Probabilisic Methods in Combinatorics
Exercise Sheet 9

Question 1. Given two sets X and Y of size n and m respectively. Show using the basic probabilistic method that if $m > \binom{n}{2}$ then an injective function $f : X \to Y$ exists.

Show that, if we instead use the Local Lemma, then we only need $m > 2en$.

Question 2. Does the Local Lemma (say the symmetric version) hold for an infinite set of events? More precisely can we conclude from the same assumptions that the probability of all the events not happening is non-zero?

(* What if we just ask that there exists some point in the probability space where they all don't happen?)

Question 3. Suppose we are given n pairs of points in some graph G, and for each pair x_i, y_i a collection F_i of a least m paths betweens x_i and y_i. Suppose further that for every i and j, each path in F_i shares an edge with at most k paths in F_j.

Show that if $m \geq 6nk$ it is possible to find a disjoint family of paths joining the pairs together.

Question 4. The *Van der Waerden number* $W(k, r)$ is the least number n such that any r-colouring of $[n]$ contains an arithmetic progression of length at least k. Use the union bound or first moment method to prove a lower bound on $W(k, r)$.

How much can this be improved by using instead the Local Lemma?

(* Prove that the numbers $W(k, r)$ actually exist, i.e, any upper bound. How does this compare to the lower bounds?)

Question 5. Let G be a d-regular graph with girth at least 6. Show that we can colour G with $cd^{\frac{4}{3}}$ colours, for some c sufficiently large, such that no cycle is 2-coloured (not insisting the the colouring is proper).

(Hint: Consider the set of events, for each induced path with 4 edges, that the path is 2-coloured.)

Show further that we can find such a colouring which is proper.

(Hint: We want to also consider the events that each edge is improperly coloured, so we will want to use the non-symmetric local lemma. A good choice for the weightings x and y for the two types of events will be just twice the probability that the event happens)