map takes values in $B_0 \subset B$, where*

$$B_0 = \bigoplus_{i=1}^r H^0(\mathbf{P}^1, K^i \otimes \mathcal{O}(D)^{i-1}). \quad (5.11)$$

Proof. Since the residue of ϕ is strictly parabolic at each of the marked points, it is in particular nilpotent, and hence the poles of $\text{Tr}(\phi^i)$ have order at most $i-1$. Hence $\text{Tr}(\phi^i)$ may be naturally identified with a section of $K(D)^i \otimes \mathcal{O}(-D) \cong K^i \mathcal{O}(D)^{i-1}$. \square

Proposition 5.2.3. *We have $\dim B_0 = \frac{1}{2}((n-2)r^2 - nr + 2)$. In the case of a complete flag quiver variety \mathfrak{M}_α , we have $\dim \mathfrak{M}_\alpha = 2 \dim B_0$.*

Proof. The line bundle $K^i \otimes \mathcal{O}(D)^{i-1}$ has degree $-2i + n(i-1)$, and hence

$$h^0(K^i \otimes \mathcal{O}(D)^{i-1}) = (n-2)i - n + 1. \quad (5.12)$$

Summing these from $i=2$ to $i=r$, we obtain the result. \square

Next we will show that the Hitchin map defines a coisotropic fibration. For convenience, in this section we will work on $M = \prod_{i=1}^n T^*\mathfrak{F}_i$. Define the components I_m of the Hitchin map via

$$I_m(dz)^m = \text{Tr}((\phi(z)dz)^m) \in H^0(\mathbf{P}^1, K(D)^m \otimes \mathcal{O}(D)^{m-1}). \quad (5.13)$$

By construction, the I_m are rational functions in z , and hence we may expand them as formal power series $I_m(z) = \sum_n I_m^n z^n$. (By a change of coordinate if necessary, we can assume that none of the a_i is zero, so that $\phi(z)$ is regular at 0.) It will be convenient to treat z as a formal parameter, so that $I_m(z)$ can be thought of as a generating function for the coefficients I_m^n .⁴ Trivially, we have the following.

Lemma 5.2.4. *We have $\{I_m^i, I_n^j\} = 0$ for all i, j if and only if $\{I_m(z), I_n(w)\} = 0$ as an element of $\mathbf{C}[[z, w]]$, where the formal parameters z, w are Casimirs of the Poisson bracket.*

⁴We think of the coefficients I_m^n as giving coordinates on the Hitchin base B . Since they are the Taylor coefficients of a rational function, only finitely many of them are algebraically independent.

Next, we define a matrix-valued formal power series as follows:

$$\Delta(z, w) = \frac{\phi(z)}{w - z} + \frac{\phi(w)}{z - w}. \quad (5.14)$$

This matrix encodes the Poisson structure, in the following sense.

Lemma 5.2.5. *We have $\{\phi_{ij}(z), \phi_{kl}(w)\} = \delta_{jk}\Delta_{il}(z, w) - \delta_{il}\Delta_{kj}(z, w)$.*

Proof. We simply compute

$$\{\phi_{ij}(z), \phi_{kl}(w)\} = \sum_{m,n} \frac{\{(\mu_m)_{ij}, (\mu_n)_{kl}\}}{(z - a_m)(w - a_n)} \quad (5.15)$$

$$= \sum_{m,n} \frac{\delta_{mn}\delta_{jk}(\mu_m)_{il} - \delta_{mn}\delta_{il}(\mu_m)_{kj}}{(z - a_m)(w - a_n)} \quad (5.16)$$

$$= \sum_m \frac{\delta_{jk}(\mu_m)_{il} - \delta_{il}(\mu_m)_{kj}}{(z - a_m)(w - a_m)} \quad (5.17)$$

$$= \delta_{jk}\Delta_{il}(z, w) - \delta_{il}\Delta_{kj}(z, w), \quad (5.18)$$

as claimed. \square

Proposition 5.2.6. *The components of the Hitchin map Poisson commute, hence the Hitchin map $h : \mathfrak{M}_\alpha \rightarrow B_0$ is a coisotropic fibration.*

Proof. First, note that

$$\mathrm{Tr}(\phi(z)^m) = \sum_{i_1, \dots, i_m} \phi_{i_1 i_2}(z) \phi_{i_2 i_3}(z) \cdots \phi_{i_m i_1}(z). \quad (5.19)$$

Hence, taking the Poisson bracket $\{I_m(z), I_n(w)\}$, expanding, and applying the Leibniz rule, we have

$$\begin{aligned} \{I_m(z), I_n(w)\} &= \sum_{\substack{a, i_1, \dots, i_m \\ b, j_1, \dots, j_n}} \phi_{i_1 i_2}(z) \cdots \widehat{\phi_{i_a i_{a+1}}(z)} \cdots \phi_{i_m i_1}(z) \\ &\quad \times \phi_{j_1 j_2}(w) \cdots \widehat{\phi_{j_b j_{b+1}}(w)} \cdots \phi_{j_n j_1}(w) \\ &\quad \times (\delta_{i_{a+1} j_b} \Delta_{i_a j_{b+1}}(z, w) - \delta_{i_a j_{b+1}} \Delta_{j_b i_{a+1}}(z, w)) \end{aligned} \quad (5.20)$$

$$\begin{aligned} &= \sum_{a,b} \mathrm{Tr}(\phi^{m-1}(z) \Delta(z, w) \phi^{n-1}(w)) \\ &\quad - \sum_{a,b} \mathrm{Tr}(\phi^{n-1}(w) \Delta(z, w) \phi^{m-1}(z)) \end{aligned} \quad (5.21)$$

$$= mn \mathrm{Tr}(\Delta(z, w) [\phi(w)^{n-1}, \phi(z)^{m-1}]). \quad (5.22)$$

Now recall that $\Delta(z, w) = \phi(z)/(w - z) + \phi(w)/(z - w)$. This is a sum of two terms, one commuting with $\phi(z)$ and the other commuting with $\phi(w)$, hence the above trace is identically zero. \square

5.2.2 Spectral Curves

It remains to show that the generic fibers of $h : \mathfrak{M}_\alpha \rightarrow B_0$ are Lagrangian subvarieties. To do this, we will use the standard technique of spectral curves [Hit87a, BNR89]. Let $L \rightarrow K(D)$ be the pullback of $K(D)$ to the total space of $K(D)$. Then L has a tautological section, which we denote by λ .

Definition 5.2.7. Let $b \in B$ and denote the components of b by b_i for $i = 1, \dots, n$. Then we define the *spectral curve* $X_b \subset K(D)$ to be the subvariety defined by

$$X_b = \{(\lambda, z) \in K(D) \mid \lambda^r + b_1(z)\lambda^{r-1} + \dots + b_r(z) = 0\}. \quad (5.23)$$

Proposition 5.2.8. *Suppose that $n \geq 3$, and let $b \in B_0$ be generic. Then the spectral curve X_b is smooth and of genus $g(X_b) = \frac{1}{2}((n-2)r^2 - nr + 2)$.*

Proof. We will adapt Hitchin's argument [Hit87a] to our setting. By the general Bertini theorem [Har77, Chapter III], a generic spectral curve will be smooth except possibly at the base points of the linear system defined by B_0 . By construction, B_0 is the subspace of $\bigoplus_{i=2}^r H^0(\mathbf{P}^1, K(D)^i)$ consisting of sections that vanish on D , so the base locus of B_0 is exactly the divisor D . Let $f(\lambda, z)$ be the equation defining $X_b \subset S$. At any point $a_i \in D$, all of the coefficients of the characteristic polynomial vanish, so that $\lambda = 0$. At such a point we have $df = db_r$, and hence $df \neq 0$ as long as b_r has only simple zeroes on D . Since this is the generic behavior, we find that a generic spectral curve is smooth.

Let S be the complex surface $S = \mathbf{P}(\mathcal{O} \oplus K(D))$, which is a compactification of the total space of $K(D)$. Then $X_b \subset S$, and there is a curve $C \subset S$ isomorphic to \mathbf{P}^1 corresponding to the zero section of $K(D)$. Note that as divisors on S , we have $X_b \sim rC$. From the basic theory of ruled surfaces [Har77, Chapter V], $C.C = \deg(K(D)) = n - 2$, and hence by the adjunction formula $C.K_S = -n$. Applying the adjunction formula to X_b , we find

$$2g(X_b) - 2 = X_b.X_b + X_b.K_S = r^2C.C + rC.K_S = r^2(n-2) - rn, \quad (5.24)$$

and hence $g(X_b) = \frac{1}{2}((n-2)r^2 - nr + 2)$. \square

Remark 5.2.9. For incomplete quivers, it is not necessarily the case that generic spectral curves are smooth. In the rank 3 incomplete case studied in Chapter 3, one may check explicitly that $\det(\phi)$ vanishes to *second order* on D , and hence the spectral curve is always singular on D . This complicates the situation, and for this reason we restrict our attention to the complete flag case.

Theorem 5.2.10. *Let $\mathfrak{M}_\alpha(r, n)$ be the complete flag star quiver of rank r with n incoming paths. Then the Hitchin map $h : \mathfrak{M}_\alpha(r, n) \rightarrow B_0$ is a coisotropic fibration whose generic fibers are Lagrangian subvarieties of $\mathfrak{M}_\alpha(r, n)$. In particular, $\mathfrak{M}_\alpha(r, n)$ is an algebraic completely integrable system.*

Proof. By Lemma 5.2.2 and Proposition 5.2.6, this is a coisotropic fibration whose base is half the dimension of $\mathfrak{M}_\alpha(r, n)$, so we just have to show that dh generically has full rank. Since \mathfrak{M}_α may be identified with a GIT quotient of $Z \subset \prod_{i=1}^n T^*\mathfrak{F}_i$, it suffices to show that $h : Z \rightarrow B_0$ generically has full rank. Let $b \in B_0$ be a generic point such that the spectral curve X_b is smooth. Let $p : X_b \rightarrow \mathbf{P}^1$ be the projection. By standard arguments (and in particular, those of [LM10]), any line bundle $L \in \text{Jac}(X_b)$ determines a degree 0 vector bundle $E := p_*L$ on \mathbf{P}^1 , together with a canonical Higgs field $\phi \in H^0(\mathbf{P}^1, \text{End}E \otimes K(D))$. For a generic L , the induced bundle E will be trivial, and choosing a trivialization we may write

$$\phi(z)dz = \sum_{i=1}^n \frac{\phi_i dz}{z - a_i}. \tag{5.25}$$

Since the coefficients of the characteristic polynomial vanish on D (since $b \in B_0 \subset B$), the residues ϕ_i are nilpotent. Hence by the Proposition 5.1.3, we may find points $m_i \in T^*\mathfrak{F}_i$ such that for each i , $\phi_i = \mu_i(m_i)$. Since ϕ is a section of $\text{End}E \otimes K(D)$, the sum of the residues is 0, and hence we have $(m_1, \dots, m_n) \in Z$. In particular, the fiber $h^{-1}(b) \subset Z$ is non-empty.

This shows that the image of $h : Z \rightarrow B_0$ contains a Zariski open and hence dense subset of B_0 . By Sard's theorem, the image of $h : Z \rightarrow B_0$ must therefore contain at least one regular value. Hence the set of $z \in Z$ such that dh has full rank is non-empty, and since this set is Zariski open, it is dense in Z . \square

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